Singularities of solutions for first order partial differential equations

Shyuichi Izumiya

Series #357. November 1996

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#333 T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.

#334 B. Khanedani and T. Suwa, First variation of horomorphic forms and some applications, 11 pages. 1996.

#335 J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations[1], 28 pages. 1996.

#336 Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.

#337 I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.

#338 F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.

#339 N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.

#340 A. Inoue, Abel-Tauber theorems for Fourier-Stieltjes coefficients, 17 pages. 1996.

#341 G. Ishikawa, Topological classification of the tangent developables of space curves, 19 pages. 1996.

#342 Y. Shimizu, A remark on estimates of bilinear forms of gradients in Hardy space, 8 pages. 1996.

#343 N. kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, 11 pages. 1996.

#344 M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, 18 pages. 1996.

#345 A. Douai and H. Terao, The determinant of a hypergeometric period matrix, 20 pages. 1996.

#346 H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in even space dimensions, 66 pages. 1996.

#347 T. Nakazi and T. Yamamoto, Weighted Norm Inequalities For Some Singular Integral Operators, 17 pages. 1996.

#348 Y. Ito and I. Nakamura, Hilbert schemes and simple singularities A_n and D_n, 22 pages. 1996

#349 R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, 42 pages. 1996

#350 F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field, 21 pages. 1996.

#351 T. Nakazi and Y. Watatani, Invariant subspace theorems for subdiagonal algebras, 22 pages. 1996.

#352 Y. Nishiura and H. Suzuki, Nonexistence of stable turing patterns with smooth limiting interfacial configurations in higher dimensional spaces, 21 pages. 1996.

#353 Y.-G. Chen, Y. Giga and K. Sato, On instant extinction for very fast diffusion equations, 9 pages. 1996.

#354 A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(\mathfrak{n})$-action on Harish-Chandra modules, 25 pages. 1996.

#355 G. Ishikawa, Topology of plane trigonometric curves and the strangeness of plane curves derived from real pseudo-line arrangements, 18 pages. 1996.

#356 N.H. Bingham and A. Inoue, The Drasin-Shea-Jordan theorem for Hankel transforms of arbitrarily large order, 13 pages. 1996.

Singularities of solutions for first order partial differential equations

Shyuichi IZUMIYA

1 Introduction

This is a survey article on recent results about the singularities for the solution of the Cauchy problem of first order partial differential equation in the following forms:

$$\text{(H)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + H(t, x_1, \ldots, x_n, \dfrac{\partial y}{\partial x_1}, \ldots, \dfrac{\partial y}{\partial x_n}) = 0 \ (t > 0) \\ y(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n), \end{cases}$$

and

$$\text{(C)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + \displaystyle\sum_{i=1}^{n} \dfrac{\partial f_i(y)}{\partial x_i} = 0 \ (t > 0) \\ y(0, x_1, \cdots, x_n) = \phi(x_1, \cdots, x_n), \end{cases}$$

where H, f_i's and ϕ are C^∞-functions.

The former equation is called *a Hamilton-Jacobi equation* and the latter one is called *a single conservation law.* These equations play an important role in various fields (e.g., calculus of variations ([41]), optimal control theory [17], differential games [16] for (H) and gas dynamics [46] and oil reservoir problems [20] for (C)).

For a sufficiently small t the solution y is classically determined using the characteristic method. Although y is initially smooth there is in general a critical time beyond which characteristics cross. The solution y past the critical time is multi-valued, that is singularities appear. In §2 (respectively, §4) we give a survey on the classifications of those singularities for (H) (respectively, (C))

The notion of *viscosity solutions*[11] (respectively, *entropy solutions*[36]) has provided the right weak setting for the study of (H) (respectively, (C)). Existence and uniqueness of the solution of (H) in the viscosity sense (respectively, (C) in the entropy sense) have been established in [11] (respectively, [36]). The single-valued viscosity (respectively, entropy) solution coincides with the smooth geometric solution until the first critical time. After the characteristics cross, the viscosity (respectively, entropy) solution develops *shock waves* i.e., curves across which the gradient of the viscosity solution (respectively, the value of the entropy solution) is discontinuous.

The method of constructing the weak solution by selecting the proper single-valued branch was introduced by Tsuji [43,44] for (H). For a convex Hamiltonian function, Nakane [39] has constructed the semi-concave solution (i.e, the viscosity solution) past the first critical time and Bogaevskii[6] has classified the generic perestroikas of shocks of the viscosity solutions for $n = 1, 2, 3$. The viscosity solution of (H) for non-convex Hamiltonian in a neighborhood of the first critical time has been constructed by Kossioris [34,35] by selecting a continuous single-valued branch of the graph of the geometric solution. In which the shock curves of the viscosity solution corresponds to the intersection of the branches of the graph of the multi-valued geometric solution. In order to study the evolution of the shock curves we follow the evolution of the intersections of the branches defining the shock. After that we solve the local Riemann problem for each stage. In the case when $n = 1$, we have proceeded the above arguments in [31] which is described in §3, however for $n \geq 2$, we have no idea to solve the Riemann problem still now.

On the other hand, the study for the shocks of entropy solutions of (C) has long history. For example, the case $n = 1$ has been studied by Ballow[3], Guckenheimer [21], Chen [10], Jennings [32] and Dafermos [14] etc., however, there are few results in the case when $n \geq 2$. Guchenheimer [22], Wagner[46] and Nakane [38] have studied the entropy solution for (C) in the several space dimensions.

Acknowledgment The author wishes to thank Terry and Sandra Wall for their hospitality and encouragement during the author's stay at their home in the summer of 1996. The author has been inspired very much by Terry's survey articles on the singularity theory, he will be very happy if this survey will impress something the next generation like as Terry's surveys.

All maps considered here are class C^∞ unless stated otherwise.

2 Geometric solutions for Hamilton-Jacobi equations

In this section we give a survey on the geometric framework for the study of (H) which was described in [25,26,27,28,29]. We consider the 1-jet bundle $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t, x_1, \ldots, x_n)$ be the canonical coordinate system on $\mathbb{R}\times\mathbb{R}^n$ and $(t, x_1, \ldots, x_n, y, s, p_1, \ldots, p_n)$ the corresponding coordinate system on $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$. Then, the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i \cdot dx_i - s\cdot dt = \theta - s \cdot dt$. We define the natural projection $\Pi : J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}) \to (\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}$ by $\Pi(t, x, y, s, p) = (t, x, y)$. *A Hamilton-Jacobi equation* is considered to be a hypersurface

$$E(H) = \left\{(t, x, y, s, p) \in J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}) | s + H(t, x, p) = 0\right\}$$

in $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$. *A geometric (multi-valued) solution* of $E(H)$ is a Legendrian submanifold L in $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ lying in $E(H)$. In this case the wave front set $W(i) = \Pi(L)$ is "the graph" of the geometric solution which is generally a hypersurface with singularities.

In order to study (H) we need the following framework: For any $c \in (\mathbb{R}, 0)$, we define

$$E(H)_c = \left\{(c, x, y, -H(c, x, p), p) | (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R})\right\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\Theta | E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to E(H)_c$ by $\iota_c(x, y, p) = (c, x, y, -H(c, x, p), p)$. The mapping i_c is a contact diffeomorphism.

We say that *a geometric Cauchy problem (with initial condition L') associated with the time parameter(GCPT) is given for an equation* $E(H)$ if there is given an n-dimensional submanifold $i : L' \subset E(H)$ with $i^*\Theta = 0$ and $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R}, 0)$. Since $X_H \notin TE(H)_c$, we have $X_H \notin TL'$, where X_H is the characteristic vector field given by

$$X_H = \frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} + (\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} - H)\frac{\partial}{\partial y} - \frac{\partial H}{\partial t}\frac{\partial}{\partial s} - \sum_{i=1}^n \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}.$$

By using the classical characteristic method, we can show that there exists a unique geometric solutions around L'. It is clear that the classical Cauchy problem (H) is a GCPT. The geometric part of our purpose is to classify the generic perestroikas of wave fronts (graphs) of geometric solutions of (H) along the time parameter.

