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Equivalent conditions on the central limit theorem
for a sequence of probability measures on R *

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ABSTRACT
In this paper we give equivalent conditions on the central limit theorem in total variation norm for a sequence of probability measures on R. This generalizes Cacoullos, Papathanasiou and Utev's central limit theorem in L^1-norm for a sequence of probability density functions on R. We also give equivalent conditions on the central limit theorem in weak convergence and those on the local limit theorem.

1. Introduction.

Let f be a probability density function on R such that $\int_R yf(y)dy = \mu$ and $\int_R (y-\mu)^2 f(y)dy = \sigma^2 < \infty$. Cacoullos and Papathanasiou (1989) introduced the following function called a covariance kernel or ω-function of f to study the characterization of probability distributions;

$$\omega(x) \equiv \int_{-\infty}^{x} (\mu - y)f(y)dy/[\sigma^2 f(x)] \qquad (1).$$

on the set $\{y \in R : f(y) > 0\}$.

It is known that $\omega(x)f(x)$ is also a probability density function on R and that $f(x)dx$ is normal iff $\omega(x) \equiv 1$ (e.g. also Cacoullos, Papathanasiou and Utev, 1992).

Put

$$\phi(x) \equiv (2\pi)^{-1/2} \exp(-x^2/2),$$
$$\Phi(x) \equiv \int_{-\infty}^{x} \phi(y)dy. \qquad (2).$$

Then the following was given in Cacoullos, Papathanasiou and Utev, 1994, Theorem 1.2 by Stein's method (e.g. Stein, 1972 and also Bolthausen, 1984).

Theorem 1.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of probability density functions on R with $\int_R yf_n(y)dy = 0$ and $\int_R y^2 f_n(y)dy = 1$ $(n \geq 1)$ and with interval supports. Denoting by ω_n the ω-function of f_n $(n \geq 1)$, the following holds;*

$$\lim_{n\to\infty} \int_R |\phi(x) - f_n(x)|dx = 0 \quad \text{iff} \lim_{n\to\infty} \int_{|\omega_n(x)-1|>\delta} f_n(x)dx = 0 \quad \text{for any } \delta > 0. \qquad (3).$$

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* Key words and phrases; central limit theorem, total variation norm

Remark 1.1. It is easy to see that the following is true (e.g. Cacoullos, Papathanasiou and Utev, 1994, PROOF OF THEOREM 1.2),

$$\lim_{n\to\infty}\int_R |\omega_n(x)-1|f_n(x)dx = 0 \quad \text{iff} \lim_{n\to\infty}\int_{|\omega_n(x)-1|>\delta} f_n(x)dx = 0 \quad \text{for any } \delta > 0. \quad (4).$$

In this paper we generalize Theorem 1.1 to a sequence of Borel probability measures on $(R, \mathbf{B}(R))$, using a different method from Cacoullos, Papathanasiou and Utev, 1994. We also give equivalent conditions on central and local limit theorems.

In section 2 we state our results which will be proved in section 3.

2. Main results.

In this section we state our results.

Let us first introduce our assumption.

(A.1). $\{P_n\}_{n=1}^{\infty}$ is a sequence of probability measures on $(R, \mathbf{B}(R))$ such that $\int_R xP_n(dx) = 0$ and that $\int_R x^2 P_n(dx) = 1$ $(n \geq 1)$.

To generalize Theorem 1.1, we give the following definition.

Definition 2.1. For a probability measure P on $(R, \mathbf{B}(R))$ such that $\int_R yP(dy) = 0$ and $\int_R y^2 P(dy) = 1$, put for $x \in R$

$$W(P)(x) \equiv \int_{-\infty}^{x} -yP(dy). \quad (5).$$

It is easy to see that $W(P)(x)$ is a probability density function on R since $\int_R yP(dy) = 0$ and $\int_R y^2P(dy) = 1$. We would also like to point out that $W(P)(\cdot)$ is defined on the whole real line, though $\omega(\cdot)$ is not (e.g. (1)).

The following is the first result (e.g. (2)) and will be used in the proof of Theorem2.3.

Theorem 2.1. *Suppose that (A.1) holds. Then the following are equivalent to one another.*

(I). $\lim_{n\to\infty} P_n(dx) = \Phi(dx)$ weakly.
(II). $W(P_n)(\cdot)$ converges to $\phi(\cdot)$, as $n \to \infty$, uniformly on every compact subsets of R.
(III). For any $\varphi \in C_0^{\infty}(R; R)$

$$\lim_{n\to\infty}\int_R \varphi(x)(W(P_n)(x)dx - P_n(dx)) = 0.$$

For probability measures $P(dx)$ and $Q(dx)$ on $(R, \mathbf{B}(R))$, put

$$\rho(P, Q) \equiv \sup_{A\in\mathbf{B}(R)} |P(A) - Q(A)|. \quad (6).$$

For the sake of simplicity, we also write

$$\rho(P, W(P)) \equiv \sup_{A\in\mathbf{B}(R)} |P(A) - \int_A W(P)(x)dx|. \quad (7).$$

Then we get the following result which plays a crucial role in the proof of Theorem 2.3.

