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Title	On instant extinction for very fast diffusion equations
Author(s)	Chen, Y.-G; Giga, Y.; Sato, K.
Citation	Hokkaido University Preprint Series in Mathematics, 353, 1-9
Issue Date	1996-10-01
DOI	https://doi.org/10.14943/83499
Doc URL	https://hdl.handle.net/2115/69103
Type	departmental bulletin paper
File Information	re353.pdf



**ON INSTANT EXTINCTION
FOR VERY FAST DIFFUSION
EQUATIONS**

Y.-G. Chen, Y. Giga and K. Sato

Series #353. October 1996

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ON INSTANT EXTINCTION FOR VERY FAST DIFFUSION EQUATIONS

Yun-Gang Chen
Hokkaido Tokai University
Sapporo 005, Japan

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Koh Sato
Department of Social Science and Information
Otaru University of Commerce
Otaru 047, Japan

Dedicated to Professor Rentaro Agemi on the occasion of his sixtieth birthday

Abstract. In this paper we prove instant extinction of the solutions to Dirichlet and Neumann boundary value problem for some quasilinear parabolic equations whose diffusion coefficient is singular when the spatial gradient of unknown function is zero.

1. Introduction.

There are many works on quasilinear parabolic equations whose diffusion coefficient is singular when the gradient ∇u of unknown function $u = u(t, x)$ is zero. Examples include the level set equations of surface evolution equations ([2], [5], [11], [12]) and the p -Laplace diffusion equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad (1.1)$$

with $1 < p < 2$ ([3], [14]). Another example is the equation

$$u_t - u_x^{-2} u_{xx} = 0, \quad (1.2)$$

for which the inverse function $x = x(t, u)$ of a solution $u(t, x)$ solves the heat equation $x_t = x_{uu}$. Except the last example (1.2), the initial value problem is globally well-posed for any bounded uniformly continuous initial data at least in the viscosity sense ([2], [5], [11], [12], [14]).

In this paper we study

$$u_t - |u_x|^{-\alpha} u_{xx} = 0 \quad (1.3)$$

for $\alpha > 0$ or its general form

$$u_t - a(u_x) u_{xx} = 0, \quad (1.4)$$

where $a > 0$ is C^1 except at the origin. If $\alpha = 2$, (1.3) is the same as (1.2) and if $\alpha < 1$, (1.3) is the same as (1.1) with $\alpha = 2 - p$ up to the dilation of u . We consider the homogeneous Dirichlet and Neumann

boundary value problem for (1.4) on an interval Ω . We postulate that all reasonable solutions should be some limit of solutions of regularized equation of (1.4):

$$u_t - a^\varepsilon(u_x)u_{xx} = 0 \text{ in } Q = (0, T) \times \Omega \quad (1.5)$$

where $a^\varepsilon > 0$ is smooth everywhere and $a^\varepsilon \rightarrow a$ in some sense. Let u^ε be the solution of (1.5) with the zero boundary condition on $\partial\Omega$. Under very mild assumptions on a^ε and u_0 we show that if a is not integrable near zero, i.e.,

$$\int_{-1}^1 a(p)dp = \infty, \quad (1.6)$$

then

$$u^\varepsilon(t, x) \rightarrow 0 \quad \text{for } t > 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (1.7)$$

uniformly in $x \in \bar{\Omega}$ provided that $u^\varepsilon(0, x) = u_0(x)$. In other words any reasonable solution extincts instantaneously irrelevant to the initial data u_0 . Let v^ε be the solution of (1.5) with the Neumann condition $v_x^\varepsilon = 0, x \in \partial\Omega$ and $v^\varepsilon(0, x) = u_0$. Then the structure condition (1.6) now implies (instead of (1.7))

$$v^\varepsilon(t, x) \rightarrow u_{0\#} \quad \text{for } t > 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (1.8)$$

