COMPLEXES OF EXACT HERMITIAN CUBES AND THE ZAGIER CONJECTURE

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Let F be an algebraic number field. Let $\Sigma_F = \mathrm{Hom}(F, \mathbb{C})$ and X_F the free abelian group generated by Σ_F. Let ι denote the complex conjugation on Σ_F and X_F. Then one can define a map called regulator

$$\rho : K_{2m-1}(F) \to (X_F \otimes \mathbb{R}(m-1))^{\bar{\iota}=\mathrm{id}}.$$

In [9], Zagier conjectured that the image of ρ can be expressed in terms of the m-th polylogarithm function

$$Li_m(z) = \sum_{k \geq 1} \frac{z^k}{k^m}, \ |z| < 1.$$

More precisely, he conjectured that $K_{2m-1}(F)$ is isomorphic modulo torsion to a certain subquotient $\mathcal{B}_m(F)$ of $\mathbb{Z}[F^\times - \{1\}]$ called the *m-th Bloch group* of F, and that the composite of the isomorphism with the regulator ρ can be expressed by values of $Li_m(z)$ at algebraic numbers. The conjecture was solved affirmatively for $m \leq 3$, but it is still open when $m \geq 4$.

As for the construction of the map $\mathcal{B}_m(F) \to K_{2m-1}(F)_\mathbb{Q}$, two methods are well-known. Beilinson and Deligne showed that the above map can be constructed under the existence of an appropriate category of mixed Tate motives [1]. On the other hand, de Jeu also constructed the map by using the wedge complexes developed by himself [5]. In this note, we will present an alternative approach. The main tool here is higher Bott-Chern forms developed by Burgos and Wang [4].

1. The Bloch group

First we introduce a one-valued version of the polylogarithm function:

$$P_m(z) = \mathcal{R}_m \left(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} (\log |z|)^k Li_{m-k}(z) \right),$$

where $\mathcal{R}_m$ is Re or Im according as m is odd or even and B_k is the k-th Bernoulli number. This is a one-valued real analytic function on $\mathbb{P}^1_\mathbb{C} - \{0, 1, \infty\}$.

In this note, we are interested in K-theory in rational coefficients, therefore we will define Bloch group with rational coefficients. Let $F^\times_{\mathbb{Q}} = F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and define a map

$$\beta_2 : \mathbb{Q}[F^\times - \{1\}] \to F^\times_{\mathbb{Q}} \wedge F^\times_{\mathbb{Q}}$$

by $[x] \mapsto x \wedge (1-x)$. Set $\mathcal{A}_2(F) = \operatorname{Ker} \beta_2$ and

$$\mathcal{C}_2(F) = \left\{ \sum_i n_i[x_i] \in \mathcal{A}_2(F); \sum_i n_i P_2(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.$$

It turns out that $\mathcal{C}_2(F)$ is generated by

$$[x] + [y] + [\tfrac{1-x}{1-xy}] + [1-xy] + [\tfrac{1-y}{1-xy}]$$

for all $x, y \in F^\times - \{1\}$ with $xy \neq 1$. The quotient group $\mathcal{B}_2(F) = \mathcal{A}_2(F)/\mathcal{C}_2(F)$ is called *Bloch group* of F.

Suppose $m \geq 3$ and subgroups $\mathcal{C}_{m-1}(F) \subset \mathcal{A}_{m-1}(F) \subset \mathbb{Q}[F^\times - \{1\}]$ are given. Define a map

$$\beta_m : \mathbb{Q}[F^\times - \{1\}] \to F^\times_{\mathbb{Q}} \otimes \big(\mathbb{Q}[F^\times - \{1\}]/\mathcal{C}_{m-1}(F)\big)$$

by $\beta_m([x]) = x \otimes [x]$. Set $\mathcal{A}_m(F) = \operatorname{Ker} \beta_m$ and

$$\mathcal{C}_m(F) = \left\{ \sum_i n_i[x_i] \in \mathcal{A}_m(F); \sum_i n_i P_m(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.$$

The quotient group $\mathcal{B}_m(F) = \mathcal{A}_m(F)/\mathcal{C}_m(F)$ is called *m-th Bloch group* of F.