In order to study the singularities of the geometric solution we introduce the notion of one-parameter Legendrian unfoldings [25,26]. Let R be an $(n+1)$-dimensional smooth manifold, $\mu : (R, u_0) \to (\mathbb{R}, t_0)$ be a submersion germ and $\ell : (R, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a smooth map germ. We say that the pair (μ, ℓ) is *a Legendrian family* if $\ell_t = \ell | \mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, t_0)$. By definition, for any (μ, ℓ) Legendrian family, there exists a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h \cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0. So we define a map germ $\mathcal{L} : (R, u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by $\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u))$. We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ). It is clear that the solution germ at each point of the GCPT for (H) is a Legendrian unfolding.

In order to study the generic classifications of perestroikas of singularities (i.e., how a singularity is generated, how one type can change into another and how different types of singularities interact), we now study how various branches of the multi-valued graph $W_t = (\{t\} \times \mathbb{R}^n \times \mathbb{R}) \cap W(i)$ intersecting at a point bifurcate in time for an arbitrary Hamiltonian $H(t, x, p)$. For the purpose, we classify the perestroikas of singularities of multi-Legendrian unfoldings which are expressed in terms of multi-germs.

Let $\mathcal{L}_i : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i)$ $(i = 1, \ldots, r)$ be Legendrian unfoldings with $\Pi(z_i) = 0$ where $z_1, \ldots, z_r$ are distinct. We call $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ *a multi-Legendrian unfolding.* Let $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ and $(\mathcal{L}'_1, \ldots, \mathcal{L}'_r)$ be multi-Legendrian unfoldings. We say that these are $P_{(r)}$*-Legendrian equivalent* if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_i) \quad (i = 1, \ldots, r)$$

of the form $K_i(t,x,y,s,p) = (\phi_1(t), \phi_2(t,x,y), \phi_3(t,x,y), \phi_4^i(t,x,y,s,p), \phi_5^i(t,x,y,s,p))$ and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u_0')$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}_i' \circ \Psi$ for any $i = 1, \ldots, r$. It is clear that if two multi-Legendrian unfoldings are $P_{(r)}$-Legendrian equivalent, then there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t,x,y) = (\phi_1(t), \phi_2(t,x,y), \phi_3(t,x,y))$ such that $\Phi(\cup_{i=1}^r W(\mathcal{L}_i)) = \cup_{i=1}^r W(\mathcal{L}_i)$. Thus the above equivalence describes how bifurcations of wavefronts (i.e. graphs of solutions) interact. We can define the notion of stability with respect to the $P_{(r)}$-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [2],[47]). Thanks to Arnol'd-Zakalyukin's theory ([2],[47]), we can construct multi-generating families of multi-Legendrian unfoldings and give a classification of $P_{(r)}$-Legendrian stable Legendrian unfoldings by using the classification of multi-families of function germs in Zakalyukin [47]. However, we only present the list of classifications for $n = 1$. For the case $n = 2, 3$, see ([2,27,28,47]).

Theorem 2.1. **[28,47]** *Suppose that $n = 1$. Then a generic multi-Legendrian unfolding is $P_{(r)}$-Legendrian equivalent to one of the multi-Legendrian unfoldings in the following list :*

$^0A_1$: $(t, u, 0, 0, 0)$;
$^0A_2$: $(t, 3u^2, 2u^3, 0, u)$;
$^1A_3$: $(t, 4u^3 + 2ut, 3u^4 + u^2t, -u^2, u)$.
$^0(^0A_1\,^0A_1)$: $((t, u, -u, 0, -1), (t, u, u, 0, 1))$;
$^1(^0A_1\,^0A_1)$: $((t, u, t \pm u^2, 1, \pm 2u), (t, u, 0, 0, 0))$;
$^1A_2\,^0A_1$: $((t, 3u^2 - t, 2u^3, u, u), (t, u, -u, 0, -1))$.
$^0A_1\,^0A_1\,^0A_1$: $((t, u, t - u, 1, -1), (t, u, 0, 0, 0), (t, u, u, 0, 1))$.

When we consider the geometric solution, we can get rid of the germ $^1(^0A_1\,^0A_1)$ from the above list because the geometric solution is a one-to-one immersion into the unfolded 1-jet space. These multi-germs have been called *standard perestroikas of fronts* by Bogaevskii [7]. The standard perestroikas of fronts in the case $n = 1$ is depicted in Figure 1.

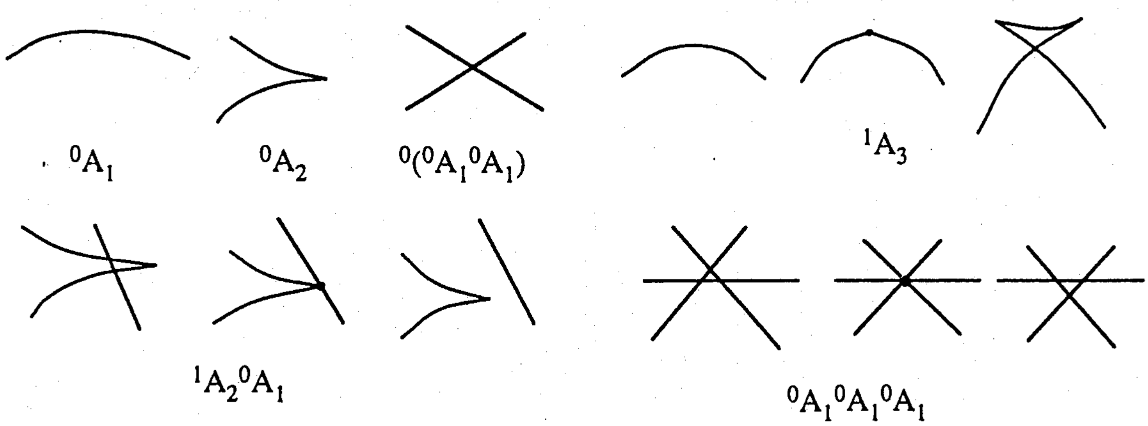


Figure 1

We now concern what are the types of singularities that the geometric solution to a given (fixed) Hamilton-Jacobi equation might exhibit. For the purpose we refer the following theorem of Bogaevskii [7].

Theorem 2.2. [7] *Suppose that $n \leq 3$. There exists a dense subset $\mathcal{O} \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ with the following properties: For any $\phi \in \mathcal{O}$, the each multi-Legendrian unfolding which is given by the geometric solution for GCPT of (H) is $P_{(r)}$-Legendrian equivalent to one of the standard perestroikas of fronts, where we consider the Whitney fine topology on the space $C^\infty(\mathbb{R}^n, \mathbb{R})$*

This theorem denotes that the standard perestroikas of fronts is the exhaustive list for the generic perestroikas of the graphs of geometric solutions for a fixed Hamilton-Jacobi equations. However, we have to consider the realization problem for the standard perestroikas of fronts as a geometric solution for a given Hamilton-Jacobi equation. For the purpose, we need a kind of non-degeneracy condition on the Hamiltonian function. We say that a Hamiltonian function $H(t, x, p)$ is *non-degenerate at* (t_0, x_0, p_0) if it $\frac{\partial^2 H}{\partial p_i \partial p_j}(t_0, x_0, p_0) \neq 0$ for some $1 \leq i, j \leq n$. This condition is weaker than the condition that $H(t, x, p)$ is convex (or concave) with respect to $(p_1, \dots, p_n)$-variables at (t_0, x_0, p_0) for $n \geq 2$. The following theorem is a realization theorem for generic singularities for a given Hamilton-Jacobi equation.

Theorem 2.3. **[28,29]** *Let $H(t, x, p)$ be a non-degenerate Hamiltonian function germ at (t_0, x_0, p_0) and $\mathcal{L} : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), (t_0, x_0, y_0, s_0, p_0))$ be a $P_{(1)}$-Legendrian stable Legendrian unfolding associated with (μ, ℓ). Then there exists a Legendrian unfolding $\mathcal{L}'$ which is a geometric solution of the Hamilton-Jacobi equation $s + H(t, x, p) = 0$ such that $\mathcal{L}$ and $\mathcal{L}'$ are $P_{(1)}$-Legendrian equivalent.*

We remark that ${}^1A_3$ singularity (even for general n) describes how the singularity appears from a smooth solution. These are $P_{(1)}$-Legendrian stable Legendrian unfoldings, so that these can be realized as geometric solutions at the non-degenerated point for a given Hamilton-Jacobi equation. We can also specify the point at where the ${}^1A_3$-singularity appears.