uniformly in $x \in \bar{\Omega}$, where $u_{0\#}$ is the average of u_0 , i.e.,

$$u_{0\#} = \int_{\Omega} u_0 dx / (\text{length of } \Omega). \quad (1.9)$$

From the study of [14] it follows that if a fails to satisfy (1.6), then the set of admissible functions in [14] is not empty so that the theory of viscosity solutions applies to (1.4) and the instant extinction does not occur. The theory in [14] also applies to (1.1) so the instant extinction does not occur for (1.1) with $1 < p < 2$. So the condition (1.6) is optimal. Several parts of our argument are extendable to a multi-dimensional domain. We first discuss extinction problem in L^2 for equations of divergence type and then apply it for one dimensional problem. Our proof depends on integration by parts and is elementary. In [4] Evans studied (1.2) with the boundary condition $u(t, a) = -\infty, u(t, b) = \infty$ for $\Omega = (a, b)$. An approximate equation (1.5) with $a^\varepsilon(p) = 1/(p^2 + \varepsilon^2)$ is considered there. Let w^ε be the solution of (1.5) with $w^\varepsilon(0, x) = u_0(x)$. Then it is shown in [4] that

$$\lim_{t \rightarrow 0} \lim_{\varepsilon \rightarrow 0} w^\varepsilon(t, x) = (u_0)^*$$

where $(u_0)^*$ is the derivative of the convex hull of a primitive of u_0 . Although the proof and the situation is different, this is interpreted as a similar phenomenon that there is no reasonable solution for (1.3) with $\alpha = 2$ continuous up to initial data.

For equations with very fast diffusion there are several results different from ours. The reader is referred to [16] and papers cited there. In our context the Cauchy problem for (1.3) is considered with a nondecreasing bounded initial data. A typical result says that the solution becomes unbounded (near $x = \infty$) instantaneously if and only if $\alpha \geq 2$. This also explains the effect of strong diffusion.

Recently, M.- H. Giga and the second author ([7], [8], [9]), constructed a continuous solution for (1.4) for any continuous (periodic) initial data even if a contains Dirac's delta function by extending the theory

of viscosity solutions. Their results apply to the case when a is Dirac's delta function so that the instant extinction does not occur. Although the singularity of a is strong, it violates (1.6) so their theory is consistent with our theory. Such a type of singularity of a is typical in describing motion by crystalline curvature as introduced by Angenent and Gurtin [1] and independently by Taylor [15].

The first and second authors were partially supported by the Grant-in-Aid for Scientific Research of The Ministry of Education, Science and Culture of Japan, and partially supported by Environmental Research Institute of Hokkaido Tokai University.

2. Instant extinction.

Let Ω be a smoothly bounded domain in $\mathbf{R}^n$ and let $T > 0$. We consider

$$u_t - \operatorname{div}(\mathbf{A}^\varepsilon(\nabla u, t)) = 0 \quad \text{in } Q = (0, T) \times \Omega \quad (2.1)$$

$$u(0, x) = u_0(x) \quad \text{at } t = 0 \quad (2.2)$$

with either

$$u = 0 \quad \text{on } \partial\Omega \quad (2.3)$$

or

$$\mathbf{A}^\varepsilon(\nabla u, t) \cdot \nu = 0 \quad \text{on } \partial\Omega \quad (2.4)$$

where ν is the outward normal on $\partial\Omega$. We assume for $0 < \varepsilon < 1$ that

- (A1) $\mathbf{A}^\varepsilon = \mathbf{A}^\varepsilon(p, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$ is continuous with its derivatives $\frac{\partial \mathbf{A}^\varepsilon}{\partial p_i}, \frac{\partial^2 \mathbf{A}^\varepsilon}{\partial p_i \partial p_j}, 1 \leq i, j \leq n$,
(A2) for each $M > 0$ there is a constant $K_\varepsilon(M) > 0$ such that $\mathbf{A}^\varepsilon(p, t) \cdot p \geq K_\varepsilon(M)|p|^2$ for $|p| < M$, $0 \leq t < T$, and $\lim_{\varepsilon \rightarrow 0} K_\varepsilon(M) = \infty$.

Our first observation is on the behavior of L^2 norm $\|u^\varepsilon\|_2(t) = (\int_\Omega |u^\varepsilon(x, t)|^2 dx)^{1/2}$.