Zagier Conjecture ([9]). *For $m \geq 2$, the rational algebraic K-theory $K_{2m-1}(F)_{\mathbb{Q}}$ is isomorphic to $\mathcal{B}_m(F)$, and the composite*

$$\mathcal{B}_m(F) \simeq K_{2m-1}(F)_{\mathbb{Q}} \xrightarrow{\rho} (X_F \otimes \mathbb{R}(m-1))^{\bar{\iota}=\mathrm{id}}$$

is written as

$$\sum_i n_i[x_i] \mapsto \left((\sqrt{-1})^{\alpha_m} \sum_i n_i P_m(x_i^\sigma) \right)_{\sigma \in \Sigma_F},$$

where α_m is 0 or 1 according as m is odd or even.

2. The complex of exact hermitian cubes

Let $< -1, 0, 1 >$ be the ordered set consisting of three elements and $< -1, 0, 1 >^n$ its n-th power. For a small exact category $\mathfrak{A}$ with a fixed zero object 0, a functor $\mathcal{F} :< -1, 0, 1 >^n \to \mathfrak{A}$ is called an *n-cube* of $\mathfrak{A}$. Let $\mathcal{F}_{\alpha_1, \cdots, \alpha_n}$ denote the image of an object $(\alpha_1, \cdots, \alpha_n)$ of $< -1, 0, 1 >^n$. For integers $1 \leq i \leq n$ and $-1 \leq j \leq 1$, an $(n-1)$-cube $\partial_i^j \mathcal{F}$ is defined by $(\partial_i^j \mathcal{F})_{\alpha_1, \cdots, \alpha_{n-1}} = \mathcal{F}_{\alpha_1, \cdots, \alpha_{i-1}, j, \alpha_i, \cdots, \alpha_{n-1}}$. It is called a *face* of $\mathcal{F}$. For an object α of $< -1, 0, 1 >^{n-1}$ and an integer $1 \leq i \leq n$, a 1-cube $\partial_{i^c}^{\alpha}\mathcal{F}$ called an *edge* of $\mathcal{F}$ is defined by

$$\mathcal{F}_{\alpha_1,\cdots,\alpha_{i-1},-1,\alpha_i,\cdots,\alpha_{n-1}} \to \mathcal{F}_{\alpha_1,\cdots,\alpha_{i-1},0,\alpha_i,\cdots,\alpha_{n-1}} \to \mathcal{F}_{\alpha_1,\cdots,\alpha_{i-1},1,\alpha_i,\cdots,\alpha_{n-1}}.$$

An n-cube $\mathcal{F}$ is said to be *exact* if all edges of $\mathcal{F}$ are short exact sequences.

Let $C_n\mathfrak{A}$ denote the set of all exact n-cubes of $\mathfrak{A}$. If $\mathcal{F}$ is an exact n-cube, then any face $\partial_i^j\mathcal{F}$ is also exact. Hence ∂_i^j induces a map

$$\partial_i^j : C_n\mathfrak{A} \to C_{n-1}\mathfrak{A}.$$

Let $\mathcal{F}$ be an exact n-cube of $\mathfrak{A}$. For an integer $1 \leq i \leq n+1$, let $s_i^1\mathcal{F}$ be an exact $(n+1)$-cube such that its edge $\partial_{i^c}^{\alpha}(s_i^1\mathcal{F})$ is $\mathcal{F}_\alpha \xrightarrow{\text{id}} \mathcal{F}_\alpha \to 0$. Similarly, let $s_i^{-1}\mathcal{F}$ be an exact $(n+1)$-cube such that $\partial_{i^c}^{\alpha}(s_i^{-1}\mathcal{F})$ is $0 \to \mathcal{F}_\alpha \xrightarrow{\text{id}} \mathcal{F}_\alpha$. An exact cube written as $s_i^j\mathcal{F}$ is said to be *degenerate*.