Theorem 2.4. [29] *If an ${}^1A_3$-singularity appears at (t_0, x_0, p_0), then $H(t, x, p)$ is non-degenerate at (t_0, x_0, p_0).*

3 Viscosity solutions for Hamilton-Jacobi equations

For convenience, we stick to the case when the Hamiltonian $H(p_1, \dots, p_n)$ depends only on the momentum variables in this section. Some of the results can be generalized for more general Hamiltonian. The viscosity solutions for nonlinear equations of first order have been introduced by Crandall and Lions [11].

A continuous function $y_v \in C([0,\infty) \times \mathbb{R}^n)$ is *a viscosity solution* of (H) provided

$$\frac{\partial \psi}{\partial t}(t,x) + H(\frac{\partial \psi}{\partial x_1}(t,x), \ldots, \frac{\partial \psi}{\partial x_n}(t,x)) \leq 0, (resp. \geq 0)$$

for any $\psi \in C^1([0,\infty) \times \mathbb{R}^n)$ for which $y_v - \psi$ attains a local maximum (resp. local minimum) at the point $(t,x) \in \mathcal{O}$ and satisfies the initial condition $\lim_{t\to 0+} y_v(t,x) = \phi(x)$.

Firstly we consider the case when the Hamiltonian $H(p_1,\ldots,p_n)$ is uniformly convex (or concave). In this case we refer the following result of Bardi-Evans [4].

Theorem 3.1.[4] *Assume that the Hamiltonian $H(p_1,\ldots,p_n)$ is uniformly convex, then*

$$y(t,x) \equiv \inf_q \sup_p \{\phi(q) + < p, (x-q) > -H(p)t\}$$

is the unique viscosity solution of (H), *where $<,>$ is the canonical inner product on $\mathbb{R}^n$.*

We remark that the above exact formula of the viscosity solution is called *the Hopf-Lax type formula,* there are some results related to this formula in [18,23,37]. We now consider the family of functions $F(t,x,p,q) = \phi(q) + < p, x-q > -H(p)t$. Since $H(p)$ is uniformly convex, we have $\sup_p\{\phi(q) + < p, (x-q) > -H(p)t\} = F|\Sigma_p(F)$, where $\Sigma_p(F) = \{(t,x,p,q) | \frac{\partial F}{\partial p_i} = x_i - q_i - \frac{\partial H}{\partial p_i}(p)t = 0 \ i = 1,\ldots,n\}$. If $\phi(q)$ has the minimum, then we have $\inf_q \sup_p\{\phi(q) + < p, (x-q) > -H(p)t\} = \min_q\{\phi(q) + < p, \frac{\partial H}{\partial p}(p) - H(p)t\}$. So the viscosity solution of (H) in this case is the minimum function of a certain family of smooth functions. In [6], Bogaevskii has classified the perestroikas of shocks of viscosity solutions for $n = 1,2,3$ by using this fact. The picture for $n = 2$ is given in Figure 2.

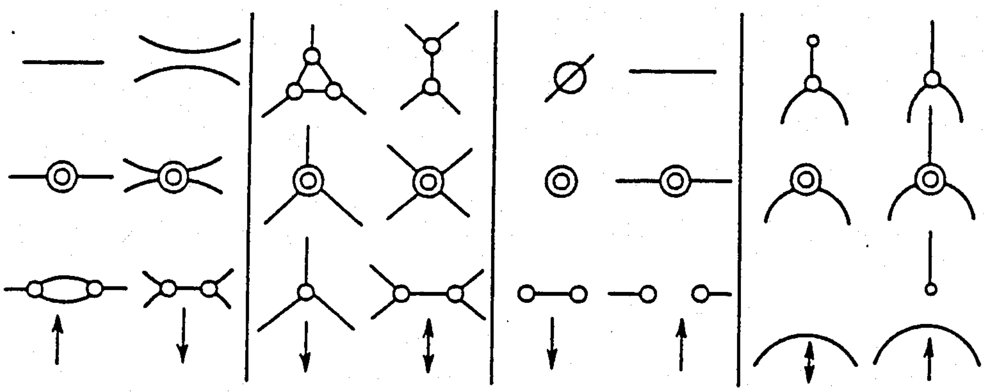


Figure 2

In [9] Chaperon defined the notion of minimax solutions for (H) using the notion of global generating families. Joukovskaia [33] has classified generic singularities of the minimax solutions for $n = 1,2$. If the Hamiltonian is convex, it is known that the minimax solution is coincide with the viscosity solution.

For general (non-convex) Hamiltonian, the situations are quite different. For convenience, we only consider the case when $n = 1$. Since $H(p)$ is not assumed to be uniformly convex (or concave), we cannot use Theorem 3.1, so that the situation should be quite complicated even for this case. We next state the criterion in a form which is useful for the construction of the solution. To this end, assume that $\mathcal{O} \subset (0,\infty)\times I\!R$ is open and that there is a smooth curve $t \to x = \chi(t)$, $t \in (t_1, t_2) \subset I\!R^+$, with negative slope, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-$, $\Gamma = \{(t,x) : x = \chi(t)\}$, where $\mathcal{O}^+$ lies on the right of Γ. Then we have the following theorem.

Theorem 3.2. [12,34] *Let $y_v \in C(\mathcal{O})$ and $y_v = y_v^+$ in $\mathcal{O}^+ \cup \Gamma, y_v = y_v^-$ in $\mathcal{O}^- \cup \Gamma$ where $y_v^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma)$. Then y_v is a viscosity solution of* (H) *in $\mathcal{O}$ if and only if the following conditions hold:*

a) y_v^+ and y_v^- are classical solutions of (H) *in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively,*

b) If $y_{v,x}^- > y_{v,x}^+$ (resp. $y_{v,x}^- < y_{v,x}^+$) across Γ, then

$$H((1-\lambda)y_{v,x}^+ + \lambda y_{v,x}^-) - (1-\lambda)H(y_{v,x}^+) - \lambda H(y_{v,x}^-) \le 0 \ (resp.\ \ge 0),$$

where $\lambda \in [0,1]$ and $y_{v,x}^\pm = \frac{\partial y_v^\pm}{\partial x}$. That is, the graph of H lies respectively below or above the line segment joining the points $(y_{v,x}^+, H(y_{v,x}^+))$ and $(y_{v,x}^-, H(y_{v,x}^-))$.

The condition b) will be referred in the sequel as the *viscosity criterion.* The hypersurface Γ in the neighbourhood of which y_v has the properties specified in the above theorem is the *shock curve.* In this case the characteristic equation (i.e., the equation corresponding to the characteristic vector field) is given as follows:

$$\begin{cases} \dfrac{dx}{dt} = H'(p), \quad x(0) = u, \\ \dfrac{dp}{dt} = 0, \quad p(0) = \phi'(u), \\ \dfrac{dy}{dt} = -H(p) + p\cdot H'(p), \quad y(0) = \phi(u). \end{cases}$$

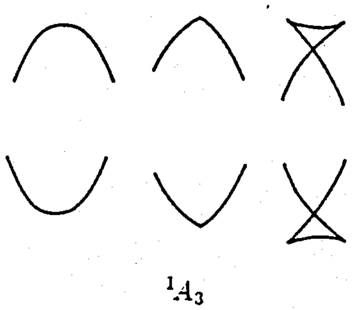
It can be solved exactly, so that the geometric solution is given by