Remark. Since the equation (2.1) is unchanged by replacing $\mathbf{A}^\varepsilon$ by $\mathbf{A}^\varepsilon + \mathbf{C}$ with $\mathbf{C}$ independent of p , the condition

$$\mathbf{A}^\varepsilon(p, t) \cdot p \geq K_\varepsilon(M)|p|^2$$

can be weakened as

$$(\mathbf{A}^\varepsilon(p, t) + \mathbf{C}) \cdot p \geq K_\varepsilon(M)|p|^2$$

with some vector $\mathbf{C}$ (independent of p).

It is not difficult to check that

$$\mathbf{A}^\varepsilon(p) = (|p|^2 + \varepsilon)^{(q-2)/2} p \quad (q > 0)$$

satisfies (A2) or its modification remarked here if and only if $0 < q \leq 1$ (cf. §3).

2.1. Lemma on L^2 estimates(Dirichlet problem).

Assume (A1)-(A2). Let $u^\varepsilon \in C(\overline{Q})$ be a solution of (2.1) and (2.2) with (2.3) such that $u_t^\varepsilon, \nabla u^\varepsilon, \nabla^2 u^\varepsilon \in C(\overline{Q})$. (For brevity here and hereafter we shall simply say that u^ε is a $C^{1,2}$ solution.) Assume

that ∇u^ε is bounded on Q , say $|\nabla u^\varepsilon| \leq M$ uniformly for $0 < \varepsilon < 1$ in Q . Then

$$\|u^\varepsilon\|_2^2(t) \leq \exp(-2\lambda_1 K_\varepsilon(M)t) \|u_0\|_2^2 \quad (2.5)$$

with $0 < \varepsilon < 1$, where λ_1 is the first eigenvalue of the minus Laplacian on Ω with the Dirichlet condition (2.3). In particular $\|u^\varepsilon\|_2^2(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $t > 0$.

Proof. Multiplying u^ε with (2.1) yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^\varepsilon|^2 dx = - \int_{\Omega} \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t) \cdot \nabla u^\varepsilon dx$$

by (A2) and the Poincaré inequality we have

$$\begin{aligned} \int_{\Omega} \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t) \cdot \nabla u^\varepsilon dx &\geq K_\varepsilon(M) \int_{\Omega} |\nabla u^\varepsilon|^2 dx \\ &\geq K_\varepsilon(M) \lambda_1 \|u^\varepsilon\|_2^2. \end{aligned}$$

This now yields

$$\frac{d}{dt} \|u^\varepsilon\|_2^2 \leq -2\lambda_1 K_\varepsilon(M) \|u^\varepsilon\|_2^2.$$

We thus obtain (2.5). The last assertion follows from (2.5) since $K_\varepsilon(M) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

2.2. Lemma on L^2 estimates (Neumann problem).

Assume (A1)-(A2). Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Assume that $|\nabla v^\varepsilon| \leq M$ on Q with some $M > 0$ independent of ε , $0 < \varepsilon < 1$. Then

$$\|v^\varepsilon - u_{0\#}\|_2^2(t) \leq \exp(-2\mu K_\varepsilon(M)t) \|u_0 - u_{0\#}\|_2^2, \quad (2.6)$$

where $\mu > 0$ is the second eigenvalue of the (minus) Laplacian on Ω with the Neumann condition (2.4). In particular, $\|v^\varepsilon - u_{0\#}\|_2^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $t > 0$. Here $u_{0\#} = \int_{\Omega} u_0 dx / |\Omega|$ where $|\Omega|$ is the volume of Ω .