Let $\mathbb{Q}C_n\mathfrak{A}$ be the free $\mathbb{Q}$-module generated by $C_n\mathfrak{A}$ and $D_n \subset \mathbb{Q}C_n\mathfrak{A}$ the submodule generated by all degenerate exact n-cubes. Let $\widetilde{\mathbb{Q}}C_n\mathfrak{A} = \mathbb{Q}C_n\mathfrak{A}/D_n$ and

$$\partial = \sum_{i=1}^{n}\sum_{j=-1}^{1}(-1)^{i+j+1}\partial_i^j : \widetilde{\mathbb{Q}}C_n\mathfrak{A} \to \widetilde{\mathbb{Q}}C_{n-1}\mathfrak{A}.$$

Then $\widetilde{\mathbb{Q}}C_*\mathfrak{A} = (\widetilde{\mathbb{Q}}C_n\mathfrak{A}, \partial)$ becomes a homological complex.

Theorem 2.1 ([6]). *The homology of $(\widetilde{\mathbb{Q}}C_n\mathfrak{A}, \partial)$ is isomorphic to the rational algebraic K-theory of $\mathfrak{A}$:*

$$H_n(\widetilde{\mathbb{Q}}C_*\mathfrak{A}, \partial) \simeq K_n(\mathfrak{A})_{\mathbb{Q}}.$$

This isomorphism preserves products on the both sides if $\mathfrak{A}$ is equipped with a strictly associative tensor product.

3. The higher Bott-Chern forms

Let M be a compact complex algebraic manifold, namely, the analytic space consisting of all $\mathbb{C}$-valued points of a smooth proper algebraic variety over $\mathbb{C}$. Let $\mathcal{E}_{\mathbb{R}}^p(M)$ be the space of real smooth differential forms of degree p on M and $\mathcal{E}^p(M) = \mathcal{E}_{\mathbb{R}}^p(M) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{E}^{p,q}(M)$ be the space of complex differential forms of type (p,q) on M. Set

$$\mathcal{D}^n(M,p) = \begin{cases} \mathcal{E}_{\mathbb{R}}^{n-1}(M)(p-1) \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p, q'<p}} \mathcal{E}^{p',q'}(M), & n < 2p, \\ \mathcal{E}_{\mathbb{R}}^{2p}(M)(p) \cap \mathcal{E}^{p,p}(M) \cap \operatorname{Ker} d, & n = 2p, \\ 0, & n > 2p \end{cases}$$

and define a differential $d_{\mathcal{D}} : \mathcal{D}^n(M,p) \to \mathcal{D}^{n+1}(M,p)$ by

$$d_{\mathcal{D}}(\omega) = \begin{cases} -\pi(d\omega), & n < 2p-1, \\ -2\partial\overline{\partial}\omega, & n = 2p-1, \\ 0, & n > 2p-1, \end{cases}$$

where $\pi : \mathcal{E}^n(M) \to \mathcal{D}^n(M,p)$ is the canonical projection. Then it is shown in [3, Thm.2.6] that $(\mathcal{D}^*(M,p), d_{\mathcal{D}})$ becomes a complex of $\mathbb{R}$-vector spaces computing the real Deligne cohomology, that is, for $n \leq 2p$ we have

$$H^n(\mathcal{D}^*(M,p), d_{\mathcal{D}}) \simeq H^n_{\mathcal{D}}(M, \mathbb{R}(p)).$$

By a *hermitian vector bundle* $\overline{E} = (E, h)$ on M we mean an algebraic vector bundle E on M with a smooth hermitian metric h. Let $K_{\overline{E}}$ denote the curvature form of the unique connection on $\overline{E}$ that is compatible with both the metric and the complex structure. The Chern form of $\overline{E}$ is defined as

$$\mathrm{ch}_0(\overline{E}) = \mathrm{Tr}(\exp(-K_{\overline{E}})) \in \bigoplus_p \mathcal{D}^{2p}(M,p).$$

An *exact hermitian n-cube* on M is an exact n-cube made of hermitian vector bundles on M. Let $\mathcal{F} = \{\overline{E}_\alpha\}$ be an exact hermitian n-cube on M. We call $\mathcal{F}$ an *emi-n-cube* if the metric on any $\overline{E}_\alpha$ with $\alpha_i = 1$ coincides with the metric induced from $\overline{E}_{\alpha_1,\cdots,\alpha_{i-1},0,\alpha_{i+1},\cdots,\alpha_n}$ by the surjection $\overline{E}_{\alpha_1,\cdots,\alpha_{i-1},0,\alpha_{i+1},\cdots,\alpha_n} \to \overline{E}_\alpha$.

