Hilbert Schemes and Simple Singularities E_6, E_7 and E_8

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Hilbert Schemes and Simple Singularities E_6, E_7 and E_8

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ABSTRACT. For any finite subgroup G of $SL(2,\mathbb{C})$ of order n, we consider $\mathrm{Hilb}^G(\mathbb{A}^2)$ "a Hilbert scheme" parametrising G-invariant 0-dimensional subschemes of length n. $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of a simple singularity $\mathbb{A}^2/G$ in the canonical manner. We study $\mathrm{Hilb}^G(\mathbb{A}^2)$ in the cases of E_6, E_7 and E_8 in detail. As a consequence we have a new explanation for the (classical) two-dimensional McKay correspondence in these cases as in the same manner as in [IN2].

Introduction. It is well known that to finite subgroups of $SL(2,\mathbb{C})$, there correspond certain hypersurface singularities as the quotients of a complex affine plane $\mathbb{A}^2$ by the finite groups,which we call simple singularities.

Let G be a finite subgroup of $SL(2,\mathbb{C})$ and n the order of G. Then $\mathrm{Hilb}^G(\mathbb{A}^2)$, the Hilbert scheme of G-orbits, is by defintion a unique connected component of the fixed locus of $\mathrm{Hilb}^n(\mathbb{A}^2)$ under the naturally induced action of G which dominates $\mathrm{Chow}^n(\mathbb{A}^2)^G$ ($\simeq \mathbb{A}^2/G$) [IN2]. $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a minimal resolution of a normal surface $\mathbb{A}^2/G$ with a simple singularity [IN2]. In [IN2] $\mathrm{Hilb}^G(\mathbb{A}^2)$ has been studied in detail in the cases A_n and D_n.

The purpose of the present article is to study, as the second half of [IN2], the structure of $\mathrm{Hilb}^G(\mathbb{A}^2)$ in the cases where G is one of binary tetra-, octa- and icosa-hedral groups $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$. The corresponding singularity of $\mathbb{A}^2/G$ is of type E_6, E_7 and E_8 as is well known.

Let π be the natural morphism of $\mathrm{Hilb}^G(\mathbb{A}^2)$ onto $\mathbb{A}^2/G$, $S := \mathbb{A}^2/G$ and E the exceptional set of π. Let $\mathfrak{m}$ (resp. $\mathfrak{m}_S$) be the maximal ideal of the origin of $\mathbb{A}^2$ (resp. $\mathbb{A}^2/G$) and let $\mathfrak{n} = \mathfrak{m}_S\mathcal{O}_{\mathbb{A}^2}$. Any point $\mathfrak{p}$ of E is a G-invariant 0-dimensional subscheme Z of $\mathbb{A}^2$ with support the origin, to which we associate a G-invariant ideal subsheaf I of $\mathfrak{m}$ defining Z. Since I contains $\mathfrak{n}$ [IN2, Corollary 2.3], we set $V(I) := I/(\mathfrak{m}I + \mathfrak{n})$.

If $\mathfrak{p}$ is a smooth point of E, $V(I)$ is a nontrivial irreducible G-module and vice versa. For any equivalence class of a nontrivial irreducible G-module ρ we define a subset $E(\rho)$ of E consisting of all $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ such that $V(I)$ contains ρ as a G-submodule. The map $\rho \mapsto E(\rho)$ gives a bijective correpondence [IN2, Theorem 3.1] between the set $\mathrm{Irr}(G)$ of all the equivalence classes of irreducible G-modules and the set $\mathrm{Irr}(E)$ of all the irreducible components of E. This turns out to be the classical McKay correpondence [Mc]. The reader can consult [IN2, 5.8] for an outline of the story.

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The present paper is organized as follows. In section 6, the first section of the present article, through 8 we study $\mathrm{Hilb}^G(\mathbb{A}^2)$ and prove the main theorem [IN2, Theorem 3.1] in the cases E_n $(n = 6, 7, 8)$ respectively.

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6. E_6.

6.1. Character table. Let G be a binary tetrahedral group $\mathbb{T}$. G is by definition a subgroup of $SL(2, \mathbb{C})$ of order 24 generated by $\mathbb{D}_2 = \langle\sigma, \tau\rangle$ and μ;

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7, & \varepsilon^7 \\ \varepsilon^5, & \varepsilon \end{pmatrix}$$

where $\varepsilon = e^{2\pi i/8}$ [Slodowy, p. 74]. G operates upon $\mathbb{A}^2$ from the right by $(x, y) \mapsto (x, y)g$ $(g \in G)$. $\mathbb{D}_2$ is a normal subgroup of G and the following is exact;

$$1 \to \mathbb{D}_2 \to G \to \mathbb{Z}/3\mathbb{Z} \to 1$$

We recall the character table of G [Schur] and the other relevant invariants.

The Coxeter number $2\bar{h}$ of E_6 is equal to 12. Let $\omega = (-1 + \sqrt{3}i)/2$.

Table 6.1

ρ	1	2	3	4	5	6	7	d	$(\bar{h} \pm d)$
	1	-1	τ	μ	μ^2	μ^4	μ^5		
$\sharp$	1	1	6	4	4	4	4		
ρ_0	1	1	1	1	1	1	1	(2)	$-$
ρ_2	2	-2	0	1	-1	-1	1	1	(5,7)
ρ_3	3	3	-1	0	0	0	0	0	(6,6)
ρ_2'	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω	1	(5,7)
ρ_1'	1	1	1	ω^2	ω	ω^2	ω	2	(4,8)
ρ_2''	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2	1	(5,7)
ρ_1''	1	1	1	ω	ω^2	ω	ω^2	2	(4,8)

6.2. Symmetric tensors $\mathrm{mod}\,\mathfrak{n}$**.** Let S_m be the space of symmetric m-tensors by $\rho_{\mathrm{nat}} := \rho_2$. The G-modules S_m and $\bar{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by ρ_2 are decomposed into irreducible G-modules. We define a G-submodule of $\mathfrak{m}/\mathfrak{n}$ by $\bar{V}_i(\rho_j) := S_i(\mathfrak{m}/\mathfrak{n})[\rho_j]$ the sum of all copies of ρ in $S_i(\mathfrak{m}/\mathfrak{n})$, and define $V_i(\rho_j)$ to be a G-submodule of S_i such that $V_i(\rho_j) \simeq \bar{V}_i(\rho_j)$, $V_i(\rho_j) \equiv \bar{V}_i(\rho_j) \,\mathrm{mod}\, \mathfrak{n}$. We use $V_i(\rho_j)$ and $\bar{V}_i(\rho_j)$ without distinction when harmless. For a G-module W we define $W[\rho]$ to be the sum of all the copies of ρ in W.

There are G-invarinat polynomials A_6, A_8, A_6^2 and A_{12} respectively of homogeneous degrees 6, 8 and 12, which are known by [Klein, p.51]. In his notation we may assume $A_6 = T$, $A_8 = W$ and $A_{12} = \phi^3$. See the paragraph 6.4.

Since $\bar{S}_m$ is the quotient of S_m by a submodule of S_m generated by all G-invariants, $\bar{S}_m$ is computed by [IN3] as follows.

$$\begin{aligned}
\bar{S}_6 &= S_6/S_0 \cdot A_6 = 2\rho_3\\
\bar{S}_7 &= S_7/S_1 \cdot A_6 = \rho_2 + \rho_2' + \rho_2''\\
\bar{S}_8 &= S_8/S_0 \cdot A_8 + S_2 \cdot A_6 = \rho_1' + \rho_1'' + \rho_3\\
\bar{S}_9 &= S_9/S_1 \cdot A_8 + S_3 \cdot A_6 = \rho_2' + \rho_2''\\
\bar{S}_{10} &= S_{10}/S_2 \cdot A_8 + S_4 \cdot A_6 = \rho_3\\
\bar{S}_{11} &= S_{11}/S_3 \cdot A_8 + S_5 \cdot A_6 = \rho_2\\
\bar{S}_{12} &= S_{12}/S_0 \cdot A_{12} + S_4 \cdot A_8 + S_6 \cdot A_6 = 0
\end{aligned}$$

where $S_0 \cdot A_{12} + S_4 \cdot A_8 + S_6 \cdot A_6 = \rho_0 + (\rho_1' + \rho_1'' + \rho_3) + (\rho_0 + 2\rho_3)$.

For small values of m the decomposition of S_m and $\bar{S}_m$ are as follows. The factors of $\bar{S}_m$ in the brackets are those in S_{McKay}.

Table 6.2

m	S_m	$\bar{S}_m$
0	ρ_0	0
1	ρ_2	ρ_2
2	ρ_3	ρ_3
3	$\rho_2' + \rho_2''$	$\rho_2' + \rho_2''$
4*	$\rho_1' + \rho_1'' + \rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
5*	$\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
6*	$\rho_0 + 2\rho_3$	$(2\rho_3)$
7*	$2\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
8*	$\rho_0 + \rho_1' + \rho_1'' + 2\rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
9	$\rho_2 + 2\rho_2' + 2\rho_2''$	$\rho_2' + \rho_2''$
10	$\rho_1' + \rho_1'' + 3\rho_3$	ρ_3
11	$2\rho_2 + 2\rho_2' + 2\rho_2''$	ρ_2
12	$2\rho_0 + \rho_1' + \rho_1'' + 3\rho_3$	0

Thus we see

Lemma 6.3. *As G-modules, we have*

(1) $V_{6\pm d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ *or* ρ *if* $d(\rho) = 0$ *resp. if* $d(\rho) \geq 1$.

(2) $\bar{S}_{6-k} \simeq \bar{S}_{6+k}$ *for any* k.

Theorem 3.2 for $G = \mathbb{T}$ follows from Table 6.2 immediately.

