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GENERIC AFFINE DIFFERENTIAL GEOMETRY OF SPACE CURVES

SHYUICHI IZUMIYA AND TAKASI SANO

ABSTRACT. We study affine invariants of space curves from the view point of the singularity theory of smooth functions. By the aid of the singularity theory we define a new equi-affine frame for space curves. We also introduce two surfaces associated with this equi-affine frame and give a generic classification of the singularities of those surfaces.

1. Introduction

There are several articles which study about "the generic differential geometry" in the Euclidian space ([2,3,4,5,6,7,11,12 etc.]). The main tools in these articles are the distance-squared functions and the height functions on submanifolds. The classical invariants of the extrinsic differential geometry can be treated as "singularities" of these functions (i.e.,this is a philosophy of R. Thom), however, as Fidal[7] pointed out, the geometric interpretation of sextactic points of a convex plane curve is quite complicated from this point of view. It has been classically known that the sextactic point is an equi-affine invariant. The affine differential geometry is a classical area in the differential geometry [1,14,15], however, it has been revived recently [13]. In [10] we introduced the notion of affine distance-cubed functions and affine height functions on convex plane curves. These functions are quite useful for the study of generic properties of invariants of the extrinsic affine differential geometry on convex plane curves. As a consequence, we applied ordinary techniques of the singularity theory for these functions and described equi-affine invariants for a convex plane curve. Moreover, we can say that the complete analogy of the generic euclidian differential geometry holds for convex plane curves. Giblin and Sapiro[8,9] also studied the affine invariant symmetric set for a convex plane curve by using the affine distance-cubed function.

In this paper we proceed the similar arguments on the space curve, however the situation is quite different. There are still some vagueness in the classical affine differential geometry of space curves. For example, the classical notion of affine torsion is not a torsion in the geometric sense (cf., §2). In this paper, by the aid of the method of the singularity theory, we define the new notions of the affine torsion and the affine binormal (i.e., *the intrinsic affine torsion* and *the intrisic affine binormal*). We also introduce equi-affine invariant surfaces (i.e., *the intrinsic affine binormal developable* and *the affine rectifying Gaussian surface*) associated with space curves which are singular surfaces in §4. The main result is Theorem 4.1 which gives local classifications of these singular surfaces. The proof of Theorem 4.1 is given in §5

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and §6. The basic techniques we used in this paper depend heavily on those in the attractive book of Bruce and Giblin [5].

All curves and maps considered here are of class C^∞ unless stated otherwise.

2. Basic notions

We now present basic concepts on affine differential geometry of space curves. The classical theory has been established in [1,14,15], however, there are still some vagueness compared with the theory for plane curves. Let $\mathbb{R}^3$ be the affine space which adopt the coordinate such that the volume of the parallelogram spanned by three vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ is given by the determinant

$$|a\ b\ c| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We fix the above coordinate in this paper. Let $\gamma : I \to \mathbb{R}^3$ be a curve with $|\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)| > 0$, where $\dot\gamma(t) = \frac{d\gamma}{dt}(t)$. *The affine arc-length* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{\frac{1}{6}} dt.$$

Then a parameter s is determined such that $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$, where $\gamma'(s) = \frac{d\gamma}{ds}(s)$. So we say that a curve γ is *parameterized by the affine arc-length* if it satisfies that

$$|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1.$$

We say that a quantity or property of the curve γ is *an equi-affine invariant* if it is invariant under the action of the special linear group and the group of parallel transportations. Affine differential geometry is a geometry which studies the equi-affine invariants in the sense of Klein's program.

Let us denote $T(s) = \gamma'(s), N(s) = \gamma''(s)$ and $B(s) = \gamma'''(s)$ and call $T(s)$ *an affine tangent, $N(s)$ an affine principal normal and $B(s)$ an affine binormal.* We also define *the affine curvature* by $\kappa_a(s) = |\gamma'(s)\ \gamma'''(s)\ \gamma^{(4)}(s)| = |T(s)\ B(s)\ B'(s)|$ and *the affine torsion* by $\tau_a(s) = -|\gamma''(s)\ \gamma'''(s)\ \gamma^{(4)}(s)| = -|N(s)\ B(s)\ B'(s)|$. By the similar arguments as those of in the Euclidian differential geometry, we have the following *Frenet-Serret type formula:*

$$\text{(2.1)} \qquad \begin{cases} T'(s) = N(s) \\ N'(s) = B(s) \\ B'(s) = -\kappa_a(s)N(s) - \tau_a(s)T(s). \end{cases}$$

It is known that this frame is not a good frame that the functions $\kappa_a(s)$ and $\tau_a(s)$ do not give the natural equation [1,14]. We can also say that the function $\tau_a(s)$ is not the torsion in the following sense: Remember the original Frenet-Serret formula in the Euclidian differential geometry:

$$
\begin{cases} \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) = -\tau(s)\mathbf{n}(s), \end{cases} \tag{2.2}
$$

where s is the Euclidian arc-length parameter, $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}$ is the principal normal vector and $\mathbf{b}$ is the binormal vector. Under this notation, suppose that the Euclidian torsion $\tau(s)$ is constantly equal to zero, then the binormal vector $\mathbf{b}$ is constant along s. So the meaning of the torsion function is to estimate the torsion of the binormal direction along $\gamma(s)$. For the affine case, even if the affine torsion $\tau_a(s)$ constantly vanishes the affine binormal is not constant.