For an emi-1-cube $\mathcal{E} : \overline{E}_{-1} \to \overline{E}_0 \to \overline{E}_1$, a canonical way of constructing a hermitian vector bundle $\mathrm{tr}_1\,\mathcal{E}$ on $M \times \mathbb{P}^1$ connecting $\overline{E}_0$ with $\overline{E}_{-1} \oplus \overline{E}_1$ is given in [4]. If $(x : y)$ denotes the homogeneous coordinate of $\mathbb{P}^1$ and $z = x/y$, then $\mathrm{tr}_1\,\mathcal{E}$ fulfills the conditions $(\mathrm{tr}_1\,\mathcal{E})|_{z=0} \simeq \overline{E}_0$ and $(\mathrm{tr}_1\,\mathcal{E})|_{z=\infty} \simeq \overline{E}_{-1} \oplus \overline{E}_1$. For an emi-$n$-cube $\mathcal{F}$, let $\mathrm{tr}_1(\mathcal{F})$ be an emi-$(n-1)$-cube on $M \times \mathbb{P}^1$ defined by $\mathrm{tr}_1(\mathcal{F})_\alpha = \mathrm{tr}_1(\partial^\alpha_{n^c}(\mathcal{F}))$ for any object α of $< -1, 0, 1 >^{n-1}$, and $\mathrm{tr}_n(\mathcal{F})$ a hermitian vector bundle on $M \times (\mathbb{P}^1)^n$ given by taking tr_1 n times.

Let $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ be the i-th projection and $z_i = \pi_i^* z$. Let $\mathfrak{S}_n$ be the symmetric group of n-letters. For an integer $1 \leq i \leq n$, a differential form with logarithmic poles S^i_n on $(\mathbb{P}^1)^n$ is defined as

$$S^i_n = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log|z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}},$$

and T_n is defined as

$$T_n = \frac{(-1)^n}{2n!} \sum_{i=1}^{n} (-1)^i S^i_n.$$

Let us define the Bott-Chern form of an emi-n-cube $\mathcal{F}$ as

$$\mathrm{ch}_n(\mathcal{F}) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{(\mathbb{P}^1)^n} \mathrm{ch}_0(\mathrm{tr}_n(\mathcal{F})) \wedge T_n \in \bigoplus_p \mathcal{D}^{2p-n}(M,p).$$

A process to make an emi-n-cube $\lambda\mathcal{F}$ from an arbitrary exact hermitian n-cube $\mathcal{F}$ has been given in [4]. By virtue of this process, we can extend the definition of the Bott-Chern form to an arbitrary exact hermitian n-cube.

Definition 3.1. *The Bott-Chern form of an exact hermitian n-cube $\mathcal{F}$ is an element of $\bigoplus_p \mathcal{D}^{2p-n}(M,p)$ defined as*

$$\mathrm{ch}_n(\mathcal{F}) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{(\mathbb{P}^1)^n} \mathrm{ch}_0(\mathrm{tr}_n(\lambda\mathcal{F})) \wedge T_n.$$

Theorem 3.2 ([4]). *Let $\widehat{\mathcal{P}}(M)$ denote the category of hermitian vector bundles on M and let $\widetilde{\mathbb{Q}}\widehat{C}_*(M) = \widetilde{\mathbb{Q}}C_*\widehat{\mathcal{P}}(M)$. Then the higher Bott-Chern forms induce a homomorphism of complexes*

$$\mathrm{ch} : \widetilde{\mathbb{Q}}\widehat{C}_*(M) \to \bigoplus_p \mathcal{D}^*(M,p)[2p].$$

Moreover, the following map

$$K_n(M)_{\mathbb{Q}} \simeq H_n(\widetilde{\mathbb{Q}}\widehat{C}_*(M)) \xrightarrow{\mathrm{ch}} \bigoplus_p H_{\mathcal{D}}^{2p-n}(M,\mathbb{R}(p))$$

agrees with the higher Chern character with values in the real Deligne cohomology.