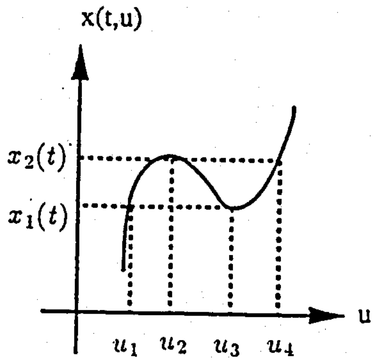
$$L_{\phi,t} = \{(t, x(t,u), y(t,u), -H(p(t,u)), p(t,u)) | u \in I\!R\},$$

where

$$\begin{cases} x(t,u) = u + tH'(\phi'(u)), \\ p(t,u) = \phi'(u) \\ y(t,u) = t\{-H(\phi'(u)) + \phi'(u)H'(\phi'(u))\} + \phi(u). \end{cases}$$

Before the first critical time that characteristics cross in the (t,x)–plane, $W_t = \Pi(L_{\phi,t})$ is the graph of the viscosity solution y_v. After the characteristics cross, W_t becomes singular. Theorem 2.1 describes the generic singularities of W_t. The first singularity appears in the form of ${}^1A_3$. See Figure 3a, where we show the shape of the appearing singularity. By Theorem 2.4, these appear at the convex or the concave points of the Hamiltonian function. Away from the singularity, the viscosity solution is given by W_t. In [31,34] we have constructed the unique viscosity solution past the first critical time by selecting a single-valued branch of W_t. Assume that the singularity of type ${}^1A_3$ appears at the point (t_0, x_0, p_0). After the critical time t_0, the wave front W_t is three-valued on an interval $(x_1(t), x_2(t))$; see Figure 3b. We can choose the viscosity solution past t_0 by selecting a continuous single-valued branch of W_t as in Figure 3c. In view of Theorem 2.5 the viscosity criterion is satisfied across $\chi(t)$ while y_v is a classical solution away from $\chi(t)$. Hence, by the uniqueness of the viscosity solution, this gives the viscosity solution of (H) past t_0.





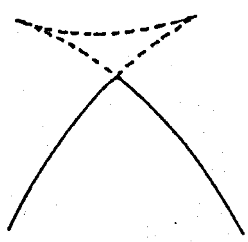


Figure 3a Figure 3b Figure 3c

By this construction, we have extended the viscosity solution beyond the first critical time t_0. According to Theorem 2.5 the shock is generated in a convex or concave domains of $H(p)$, so the viscosity criterion is automatically satisfied. The graph of the viscosity solution past the first critical time is depicted by a full line in Figure 1c, where we assume that H is convex in the neighborhood of the appearing singularity ${}^1A_3$. The shock corresponds to the intersection of the two branches and it is called a *genuine shock*. The genuine shock is defined as the intersection of two incoming characteristics (or waves) and its speed is given by the Rankine-Hugoniot condition

$$\chi'(t) = \frac{H(y^+_{v,x}(t,\chi(t))) - H(y^-_{v,x}(t,\chi(t)))}{y^+_{v,x}(t,\chi(t)) - y^-_{v,x}(t,\chi(t))},$$

where $\chi'(t) = \frac{dx}{dt}(t)$.

Therefore in order to follow the evolution of the shock we have to study the following questions:

a) How different branches of the multi-valued graph of W_t intersecting at one point bifurcate in time.

b) If the two branches initially defining the shock continue to cross, whether the viscosity criterion is satisfied across the intersection.

The normal forms of the generic perestroikas of different branches of W_t are given in Theorem 2.1. If the viscosity criterion is satisfied at the time $t_\alpha = t_0 + \varepsilon$, we can choose the correct branch of the graphs of the geometric solutions as viscosity solutions. See Figure 4 (All pictures can be turned upside down).

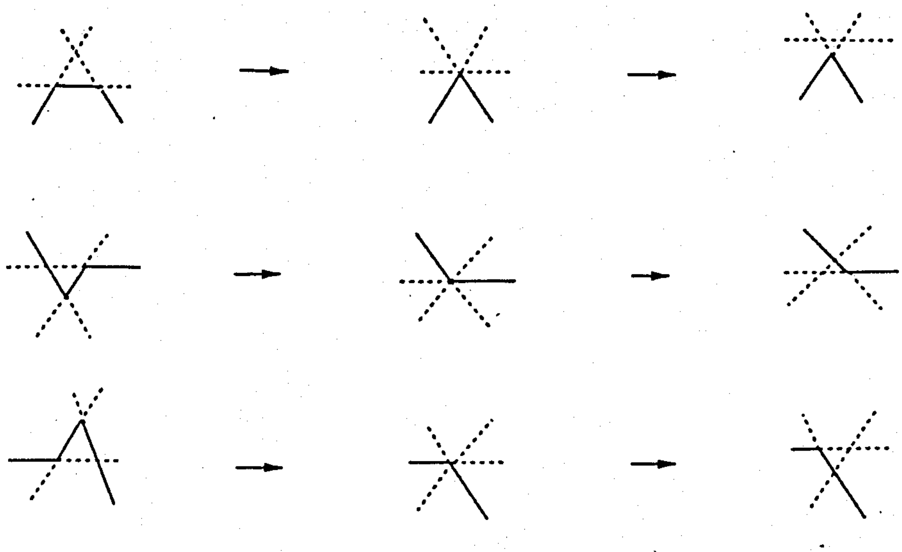


Figure 4

We will now investigate how the viscosity criterion can be violated across the intersection of two branches. Assume that a generated shock is defined by two intersecting branches y^- and y^+. We denote by y^- (resp. y^+) the branch representing the viscosity solution for $x < \chi(t)$ (resp. $x > \chi(t)$). If the two branches remain intersected they evolve according to ${}^0({}^0A_1\,{}^0A_1)$. We denote by $\chi(t)$ the intersection of the two branches. It is clear that for generic Hamiltonian function $H(p)$, H has only Morse type critical points and no tritangent lines. So we assume that the Hamiltonian has the above properties. By Theorems 2.1 and 2.2, we have the following theorem.

Theorem 3.3.[31] *For a generic initial function ϕ, if the viscosity criterion is violated at t_α, then the only following 3 cases may occur:*

(1) *The normal form is ${}^0({}^0A_1\,{}^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*

(2) *The normal form is ${}^0({}^0A_1\,{}^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*

(3) *The normal form is ${}^1A_2\,{}^0A_1$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*

Here, $P^+ = (y_x^+(t_\alpha,\chi(t_\alpha)), H(y_x^+(t_\alpha,\chi(t_\alpha))),\ P^- = (y_x^-(t_\alpha,\chi(t_\alpha)), H(y_x^-(t_\alpha,\chi(t_\alpha)))$ and $\overline{P^+P^-}$ denotes the line through P^+, P^- in the $(p, H(p))$-plane.

In [31] we have solved local Riemann problems and constructed viscosity solutions for each case in the above theorem. We have considered extra five cases, however, Bogaevskii pointed out that those extra five cases are not generic. Furthermore, he has shown that the case 3) cannot occur if the viscosity criterion is satisfied before the perestroika time t_α. So we may consider the cases 1) and 2). Here we only mention the case 1) in order to introduce the technique which is used in [31,34]. We assume that the graph of the viscosity solution at the time $t \leq t_\alpha$ is depicted as in Figure 5a.