6.4. Generators of $V_j(\rho)$. Let

$$\begin{gathered} p_1 = x^2 - y^2, p_2 = x^2 + y^2, p_3 = xy \\ q_1 = x^3 + (2\omega+1)xy^2, q_2 = y^3 + (2\omega+1)x^2y \\ s_1 = x^3 + (2\omega^2+1)xy^2, s_2 = y^3 + (2\omega^2+1)x^2y \\ \gamma_1 = x^5 - 5xy^4, \gamma_2 = y^5 - 5x^4y \\ T = p_1p_2p_3, \phi = p_2^2 + 4\omega p_3^2, \psi = p_2^2 + 4\omega^2 p_3^2, W = \phi\psi. \end{gathered}$$

We note that $\mathfrak{n}$ is generated by T, W and ϕ^3 (or ψ^3) by [Klein, p.51].
By computations we have

Table 6.3

$$\begin{array}{ll} V_1(\rho_2) = \{x, y\} & V_7(\rho_2) = V_3(\rho_2'')\phi \equiv V_3(\rho_2')\psi \\ V_2(\rho_3) = \{x^2, xy, y^2\} & V_7(\rho_2') = V_3(\rho_2'')\psi \\ V_3(\rho_2') = \{q_1, q_2\} & V_7(\rho_2'') = V_3(\rho_2')\phi \\ V_3(\rho_2'') = \{s_1, s_2\} & V_8(\rho_1') = \{\psi^2\} \\ V_4(\rho_1') = \{\phi\} & V_8(\rho_1'') = \{\phi^2\} \\ V_4(\rho_1'') = \{\psi\} & V_8(\rho_3) = V_4(\rho_3)\phi \equiv V_4(\rho_3)\psi \\ V_4(\rho_3) = \{p_1p_2, p_2p_3, p_3p_1\} & V_9(\rho_2') = V_1(\rho_2)\psi^2 \\ V_5(\rho_2) = \{\gamma_1, \gamma_2\} & V_9(\rho_2'') = V_1(\rho_2)\phi^2 \\ V_5(\rho_2') = V_1(\rho_2)\phi & V_{10}(\rho_3) = V_2(\rho_3)\phi^2 \equiv V_2(\rho_3)\psi^2 \\ V_5(\rho_2'') = V_1(\rho_2)\psi & V_{11}(\rho_2) = V_3(\rho_2')\phi^2 \equiv V_3(\rho_2'')\psi^2 \\ V_6(\rho_3) = V_2(\rho_3)\phi \oplus V_2(\rho_3)\psi & \end{array}$$

where $\equiv$ stands for congruence $\operatorname{mod} \mathfrak{n}$ and we use the relations

$$\begin{gathered} \rho_2' = \rho_1' \cdot \rho_2 = \rho_1'' \cdot \rho_2'', \quad \rho_2'' = \rho_1' \cdot \rho_2' = \rho_1'' \cdot \rho_2, \\ \rho_2 = \rho_1' \cdot \rho_2'' = \rho_1'' \cdot \rho_2', \quad \rho_3 = \rho_1' \cdot \rho_3 = \rho_1'' \cdot \rho_3. \end{gathered}$$

In view of Table 6.2 each ieeducible G-factor appears in $\bar{S}_m$ with multiplicity at most one except when $m = 6$, $\rho = \rho_3$. Therefore the following congruence of G-modules $\operatorname{mod} \mathfrak{n}$ are clear from the fact that these G-modules $\operatorname{mod} \mathfrak{n}$ are nontrivial.

$$\begin{array}{ll} V_3(\rho_2'')\phi \equiv V_3(\rho_2')\psi, & V_4(\rho_3)\phi \equiv V_4(\rho_3)\psi \\ V_1(\rho_2)\phi^2 \equiv V_5(\rho_2)\psi, & V_5(\rho_2)\phi \equiv V_1(\rho_2)\psi^2 \\ V_2(\rho_3)\phi^2 \equiv V_2(\rho_3)\psi^2, & V_3(\rho_2')\phi^2 \equiv V_3(\rho_2'')\psi^2. \end{array}$$

For instance, $s_i\phi - q_i\psi \equiv 0 \operatorname{mod} T$ so that $V_3(\rho_2'')\phi \equiv V_3(\rho_2')\psi$. Since $p_1p_2(\phi - \psi) \equiv 0 \operatorname{mod} T$, $p_2p_3(\phi - \omega\psi) \equiv 0 \operatorname{mod} T$ and $p_3p_1(\phi - \omega^2\psi) \equiv 0 \operatorname{mod} T$ so that $V_4(\rho_3)\phi \equiv V_4(\rho_3)\psi$.

Lemma 6.5.

$$(1)\quad S_k\bar{V}_4(\rho_1') = \begin{cases} \rho_2' & (k=1) \\ \rho_3 & (k=2) \\ \rho_2+\rho_2'' & (k=3) \\ \rho_1''+\rho_3 & (k=4) \\ \bar{S}_{k+4} & (k\geq 5) \end{cases}$$

$$(2)\quad S_k\bar{V}_5(\rho_2') = S_{k+1}\bar{V}_4(\rho_1') \quad (k\geq 1)$$

$$(3)\quad S_k\bar{V}_5(\rho_2) = \begin{cases} \rho_3 & (k=1) \\ \rho_2'+\rho_2'' & (k=2) \\ \bar{S}_{k+5} & (k\geq 3) \end{cases}$$

$$(4)\quad S_1\bar{V}_7(\rho_2') = \rho_1'+\rho_3.$$

Proof. (1) is clear for $k=1,2$. Next we consider $S_3V_4(\rho_1')$. By Table 6.2 $S_3V_4(\rho_1') \simeq S_3\otimes V_4(\rho_1') \simeq \rho_2''+\rho_2$. We prove $S_1\cdot A_6 \neq \{S_3V_4(\rho_1')\}[\rho_2] = V_3(\rho_2'')V_4(\rho_1')$. Otherwise A_6 is divisible by $\phi\in V_4(\rho_1')$, whence $A_6/\phi\in V_2(\rho_1'')=\{0\}$, a contradiction. Hence we have $S_3\bar{V}_4(\rho_1')=\rho_2+\rho_2''$. Similarly $S_4V_4(\rho_1')=\rho_0+\rho_1''+\rho_3$ where $\{S_4V_4(\rho_1')\}[\rho_0] = S_0\cdot A_8$. The factors ρ_1'' and ρ_3 in $S_4V_4(\rho_1')$ are indivisible by A_6. In fact, otherwise $\{S_4V_4(\rho_1')\}[\rho_3]=S_2\cdot A_6$ because $S_2\simeq\rho_3$. It follows that A_6 is divisible by ϕ, which is a contradiction. Therefore $S_4\bar{V}_4(\rho_1')=\rho_1''+\rho_3$. Finally we see $S_5V_4(\rho_1')=\rho_2+\rho_2'+\rho_2''$ where $\{S_5V_4(\rho_1')\}[\rho_2]=S_1\cdot A_8$. The factors ρ_2' and ρ_2'' in $S_5V_4(\rho_1')$ are indivisible by A_6. For instance if $\{S_5V_4(\rho_1')\}[\rho_2'] = V_3(\rho_2')\cdot A_6$, then since the generators of $V_3(\rho_2')$ are coprime, A_6 is divisible by ϕ, a contradiciton. It follows that $S_5\bar{V}_4(\rho_1')=\rho_2'+\rho_2''=\bar{S}_9$. The rest of (1) is clear. (2) is clear from (1).

Next we prove (3). First we see $\dim S_1V_5(\rho_2)=4$ by Table 6.3. Therefore $S_1V_5(\rho_2)\simeq \rho_2\otimes\rho_2\simeq \mathrm{Hom}(\rho_2,\rho_2)\simeq\rho_0+\rho_3$. Hence $\{S_1V_5(\rho_2)\}[\rho_0]=S_0\cdot A_6$. It follows that $S_1\bar{V}_5(\rho_2)=\rho_3$. Next we consider $S_2V_5(\rho_2)$. Since $\dim S_1\otimes V_5(\rho_2)=4$, we have $\dim S_1\otimes V_5(\rho_2)\geq 5$. We see that $S_2V_5(\rho_2)=S_2\otimes V_5(\rho_2)=\rho_2+\rho_2'+\rho_2''$, and that $\rho_2\simeq S_1\cdot A_6\subset S_2V_5(\rho_2)$, $V_3(\rho_2'')V_4(\rho_1'')=V_7(\rho_2')\simeq\rho_2'$ and $V_3(\rho_2')V_4(\rho_1')=V_7(\rho_2'')\simeq\rho_2''$. Hence $S_2\bar{V}_5(\rho_2)=\rho_2'+\rho_2''$.

Meanwhile $S_1V_3(\rho_2'')=S_1\otimes V_3(\rho_2'')=\rho_1''+\rho_3$, so that $S_1V_7(\rho_2')=S_1V_3(\rho_2'')V_4(\rho_1'')=\rho_1'+\rho_3$. We will prove $S_1\bar{V}_7(\rho_2')=\rho_1'+\rho_3$. Otherwise, by Table 6.2 we have $\{S_1\bar{V}_7(\rho_2')\}[\rho_3]=0$ so that $\{S_1V_7(\rho_2')\}[\rho_3]=S_2A_6$. Since $\dim S_1<\deg\rho_3$, there exists a nontrivial element in $\{S_1V_7(\rho_2')\}[\rho_3]$ divisible by x and A_6. This implies that there exists a nontrivial element of $V_7(\rho_2')$ divisible by A_6. It follows that $V_7(\rho_2')$ is divisible by A_6, hence $V_1(\rho_2')=A_6^{-1}V_7(\rho_2')\neq\{0\}$, a contradiction. Therefore $\{S_1V_7(\rho_2')\}[\rho_3]=\rho_3$ and $S_1\bar{V}_7(\rho_2')=\rho_1'+\rho_3$. Similarly $S_1\bar{V}_7(\rho_2'')=\rho_1''+\rho_3$. This proves (4). Moreover $S_3\bar{V}_5(\rho_2)=S_1S_2\bar{V}_5(\rho_2)=S_1(\bar{V}_7(\rho_2')+\bar{V}_7(\rho_2''))$ so that $S_3\bar{V}_5(\rho_2)\supset\rho_1'+\rho_1''+\rho_3=\bar{S}_8$. This proves (3). □

Lemma 6.6. *Let $W_k=S_1\cdot\bar{V}_5(\rho_2^{(k)})$ $(\simeq\rho_3)$ for any $k=0,1,2$ where $\rho_2^{(k)}=\rho_2,\rho_2',\rho_2''$. Let $W\in\mathbb{P}(V_6(\rho_3))$. Then $S_1W=\rho_2+\rho_2'+\rho_2''$ if and only if $W\neq W_k$ $(k=1,2,3)$.*

Proof. We see $S_1\cdot W_1=S_2\cdot\bar{V}_5(\rho_2')=S_3\cdot\bar{V}_4(\rho_1')=\rho_2+\rho_2''$ by Lemma 6.5. Similarly $S_1\cdot W_2=S_3\cdot\bar{V}_4(\rho_1'')=\rho_2+\rho_2'$. Also by Lemma 6.5 (3) we have $S_1\cdot W_0=\rho_2'+\rho_2''$.

Conversely assume $W\neq W_k$ for any k. Choose and fix a G-module isomorphism $h:W_1\to W_2$. For instance, $h(p_k\phi)=\omega^{-k}p_k\psi$. Then h induces a natural isomorphism $\{S_1\otimes h\}[\rho_2]:\{S_1\otimes W_1\}[\rho_2]\to\{S_1\otimes W_2\}[\rho_2]$, which induces an isomorphism $\{S_1\cdot h\}[\rho_2]:$

$\{S_1 \cdot W_1\}[\rho_2] \to \{S_1 \cdot W_2\}[\rho_2]$. Since $\bar{S}_7$ contains a single ρ_2, we have $\{S_1 \cdot W_1\}[\rho_2] = \{S_1 \cdot W_2\}[\rho_2]$ ($\simeq \rho_2$). It follows that $\{S_1 \cdot h\}[\rho_2]$ is a nonzero constant multiple of the identity. Since $V_6(\rho_3) = W_1 \oplus W_2$, this proves uniqueness of a G-submodule W ($\simeq \rho_3$) of $V_6(\rho_3)$ such that $\{S_1 \cdot W\}[\rho_2] = 0$. Since $\{S_1 \cdot W_0\}[\rho_2] = 0$, we have $\{S_1 \cdot W\}[\rho_2] \neq 0$ by the assumption $W \neq W_0$. Similarly there exists a unique proper G-submodule $W \in V_6(\rho_3)$ such that $\{S_1 \cdot W\}[\rho_2'] = 0$ or $\{S_1 \cdot W\}[\rho_2''] = 0$. As we saw above, $\{S_1 \cdot W_1\}[\rho_2'] = 0$ and $\{S_1 \cdot W_2\}[\rho_2''] = 0$. Therefore $S_1 \cdot W = \rho_2 + \rho_2' + \rho_2''$ if $W \neq W_k$ $(k = 0, 1, 2)$. □

[IN2, Theorem 3.6] is restated as follows. This is true also for D_n, E_7 and E_8.