Proof. We set $u^\varepsilon = v^\varepsilon - u_{0\#}$ and observe that u^ε solves (2.1) and (2.4) with

$$u^\varepsilon(0, x) = u_0 - u_{0\#}.$$

Since u^ε solves (2.1) and (2.4), we see

$$\frac{d}{dt} \int_{\Omega} u^\varepsilon(t, x) dx = 0.$$

From $(u^\varepsilon(0, x))_{\#}$ it now follows that $(u^\varepsilon(0, \cdot))_{\#} = 0$. Multiplying u^ε with (2.1) and integrating by parts with (2.4) yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^\varepsilon|^2 dx = - \int_{\Omega} \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t) \cdot \nabla u^\varepsilon dx$$

Since $(u^\varepsilon(t, \cdot))_{\#} = 0$, the Poincaré inequality ([10], §7.8, (7.45)) with (A2) yields

$$\begin{aligned} \int_{\Omega} \mathbf{A}^\varepsilon(\nabla u^\varepsilon, t) \cdot \nabla u^\varepsilon dx &\geq K_\varepsilon(M) \int_{\Omega} |\nabla u^\varepsilon|^2 dx \\ &\geq K_\varepsilon(M) \mu \|u^\varepsilon\|_2^2 \end{aligned}$$

As in the proof of Lemma 2.1 this now yields (2.6). The last assertion follows from (2.6) and $K_\varepsilon(M) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

2.3. Theorem on uniform convergence.

Assume (A1) - (A2).

- (i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.3). Assume that $|\nabla u^\varepsilon| \leq M$ on Q for $0 < \varepsilon < 1$. Then there is $C = C(n)$ such that

$$\|u^\varepsilon\|_\infty(t) \leq CM^{1-\theta} \exp(-\lambda_1 K_\varepsilon(M)\theta t) \|u_0\|_2^\theta, \quad \theta = 2/(n+2).$$

In particular $\|u^\varepsilon\|_\infty(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $t > 0$.

- (ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Assume that $|\nabla v^\varepsilon| \leq M$ on Q for $0 < \varepsilon < 1$. Then there is $C = C(n)$ such that

$$\|v^\varepsilon - u_{0\#}\|_\infty(t) \leq CM^{1-\theta} \exp(-\mu K_\varepsilon(M)\theta t) \|u_0 - u_{0\#}\|_2^\theta.$$

In particular $\|v^\varepsilon - u_{0\#}\|_\infty(t) \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $t > 0$. Here $\|f\|_\infty = \sup\{|f(x)|; x \in \Omega\}$.

Proof. Our estimates follows from (2.5) and (2.6) with the Gagliardo-Nirenberg inequality

$$\|f\|_\infty \leq C\|f\|_2^\theta \cdot \|\nabla f\|_\infty^{1-\theta}, \quad \theta = \frac{2}{n+2}. \quad (2.7)$$

for f with $f = 0$ on $\partial\Omega$ or f with $f_\# = 0$. For $f \in C^1(\overline{\Omega})$ vanishing on $\partial\Omega$ the inequality (2.7) immediately follows from that for $f \in C_0^\infty(\mathbf{R}^n)$ which is standard ([6], Part 1, Theorem 9.3).

The weaker version of the inequality (2.7), i.e.,

$$\|f\|_\infty \leq C_0\|f\|_2^\theta (\|\nabla f\|_\infty^{1-\theta} + \|f\|_\infty^{1-\theta}) \quad (2.8)$$

holds for all $f \in C^\infty(\overline{\Omega})$ with C_0 independent of f . This inequality may be familiar ([6], Part1, Theorem 10.1). We note that (2.7) follows from (2.8) for f with $f_\# = 0$. Since (2.7) for f with $f_\# = 0$ is less familiar, we give the proof that (2.8) implies (2.7) with $f_\# = 0$ for convenience.

Suppose that (2.7) were false. Then there would exist $\{f_j\} \subset C^\infty(\overline{\Omega})$ with $\|f_j\|_\infty = 1$ and $f_\# = 0$ that satisfies

$$\frac{1}{j} = \frac{\|f_j\|_\infty}{j} > \|f_j\|_2^\theta \|\nabla f_j\|_\infty^{1-\theta} \geq \frac{\|f_j\|_\infty}{C_0} - \|f_j\|_2^\theta \|f_j\|_\infty^{1-\theta}; \quad (2.9)$$

the second inequality follows from (2.8). This yields either $\|f_j\|_2 \rightarrow 0$ or $\|\nabla f_j\|_\infty \rightarrow 0$ as $j \rightarrow \infty$. If $\|f_j\|_2 \rightarrow 0$, then by (2.9) it contradicts $\|f_j\|_\infty = 1$ for all j . If $\|\nabla f_j\|_\infty \rightarrow 0$, then the Ascoli-Arzelà theorem says that $\{f_j\}$ converges subsequently to a constant function g in $C(\overline{\Omega})$. Since $f_{j\#} = 0$ implies $g = 0$, this contradicts the property $\|f_j\|_\infty = 1$ for all j .