Let X be a smooth proper variety defined over $\mathbb{Q}$. By a *hermitian vector bundle* $\overline{E} = (E,h)$ on X, we mean a vector bundle E on X with an ι-invariant smooth hermitian metric h on the holomorphic vector bundle $E(\mathbb{C})$. In the same way as above, one can consider an exact hermitian n-cube $\mathcal{F}$ on X and define its Bott-Chern form $\mathrm{ch}_n(\mathcal{F})$. Let $\widetilde{\mathbb{Q}}\widehat{C}_*(X)$ denote the complex of exact hermitian cubes on X. Then we have an isomorphism preserving products

$$K_*(X)_{\mathbb{Q}} \simeq H_*(\widetilde{\mathbb{Q}}\widehat{C}_*(X))$$

and the Bott-Chern forms leads to the regulator map for X:

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \bigoplus_p H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) := \bigoplus_p H_{\mathcal{D}}^{2p-n}(X(\mathbb{C}),\mathbb{R}(p))^{\bar{\iota}=\mathrm{id}}$$

4. The main theorem

We introduce a new real analytic function on $\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ coming from the polylogarithm function. Let

$$I_m(z) = \sum_{j=0}^{m-1} \frac{(-\log|z|)^j}{j!} Li_{m-j}(z),$$

and

$$L_m(z) = \mathfrak{R}_m \left(\sum_{0 \le 2r < m} \frac{(-1)^r}{2^r r!} \frac{(\log|z|)^{2r}}{(2m-3)(2m-5)\cdots(2m-2r-1)} I_{m-2r}(z) \right).$$

When $m \le 3$, $L_m(z)$ is equal to $P_m(z)$, but is not so when $m \ge 4$. However, for $\sum_i n_i[x_i] \in \mathcal{A}_m(F)$, we have

$$\sum_i n_i L_m(x_i^\sigma) = \sum_i n_i P_m(x_i^\sigma).$$

The function $L_m(z)$ satisfies the following differential equation, which is obtained by a direct calculation.

Theorem 4.1. [8, Thm.5.4] *If $m \ge 2$, then*

$$(-1)^m dL_m(z) = Im\left(\frac{dz}{z}\right) L_{m-1}(z) - \frac{\sqrt{-1}}{2m-3} \log|z| (\bar{\partial} L_{m-1}(z) - \partial L_{m-1}(z)).$$

Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$ and let z be the absolute coordinate of X. Hence we can write $X = \operatorname{Spec} \mathbb{Q}[z, 1/z, 1/(1-z)]$. We want to apply the theory of Bott-Chern forms to X. But since X is not proper over $\mathbb{Q}$, we can not apply directly the results mentioned in the preceding section.

For an exact hermitian n-cube $\mathcal{F}$ on X, one can define $\operatorname{ch}_n(\mathcal{F})$ as a differential form on $X(\mathbb{C})$ by the same integral expression. Moreover, for $n \ge 2$, we have $d \operatorname{ch}_n(\mathcal{F}) = -\operatorname{ch}_{n-1}(\partial\mathcal{F})$. Hence when $n \ge 2$, $\operatorname{ch}_n(\mathcal{F})$ induces a map from the rational K-theory of X to the de Rham cohomology of $X(\mathbb{C})$.

For $f \in \mathcal{O}_X^\times$, let $\langle f \rangle$ be an exact hermitian 1-cube on X given as

$$0 \to \overline{\mathcal{O}_X} \xrightarrow{f} \overline{\mathcal{O}_X}.$$

Proposition 4.2. [8, Prop.6.1] *There exists an element $h_n(z) \in \widetilde{\mathbb{Q}}\widehat{C}_{2n-1}(X)$ for each $n \ge 1$ satisfying the following conditions:*

(1) $h_1(z) = \langle z \rangle$.
(2) $\partial h_n(z) = \sum_{i=1}^{n-1} h_i(z) \otimes h_{n-i}(z)$.
(3) $\operatorname{ch}_{2n-1}(h_n(z)) = 0$ *for* $n \ge 2$.