Proposition 2.2. *For any probability measures $P(dx)$ such that $\int_R xP(dx) = 0$ and that $\int_R x^2P(dx) < \infty$ and $Q(dx) = q(x)dx$ on $(R, \mathbf{B}(R))$ and $r > 1$,*

$$|\rho(P, W(P)) - \rho(P, Q)| \tag{8}$$
$$\leq r \sup_{|x|\leq r} |q(x) - W(P)(x)| + \int_{|x|\geq r} q(x)dx + \int_{|x|\geq r} x^2P(dx).$$

Finally we state our main result. It turns out that the following equivalence can be shown via the weak convergence on the central limit theorem by way of Proposition 2.2.

Theorem 2.3. *Suppose that (A.1) holds. Then the following are equivalent to each other.*

(I). $\lim_{n\to\infty} \rho(P_n, W(P_n)) = 0$.
(II). $\lim_{n\to\infty} \rho(\Phi, P_n) = 0$.
Remark 2.1. Theorem 2.3 generalize Theorem 1.1 from the following; for any probability density functions f and g on R

$$2 \sup_{A\in\mathbf{B}(R)} |\int_A f(x)dx - \int_A g(x)dx| = \int_R |f(y) - g(y)|dy.$$

Let us state another assumption and definition to state our final result.
(A.2). $P_n(dx) = f_n(x)dx$ for $n \geq 1$.
Definition 2.2. For a probability density function f on R such that $\int_R yf(y)dy = 0$ and $\int_R y^2f(y)dy = 1$, put for $x \in R$,

$$W(f)(x) \equiv \int_{-\infty}^{x} -yf(y)dy. \tag{9}$$

$W(f)$ is defined on R and $W(f)(x) = \omega(x)f(x)$ on the set $\{y \in R : f(y) > 0\}$ (e.g. (1)).

Proposition 2.4. *Suppose that (A.1)-(A.2) hold. Then the following are equivalent to each other.*

(I). $\lim_{n\to\infty} f_n(\cdot) = \phi(\cdot)$, uniformly on every compact subsets of R.
(II). $\lim_{n\to\infty} W(f_n)(\cdot)/f_n(\cdot) = 1$, uniformly on every compact subsets of R.
We close this section by proving Proposition 2.4.
Proof of Proposition 2.4.
(Proof of (II) from (I)). From (I), by Theorem 2.1, $\lim_{n\to\infty} W(f_n)(\cdot) = \phi(\cdot)$, uniformly on every compact subsets of R. This and (I) implies (II).

Q. E. D.

(Proof of (I) from (II)). By (II), for any $\varphi \in C_0^\infty(R; R)$,

$$\lim_{n\to\infty} \int_R \varphi(x)[W(f_n)(x) - f_n(x)]dx = 0.$$

Therefore by Theorem 2.1, $\lim_{n\to\infty} W(f_n)(\cdot) = \phi(\cdot)$, uniformly on every compact subsets of R. Hence as $n \to \infty$,

$$f_n(\cdot) = W(f_n)(\cdot)\{W(f_n)(\cdot)/f_n(\cdot)\}^{-1} \to \phi(\cdot),$$

uniformly on every compact subsets of R from (II).

Q. E. D.

3. Proof of results in section 2.

In this section we prove Theorems 2.1-2.3.

Let us first prove Theorem 2.1.

Proof of Theorem 2.1.

(Proof of (II) from (I)). For $x \in R$, take r for which $r > |x|$. Then

$$W(P_n)(x) = \int_{-r}^{x} -yP_n(dy) + \int_{-\infty}^{-r} -yP_n(dy) \equiv I_{n,r} + II_{n,r}.$$

From (I),

$$I_{n,r} \to \int_{-r}^{x} -y\phi(y)dy \quad (\text{as } n \to \infty) \;\to \int_{-\infty}^{x} -y\phi(y)dy = \phi(x) \quad (\text{as } r \to \infty)$$

; and from (A.1), by Chebychev's inequality

$$II_{n,r} \le 1/r \to 0 \quad (\text{as } r \to \infty).$$

The convergence of $W(P_n)$ at each point implies the uniform convergence of $W(P_n)$ on every compact subsets of R, since $W(P)(\cdot)$ is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$, and since $\phi(x)$ is continuous on R.