Theorem 6.7. *Let E be the exceptional set of the morphism $\pi : X_G \to S_G$, $\mathrm{Sing}(E)$ the singular points of E. Let $E(\rho)$ be an irreducible component of E and $E^0(\rho) := E(\rho) \setminus \mathrm{Sing}(E)$, $P(\rho, \rho') = E(\rho) \cap E(\rho')$ $(\rho, \rho' \in \mathrm{Irr}(G))$. $E^0(\rho)$ is given by the following*

(1) *if $d(\rho) = 0$, then*

$$E^0(\rho) = \{\mathcal{I}(W);\ W \subset V_{\bar{h}}(\rho), W \neq 0, S_1 V_{\bar{h}-1}(\rho')\ (\forall \rho' \in \mathrm{Irr}(G), d(\rho') = 1)\}$$

(2) *if $d(\rho) \geq 1$ and if there is no adjacent ρ' with $d(\rho') > d(\rho)$, then*

$$E^0(\rho) = \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho)}(\rho) \\ W \neq 0, V_{\bar{h}+d(\rho)}(\rho) \end{matrix}\right\},$$

(3) *if $d(\rho) \geq 1$ and if there is an adjacent ρ' with $d(\rho') > d(\rho)$, then*

$$E^0(\rho) = \left\{\mathcal{I}(W);\ \begin{matrix} W \subset V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho)}(\rho) \\ W \neq 0, V_{\bar{h}-d(\rho)}(\rho), V_{\bar{h}+d(\rho)}(\rho) \end{matrix}\right\}$$

where W is a nontrivial proper G-submodule in respective case. Moreover $\mathrm{Sing}(E) = \{P(\rho, \rho'); \rho$ and ρ' adjacent, $(\rho, \rho' \in \mathrm{Irr}(G))\}$ and the singular points $P(\rho, \rho')$ are given by

$$P(\rho, \rho') = \mathcal{I}(W(\rho, \rho'))$$

$$W(\rho, \rho') = \begin{cases} V_{\bar{h}-d(\rho)}(\rho) \oplus V_{\bar{h}+d(\rho')}(\rho') & (d(\rho') = d(\rho) + 1 \geq 2) \\ \{S_1 \cdot V_{\bar{h}-1}(\rho')\}[\rho] \oplus V_{\bar{h}+1}(\rho') & (d(\rho) = 0, d(\rho') = 1) \end{cases}$$

Remark. $\rho \in \mathrm{Irr}(G)$ belongs to (1) if $\rho = \rho_3$, (2) if $\rho = \rho_2, \rho_1', \rho_1''$, and (3) if $\rho = \rho_2', \rho_2''$. The set of adjacent pairs in the case E_6 is as follows;

$$(\rho_1^{(k)}, \rho_2^{(k)}), (\rho_2^{(k)}, \rho_3), (\rho_2, \rho_3) \quad (k = 1, 2)$$

Proof—E_6 case. We consider those $I \in X_G$ in the exceptional set E, equivalently $I \in X_G$ with $I \subset \mathfrak{m}$. For a finite submodule W of $\mathfrak{m}$ we define $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ and $V(\mathcal{I}(W)) := \mathcal{I}(W)/\mathfrak{m}\mathcal{I}(W) + \mathfrak{n}$. The symbol $\equiv$ stands for the congruence mod $\mathfrak{n}$.

The case $\mathcal{I}(W) \in E^0(\rho_1')$.

Let $W (\simeq \rho'_1) \in \mathbb{P}(V_4(\rho'_1) \oplus V_8(\rho'_1))$. Suppose $W \neq V_8(\rho'_1)$ and set $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$. Since $\bar{S}_{12} = 0$, by Lemma 6.5 we have $S_k \cdot W \equiv S_k \cdot \bar{V}_4(\rho'_1)$ for $k \geq 4$. Also by Lemma 6.5 $S_k \cdot \bar{V}_4(\rho'_1) = \bar{S}_{k+4}$ for $k \geq 5$. Hence $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 9$. Since $S_k \cdot W = S_k \cdot \bar{V}_4(\rho'_1) \bmod \bar{S}_9$ for $k \geq 1$, we infer that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_4(\rho'_1) = W + \sum_{k=1}^{4} S_k \cdot \bar{V}_4(\rho'_1) + \sum_{k=9}^{11} \bar{S}_k.$$

We see by Lemma 6.5

$$\begin{aligned} W + S_4\bar{V}_4(\rho'_1) &= \rho'_1 + \rho''_1 + \rho_3 = \frac{1}{2}(\bar{S}_4 + \bar{S}_8), \\ S_1\bar{V}_4(\rho'_1) + S_3\bar{V}_4(\rho'_1) &= \rho_2 + \rho'_2 + \rho''_2 = \frac{1}{2}(\bar{S}_5 + \bar{S}_7), \\ S_2\bar{V}_4(\rho'_1) &= \rho_3 = \frac{1}{2}\bar{S}_6. \end{aligned}$$

By duality $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ and $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in E^0(\rho'_2)$.

Let $W (\simeq \rho'_2) \in \mathbb{P}(V_5(\rho'_2) \oplus V_7(\rho'_2))$. Suppose $W \neq V_5(\rho'_2), V_7(\rho'_2)$. Since $\bar{S}_{12} = 0$, we have $S_k \cdot W \equiv S_k \cdot \bar{V}_5(\rho'_2) = \bar{S}_{k+5}$ for $k \geq 5$ by the condition $W \neq V_7((\rho'_2)$. We also see that $S_4 \cdot W = S_4 \cdot \bar{V}_5(\rho'_2) \bmod \bar{S}_{11} = \bar{S}_9$. Therefore $\bar{S}_9 \subset \mathcal{I}(W)/\mathfrak{n}$. Hence $S_k \cdot W = S_k \cdot \bar{V}_5(\rho'_2) \bmod \bar{S}_9$ for $k \geq 2$. Since $S_1 \cdot \bar{V}_5(\rho'_2) = \rho_3$ and $S_1 \cdot \bar{V}_7(\rho'_2) = \rho'_1 + \rho_3$, we have $S_1 \cdot W \equiv \rho'_1 + \rho_3$ and $\{S_1 \cdot W\}[\rho'_1] \equiv \bar{V}_8(\rho'_1) \subset \mathcal{I}(W)/\mathfrak{n}$ by the assumption $W \neq V_5(\rho'_2)$. Since $S_3\bar{V}_5(\rho'_2) = \rho''_1 + \rho_3$, we have $\bar{S}_8 = \bar{V}_8(\rho'_1) \oplus S_3\bar{V}_5(\rho'_2) \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_5(\rho'_2) = W + \sum_{k=1}^{2} S_k \cdot \bar{V}_5(\rho'_2) + \sum_{k=8}^{11} \bar{S}_k,$$

$$W + S_1\bar{V}_5(\rho'_2) + S_2\bar{V}_5(\rho'_2) = \rho_2 + \rho'_2 + \rho''_2 + \rho_3 = \frac{1}{2}(\bar{S}_5 + \bar{S}_6 + \bar{S}_7)$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASES $\mathcal{I}(W) \in E^0(\rho''_1)$ OR $\mathcal{I}(W) \in E^0(\rho''_2)$. Similar.

THE CASE $\mathcal{I}(W) \in E^0(\rho_2)$.

Let $W (\simeq \rho_2) \in \mathbb{P}(V_5(\rho_2) \oplus V_7(\rho_2))$. Suppose that $W \neq V_7(\rho_2)$. In the same manner as above we see that $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 10$. It follows that $S_3 \cdot W = S_3 \cdot \bar{V}_5(\rho_2) \bmod \bar{S}_{10} = \bar{S}_8$. Therefore $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 8$. Similarly $S_2 \cdot W \equiv S_2 \cdot \bar{V}_5(\rho_2) = \rho'_2 + \rho''_2 \bmod \bar{S}_8$ and $S_1 \cdot W \equiv S_1 \cdot \bar{V}_5(\rho_2) = \rho_3 \bmod \bar{S}_8$. It follows

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_5(\rho_2) = W + \sum_{k=1}^{2} S_k \cdot \bar{V}_5(\rho_2) + \sum_{k=8}^{11} \bar{S}_k,$$

$$W + S_1\bar{V}_5(\rho_2) + S_2\bar{V}_5(\rho_2) = \rho_2 + \rho'_2 + \rho''_2 + \rho_3 = \frac{1}{2}(\bar{S}_5 + \bar{S}_6 + \bar{S}_7)$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Thus we see $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in E^0(\rho_3)$.

Let $W \in \mathbb{P}(V_6(\rho_3))$. Let $W_k = S_1 \cdot V_5(\rho_2^{(k)})$ for any $k = 0,1,2$ where $\rho_2^{(k)} = \rho_2, \rho_2', \rho_2''$. Now we suppose that $W \neq W_k$. Then $S_1 \cdot W \equiv \bar{S}_7$ by Lemma 6.6 so that $\mathcal{I}(W)$ contains $\bar{S}_k$ for any $k \geq 7$. It follows

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k W = W + \sum_{k=7}^{11} \bar{S}_k.$$

Hence $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Therefore $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in P(\rho_1', \rho_2')$.

Let $W = W(\rho_1', \rho_2') := V_8(\rho_1') \oplus V_5(\rho_2')$. Recall that $W = \{S_1 \cdot V_7(\rho_2')\}[\rho_1'] \oplus V_5(\rho_2') = V_8(\rho_1') \oplus S_1 \cdot V_4(\rho_1')$. By Lemma 6.5, we see that $S_1 \cdot \bar{V}_5(\rho_2') = \rho_3$, $S_2 \cdot \bar{V}_5(\rho_2') = \rho_2 + \rho_2''$, $S_3 \cdot \bar{V}_5(\rho_2') = \rho_1'' + \rho_3$ and $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 8$. It follows that $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$ by the table. Therefore $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in P(\rho_2', \rho_3)$.