Theorem 4.3. [8, Thm.6.2] *For each $m \ge 1$, there exists $\mathcal{L}_m(z) \in \widetilde{\mathbb{Q}}\widehat{C}_{2m-1}(X)$ satisfying the following conditions:*

(1) $\mathcal{L}_1(z) = -2\langle 1-z \rangle$.
(2) $\partial \mathcal{L}_m(z) = \sum_{i=1}^{m-1} 2^i h_i(z) \otimes \mathcal{L}_{m-i}(z)$ *for* $m \ge 2$.
(3) *If* $\operatorname{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)}$ *denotes the part of degree* 0 *of* $\operatorname{ch}_{2m-1}(\mathcal{L}_m(z))$, *then*

$$\operatorname{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = (\sqrt{-1})^{\alpha_m} L_m(z),$$

where α_m *is* 0 *or* 1 *according as* m *is odd or even.*

Outline of the proof: We will prove the theorem by induction on m. Assume that $\mathcal{L}_1(z), \cdots, \mathcal{L}_{m-1}(z)$ exist. By the product formula for Bott-Chern forms [7, Prop.4.2],

$$\begin{aligned}
&\mathrm{ch}_{2m-2}\left(\sum_{i=1}^{m-1} 2^i h_i(z) \otimes \mathcal{L}_{m-i}(z)\right) \\
&= (-\sqrt{-1})^{\alpha_m}\left(Im\left(\frac{dz}{z}\right)L_{m-1}(z) - \frac{\sqrt{-1}}{2m-3}\log|z|(\bar{\partial}L_{m-1}(z) - \partial L_{m-1}(z))\right) \\
&= -(\sqrt{-1})^{\alpha_m} dL_m(z).
\end{aligned}$$

It can be shown that the map

$$\mathrm{ch}_{2m-2} : K_{2m-2}(X)_{\mathbb{Q}} \to H^1_{dR}(X(\mathbb{C}), \mathbb{R}(m-1))^{\bar{\iota}=\mathrm{id}}$$

is injective. Hence there exists $\mathcal{L}_m(z) \in \widetilde{\mathbb{Q}}\widehat{C}_{2m-1}(X)$ satisfying the equation (2). Then

$$d\,\mathrm{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = -\,\mathrm{ch}_{2m-2}(\partial\mathcal{L}_m(z)) = (\sqrt{-1})^{\alpha_m} dL_m(z),$$

therefore

$$\mathrm{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = (\sqrt{-1})^{\alpha_m} L_m(z) + a_m$$

for some constant a_m. We can eliminate this constant term by using the Borel's theorem for the regulator map of $\mathbb{Q}$ [2]. □

Theorem 4.4. [8, Thm.7.2, Thm.7.5] *There exists a homomorphism*

$$\mathcal{P}_m : \mathcal{A}_m(F) \to \widetilde{\mathbb{Q}}\widehat{C}_{2m-1}(F)$$

satisfying the following conditions:

(1) $\mathrm{ch}_{2m-1}(\mathcal{P}_m(\xi)) = 0$ *for any* $\xi \in \mathcal{A}_m(F)$.
(2) $\partial(\mathcal{L}_m(\xi) + \mathcal{P}_m(\xi)) = 0$ *for any* $\xi \in \mathcal{A}_m(F)$.

By virtue of the above theorems, one can define a map

$$\mathcal{B}_m(F) = \mathcal{A}_m(F)/\mathcal{C}_m(F) \to K_{2m-1}(F)_{\mathbb{Q}}$$

by $[\xi] \mapsto \mathcal{L}_m(\xi) + \mathcal{P}_m(\xi)$. It is easy to see that the composite of this map with the regulator satisfies the condition of the Zagier conjecture.

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