Q. E. D.

(Proof of (I) from (II)). By (A.1), $\{P_n\}_{n\ge1}$ is tight. Therefore there exist a probability measure Q on $(R, \mathbf{B}(R))$ and a subsequence $\{P_{n_k}\}_{k\ge1}$ which converges weakly to Q as $k \to \infty$.

In the same way as in (Proof of (II) from (I)), for $x \in R$, $W(P_{n_k})(x)$ converges to $W(Q)(x)$ as $k \to \infty$, and

$$W(Q)(x) = \int_{-\infty}^{x} -yQ(dy) = \phi(x),$$

from (II). This completes the proof.

Q. E. D.

(Proof of (III) from (I)). (II) implies that $W(P_n)(x)dx$ converges to $\phi(x)dx$ as $n \to \infty$, weakly. Since (I) and (II) are equivalent to each other, the proof is over.

Q. E. D.

(Proof of (I) from (III)). For any $\varphi \in C_0^\infty(R; R)$

$$\int_R [\varphi''(x) - x\varphi'(x)] P_n(dx) = \int_R \varphi''(x)[P_n(dx) - W(P_n)(x)dx] \to 0$$

as $n \to \infty$, from (III). This implies (I), since $\{P_n\}_{n\geq 1}$ is tight from (A.1) and since $\phi(x)$ is a unique solution of the following PDE; for any $\varphi \in C_0^\infty(R; R)$

$$\int_R [\varphi''(x) - x\varphi'(x)]\phi(x)dx = 0,$$
$$\int_R \phi(x)dx = 1.$$

Q. E. D.

Next let us prove Proposition 2.2.
Proof of Proposition 2.2. Put $U_r(o) \equiv \{y \in R; |y| \leq r\}$. Then for $r > 1$,

$$\begin{aligned} &\rho(P, Q) \\ &\leq \rho(P, W(P)) + r \sup_{|x|\leq r} |W(P)(x) - q(x)| + \int_{|x|\geq r} x^2 P(dx) + Q(U_r(o)^c), \end{aligned} \quad (10).$$

since for any $A \in \mathbf{B}(R)$,

$$\begin{aligned} &P(A) - Q(A) \\ &= P(A \cap U_r(o)) - \int_{A\cap U_r(o)} W(P)(x)dx \\ &\quad + \int_{A\cap U_r(o)} (W(P)(x) - q(x))dx + P(A \cap U_r(o)^c) - Q(A \cap U_r(o)^c), \end{aligned}$$

and since $r > 1$.

We can also show the following;

$$\begin{aligned} &\rho(P, W(P)) \\ &\leq \rho(P, Q) + r \sup_{|x|\leq r} |W(P)(x) - q(x)| + Q(U_r(o)^c) + \int_{|x|\geq r} x^2 P(dx), \end{aligned} \quad (11).$$

since for any $A \in \mathbf{B}(R)$,

$$P(A) - \int_A W(P)(x)dx$$
$$= P(A) - Q(A) + \int_{A\cap U_r(o)} (q(x) - W(P)(x))dx + \int_{A\cap U_r(o)^c} (q(x) - W(P)(x))dx,$$

and since, from the assumption on P,

$$\begin{aligned}\int_{|x|>r} W(P)(x)dx &= \int_{|x|>r} (\int_{-\infty}^{x} -yP(dy))dx \\ &= \int_r^\infty (\int_x^\infty yP(dy))dx + \int_{-\infty}^{-r} (\int_{-\infty}^{x} -yP(dy))dx \\ &= \int_r^\infty y(y-r)P(dy) + \int_{-\infty}^{-r} y(y+r)P(dy) \\ &\le \int_r^\infty y^2 P(dy) + \int_{-\infty}^{-r} y^2 P(dy).\end{aligned}$$

Q. E. D.

Finally we prove Theorem 2.3.

Proof of Theorem 2.3. (I) and (II) implies (III) and (I) in Theorem 2.1, respectively, and henceforth (II) in Theorem 2.1.

This completes the proof. In fact, by Proposition 2.2, from (A.1), for $r > 1$

$$\begin{aligned}&\le |\rho(P_n, \Phi) - \rho(P_n, W(P_n))| \qquad (12).\\ &\le r \sup_{|x|\le r} |\phi(x) - W(P_n)(x)| + \int_{|x|\ge r} \phi(x)dx + 1 - \int_{|x|\le r} x^2 P_n(dx) \\ &\to \int_{|x|\ge r} \phi(x)dx + 1 - \int_{|x|\le r} x^2\phi(x)dx \quad \text{(as } n\to\infty\text{, from Theorem 2.1, (I) and (II))} \\ &\to 0 \quad (\text{as } r \to \infty).\end{aligned}$$

Q. E. D.

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