Let $W = W(\rho_2', \rho_3) := V_7(\rho_2') \oplus S_1 V_5(\rho_2') = V_7(\rho_2') \oplus W_1$. We recall that $S_1 \cdot W_1 = \rho_2 + \rho_2''$, so that $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 7$. Since $W_1 = \rho_3$ we have $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$ by the table. Therefore $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASES $\mathcal{I}(W) \in P(\rho_2, \rho_3)$ OR $\mathcal{I}(W) \in P(\rho_2'', \rho_3)$. Similar.

By the following three Lemmas we can complete the proof of Theorem 6.7 by the same argument as in [IN2, section 5]. □

The following Lemmas are proved in the same manner as in section 5.

Lemma 6.8. *The limit of $\mathcal{I}(W)$ is $\mathcal{I}(W(\rho, \rho'))$ when $\mathcal{I}(W) \in E(\rho)$ approaches the point $P(\rho, \rho')$ for ρ' adjacent to ρ.*

Lemma 6.9. *$E(\rho)$ and $E(\rho')$ intersects at $P(\rho, \rho')$ transversally if ρ and ρ' are adjacent.*

Lemma 6.10. *$E(\rho)$ is smooth.*

6.11. Conclusion. Let $I \in X_G$. If $\mathrm{supp}(\mathcal{O}_{\mathbb{A}^2}/I)$ is not the origin, then

$$I = (T(x,y) - T(a,b), \phi^3(x,y) - \phi^3(a,b), W(x,y) - W(a,b))$$

where $(a,b) \neq (0,0)$.

By the same argument as in [IN2, section 5] we thus obtain a complete description of the G-invariant ideals in X_G. We note that $X_G = \mathrm{Hilb}^{|G|}(\mathbb{A}^2)^G$, the set of all the G-fixed points by [IN2, Corollary 5.5] and Remark to it.

7. E_7.

7.1. Character table. Let G be a binary octahedral group $\mathbb{O}$. G is by definition a subgroup of $SL(2,\mathbb{C})$ of order 48 generated by $\mathbb{T} = \langle\sigma, \tau, \mu\rangle$ and κ;

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \ \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \ \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7, & \varepsilon^7 \\ \varepsilon^5, & \varepsilon \end{pmatrix}, \ \kappa = \begin{pmatrix} \varepsilon, & 0 \\ 0, & \varepsilon^7 \end{pmatrix},$$

where $\varepsilon = e^{2\pi i/8}$ [Slodowy, p. 73]. G operates upon $\mathbb{A}^2$ from the right by $(x, y) \mapsto (x, y)g$ $(g \in G)$. $\mathbb{D}_2$ and $\mathbb{T}$ are normal subgroups of G and the following sequences are exact;

$$1 \to \mathbb{T} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$$
$$1 \to \mathbb{D}_2 \to G \to \mathcal{S}_3 \to 1$$

where $\mathcal{S}_3$ is the symmetric group of 3 letters.

We recall the character table of G (easy to derive from [Schur]) and the other relevant invariants. The Coxeter number $2\bar{h}$ of E_7 is equal to 18.

Table 7.1

ρ	1	2	3	4	5	6	7	8	d	$(\bar{h} \pm d)$
	1	-1	μ	μ^2	τ	κ	$\tau\kappa$	κ^3		
$\sharp$	1	1	8	8	6	6	12	6		
ρ_0	1	1	1	1	1	1	1	1	(3)	–
ρ_2	2	-2	1	-1	0	$\sqrt{2}$	0	$-\sqrt{2}$	2	(7, 11)
ρ_3	3	3	0	0	-1	1	-1	1	1	(8, 10)
ρ_4	4	-4	-1	1	0	0	0	0	0	(9, 9)
ρ_3'	3	3	0	0	-1	-1	1	-1	1	(8, 10)
ρ_2'	2	-2	1	-1	0	$-\sqrt{2}$	0	$\sqrt{2}$	2	(7, 11)
ρ_1'	1	1	1	1	1	-1	-1	-1	3	(6, 12)
ρ_2''	2	2	-1	-1	2	0	0	0	1	(8, 10)

7.2. Symmetric tensors mod $\mathfrak{n}$. The G-modules S_m and $\bar{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by $\rho_{\mathrm{nat}} := \rho_2$ are decomposed into irreducible G-modules for small values of m as follows. The factors of $\bar{S}_m$ in the brackets are those in S_{McKay}. We use the same notation $\bar{V}_m(\rho)$ and $V_m(\rho)$ for $\rho \in \mathrm{Irr}(G)$ as before.

Table 7.2

m	S_m	$\bar{S}_m$
1	ρ_2	ρ_2
2	ρ_3	ρ_3
3	ρ_4	ρ_4
4	$\rho_2'' + \rho_3'$	$\rho_2'' + \rho_3'$
5	$\rho_2' + \rho_4$	$\rho_2' + \rho_4$
6*	$\rho_1' + \rho_3 + \rho_3'$	$(\rho_1') + \rho_3 + \rho_3'$
7*	$\rho_2 + \rho_2' + \rho_4$	$(\rho_2 + \rho_2') + \rho_4$
8*	$\rho_0 + \rho_2'' + \rho_3 + \rho_3'$	$(\rho_2'' + \rho_3 + \rho_3')$
9*	$\rho_2 + 2\rho_4$	$(2\rho_4)$
10*	$\rho_2'' + 2\rho_3 + \rho_3'$	$(\rho_2'' + \rho_3 + \rho_3')$
11*	$\rho_2 + \rho_2' + 2\rho_4$	$(\rho_2 + \rho_2') + \rho_4$
12*	$\rho_0 + \rho_1' + \rho_2'' + \rho_3 + 2\rho_3'$	$(\rho_1') + \rho_3 + \rho_3'$
13	$\rho_2 + 2\rho_2' + 2\rho_4$	$\rho_2' + \rho_4$
14	$\rho_1' + \rho_2'' + 2\rho_3 + 2\rho_3'$	$\rho_2'' + \rho_3'$
15	$\rho_2 + \rho_2' + 3\rho_4$	ρ_4
16	$\rho_0 + 2\rho_2'' + 2\rho_3 + 2\rho_3'$	ρ_3
17	$2\rho_2 + \rho_2' + 3\rho_4$	ρ_2
18	$\rho_0 + \rho_1' + \rho_2'' + 3\rho_3 + 2\rho_3'$	0

Table 7.3

m	ρ	$V_m(\rho)$
7	ρ_2	$7x^4y^3 + y^7, -x^7 - 7x^3y^4$
11	ρ_2	$x^{10}y - 6x^6y^5 + 5x^2y^9, -xy^{10} + 6x^5y^6 - 5x^9y^2$
8	ρ_3	$-2xy^7 - 14x^5y^3, x^8 - y^8, 2x^7y + 14x^3y^5$
10	ρ_3	$4x^{10} + 60x^6y^4, 5x^9y + 54x^5y^5 + 5xy^9,$ $60x^4y^6 + 4y^{10}$
9	ρ_4	
9	W_2''	$12x^6y^3 + 12x^2y^7, x^9 - 10x^5y^4 + xy^8,$ $-x^8y + 10x^4y^5 - y^9, 12x^7y^2 + x^3y^6$
9	W_3	$21x^6y^3 + 3x^2y^7, -x^9 + 7x^5y^4 + 2xy^8,$ $-2x^8y - 7x^4y^5 + y^9, -3x^7y^2 - 21x^3y^6$
9	W_3'	x^3T, x^2yT, xy^2T, y^3T
8	ρ_3'	x^2T, xyT, y^2T
10	ρ_3'	$-3x^8y^2 - 14x^4y^6 + y^{10}, 8x^7y^3 + 8x^3y^7,$ $x^{10} - 14x^6y^4 - 3x^2y^8$

m	ρ	$V_m(\rho)$
7	ρ_2'	xT, yT
11	ρ_2'	$-11x^8y^3 - 22x^4y^7 + y^{11}, 11x^3y^8 + 22x^7y^4 - x^{11}$
6	ρ_1'	T
12	ρ_1'	$x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$
8	ρ_2''	$\psi^2, -\phi^2$
10	ρ_2''	$x^5y\psi - xy^5\phi, -x^5y\phi + xy^5\psi$

where $\phi = p_2^2+4\omega p_3^2$, $\psi = p_2^2+4\omega^2 p_3^2$, $T(x,y) = (x^4-y^4)xy$. In the above table we use the exceptional notation for $W_j^{(i)}$. $W_j^{(i)} \simeq \rho_4$ are irreducible G-submodules of $V_9(\rho_4) \simeq \rho_4^{\oplus 2}$.

Lemma 7.3. *The G-module $S_m\bar{V}_k(\rho)$ is decomposed into irreducible G-submodules as follows. We read the following table as $S_2\bar{V}_6(\rho_1') = \rho_3'$, $S_2\bar{V}_8(\rho_2'') = \rho_3 + \rho_3'$ and so on.*

m	k	ρ	$S_m\bar{V}_k(\rho)$	m	k	ρ	$S_m\bar{V}_k(\rho)$
1	6	ρ_1'	ρ_2'	2	8	ρ_2''	$\rho_3 + \rho_3'$
2	6		ρ_3'	3	8		$\rho_2 + \rho_2' + \rho_4$
3	6		ρ_4	1	7	ρ_2	ρ_3
4	6		$\rho_2'' + \rho_3$	2	7		ρ_4
5	6		$\rho_2 + \rho_4$	3	7		$\rho_2'' + \rho_3'$
6	6		$\rho_3 + \rho_3'$	4	7		$\rho_2' + \rho_4$
7	6		$\rho_2' + \rho_4$	5	7		$\rho_1' + \rho_3 + \rho_3'$
1	11	ρ_2'	$\rho_1' + \rho_3'$	1	10	ρ_3	$\rho_2 + \rho_4$
1	8	ρ_2''	ρ_4	1	10	ρ_3'	$\rho_2' + \rho_4$

Proof. The assertions for $(m,k) = (1,6),(2,6),(3,6)$ are clear. By Table 7.2 we see that there are three generators A_8, A_{12} and A_{18} of respective degrees 8, 12 and 18 for the ring of G-invariant polynomials. We know that $A_8 = \phi\psi$, $A_{12} = T^2$ by [Klein, p. 54].

First we note $S_m = S_{m-8}\cdot A_8 \oplus \bar{S}_m$ $(m = 10, 11)$ and $S_4V_6(\rho_1') = (\rho_2''+\rho_3')\otimes\rho_1' = \rho_2''+\rho_3$, $S_5V_6(\rho_1') = (\rho_2' + \rho_4)\otimes\rho_1' = \rho_2 + \rho_4$. If $\{S_4\bar{V}_6(\rho_1')\}[\rho_3] = 0$ in $\bar{S}_{10}$, then $\{S_4V_6(\rho_1')\}[\rho_3] = S_2\cdot A_8$. A_8 would be divisible by T, a generator of $V_6(\rho_1')$. It is however impossible. Hence $\{S_4\bar{V}_6(\rho_1')\}[\rho_3] = \rho_3$ so that $S_4\bar{V}_6(\rho_1') = \rho_2'' + \rho_3$. $S_5\bar{V}_6(\rho_1') = \rho_2 + \rho_4$ is similarly proved.