On the other hand, the above affine frame has the property that $|T(s)\ N(s)\ B(s)| = 1$. We call the frame which has this property *an equi-affine frame.* There are infinitely many equi-affine frames. For example, $(T(s), N(s), B(s)+\lambda(s)T(s))$ is an equi-affine frame for each function $\lambda(s)$. We call such a family *a general equi-affine frame* and each member is called *a special equi-affine frame.*

We remark that $(T(s), N(s), B(s) + \mu(s)N(s))$ is also an equi-affine frame for each μ. If we adopt this frame, we have

$$
\begin{aligned}
(B(s) + \mu(s)N(s))' =& B'(s) + \mu'(s)N(s) + \mu(s)N'(s) \\
=& - \kappa_a(s)N(s) - \tau_a(s)T(s) + \mu'(s)N(s) + \mu(s)B(s) \\
=& (-\kappa_(s) + \mu(s) - \mu^2(s))N(s) - \tau_a(s)T(s) + \mu(s)(B(s) + \mu(s)N(s)).
\end{aligned}
$$

This means that the derivative of the new binormal $B(s) + \mu(s)N(s)$ in the Frenet-Serret type formula depends on itself, so that this frame is not good. In the following sections we try to choose a good special equi-affine frame by the aid of the singularity theory.

3. Affine invariant functions

In this section we introduce two different families of functions on a non-degenerate space curve which are useful for the study of the equi-affine invariants of the curve. Let $\gamma : I \to \mathbb{R}^3$ be a space curve with $|\gamma(s)' \ \gamma''(s) \ \gamma'''(s)| = 1$.

3-1) Affine distance functions. We now define a three parameter family of smooth functions on I

$$F : I \times \mathbb{R}^3 \to \mathbb{R}$$

by

$$F(s,u) = |\gamma'(s) \ \gamma''(s) \ \gamma(s) - u|.$$

We call F *an affine distance (6-th powered) function on* γ.

Differentiating $F(s,u)$ with respect to s, we have

$$\begin{aligned}
&(3.1) \qquad f_u'(s) =&& M_{13}(s),\\
&(3.2) \qquad f_u''(s) =&& M_{23}(s) - \kappa_a(s)M_{12}(s),\\
&(3.3) \qquad f_u'''(s) =&& (\tau_a(s) - \kappa_a(s))M_{12}(s) + (\tau_a(s_-2\kappa_a'(s))M_{13}(s) - \kappa_a(s)M_{23}(s),\\
&(3.4) \qquad f_u^{(4)}(s) =&& (\tau_a''(s) - \kappa_a''(s) + \kappa_a^2(s))M_{12}(s)\\
& && +(\tau_a(s) - 2\kappa_a'(s))M_{13}(s) - \kappa_a(s)M_{23}(s)(s),\\
&(3.5) \qquad f_u^{(5)}(s) =&& (\tau_a''(s) - \kappa_a'''(s) + 4\kappa_a(s)\kappa_a'(s) - 2\kappa_a(s)\tau_a(s))M_{12}\\
& && +(2\tau_a'(s) - 3\kappa_a'(s) + \kappa_a^2(s))M_{13}(s) + (\tau_a(s) - 3\kappa_a'(s))M_{23}(s) - \kappa_a(s),
\end{aligned}$$

where $f_u(s) = F(s,u)$ for any $u \in \mathbb{R}^2$ and $M_{ij}(s) = |\gamma^{(i)}(s), \gamma^{(j)}(s), \gamma(s) - u|$.

It follows from these formulae that we have the following proposition.