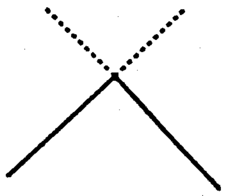


Figure 5a

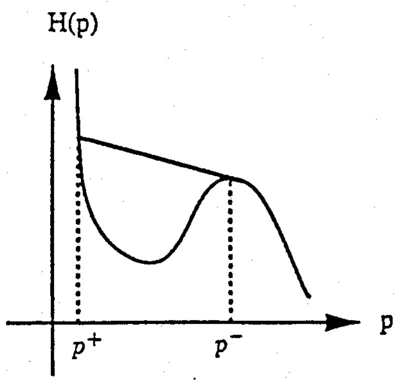


Figure 5b

For convenience, we assume that $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at the point $(y_x^-(t_\alpha, \chi(t_\alpha)), H(y_x^-(t_\alpha, \chi(t_\alpha)))$ and $H''((y_x^-(t_\alpha, \chi(t_\alpha))) < 0$ (see Figure 5b). As we already mentioned that the genuine shocks satisfies the Rankin-Hugoniot condition. So we should construct new characteristics which satisfies both of the Rankin-Hugoniot condition and the viscosity criterion. In this case we have

$$H'(y_x^-(t_\alpha, \chi(t_\alpha))) = \frac{H(y_x^+(t_\alpha, \chi(t_\alpha))) - H(y_x^-(t_\alpha, \chi(t_\alpha)))}{y_x^+(t_\alpha, \chi(t_\alpha)) - y_x^-(t_\alpha, \chi(t_\alpha))} = \chi'(t_\alpha).$$

If

$$H'(y_x^-(t, \chi(t))) < \frac{H(y_x^+(t, \chi(t))) - H(y_x^-(t, \chi(t)))}{y_x^+(t, \chi(t)) - y_x^-(t, \chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is violated for $t_\alpha < t < t_\alpha + \varepsilon$, so that a new way to build the solution is required (cf., Figure 6).

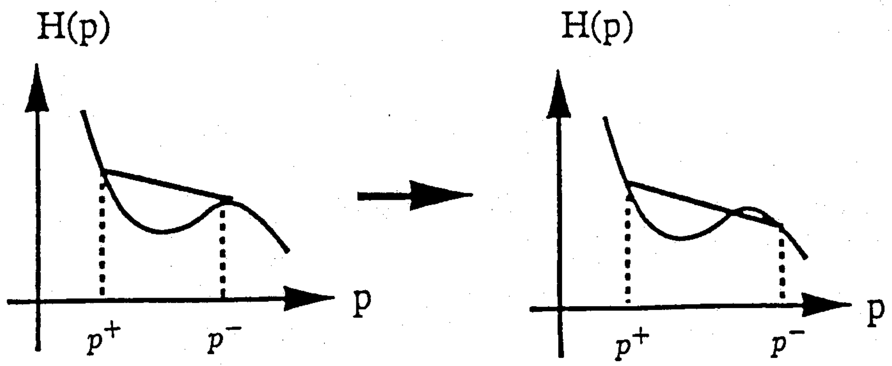


Figure 6

Lets consider the relation $H'(q) = \frac{H(p)-H(q)}{p-q}$ around (q_0, p_0) with $q_0 \neq p_0$, $H'(q_0) = \frac{H(p_0)-H(q_0)}{p_0-q_0}$ and $H''(q_0) \neq 0$. By the implicit function theorem, there exists a smooth function ψ around p_0 such that the above relation is equivalent to $q = \psi(p)$. We first construct *the contact discontinuity shock curve* as the solution of the following initial value problem.

$$\begin{cases} \chi_c'(t) = H'(\psi(y_x^+(t, \chi_c(t)))), \\ \qquad \chi_c(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau, \chi_c(\tau))$ should be satisfied the following:

$$\begin{cases} x'(t) = H'(p(t)), \\ p'(t) = 0 \\ y'(t) = -H(p(t)) + p(t)H'(p(t)), \end{cases}$$

with initial condition $x(\tau) = \chi_c(\tau)$, $y(\tau) = y^+(\tau, \chi_c(\tau))$ and $p(\tau) = \psi(y_x^+(\tau, \chi_c(\tau)))$. So the solution is exactly given as follows:

$$\begin{cases} \tilde{x}(t) = \chi_c(\tau) + (t-\tau)H'(\psi(y_x^+(\tau, \chi_c(\tau)))), \\ \tilde{p}(t) = \psi(y_x^+(\tau, \chi_c(\tau))) \\ \tilde{y}(t) = y^+(\tau, \chi_c(\tau)) + (t-\tau)\{-H(\psi(y_x^+(\tau, \chi_c(\tau)))) + \psi(y_x^+(\tau, \chi_c(\tau)))H'(\psi(y_x^+(\tau, \chi_c(\tau))))\}. \end{cases}$$

We have shown that if the viscosity criterion is violated for $t > t_\alpha$, then the contact discontinuity curve χ is convex and the viscosity solution can be constructed. Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.

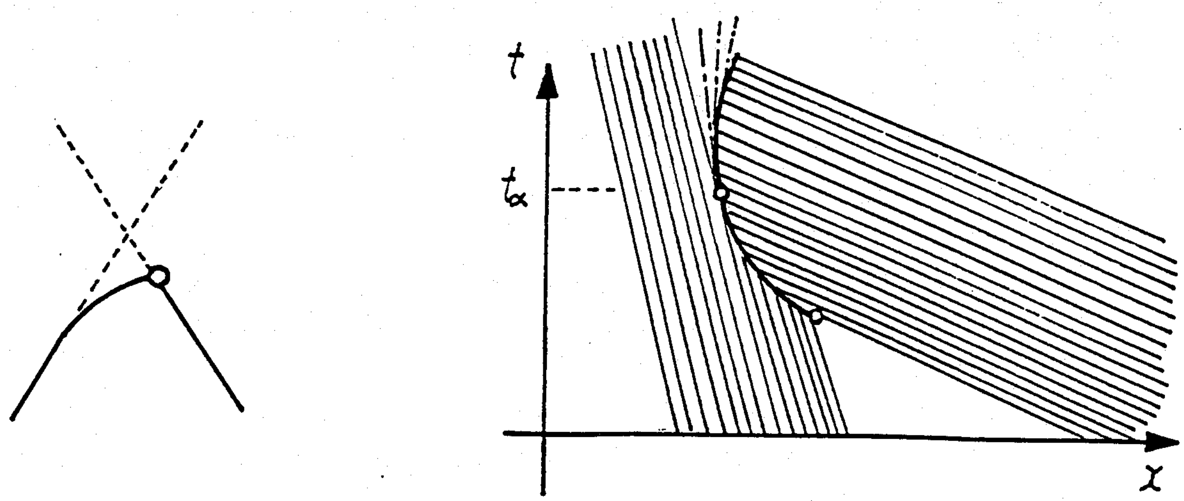


Figure 7

We have constructed the viscosity solutions for other cases. The perestroikas of the graphs of viscosity solutions is depicted as in Figure 8. In these construction there appeared not only the contact discontinuity shock curves but also the rarefaction wave type shock curve (see (3) of Figure 8). So we have two different kinds of shocks after the viscosity criterion is violated. As a consequence of the above arguments that the graph of a viscosity solution is not always a subset of the graph of the geometric solution for (H). Until a couple of years ago, there are very few people had believed this fact. Now this fact is known to many mathematicians (cf.,[45]).

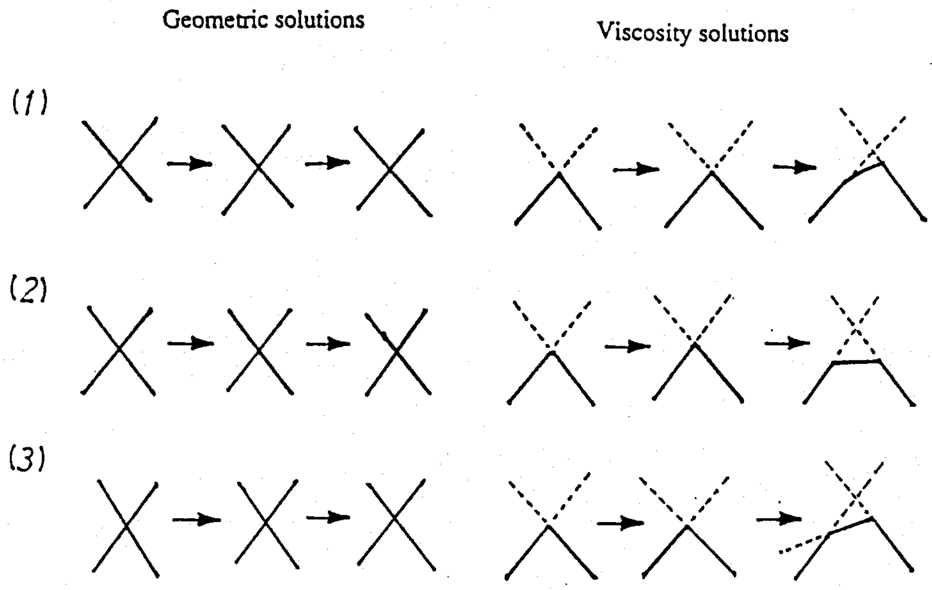


Figure 8

As a conclusion of this section, we present some problems.