Since $S_6V_6(\rho_1') = (\rho_1')^2+\rho_3+\rho_3' = \rho_0+\rho_3+\rho_3'$, $S_6\bar{V}_6(\rho_1') = \rho_3+\rho_3'$ or ρ_3. If $S_6\bar{V}_6(\rho_1') = \rho_3$, then $S_6[\rho_3]\cdot V_6(\rho_1')$ is divisible by T^2, so that $S_6[\rho_3]$ is divisible by T. Since deg $T = 6$, this is impossible. Hence $S_6\bar{V}_6(\rho_1') = \rho_3 + \rho_3'$.

Next we have $S_7V_6(\rho_1') = \rho_2'+\rho_2+\rho_4$ and $\{S_7V_6(\rho_1')\}[\rho_2] = \rho_2\cdot A_{12}$. If $\{S_7\bar{V}_6(\rho_1')\}[\rho_4] = 0$, then $\{S_7V_6(\rho_1')\}[\rho_4] = V_7[\rho_4]V_6(\rho_1') = \rho_4\cdot A_{12}$ or $\rho_4\cdot A_8$. In the first case, $V_7[\rho_4]$ is divisible by T, which is impossible because deg $T = 6$ and dim $S_1 = 2 <$ deg $\rho_4 = 4$. In the second case, $V_7[\rho_4]$ is divisible by A_8, which is impossible. It follows that $\{S_7\bar{V}_6(\rho_1')\}[\rho_4] = \rho_4$. If $\{S_7\bar{V}_6(\rho_1')\}[\rho_2'] = 0$, then $V_7[\rho_2]V_6(\rho_1') = \rho_2'\cdot A_{12}$ or $\rho_2'\cdot A_8$. In the first case $V_7[\rho_2]$ is divisible by T, which contradicts Table 7.3. In the second case $V_7[\rho_2]$ is divisible by A_8, absurd. Hence $\{S_7\bar{V}_6(\rho_1')\}[\rho_2'] = \rho_2'$. It follows that $S_7\bar{V}_6(\rho_1') = \rho_2' + \rho_4 = \bar{S}_{13}$.

We note next $\dim S_1V_{11}(\rho_2') \geq 3$. If $\dim S_1V_{11}(\rho_2') = 3$, then there exists a $f \in S_{10}$ such that $V_{11}(\rho_2') = S_1 \cdot f$. Hence $f \in S_{10}[\rho_1'] = \{0\}$, a contradiction. Hence $\dim S_1V_{11}(\rho_2') = 4$. so that $S_1V_{11}(\rho_2') = \rho_1' + \rho_3'$. If $\{S_1\bar{V}_{11}(\rho_2')\}[\rho_3'] = 0$, we have $\{S_1V_{11}(\rho_2')\}[\rho_3'] = V_4[\rho_3'] \cdot A_8$ by Table 7.2. Since $\dim S_1 < \deg \rho_3' = 3$, there exists a nontrivial element of $\{S_1V_{11}(\rho_2')\}[\rho_3']$ divisible by both x and A_8. Hence $V_{11}(\rho_2')$ contains a nontrivial element divisible by A_8. This implies that $V_{11}(\rho_2')$ is divisible by A_8. Then $V_3(\rho_2') = V_{11}(\rho_2')A_8^{-1} = \rho_2'$, which contradicts $S_3 = \rho_4$. Hence $S_1\bar{V}_{11}(\rho_2') = \rho_1' + \rho_3'$.

It is clear from $\rho_2 \otimes \rho_2'' = \rho_4$ and Table 7.2 that $S_1\bar{V}_8(\rho_2'') = \rho_4$.

Next $S_2 \otimes V_8(\rho_2'') = \rho_3 + \rho_3'$ by Table 7.1. Since $\dim S_2V_8(\rho_2'') \geq 4$, we have $S_2V_8(\rho_2'') = \rho_3 + \rho_3'$. If $\{S_2\bar{V}_8(\rho_2'')\}[\rho_3] = 0$, then $\{S_2V_8(\rho_2'')\}[\rho_3] = S_2 \cdot A_8$. Since $\deg \rho_2'' < \deg \rho_3$ and $V_8(\rho_2'')$ is generated by ϕ^2 and ψ^2, there exists a nontrivial element of $\{S_2V_8(\rho_2'')\}[\rho_3]$ divisible by both ϕ^2 and A_8. Since ϕ and ψ are coprime, S_{10} contains a nontrivial element divisible by $\phi^2\psi$, a contradiction. If $\{S_2\bar{V}_8(\rho_2'')\}[\rho_3'] = 0$, then $\{S_2V_8(\rho_2'')\}[\rho_3'] = S_2 \cdot A_8 = \rho_3$, a contradiction. Hence $S_2\bar{V}_8(\rho_2'') = \rho_3 + \rho_3'$.

Next we consider $S_3\bar{V}_8(\rho_2'')$. Since $\dim S_2V_8(\rho_2'') = 6$ by the above proof, we have $\dim S_3V_8(\rho_2'') \geq 7$. By Table 7.1 $S_3 \otimes V_8(\rho_2'') = \rho_2 + \rho_2' + \rho_4$ so that $S_3V_8(\rho_2'') = \rho_2 + \rho_2' + \rho_4$. Assume $S_3\bar{V}_8(\rho_2'') \neq \rho_2 + \rho_2' + \rho_4$. Then by Table 7.2 it is possible only that $\{S_3\bar{V}_8(\rho_2'')\}[\rho_4] = 0$. Assume $\{S_3V_8(\rho_2'')\}[\rho_4] = S_3 \cdot A_8$ so that there exists an element of $\{S_3V_8(\rho_2'')\}[\rho_4]$ divisible by both ϕ^2 and A_8. Therefore there exists a nontrivial element of S_3 divisible by ψ, which is a contradiction. Hence $S_3\bar{V}_8(\rho_2'') = \rho_2 + \rho_2' + \rho_4$.

Clearly $S_1V_7(\rho_2) = \rho_0 + \rho_3$, $S_2V_7(\rho_2) = \rho_2 + \rho_4$. Hence $S_1\bar{V}_7(\rho_2) = \rho_3$ and $S_2\bar{V}_7(\rho_2) = \rho_4$.

Next $S_3 \otimes V_7(\rho_2) = \rho_4 \otimes \rho_2 = \rho_2'' + \rho_3 + \rho_3'$ by Table 7.1. Since $\dim S_2V_7(\rho_2) = 6$, we have $\dim S_3V_7(\rho_2) \geq 7$ so that $S_3 \otimes V_7(\rho_2) = \rho_2'' + \rho_3 + \rho_3'$. It is clear that $\{S_1V_7(\rho_2)\}[\rho_0] = S_0 \cdot A_8$, $\{S_2V_7(\rho_2)\}[\rho_2] = S_1 \cdot A_8$. Hence $\{S_3V_7(\rho_2)\}[\rho_3] = S_2 \cdot A_8$. It is clear that $\{S_3V_7(\rho_2)\}[\rho_3'] \neq S_2 \cdot A_8$ and $\{S_3V_7(\rho_2)\}[\rho_2''] \neq S_2 \cdot A_8$. Hence $S_3\bar{V}_7(\rho_2) = \rho_2'' + \rho_3'$.

Next we see $\dim S_4V_7(\rho_2) = 10$, $S_4V_7(\rho_2) \simeq S_4 \otimes V_7(\rho_2) = \rho_2' + 2\rho_4$. Hence $S_4\bar{V}_7(\rho_2) = \rho_2' + \rho_4$ by Table 7.2.

It is easy to see $\dim S_5V_7(\rho_2) = 12$. Hence $S_5V_7(\rho_2) = S_5 \otimes V_7(\rho_2) = \rho_1' + \rho_2'' + \rho_3 + 2\rho_3'$ so that $S_5\bar{V}_7(\rho_2) = \rho_1' + \rho_3 + \rho_3' = \bar{S}_{12}$ by Table 7.2.

Similarly we easily see $\dim S_1V_{10}(\rho_3) = \dim S_1V_{10}(\rho_3') = 6$. Hence $S_1V_{10}(\rho_3) = \rho_2 + \rho_4$, $S_1V_{10}(\rho_3') = \rho_2' + \rho_4$. If $\{S_1\bar{V}_{10}(\rho_3)\}[\rho_4] = 0$, then $\{S_1V_{10}(\rho_3)\}[\rho_4] = S_3 \cdot A_8$. Therefore there exists a nontrivial element of $V_{10}(\rho_3)$ divisible by A_8 so that $V_{10}(\rho_3)$ is divisible by A_8. This implies that $\bar{V}_{10}(\rho_3) = 0$. But by the choice of it, $V_{10}(\rho_3) \simeq \bar{V}_{10}(\rho_3)$, a contradiction.

This completes the proof of Lemma 7.3. □

Corollary 7.4. $S_1\bar{V}_6(\rho_1') = \bar{V}_7(\rho_2')$, $S_2\bar{V}_6(\rho_1') = \bar{V}_8(\rho_3')$, $S_1\bar{V}_7(\rho_2) = \bar{V}_8(\rho_3)$.

Corollary 7.5. $S_3\bar{V}_8(\rho_2'') = \bar{S}_{11}$, $S_5\bar{V}_7(\rho_2) = \bar{S}_{12}$, $S_7\bar{V}_6(\rho_1') = \bar{S}_{13}$.

Proof. Clear from Table 7.2 and Lemma 7.3. □

Corollary 7.6. $S_2\bar{V}_8(\rho_3') = \rho_2'' + \rho_3$, $S_2\bar{V}_8(\rho_2'') = \rho_3 + \rho_3'$, $S_2\bar{V}_8(\rho_3) = \rho_2'' + \rho_3'$.

Proof. It follows from Lemma 7.3 and Corollary 7.4. □

7.7. Proof of Theorem 6.7 — E_7.

We follow the proof of Theorem 6.7 in the E_6 case words for words. For a finite submodule W of $\mathfrak{m}$ we define $\mathcal{I}(W) = W\mathcal{O}_{\mathbb{A}^2} + \mathfrak{n}$ and $V(\mathcal{I}(W)) := \mathcal{I}(W)/\mathfrak{m}\mathcal{I}(W) + \mathfrak{n}$. As in section 6, we denote congruence mod $\mathfrak{n}$ by $\equiv$.