3. One dimensional problem.

Unfortunately, in general the gradient bound may not be obtained in Q if $n > 1$ because of variety of domain, unless the equation is uniformly parabolic (see e.g. [13]). To avoid this extra difficulty we shall

concentrate on one dimensional case. In this case we may assume $\Omega = (0, L)$. The Dirichlet problem can be reduced to the periodic boundary problem on $(-L, L)$ by extending u on $(-L, 0)$ by $-u(-x, t)$. The Neumann problem can be also reduced to the periodic boundary problem on $(-L, L)$ by extending u on $(-L, 0)$ by $u(-x, t)$ provided that $a^\varepsilon = \partial A^\varepsilon / \partial p$ is even in p , where a^ε is now scalar and positive. Since $w = u_x$ of (2.1) now solves

$$w_t - a^\varepsilon(u_x, t)w_{xx} - a_p^\varepsilon(u_x, t)w_x^2 = 0 \quad \text{on} \quad (0, T) \times \mathbf{R},$$

applying the maximum principle yields the bound of u_x for the bound of u_{0x} . Here $a_p^\varepsilon(p, t) = \frac{\partial^2 A^\varepsilon}{\partial p^2}$.

3.1. Lemma on bound for u_x .

Assume (A1) and $n = 1$.

(i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.3). Then

$$\|u_x^\varepsilon\|_\infty(t) \leq \|u_{0x}\|_\infty, \quad 0 \leq t \leq T, \quad 0 < \varepsilon < 1$$

(ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (2.1) and (2.2) with (2.4). Then

$$\|v_x^\varepsilon\|_\infty(t) \leq \|u_{0x}\|_\infty, \quad 0 \leq t \leq T, \quad 0 < \varepsilon < 1$$

provided that $a^\varepsilon(p, t)$ is odd in p .

We now give a sufficient condition for (A2) when $n = 1$. We rewrite (2.1) as

$$u_t - a^\varepsilon(u_x, t)u_{xx} = 0, \tag{3.1}$$

where $a^\varepsilon(p, t) = \frac{\partial A^\varepsilon}{\partial p}$.

3.2. Proposition.

Assume that for $0 < \varepsilon < 1$,

(a1) $a^\varepsilon = a^\varepsilon(p, t) : \mathbf{R} \times [0, T] \rightarrow \mathbf{R}$ is continuous with its derivative $a_p^\varepsilon = \frac{\partial a^\varepsilon}{\partial p}$ and $a^\varepsilon > 0$,

(a2) $pa_p^\varepsilon \leq 0$,

(a3) $a^\varepsilon(p, t)$ is even in p ,

(a4) for each $M > 0$, there is $k_\varepsilon(M)$ such that

$$\int_0^M a^\varepsilon(p, t)dp \geq k_\varepsilon(M), \quad 0 \leq t \leq T$$

and

$$k_\varepsilon(M) \rightarrow \infty \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Then $A^\varepsilon(p, t) = \int_0^p a^\varepsilon(q, t) dq$ satisfies (A1). Moreover $A^\varepsilon(p, t) = \beta^\varepsilon(p, t)p$ with continuous $\beta^\varepsilon: \mathbf{R} \times [0, T] \rightarrow \mathbf{R}$. (This condition guarantees that $u_x = 0$ implies $A^\varepsilon(u_x, t) = 0$).