Problem (3.a) For $n = 1$, the last arguments describes the perestroikas of shocks around the first time t_α when the viscosity criterion is violated.

(1) What happens after that?

(2) Can we proceed the similar arguments in the case that the Hamiltonian depends also on (t, x)?

Problem (3.b) For the case $n \geq 2$, we have the following question:

(1) How the viscosity criterion should be violated?

(2) How can we construct the viscosity solution after the viscosity criterion is violated? What is the contact discontinuity in the case $n \geq 2$?

(3) Find the crucial example which describes the situation of the problem (1) and (2).

(4) (A problem posed by Bogaevskii). Are there any useful formulae to classify the perestroikas of shocks such as the Hopf-Lax type formula?

4 Geometric solutions for single conservation laws

Since $\dfrac{\partial y}{\partial t} + \sum_{i=1}^{n} \dfrac{\partial f_i}{\partial x_i}(y) = \dfrac{\partial y}{\partial t} + \sum_{i=1}^{n} \dfrac{df_i}{dy}(y)\dfrac{\partial y}{\partial x_i}$, single conservation laws are the special case of time-dependent first order quasi-linear partial differential equations. In [30] we have constructed the geometric framework of the single conservation lows in the projective cotangent bundle

$$\Pi : PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}) \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$$

Because of the trivialization $PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})^*$, we can choose $((t, x_1, \ldots, x_n, y), [\sigma; \xi_1; \ldots; \xi_n; \eta])$ as the homogeneous coordinate system, where $[\sigma; \xi_1; \ldots; \xi_n; \eta]$ is the homogeneous coordinates of the projective space $P(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})^*$. The canonical contact structure is given by the following hyperplane field:

$$K_{((t,x,y),[\sigma;\xi;\eta])} = \{X | \nu\sigma + \sum_{i=1}^{n} \mu_i \xi_i + \lambda\eta = 0, d\Pi(X) = \nu\frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \lambda\frac{\partial}{\partial y}\}.$$

An immersion $i : L \to PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ is said to be *a Legendrian immersion* if $\dim L = n + 1$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. *A single conservation law* is considered to be a hypersurface

$$E(1, f_1, \ldots, f_n) = \{((t, x, y), [\sigma; \xi; \eta]) \in PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}) | \sigma + \sum_{i=1}^{n} \frac{df_i}{dy}(y)\xi_i = 0\}.$$

A geometric solution of $E(1, f_1, \ldots, f_n)$ is a Legendrian submanifold L of $PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ lying in $E(1, f_1, \ldots, f_n)$ such that $\Pi|L$ is an embedding. We consider the meaning of the notion of geometric solutions. Let S be a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, then we have a unique Legendrian submanifold $\hat{S}$ in $PT^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ such that $\Pi(\hat{S}) = S$ which is given as follows :

$$\hat{S} = \{((t, x, y), [\sigma; \xi; \eta]) | \sigma\frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial y} \in TS^{\perp} \subset T(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})\},$$

where we adopt the canonical inner product on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$. It follows that if L is a geometric solution of $E(1, f_1, \ldots, f_n)$, then we have $L = \widehat{\Pi(L)}$.

By the classical theory of first order partial differential equations [1], the characteristic vector field of $E(1, f_1, \ldots, f_n)$ is defined to be $X(1, f_1, \ldots, f_n) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{df_i}{dy}(y)\frac{\partial}{\partial x_i}$. Then we can easily verify the following characterization theorem of geometric solutions.

Proposition 4.1. *Let S be a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$. Then $\hat{S}$ is a geometric solution of $E(1, f_1, \ldots, f_n)$ if and only if the characteristic vector field $X(1, f_1, \ldots, f_n)$ is tangent to S.*

We can solve the Cauchy problem (C) by the method of characteristics. Set $a_i(y) = \frac{df_i}{dy}(y)$, $1 \leq i \leq n$. Then the characteristic equation (i.e., the equation corresponding to the characteristic vector field) associated with (C) through $(0, x_0)$ is given as follows:

$$\begin{cases} \dfrac{dx_i}{dt}(t) = a_i(y(t, x(t)), \quad x_i(0) = x_{i,0} \\ \dfrac{dy}{dt}(t, x(t)) = 0, \quad y(0, x(0)) = \phi(x_0). \end{cases}$$

In this case the solution of the characteristic equation can be exactly solved as follows:

$$x(t) = x_0 + ta(\phi(x_0)) \text{ and } y(t, x(t)) = y(0, x(0)) = \phi(x_0).$$

We define the corresponding embedding $\mathcal{I}[a, \phi] : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ by $\mathcal{I}[a, \phi](t, u) = (t, u + ta(\phi(u)), \phi(u))$, where $a = (a_1, \ldots, a_n)$. By definition the characteristic vector field is always tangent to the image of this embedding, so that it gives the geometric solution of (C). We also define mappings $F[a, \phi] : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by $F[a, \phi](t, u) = u + ta(\phi(u))$ and $f[a, \phi; t_0] : \mathbb{R}^n \to \mathbb{R}^n$ by $f[a, \phi; t_0](u) = F[a, \phi](t_0, u)$.

By the classical theory of characteristics, it follows that the classical solution of (C) is expressed by $y(t_0, x) = \phi(f[a, \phi; t_0]^{-1}(x))$ at x which is not critical value of $f[a, \phi; t_0]$. Then our problem is to study the singularities of $f[a, \phi; t_0]$. We can calculate the Jacobian matrix of $f[a, \phi; t_0]$ at u_0 as follows:

$$J(f[a, \phi; t_o]) = \begin{pmatrix} 1 + c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & 1 + c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & 1 + c_{nn} \end{pmatrix}$$

where $c_{ij} = t_0 \frac{da_i}{dy}(\phi(u_0)) \frac{\partial \phi}{\partial u_j}(u_0)$.

In order to study the perestroikas of singularities of $f[a, \phi; t]$ along t, we introduce the notion of one-parameter unfoldings of immersion. Let R be an $(n+1)$-dimensional smooth manifold and $\mu(R, u_0) \to (\mathbb{R}, t_0)$ be a submersion germ. We call a map germ $\mathcal{I} : (R, u_0) \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ of the form $\mathcal{I}(t, x) = (\mu(u), x(u), y(u))$ *an unfolding of immersion* if $\mathcal{I}$ is an immersion germ and $\mathcal{I}|\mu^{-1}(t)$ is also an immersion for each $t \in (\mathbb{R}, t_0)$.

For the study of the perestroikas of singularities of unfoldings of immersion, we introduce the following equivalence relation. Let $\mathcal{I}_i : (R, u_i) \to (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, (t_i, x_i, y_i))$ $(i = 0, 1)$ be unfoldings of immersion. We say that $\mathcal{I}_0$ and $\mathcal{I}_1$ are *t-P-$\mathcal{A}$-equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), (t_0, x_0, y_0)) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), (t_1, x_1, y_1))$ of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x), \phi_3(t, x, y))$ and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u_1)$ such that $\hat{\Phi} \circ \mathcal{I}_0 = \mathcal{I}_1 \circ \Psi$.

Theorem 4.2.[30,47] *The generic unfolding of immersion is t-P-$\mathcal{A}$-equivalent to one of the unfoldings of immersion in the following list:*

$({}^0A_k)$ $\qquad (t, u_n^{k+1} + \sum_{i=1}^{k-1} u_i u_n^i, u_1, \ldots, u_n) \ (0 \le k \le n)$

$$({}^1A_k) \qquad (t, u_n^{k+1} + u_n^{k-1}(t \pm u_{k-1}^2 \pm \cdots \pm u_{n-1}^2) + \sum_{i=1}^{k-2} u_i u_n^i, u_1, \ldots, u_n) \ (2 \le k \le n+1).$$

We remark that the germ of type ${}^1A_2$ describes the situation that how the singularity of the geometric solution appears or vanishes. We can also define the notion of *stable unfoldings of immersion* exactly the same way as the usual definition of the stable map germs in [19]. Then the above theorem gives a generic classification of the stable unfoldings of immersion.