THE CASE $\mathcal{I}(W) \in E^0(\rho_1')$.
Let $W(\simeq \rho_1') \in \mathbb{P}(V_6(\rho_1') \oplus V_{12}(\rho_1'))$. Suppose $W \neq V_{12}(\rho_1')$. Since $\bar{S}_{18} = 0$, we have $S_k \cdot W \equiv S_k \cdot \bar{V}_6(\rho_1')$ for $k \geq 6$. Also by Corollary 7.5 $S_k \cdot \bar{V}_6(\rho_1') = \bar{S}_{k+6}$ for $k \geq 7$. Hence $\bar{S}_k \subset \mathcal{I}(W)/\mathfrak{n}$ for $k \geq 13$. We see $S_k \cdot W = S_k \cdot \bar{V}_6(\rho_1') \mod \bar{S}_{k+12}$ for $k \geq 1$. Therefore

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot \bar{V}_6(\rho_1') = W + \sum_{k=1}^{6} S_k \cdot \bar{V}_6(\rho_1') + \sum_{k=13}^{17} \bar{S}_k.$$

By Lemma 7.3 and Table 7.2 we have

$$W + \sum_{k=1}^{6} S_k \cdot \bar{V}_6(\rho_1') = \frac{1}{2}\sum_{k=6}^{12} \bar{S}_k.$$

Hence by duality $\mathcal{I}(W)/\mathfrak{n} = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho$. Thus $\mathcal{I}(W) \in X_G$ and $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in E^0(\rho_2')$.
Let $W(\simeq \rho_2') \in \mathbb{P}(V_7(\rho_2') \oplus V_{11}(\rho_2'))$. Suppose $W \neq V_7(\rho_2'), V_{11}(\rho_2')$. Since $\bar{S}_{18} = 0$, we have $S_k \cdot W \equiv S_k \cdot \bar{V}_7(\rho_2')$ for $k \geq 7$ by the condition $W \neq V_{11}((\rho_2')$. By Corollaries 7.4 and 7.5 $S_k \cdot W \equiv S_k\bar{V}_7(\rho_2') = \bar{S}_{k+7}$ for $k \geq 7$. Therefore $\bar{S}_{14} \subset \mathcal{I}(W)/\mathfrak{n}$. Hence by Corollary 7.4 $S_6 \cdot W \equiv S_6 \cdot \bar{V}_7(\rho_2') \mod \bar{S}_{14} = \bar{S}_{13}$ so that $\bar{S}_{13} \subset \mathcal{I}(W)/\mathfrak{n}$. By Lemma 7.3 $S_1 \cdot \bar{V}_7(\rho_2') = \rho_3'$, $S_2 \cdot \bar{V}_7(\rho_2') = \rho_4$, $S_1 \cdot \bar{V}_{11}(\rho_2') = \rho_1' + \rho_3'$, Therefore $S_1 W = \rho_1' + \rho_3'$, $S_2 W \equiv S_2 \bar{V}_7(\rho_2') \mod \bar{S}_{13} \simeq \rho_4$. Moreover $\bar{V}_{12}(\rho_1') \subset S_1 W \subset \mathcal{I}(W)/\mathfrak{n}$ and $S_5\bar{V}_7(\rho_2') = S_5\bar{V}_6(\rho_1') = V_{12}(\rho_3) \oplus V_{12}(\rho_3') \subset \mathcal{I}(W)/\mathfrak{n}$. Therefore $\bar{S}_{12} \subset \mathcal{I}(W)/\mathfrak{n}$. It follows that

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{5} S_k \cdot \bar{V}_7(\rho_2') + \sum_{k=12}^{17} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=7}^{11} \bar{S}_k + \sum_{k=12}^{17} \bar{S}_k = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho.\end{aligned}$$

Hence $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in E^0(\rho_3')$.
Let $W(\simeq \rho_3') \in \mathbb{P}(V_8(\rho_3') \oplus V_{10}(\rho_3'))$, and assume $W \neq V_8(\rho_3'), V_{10}(\rho_3')$. Similarly we see that $\bar{S}_{11} \subset \mathcal{I}(W)/\mathfrak{n}$. Since $S_1 W \equiv S_1\bar{V}_8(\rho_3') \mod \bar{S}_{11} = \rho_4$ and $S_2 W \equiv S_2\bar{V}_8(\rho_3') \mod \bar{S}_{11} = \rho_2'' + \rho_3$ by Corollary 7.4, it follows that

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{2} S_k \cdot \bar{V}_8(\rho_3') + \sum_{k=11}^{17} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=8}^{10} \bar{S}_k + \sum_{k=11}^{17} \bar{S}_k = \sum_{\rho\in\mathrm{Irr}(G)} \deg(\rho)\rho.\end{aligned}$$

Hence $\mathcal{I}(W) \in X_G$ with $V(\mathcal{I}(W)) \simeq W$.

THE CASE $\mathcal{I}(W) \in E^0(\rho_2)$.

Let $W\,(\simeq \rho_2) \in \mathbb{P}(V_7(\rho_2) \oplus V_{11}(\rho_2))$. Suppose that $W \neq V_{11}(\rho_2)$. In the same manner as above we see $\bar{S}_{12} \subset \mathcal{I}(W)/\mathfrak{n}$. We infer $\mathcal{I}(W) \in X_G$ and $V(\mathcal{I}(W)) \simeq W$ because

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{4} S_k \cdot \bar{V}_7(\rho_2) + \sum_{k=12}^{17} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=7}^{11} \bar{S}_k + \sum_{k=12}^{17} \bar{S}_k = \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho.\end{aligned}$$

THE CASE $\mathcal{I}(W) \in E^0(\rho_3)$ OR $\mathcal{I}(W) \in E^0(\rho_2'')$. Similar.

THE CASE $\mathcal{I}(W) \in E^0(\rho_4)$.

Let $W_2'' = S_1\bar{V}_8(\rho_2'')$, $W_3 = S_1\bar{V}_8(\rho_3)$ and $W_3' = S_1\bar{V}_8(\rho_3')$. Then by Corollary 7.6 $S_1W_2'' \equiv \rho_3+\rho_3'$, $S_1W_3 \equiv \rho_2''+\rho_3'$ and $S_1W_3' \equiv \rho_2''+\rho_3$. Moreover we see in the same manner as in Lemma 6.6 that for any $W \in \mathbb{P}(V_9(\rho_4))$ $S_1W \equiv \bar{S}_{10}$ if and only if $W \neq W_2'', W_3, W_3'$. Suppose $W \neq W_2'', W_3, W_3'$. Then we have

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k=10}^{17} \bar{S}_k = \frac{1}{2}\bar{S}_9 + \sum_{k=10}^{17} \bar{S}_k = \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho.$$

It follows that $\mathcal{I}(W) \in X_G$ and $V(\mathcal{I}(W)) \simeq W$.

Any adjacent pair (ρ, ρ') $(\rho, \rho' \in \mathrm{Irr}(G))$ for G the binary octahedral group is one of the following;

$$\begin{gathered}(\rho_1', \rho_2'), \quad (\rho_2', \rho_3'), \quad (\rho_3', \rho_4) \\ (\rho_2, \rho_3), \quad (\rho_3, \rho_4), \quad (\rho_2'', \rho_4).\end{gathered}$$

For any adjacent pair (ρ, ρ'), we have $W(\rho, \rho')$ defined in Theorem 6.7. For instance let us consider (ρ_1', ρ_2'), and $W := W(\rho_1', \rho_2') = V_{12}(\rho_1') \oplus V_7(\rho_2')$. Then by the above proof of Theorem 6.7 in the case E_7-ρ_2', we see that $\bar{S}_{13} \subset \mathcal{I}(W)/\mathfrak{n}$. We also have

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= \sum_{k=0}^{5} S_k \cdot \bar{V}_7(\rho_2') + V_{12}(\rho_1') + \sum_{k=13}^{17} \bar{S}_k \\ &= \sum_{k=0}^{4} S_k \cdot \bar{V}_7(\rho_2') + \sum_{k=12}^{17} \bar{S}_k = \sum_{\rho \in \mathrm{Irr}(G)} \deg(\rho)\rho.\end{aligned}$$

because $\bar{S}_{12} = V_{12}(\rho_1') + S_5 \cdot \bar{V}_7(\rho_2')$. The remaining cases are similar.

Once we prove the above, we can follow the proof of [IN2, Theorem 5.9 and 5.15] word for word to show Theorem 6.7. □

We also can give a complete description of G-invariant ideals in X_G. In fact, let $I \in X_G$. If $\mathrm{supp}(\mathcal{O}_{\mathbb{A}^2}/I)$ is not the origin, then we know that

$$I = (W(x,y) - W(a,b), T^2(x,y) - T^2(a,b), F(x,y) - F(a,b))$$

where $\chi = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$, $F(x,y) = \chi T$, $W(x,y) = \phi\psi$ and $(a,b) \neq (0,0)$.

We also note that $\mathrm{Hilb}^{|G|}(\mathbb{A})^G = X_G$ by the Remark to [IN2, Corollary 5.5].

8. E_8.

8.1. Character table. Let G be a binary icosahedral group $\mathbb{I}$. G is by definition a subgroup of $SL(2,\mathbb{C})$ of order 120 generated by σ and τ;

$$\sigma = -\begin{pmatrix} \varepsilon^3, & 0 \\ 0, & \varepsilon^2 \end{pmatrix}, \ \tau = \frac{1}{\sqrt{5}}\begin{pmatrix} -(\varepsilon - \varepsilon^4), & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3, & \varepsilon - \varepsilon^4 \end{pmatrix}$$

where $\varepsilon = e^{2\pi i/5}$. We note $\sigma^5 = \tau^2 = -1$. G operates upon $\mathbb{A}^2$ from the right by $(x,y) \mapsto (x,y)g$ $(g \in G)$. G is isomorphic to $SL(2,\mathbb{F}_5)$. An isomorphsim of G with $SL(2,\mathbb{F}_5)$ is given by the map $\sigma \mapsto \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \tau \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Let $\eta = \varepsilon^2 = e^{4\pi i/5}$. In Slodowy's notation [Sl, p. 74]

$$\tau = \frac{1}{\eta^2 - \eta^3}\begin{pmatrix} \eta + \eta^4, & 1 \\ -1, & -\eta - \eta^4 \end{pmatrix}.$$

We recall the character table of G [Schur] and the other relevant invariants.

The Coxeter number $2\bar{h}$ of E_8 is equal to 30. Let $\mu^{\pm} = \frac{1\pm\sqrt{5}}{2}$.

Table 8.1

ρ	1	2	3	4	5	6	7	8	9	d	$(\bar{h} \pm d)$
	1	-1	σ	σ^2	σ^3	σ^4	τ	$\sigma^2\tau$	$\sigma^7\tau$		
$\sharp$	1	1	12	12	12	12	30	20	20		
ρ_0	1	1	1	1	1	1	1	1	1	(5)	–
ρ_2	2	-2	μ^+	$-\mu^-$	μ^-	$-\mu^+$	0	-1	1	4	(11, 19)
ρ_3	3	3	μ^+	μ^-	μ^-	μ^+	-1	0	0	3	(12, 18)
ρ_4	4	-4	1	-1	1	-1	0	1	-1	2	(13, 17)
ρ_5	5	5	0	0	0	0	1	-1	-1	1	(14, 16)
ρ_6	6	-6	-1	1	-1	1	0	0	0	0	(15, 15)
ρ_4'	4	4	-1	-1	-1	-1	0	1	1	1	(14, 16)
ρ_2'	2	-2	μ^-	$-\mu^+$	μ^+	$-\mu^-$	0	-1	1	2	(13, 17)
ρ_3''	3	3	μ^-	μ^+	μ^+	μ^-	-1	0	0	1	(14, 16)

8.2. Symmetric tensors mod $\mathfrak{n}$. The G-module $\bar{S}_m := S_m(\mathfrak{m}/\mathfrak{n})$ by $\rho_{\mathrm{nat}} := \rho_2$ are decomposed into irreducible G-modules for small values of m as follows. The factors of

$\bar{S}_m$ in the brackets are those in S_{McKay}. We use the same notation $\bar{V}_m(\rho)$ and $V_m(\rho)$ for $\rho \in \mathrm{Irr}(G)$ as before.