Proof. From (a1) it follows (A1). By (a2) $A^\varepsilon(p, t)$ is concave in $p \in [0, \infty)$. Thus

$$A^\varepsilon(p, t) \geq \frac{A^\varepsilon(M, t)}{M} p \quad \text{for } 0 \leq p \leq M.$$

By (a4) we now obtain

$$A^\varepsilon(p, t) \geq \frac{k_\varepsilon(M)}{M} p \quad \text{for } 0 \leq p \leq M.$$

Since $A^\varepsilon(p, t)$ is odd by (a3), this yields

$$A^\varepsilon(p, t)p \geq \frac{k_\varepsilon(M)}{M} p^2 \quad \text{for } |p| \leq M.$$

Setting $K_\varepsilon(M) = k_\varepsilon(M)/M$ leads to (A2) where the last property follows from definition of A^ε .

3.3. Theorem on instant extinction.

Assume (a1) - (a4).

- (i) Let $u^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (3.1) in $Q = (0, T) \times \Omega$ with (2.2) and (2.3). If u_{0x} is bounded on Ω then there is a universal constant C that satisfies

$$\|u^\varepsilon\|_\infty(t) \leq C \|u_{0x}\|_\infty^{1-\theta} \exp(-\lambda_1 K_\varepsilon(\|u_{0x}\|_\infty) t \theta) \|u_0\|_2^\theta, \quad t > 0.$$

In particular

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon\|_\infty(t) = 0 \quad \text{for } t > 0.$$

- (ii) Let $v^\varepsilon \in C(\overline{Q})$ be a $C^{1,2}$ solution of (3.1) in Q with (2.2) and

$$u_x^\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{2.4'}$$

If u_{0x} is bounded on Ω then

$$\|v^\varepsilon - u_{0\#}\|_\infty(t) \leq C \|u_{0x}\|_\infty^{1-\theta} \exp(-\mu K_\varepsilon(\|u_{0x}\|_\infty) t \theta) \|u_0 - u_{0\#}\|_2^\theta, \quad t > 0.$$

In particular

$$\lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - u_{0\#}\|_\infty(t) = 0 \quad \text{for } t > 0.$$

Proof. This follows from Theorem 2.3, Lemma 3.1 and Proposition 3.2.

3.4. Remark on the periodic boundary condition.

The second statement of Theorem 3.3 is still valid even if (2.4') is replaced by the periodic boundary condition. Here $\mu > 0$ should be the second eigenvalue of the minus Laplacian on Ω with the periodic boundary condition. The proof is the same as the Neumann problem so is omitted.

3.5. Example.

We consider

$$u_t - a(u_x)u_{xx} = 0 \quad (3.2)$$

with $a \in C^1(\mathbf{R} \setminus \{0\})$, $a > 0$. We moreover assume that $a(p) = a(-p)$ and $a'(p) > 0$ for $p > 0$. Our main assumption on a is

$$\int_0^1 a(q) dq = \infty. \quad (3.3)$$

A typical example is

$$a(p) = |p|^{-\alpha} \quad \text{with} \quad \alpha > 1. \quad (3.4)$$

Let $a^\varepsilon(p) \rightarrow a(p)$ as $\varepsilon \rightarrow 0$ for a.e. p . Then the Fatou lemma with (3.3) implies

$$\lim_{\varepsilon \rightarrow 0} \int_0^M a^\varepsilon(p) dp = \infty$$

so such $a^\varepsilon(p)$ fulfills (a4). Of course there are many examples of such a sequence. For example, we may take

$$a^\varepsilon(p) = (|p|^2 + \varepsilon)^{\alpha/2} \quad \text{for} \quad a(p) = |p|^{-\alpha}.$$

Our Theorem 3.3 says if the solution of (3.2) and (2.2) with (2.3) or (2.4') is obtained as a limit of solutions of regularized problem, then it must extinct instantaneously for the Dirichlet problem and it must equal the average of the initial data instantaneously for the Neumann problem. By Remark 3.4 the statement for the Neumann problem also holds for the periodic boundary value problem. In [9] a continuous solution for (3.2) is constructed for any continuous periodic initial data even if a contains Dirac's delta function.

It is unlikely that instant extinction occurs in their problem. However, their results do not contradict ours since in their paper the integral

$$\int_{-1}^1 a dq$$

is assumed to be finite which excludes (3.3).

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