Theorem 4.3.[30] There exists a dense subset $\mathcal{O} \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ with the following properties: For any $\phi \in \mathcal{O}$ and $(t_0, u_0) \in \mathbb{R} \times \mathbb{R}^n$, the germ $I[a, \phi]$ at (t_0, u_0) is t-P-$\mathcal{A}$-equivalent to one of germs in the list of Theorem 4.2.

In [30] we have shown that ${}^0A_k$-type germs in Theorem 4.2 can always be realized as geometric solutions for a given single conservation law. However, if an unfolding of immersion has a non-trivial bifurcation, the situation is different. It is known that the unfolding of immersion of type ${}^1A_2$ is not t-P-$\mathcal{A}$-equivalent to any germs of geometric solutions of $\frac{\partial y}{\partial t} + y^2 \frac{\partial y}{\partial x_i} = 0$.

Hence, we need a kind of non-degeneracy condition on the single conservation law to realize nontrivial bifurcations. The single conservation law $E(1, f_1, \ldots, f_n)$ is *non-degenerated* at (t_0, x_0, y_0) if $\dfrac{d^2 f}{dy^2}(y) \neq 0$ at (t_0, x_0, y_0), where $\dfrac{d^2 f}{dy^2}(y) = (\dfrac{d^2 f_1}{dy^2}(y), \ldots, \dfrac{d^2 f_n}{dy^2}(y))$. In [30] we have shown the following realization theorem.

Theorem 4.4.[30] *Let $\mathcal{I} : (R, u_0) \to (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, (t_0, x_0, y_0))$ be a stable unfolding of immersion. Then*

(1) *If $\mathcal{I}$ has a trivial bifurcation, then there exists an unfolding of immersion $\mathcal{I}'$ such that* Image$\mathcal{I}'$ *is a local geometric solution of $E(1, f_1, \ldots, f_n)$ and $\mathcal{I}$, $\mathcal{I}'$ are t-P-$\mathcal{A}$-equivalent.*

(2) *If the equation $E(1, f_1, \ldots, f_n)$ is non degenerated at (t_0, x_0, y_0), then there exists an unfolding of immersion $\mathcal{I}'$ such that* Image$\mathcal{I}'$ *is a local geometric solution of $E(1, f_1, \ldots, f_n)$ and $\mathcal{I}$, $\mathcal{I}'$ are t-P-$\mathcal{A}$-equivalent.*

(3) *If an ${}^1A_k$ singularity appears at a point (t_0, x_0, y_0), then the equation $E(1, f_1, \ldots, f_n)$ is non-degenerated at (t_0, x_0, y_0).*

We have also classified the multi-unfolding of immersionin [30], however, we have no space to describe it.

5 Entropy solutions for single conservation laws

An entropy solution for (C) is a function y_e satisfying the following: for any $\psi \in C^\infty(\mathbb{R}^{1+n})$

$$(W) \qquad \iint_{\mathbb{R}^+ \times \mathbb{R}^n} (y_e \frac{\partial \psi}{\partial t} + \sum_{i=1}^n f_i(y_e) \frac{\partial \psi}{\partial x_i}) dt dx + \int_{\mathbb{R}^n} \phi \psi(0, x) dx = 0$$

and for any $\psi \in C_0^\infty(\mathbb{R}^{1+n}), g \geq 0$, and any $k \in \mathbb{R}$,

$$(E) \qquad \iint_{\mathbb{R}^+ \times \mathbb{R}^n} \operatorname{sgn}(y_e - k)\{(y_e - k)\frac{\partial \psi}{\partial t} + \sum_{i=1}^{n}(f_i(y_e) - f_i(k))\frac{\partial \psi}{\partial x_i}\}dtdx \geq 0.$$

We call the condition (E) *an entropy condition.* For the existence and uniqueness of entropy solutions, see, for example Kruzkov[36].

If we solve (C) by the characteristic method like as in §4, before the characteristics cross, the classical solution $y(t,x) = \phi(f[a,\phi;t]^{-1}(x))$ is the entropy solution. After the critical time t_0 when the characteristic cross, that solution develops discontinuities (shocks).

Firstly we consider the case when $n = 1$. Under the assumption that $f'(y)$ is uniformly convex, Lax[37] has given an explicit formula for the entropy solution. There are some results on the shocks for the entropy solution by using this formula. For non-convex $f'(y)$, Ballow[3], Guckenheimer[21], Jenning[32], Chen[10] and Dafermos[14] etc.,studied how shocks generate and propagate.

In this case the entropy condition can be replaced by the following rather a familiar condition: Assume that $\mathcal{O} \subset (0,\infty) \times \mathbb{R}$ is open and that there is a smooth curve $t \to x = \chi(t)$, $t \in (t_1,t_2) \subset \mathbb{R}^+$, with negative slope, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-$, $\Gamma = \{(t,x) : x = \chi(t)\}$, where $\mathcal{O}^+$ lies on the right of Γ. Then we have the following theorem.

Theorem 5.1. [32,42] *Let y_e be the solution satisfies* (W) *and $y_e = y_e^+$ in $\mathcal{O}^+ \cup \Gamma, y_e = y_e^-$ in $\mathcal{O}^- \cup \Gamma$ where $y_e^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma)$. Then y_e satisfies the entropy condition* (E) *in $\mathcal{O}$ if and only if the following conditions hold:*

a) y_e^+ and y_e^- are classical solutions of (C) *in $\mathcal{O}^+$ and $\mathcal{O}^-$ respectively,*

b) If $y_e^- > y_e^+$ (resp. $y_e^- < y_e^+$) across Γ, then

$$f\left((1-\lambda)y_e^+ + \lambda y_e^-\right) - (1-\lambda)f(y_e^+) - \lambda f(y_e^-) \leq 0 \ \ (resp. \ \geq 0),$$

where $\lambda \in [0,1]$. That is, the graph of f lies respectively below or above the line segment joining the points $(y_e^+, f(y_e^+))$ and $(y_e^-, f(y_e^-))$.

We can recognize that the viscosity criterion is the analogous property of the entropy condition (cf., Theorem 3.2). We also have the following Rankin-Hugoniot condition for the shock curve $\chi(t)$ of the entropy solution [32,42]:

$$\chi'(t) = \frac{f(y_e^+) - f(y_e^-)}{y_e^+ - y_e^-}.$$

On the other hand, there is a relation between (H) and (C) as follows: Differentiating the viscosity solution of $\frac{\partial y}{\partial t} + H(\frac{\partial y}{\partial x}) = 0$ with respect to x in the region of smoothness we have $\frac{\partial^2 y}{\partial t \partial x} + \frac{dH}{dp}(\frac{\partial y}{\partial x})\frac{\partial^2 y}{\partial x^2} = 0$. By Theorem 5.1, we can show that the derivative of the viscosity solution is the entropy solution of the scalar conservation law $\frac{\partial u}{\partial t} + \frac{dH}{dp}(u)\frac{\partial u}{\partial x} = 0$. So the classification of Bogaevskii[6] gives a complete list of the perestroikas of shocks for (C) in the case when $f'(y)$ is uniformly convex. Figure 8 also gives a classification of the perestroikas of shocks at the first time t_α when the entropy condition is violated.

For the several dimensional case, there are very few result on this subjects. Nakane [39] has constructed the entropy solution for (C) by selecting the proper discontinuous branch of the geometric solution past the first critical time. Here, we only refer the articles of Guckenheimer[22] and Wagner[46]. A different geometric framework for the study of shock waves for single conservation laws and systems of conservation laws using the concept of generalized characteristics has been given by Dafermos [15].

We also give some problems concerning the perestroikas of the shocks of the entropy solution of (C).