Table 8.2

m	$\bar{S}_m$	m	$\bar{S}_m$
0	0	30	0
1	ρ_2	29	ρ_2
2	ρ_3	28	ρ_3
3	ρ_4	27	ρ_4
4	ρ_5	26	ρ_5
5	ρ_6	25	ρ_6
6	$\rho_3'' + \rho_4'$	24	$\rho_3'' + \rho_4'$
7	$\rho_2' + \rho_6$	23	$\rho_2' + \rho_6$
8	$\rho_4' + \rho_5$	22	$\rho_4' + \rho_5$
9	$\rho_4 + \rho_6$	21	$\rho_4 + \rho_6$
10	$\rho_3 + \rho_3'' + \rho_5$	20	$\rho_3 + \rho_3'' + \rho_5$
11*	$(\rho_2) + \rho_4 + \rho_6$	19*	$(\rho_2) + \rho_4 + \rho_6$
12*	$(\rho_3) + \rho_4' + \rho_5$	18*	$(\rho_3) + \rho_4' + \rho_5$
13*	$(\rho_2' + \rho_4) + \rho_6$	17*	$(\rho_2' + \rho_4) + \rho_6$
14*	$(\rho_3'' + \rho_4' + \rho_5)$	16*	$(\rho_3'' + \rho_4' + \rho_5)$
15*	$(2\rho_6)$		

Table 8.3

m	ρ	$V_m(\rho)$
11	ρ_2	$x\sigma_1, -y\sigma_2$
19	ρ_2	$-57x^{15}y^4 + 247x^{10}y^9 + 171x^5y^{14} + y^{19},$
		$-x^{19} + 171x^{14}y^5 - 247x^9y^{10} - 57x^4y^{15}$
12	ρ_3	$x^2\sigma_1, -5x^{11}y - 5xy^{11}, y^2\sigma_2$
18	ρ_3	$-12x^{15}y^3 + 117x^{10}y^8 + 126x^5y^{13} + y^{18},$
		$45x^{14}y^4 - 130x^9y^9 - 45x^4y^{14}$
		$x^{18} - 126x^{13}y^5 + 117x^8y^{10} + 12x^3y^{15}$
13	ρ_4	$x^3\sigma_1, -3x^{12}y + 22x^7y^6 - 7x^2y^{11},$
		$-7x^{11}y^2 - 22x^6y^7 - 3xy^{12}, y^3\sigma_2$
17	ρ_4	$-2x^{15}y^2 + 52x^{10}y^7 + 91x^5y^{12} + y^{17}$
		$10x^{14}y^3 - 65x^9y^8 - 35x^4y^{13}$
		$-35x^{13}y^4 + 65x^8y^9 + 10x^3y^{14}$
		$-x^{17} + 91x^{12}y^5 - 52x^7y^{10} - 2x^2y^{15}$
14	ρ_5	$x^4\sigma_1, -2x^{13}y + 33x^8y^6 - 8x^3y^{11},$
		$-5x^{12}y^2 - 5x^2y^{12},$
		$-8x^{11}y^3 - 33x^6y^8 - 2xy^{13}, -y^4\sigma_2$

m	ρ	$V_m(\rho)$
16	ρ_5	$64x^{15}y + 728x^{10}y^6 + y^{16}$,
		$66x^{14}y^2 + 676x^9y^7 - 91x^4y^{12}$,
		$56x^{13}y^3 + 741x^8y^8 - 56x^3y^{13}$,
		$91x^{12}y^4 + 676x^7y^9 - 66x^2y^{14}$,
		$x^{16} + 728x^6y^{10} - 64xy^{15}$
13	ρ_2'	$y^3\tau_2, -x^3\tau_1$
17	ρ_2'	$x^{17} + 119x^{12}y^5 + 187x^7y^{10} + 17x^2y^{15}$
		$-17x^{15}y^2 + 187x^{10}y^7 - 119x^5y^{12} + y^{17}$
14	ρ_3''	$x^{14} - 14x^9y^5 + 49x^4y^{10}$,
		$7x^{12}y^2 - 48x^7y^7 - 7x^2y^{12}$,
		$49x^{10}y^4 + 14x^5y^9 + y^{14}$
16	ρ_3''	$3x^{15}y - 143x\ 10y^6 - 39x^5y^{11} + y^{16}$,
		$-25x^{13}y^3 - 25x^3y^{13}$,
		$x^{16} + 39x^{11}y^5 - 143x^6y^{10} - 3xy^{15}$
14	ρ_4'	$xy^3\tau_2, -x^4\tau_1, y^4\tau_2, -x^3y\tau_1$
16	ρ_4'	$-2x^{15}y + 77x^{10}y^6 - 84x^5y^{11} + y^{16}$,
		$3x^{12}y^4 + 110x^7y^9 + 15x^2y^{14}$,
		$-x^{16} - 84x^{11}y^5 - 77x^6y^{10} - 2xy^{15}$,
		$15x^{14}y^2 - 110x^9y^7 + 35x^4y^{12}$
15	ρ_6	
15	W_3''	$x^{15} + 84x^{10}y^5 + 77x^5y^{10} + 2y^{15}$,
		$-x^{14}y + 14x^9y^6 - 49x^4y^{11}$,
		$-7x^{13}y^2 + 48x^8y^7 + 7x^3y^{12}$,
		$7x^{12}y^3 - 48x^7y^8 - 7x^2y^{13}$,
		$-49x^{11}y^4 - 14x^6y^9 - xy^{14}$,
		$-2x^{15} + 77x^{10}y^5 - 84x^5y^{10} + y^{15}$
15	W_4'	$x^{15} + 39x^{10}y^5 - 143x^5y^{10} - 3y^{15}$
		$-2x^{14}y + 78x^9y^6 + 52x^4y^{11}$
		$x^{13}y^2 - 39x^8y^7 - 26x^3y^{12}$
		$-26x^{12}y^3 + 39x^7y^8 + x^2y^{13}$
		$52x^{11}y^4 - 78x^6y^9 - 2xy^{14}$
		$3x^{15} - 143x^{10}y^5 - 39x^5y^{10} + y^{15}$
15	W_5	$5x^{15} + 330x^{10}y^5 - 55x^5y^{10}$
		$-7x^{14}y + 198x^9y^6 - 43x^4y^{11}$
		$-19x^{13}y^2 + 66x^8y^7 - 31x^3y^{12}$
		$-31x^{12}y^3 - 66x^7y^8 - 19x^2y^{13}$
		$-43x^{11}y^4 - 198x^6y^9 - 7xy^{14}$
		$-55x^{10}y^5 - 330x^5y^{10} + 5y^{15}$

where $\sigma_1 = x^{10} + 66x^5y^5 - 11y^{10}$, $\sigma_2 = -11x^{10} - 66x^5y^5 + y^{10}$, $\tau_1 = x^{10} - 39x^5y^5 - 26y^{10}$, $\tau_2 = -26x^{10} + 39x^5y^5 + y^{10}$. In the above table we use the exceptional notation for $W_j^{(i)}$. $W_3'' = S_1V_{14}(\rho_3'')$ ($\simeq \rho_6$), $W_4' = S_1V_{14}(\rho_4')$ ($\simeq \rho_6$), $W_5 = S_1V_{14}(\rho_5)$ ($\simeq \rho_6$) and they are irreducible G-submodules of $V_{15}(\rho_6) \simeq \rho_6^{\oplus 2}$.

Lemma 8.3. *The G-module $S_m\bar{V}_k(\rho)$ is decomposed into irreducible G-submodules as follows.*

m	k	ρ	$S_m\bar{V}_k(\rho)$	m	k	ρ	$S_m\bar{V}_k(\rho)$
1	11	ρ_2	ρ_3	1	16	ρ_5	$\rho_4+\rho_6$
2	11		ρ_4	1	13	ρ_2'	ρ_4'
3	11		ρ_5	2	13		ρ_6
4	11		ρ_6	3	13		$\rho_3''+\rho_5$
5	11		$\rho_3''+\rho_4'$	4	13		$\rho_4+\rho_6$
6	11		$\rho_2'+\rho_6$	5	13		$\rho_3+\rho_4'+\rho_5$
7	11		$\rho_4'+\rho_5$	1	16	ρ_4'	$\rho_2'+\rho_6$
8	11		$\rho_4+\rho_6$	1	14	ρ_3''	ρ_6
9	11		$\rho_3+\rho_3''+\rho_5$	2	14		$\rho_4'+\rho_5$
1	18	ρ_3	$\rho_2+\rho_4$	3	14		$\rho_2'+\rho_4+\rho_6$
1	17	ρ_4	$\rho_3+\rho_5$				

Proof. We give a very rough proof of the Lemma. We recall that the ring of G-invariant polynomials is generated by three elements A_{12}, A_{20} and A_{30} of degree 12, 20, 30 respectively [Klein, p. 55]. Note that $S_1 \otimes V_{11}(\rho_2) = \rho_2 \otimes \rho_2 = \rho_0 + \rho_3$. Hence $S_1 \otimes V_{11}(\rho_2) = \rho_0 A_{12} + \rho_3$. In fact $A_{12} = xy(x^{10} + 11x^5y^5 - y^{10})$ by [Klein, p. 56]. It follows that $S_1\bar{V}_{11}(\rho_2) = \rho_3$. Similarly $S_k \otimes V_{11}(\rho_2) \supset S_{k-1}A_{12}$. Therefore $S_2 \otimes V_{11}(\rho_2) = \rho_2 + \rho_4$, $S_2\bar{V}_{11}(\rho_2) = \rho_4$, $S_3 \otimes V_{11}(\rho_2) = \rho_3 + \rho_5$, $S_3\bar{V}_{11}(\rho_2) = \rho_5$, $S_4 \otimes V_{11}(\rho_2) = \rho_4 + \rho_6$, $S_4\bar{V}_{11}(\rho_2) = \rho_6$, $S_5 \otimes V_{11}(\rho_2) = \rho_3'' + \rho_4' + \rho_5$, $S_5\bar{V}_{11}(\rho_2) = \rho_3'' + \rho_4'$. All of these are proved in the same manner as in Lemma 7.3. In fact, for instance dim $S_5V_{11}(\rho_2) = 7$ by Table 8.3, and $\rho_6 \otimes \rho_2 = \rho_3'' + \rho_4' + \rho_5$ so that $S_5\bar{V}_{11}(\rho_2) = \rho_3'' + \rho_4'$.

We see $S_6\bar{V}_{11}(\rho_2) = \rho_2' + \rho_6$ because $\bar{S}_{17} = \rho_2' + \rho_4 + \rho_6$ and $\rho_2 \otimes S_5\bar{V}_{11}(\rho_2) = \rho_2 \otimes (\rho_3'' + \rho_4') = \rho_2' + 2\rho_6$ contains no ρ_4. $\bar{S}_{18} = \rho_3 + \rho_4' + \rho_5$ and $\rho_2 \otimes S_6\bar{V}_{11}(\rho_2) = \rho_2 \otimes (\rho_2' + \rho_6)$ contains no ρ_3, whence $S_7\bar{V}_{11}(\rho_2) = \rho_4' + \rho_5$. Similarly $S_8\bar{V}_{11}(\rho_2) = \rho_4 + \rho_6$ because $\bar{S}_{19} = \rho_2 + \rho_4 + \rho_6$, $\rho_2 \otimes S_7\bar{V}_{11}(\rho_2) = \rho_2' + \rho_4 + 2\rho_6$. By Table 8.2 $\bar{S}_{20} = \rho_3 + \rho_3'' + \rho_5$. $\rho_2 \otimes S_8\bar{V}_{11}(\rho_2) = \rho_3 + 2\rho_5 + \rho_3'' + \rho_4'$. Hence $S_9\bar{V}_{11}(\rho_2) = \rho_3 + \rho_3'' + \rho_5 = \bar{S}_{20}$.

$S_1\bar{V}_{18}(\rho_3) = \rho_2 + \rho_4$ follows from comparison of $S_1 \otimes \bar{V}_{18}(\rho_3)$ and S_{19} and the fact that any polynomial in $V_{18}(\rho_3)$ is indivisible by A_{12}.

Similarly $S_1\bar{V}_{17}(\rho_4) = \rho_3 + \rho_5$, $S_1\bar{V}_{16}(\rho_5) = \rho_4 + \rho_6$ and $S_1\bar{V}_{13}(\rho_2') = \rho_4'$. Since $\rho_3 \otimes \rho_2' = \rho_6$, we see $S_2\bar{V}_{13}(\rho_2') = \rho_6$. One checks dim $S_3V_{13}(\rho_2') =$ dim $S_1W_4' = 8$ by using Table 8.3. It follows from it that $S_3\bar{V}_{13}(\rho_2') = \rho_3'' + \rho_5$. Similarly it is clear that $S_4V_{13}(\rho_2') = S_4 \otimes V_{13}(\rho_2') = \rho_4 + \rho_6$ and $S_5\bar{V}_{13}(\rho_2') = S_5 \otimes \bar{V}_{13}(\rho_2') = \bar{S}_{18}$. Note dim $S_kV_{14}(\rho_3'') = 3(k+1)$ $(k = 1, 2, 3)$ so that $S_kV_{14}(\rho_3'') = S_k \otimes V_{14}(\rho_3'')$. It follows from it that $S_k\bar{V}_{14}(\rho_3'') = S_k \otimes \rho_3''$ $(k = 1, 2, 3)$. In particular, $S_2\bar{V}_{14}(\rho_3'') = \rho_3 \otimes \rho_3' = \rho_4' + \rho_5$, $S_3\bar{V}_{14}(\rho_3'') = \rho_2' + \rho_4 + \rho_6 = \bar{S}_{17}$. □

Corollary 8.4. $S_k\bar{V}_{11}(\rho_2) = \bar{V}_{11+k}(\rho_{k+2})$ $(1 \le k \le 3)$, $S_1\bar{V}_{13}(\rho_2') = \bar{V}_{14}(\rho_4')$.

Corollary 8.5. $S_9\bar{V}_{11}(\rho_2) = \bar{S}_{20}$, $S_5\bar{V}_{13}(\rho_2') = \bar{S}_{18}$, $S_3\bar{V}_{14}(\rho_3'') = \bar{S}_{17}$.

Corollary 8.6. $S_2\bar{V}_{14}(\rho_5) = \rho_3'' + \rho_4'$, $S_2\bar{V}_{14}(\rho_4') = \rho_3'' + \rho_5$, $S_2\bar{V}_{14}(\rho_3'') = \rho_4' + \rho_5$.

Proof. By table 8.3 dim $S_1W_3'' = 9$, dim $S_1W_4' = 8$, dim $S_1W_5 = 7$. Hence $S_2V_{14}(\rho_3'') = S_1W_3'' = \rho_4 + \rho_5$, $S_2\bar{V}_{18}(\rho_4') = S_2V_{18}(\rho_4') = S_1W_4' = \rho_3'' + \rho_5$, $S_5\bar{V}_{11}(\rho_2) = S_2\bar{V}_{14}(\rho_5) = S_1W_5 = \rho_3'' + \rho_4'$. □

8.7. Proof of Theorem 6.7 — E_8.

We have only to follow the proof of Theorem 6.7 — E_6 case word for word. We give a very rough proof in this case.

Let $W\,(\simeq \rho_2) \in \mathbb{P}(V_{11}(\rho_2) \oplus V_{19}(\rho_2))$. Suppose $W \neq V_{19}(\rho_2)$. Then we have

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{8} S_k \cdot \bar{V}_{11}(\rho_2) + \sum_{k=20}^{29} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=11}^{19} \bar{S}_k + \sum_{k=20}^{29} \bar{S}_k = \sum_{\rho\in \mathrm{Irr}(G)} \deg(\rho)\rho\end{aligned}$$

For $W \in \mathbb{P}(V_{9+i}(\rho_i) \oplus V_{21-i}(\rho_i))$ with $W \neq V_{9+i}(\rho_i), V_{21-i}(\rho_i))$ $(i = 3, 4, 5)$, we have

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k=1}^{12-2i} S_k \cdot \bar{V}_{9+i}(\rho_i) + \sum_{k=22-i}^{29} \bar{S}_k \\ &= W + \sum_{k=i-1}^{10-i} S_k \cdot \bar{V}_{11}(\rho_2) + \sum_{k=22-i}^{29} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=9+i}^{21-i} \bar{S}_k + \sum_{k=22-i}^{29} \bar{S}_k = \sum_{\rho\in \mathrm{Irr}(G)} \deg(\rho)\rho\end{aligned}$$

For $W \in \mathbb{P}(V_{15}(\rho_6))$ with $W \neq W_3'', W_4', W_5$ we have

$$\mathcal{I}(W)/\mathfrak{n} = W + \sum_{k\geq 1} S_k \cdot W = W + \sum_{k=16}^{29} \bar{S}_k = \sum_{\rho\in \mathrm{Irr}(G)} \deg(\rho)\rho$$

because $S_1W \equiv \bar{S}_{16}$ by combining Lemma 8.6 with the proof of Lemma 6.6.

For $W \in \mathbb{P}(V_{13}(\rho_2') \oplus V_{17}(\rho_2'))$ with $W \neq V_{17}(\rho_2')$ we have

$$\begin{aligned}\mathcal{I}(W)/\mathfrak{n} &= W + \sum_{k\geq 1} S_k \cdot \bar{V}_{13}(\rho_2') \\ &= W + \sum_{k=1}^{4} S_k \cdot \bar{V}_{13}(\rho_2') + \sum_{k=18}^{29} \bar{S}_k \\ &= \frac{1}{2}\sum_{k=13}^{17} \bar{S}_k + \sum_{k=18}^{29} \bar{S}_k = \sum_{\rho\in \mathrm{Irr}(G)} \deg(\rho)\rho\end{aligned}$$

The remaining cases are similar. We can prove Theorem 6.7 in the same manner as before. □.

9. Finé.[1]

The fundamental divisor of the singularity $S := \mathbb{A}^2/G$ is the highest root of the adoint representation, in other words, The scheme-theoretic fiber of π at the origin $Z^* := \mathrm{Hilb}^G(\mathbb{A}^2) \times_S \{0\}$ is a Cartier divisor of $\mathrm{Hilb}^G(\mathbb{A}^2)$ with $Z^* = \sum_{\rho \in \mathrm{Irr}(G)} (\deg\ \rho) E(\rho)$.

See [GSV]. This fact is proved directly by using the equations of the singularities. On the other hand this is proved in our context by using representations of G. This will be discussed in detail elsewhere. Any vertex of the Dynkin diagram has an interpretation of a certain nontrivial G-submodule of $\mathfrak{m}/\mathfrak{n}$. The new vertex in an extended Dynkin diagram (added to the Dynkin diagram) is also naturally interpreted as a trivial G-module and it corresponds to the 0-module of $\mathfrak{m}/\mathfrak{n}$ so that the fundamental divisor of the singularity, the scheme-theoretic fiber of π over the singular point, corresponds to the highest root of the root system and we recover the extended Dynkin diagram this way.

We would like to mention once again the dualities on the coinvariant algebra $O_{\mathbb{A}^2}/\mathfrak{n}$ [IN2, Theorem 3.1]. The first duality was observed in [IN2] and in the present article by case-by-case examinations. However after completing [IN2] and the persent work, we were informed by Shinoda that the first duality also follows from Steinberg [St].

In the last, we would like to mention a connection of our work with Kronheimer's quivers. The coinvariant algebra $O_{\mathbb{A}^2}/\mathfrak{n}$ with a natural algebra structure is a kind of quiver, whose quotient quivers are Kronheimer's ones [Kr]. All of the Kronheimer's would be obtained this way so far as the group algebra $\mathbb{C}[G]$ is concerned. These will be discussed in [IN3].

Next we would like to mention unsolved or under-research related problems.

Conjecture 9.1. *Let G be any finite subgroup of $SL(3, \mathbb{C})$. Then $\mathrm{Hilb}^G(\mathbb{A}^3)$ is a crepant smooth resolution of $\mathbb{A}^3/G$.*

The conjecture is solved affirmatively in the abelian case [N2]. The other cases will be discussed elsewhere, for instance in the papers of Gomi-Nakamura-Shinoda in preparartion.

If G is a cyclic group of order two generated by $(-1)^{\oplus 4}$ in $SL(4, \mathbb{C})$, $\mathrm{Hilb}^G(\mathbb{A}^4)$ is nonsingular, which is however not a crepant resolution of $\mathbb{A}^3/G$. There are also many examples of singular $\mathrm{Hilb}^G(\mathbb{A}^4)$ for G abelian. Miles Reid also computed various examples, who is trying to understand higher dimensional McKay correpondence.

The following would be perhaps important for the future application.

Conjecture 9.2. *Let G is a finite subgroup of $SL(n, \mathbb{C})$, N a normal subgroup of G. Then $\mathrm{Hilb}^G(\mathbb{A}^n) \simeq \mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbb{A}^n))$.*

Definition. G is an irreducible subgroup of $SL(n, \mathbb{C})$ if the natural inclusion of G into $SL(n, \mathbb{C})$ is an irreducible representation of G.

The most interesting is the following

[1] This section was named by Y. Ito.

Conjecture 9.3. *If G is a finite irreducible simple subgroup of $SL(n, \mathbb{C})$, then* $\mathrm{Hilb}^G(\mathbb{A}^n)$ *is a smooth resolution of* $\mathbb{A}^n/G$.

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