Problem (5.a) What is the notion of "convexity" of $(f_1(y), \ldots, f_n(y))$ for $n \geq 2$? One of the candidate is that each $f_i(y)$ is uniformly convex for $i = 1, 2, \ldots, n$. Under this condition, can we have a (useful) explicit formula for the entropy solution of (C) such as the Hopf-Lax type formula for the viscosity solution?

Problem (5.b) Solve the Riemann problem for each normal form of the multi-unfoldings of immersion given in [30].

Problem (5.c) Consider the system case. The geometric singularities of multi-valued solutions for first order systems have been studied by Rakhimov [40] and Caflisch et al [8], however,they have not classified the perestroikas of singularities for the Cauchy problem.

(1) Classify the perestroikas of geometric singularities.

(2) How can we solve the Riemann problem for each normal forms?

(3) (Important!) What is the correct notion of weak solutions?

References

[1] V. I. Arnol'd, *Geometric Methods in the Theory of Ordinary Differential Equations,* Springer-Verlag, 1983

[2] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps,* Birkhauser, (1986).

[3] D. P. Ballou, *Solutions to nonlinear hyperbolic Cauchy problems without convexity conditions,* Trans. Amer.Math. Soc. **152** (1970), 441–460.

[4] M. Bardi and L. C. Evans, *On Hopf's formulas for solutions of hamilton-Jacobi equations,* Nonlinear Analysis, **8** (1984), 1373–1389

[5] P. Bernhard, *Singular surfaces in differential games, an introduction; in Differential Games and Applications, Lecture Notes in Control and Information Sciences ed. by P. Hagedorn et al.* **3** Springer Verlag (1977), 1–33

[6] I. A. Bogaevskii, *Modifications of singularities of minimum functions and bifurcations of shock waves at the Burgers equation with vanishing viscosity,* Leningrad Math. J. **1** (1990) 807-823

[7] I. A. Bogaevskii, *Perestroikas of fronts in evolutionary families,* Proceedings of the Steklov Institute of Math., **209** (1995), 57–72

[8] R. E. Caflisch, N. Ercolani, T. Y. Hou and Y. Landis, *Multi-valued solutions and branch point singularities for nonlinear hyperbolic systems,* Comm. Pure Appl. Math. **46** (1993), 453-499

[9] M. Chaperon, *Lois de conservation er géoétrie symplectique* , C. R. Aca. Sc. Paris **312** (1991), 345–348

[10] N. M. Chen, *On types of singularities for solutions of nonlinear hyperbolic systems,* Bull. Inst. Math. Acad. Sinica **10** (1982), 405-416

[11] M. G. Crandall and P. -Lions, *Viscosity solutions of Hamilton-Jacobi equations,* Trans. Amer. Math. Soc. **277** (1983),1–42

[12] M. G. Crandall, L.C. Evans and P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations,* Trans.Amer. Math. Soc **282** (1984), 487–502

[13] M. G. Crandall, H. Ishii and P.-Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc. **27** (1992), 1–67

[14] C. M. Dafermos, *Regularity and large time behavior of solutions of a conservation law without convexity,* Proc. Roy. Soc. Edinburgh **99A** (1985), 201-239

[15] C. M. Dafermos, *Generalized characteristics in hyperbolic systems of conservation laws,* Arch. Rational Mech. Anal. **107** (1989), 127-155

[16] L. C. Evans and P. E. Souganides, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations,* Indiana Univ. Math. J. **33** (1984), 773–797

[17] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions,* Springer-Verlag (1993)

[18] V. A. Florin, *Some simplest nonlinear problems of the consolidation of an aqueously saturated earthen medium,* Izv. Akad. Nauk SSSR Otdel.Tekhn. Nauk **9** (1948), 1389-1397

[19] C. G. Gibson, *Singular points of Smooth Mappings,* Pitman, London 1979

[20] J. Glim, D.Marchesin and O. Mcbryan, *Unstable finger in two phase flows,* Comm. Pure Appl. Math. **34** (1981), 53–75

[21] J. Guckenheimer, *Solving a single conservation law,* Lecture notes in Mathematics **468** Springer Verlag, New York, 1975, 108-134

[22] J. Guckenheimer, *Shocks and rarefaction in two space dimensions,* Arch. Rational Mech. Anal. **59** (1975), 281–291.

[23] E. Hopf *Generalized solution of non-linear equations of first order,* Jour. of Math. and Mech. **14** (1965), 951–973

[24] R. Isaacs *Differential Games,* John Wiley, New York (1965)

[25] S. Izumiya, *The theory of Legendrian unfoldings and first order differential equations,* Proc. Royal Soc. Edingburh, **123A** (1993), 517–532.

[26] S. Izumiya, *Perestroikas of optical wave fronts and graphlike Legendrian unfoldings,* J. of Differential Geometry, **38** (1993), 485–500

[27] S. Izumiya *Geometric singularities for Hamilton-Jacobi equation,* Advanced Studies in Pure Math, **22** (1993), 89–100

[28] S. Izumiya and G. T. Kossioris *Semi-local classification of geometric singularities for Hamilton-Jacobi equations,* J. of Diff. Equations, **118** (1995), 166–193

[29] S. Izumiya and G. T. Kossioris *Realization theorem of geometric singularities for Hamilton-Jacobi equations,* to appear in Commun. Anal. and Geom.

[30] S. Izumiya and G. T. Kossioris, *Geometric singularities for solutions of single conservation laws,* to appear in Arch. Rational Mech. Anal.

[31] S. Izumiya and G. T. Kossioris, *Bifurcations of shock waves for viscosity solutions of Hamilton-Jacobi equations of one space variables,* to appear in Bull. Sciences Math.

[32] G. Jennings, *Piecewise smooth solutions of a single conservation law exist,* Adv. in Math **33** (1979), 192-205.

[33] T. Joukovskaia, *Singularités de minimax et solutions faibles d'équations aux dérivées partielles,* These, Université Denis Dierot (Paris 7), (1994)

[34] G. T. Kossioris, *Propagation of singularities for viscosity solutions of Hamilton-Jacobi equations in one space variable,* Comm. P.D.E., **18** (1993), 747–770

[35] G. T. Kossioris, *Formation of singularities for viscosity solutions of Hamilton-Jacobi equations in higher dimensions,* Comm. P.D.E., **18** (1993), 1085–1108

[36] S. N. Kruzkov, *First order quasilinear equations in several independent variables,* Math. USSR Sb. **10** (1970), 217-243.

[37] P.D. Lax, *Hyperbolic systems of conservation laws II,* Comm. Pure Appl. **10** (1957), 537–566

[38] S. Nakane, *Formation of shocks for a single conservation law,* SIAM J. Math. Anal. **19** (1988), 1391–1408.

[39] S. Nakane, *Formation of singularities for Hamilton-Jacobi equations in several space variables,* J. Math. Soc. Japan **43** (1991), 89–100

[40] A. KH. Rakimov, *Singularities of Riemannian Invariants,* Funct. Anal. Appl., **27** (1993), 39-50

[41] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations,* D. Van Nostrand, London (1966)

[42] J. Smoller, *Sock waves and Reaction-Diffusion Equations,* Grund. der Math. Wiss. **258** (1980), Springer

[43] M. Tsuji, *Solution globale et propagation des singularites pour l'equation de Hamilton-Jacobi,* C. R. Acad. Sc. Paris **289** (1979), 397–400

[44] M. Tsuji, *Formation of singularities for Hamilton-Jacobi equation II,* J. Math. Kyoto Univ., **26** (1986) 299–308

[45] C. Viterbo, *Generating functions, symplectic geometry and applications,* Proceedings of the Intern. Cong. Math., (1994) 537–547

[46] D. Wagner, *The Riemann problem in two space dimensions for a single conservation law,* SIAM J. Math. Ann. **14** (1983), 534–559.

[47] V. M. Zakalyukin *Reconstructions of fronts and caustics depending on a parameter and versality of mappings,* J. of Soviet Math. **27** (1984),2713–2735.

Shyuichi IZUMIYA

Department of Mathematics,

Hokkaido University,

Sapporo 060, JAPAN

e-mail: izumiya@math.hokudai.ac.jp