Proposition 3.1. *Let $\gamma : I \to \mathbb{R}^3$ be a space curve with $|\gamma'(s) \ \gamma''(s) \ \gamma'''(s)| = 1$. Then*
(1) $f_u'(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \lambda\gamma'(s_0) - \mu\gamma'''(s_0)$.
(2) $f_u'(s_0) = f_u''(s_0) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$.
(3) $f_u'(s_0) = f_u''(s_0) = f_u'''(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa_a'(s_0) \neq 0$ and $u = \gamma(s_0) + \frac{1}{\tau - a(s_0) - \kappa_a'(s_0))}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0)$.
(4) $f_u'(s_0) = f_u''(s_0) = f_u'''(s_0) = f^{(4)}(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa_a'(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\tau_a(s_0) - \kappa_a'(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $\tau_a'(s_0) - \kappa_a''(s_0) = 0$.
(5) $f_u'(s_0) = f_u''(s_0) = f_u'''(s_0) = f^{(4)}(s_0) = f^{(5)}(s_0) = 0$ if and only if $\tau_a(s_0) - \kappa_a'(s_0) \neq 0$, $u = \gamma(s_0) + \frac{1}{\tau_a(s_0) - \kappa_a'(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $\tau_a'(s_0) - \kappa_a''(s_0) = \tau_a''(s_0) - \kappa_a'''(s_0) = 0$.

Proof. (1) By the formula (3.1), $f_u'(s_0) = 0$ if and only if $\gamma'(s_0)$, $\gamma'''(s_0)$ and $\gamma(s_0) - u$ are linearly dependent and $\gamma'(s_0)$ and $\gamma'''(s_0)$ are linearly independent.
(2) It follows from the formula (1) and (3.2) that $f_u'(s_0) = f_u''(s_0) = 0$ if and only if $\lambda, \mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \lambda\gamma'(s_0) - \mu\gamma'''(s_0)$ and $0 = M_{23}(s_0) - \kappa_a(s_0)M_{12}(s_0) = \lambda - \mu\kappa_a(s_0)$.
(3) By the consequence (2) and (3.3), $f_u'(s_0) = f_u''(s_0) = f_u'''(s_0) = 0$ if and only if there exists $\mu \in \mathbb{R}$ such that $u = \gamma(s_0) - \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau_a(s_0) - \kappa_a'(s_0))M_{12}(s_0) -$

$\kappa_a(s_0)M_{13}(s_0)+1=(\tau_a(s_0)-\kappa_a'(s_0))\mu+1$. The last condition is equivalent to the condition that $\tau_a(s_0)-\kappa_a'(s_0)\neq 0$ and $\mu=-\frac{1}{\tau_a(s_0)-\kappa_a'(s_0)}$.

(4) By the consequence (3) and (3.4), $f_u'(s_0)=f_u''(s_0)=f_u'''(s_0)=f_u^{(4)}(s_0)=0$ if and only if $\tau_a(s_0)-\kappa_a'(s_0)\neq 0$, $u=\gamma(s_0)+\frac{1}{\tau_a(s_0)-\kappa_a'(s_0)}(\kappa_a(s_0)\gamma'(s_0)+\gamma'''(s_0)$ and

$$\begin{aligned}0&=(\tau_a'(s_0)-\kappa_a''(s_0)+\kappa_a^2(s_0))M_{12}(s_0)+(\tau_a(s_0)-2\kappa_a'(s_0))M_{13}(s_0)-\kappa_a'(s_0)M_{23}(s_0)\\&=-\frac{1}{\tau_a(s_0)-\kappa_a'(s_0))}(\tau_a'(s_0)-\kappa_a''(s_0)+\kappa_a^2(s_0))+\frac{\kappa_a^2(s_0)}{\tau_a(s_0)-\kappa_a'(s_0)}.\end{aligned}$$

The last condition is equivalent to the condition that $\tau_a'(s_0)-\kappa_a''(s_0)=0$.

(5) By the consequence (4) and (3.5), $f_u'(s_0)=f_u''(s_0)=f_u'''(s_0)=f_u^{(4)}(s_0)=f_u^{(5)}(s_0)=0$ if and only if $\tau_a(s_0)-\kappa_a'(s_0)\neq 0$, $u=\gamma(s_0)+\frac{1}{\tau_a(s_0)-\kappa_a'(s_0)}(\kappa_a(s_0)\gamma'(s_0)+\gamma'''(s_0)$, $\tau_a'(s_0)-\kappa_a''(s_0)=0$ and $0=(\tau_a''(s_0)\kappa_a'''(s_0)+4\kappa_a(s_0)\kappa_a'(s_0)-2\kappa_a(s_0)\tau_a(s_0))M_{12}(s_0)+(2\tau_a'(s_0)-3\kappa_a'(s_0)+\kappa_a^2(s_0))M_{13}(s_0)+(\tau_a(s_0)-3\kappa_a'(s_0))M_{23}(s_0)-\kappa_a(s_0)=-\frac{1}{\tau_a(s_0)-\kappa_a'(s_0)}(\tau_a''(s_0)-\kappa_a'''(s_0)+4\kappa_a(s_0)\kappa_a'(s_0)-2\tau_a(s_0)\kappa_a(s_0))-\frac{1}{\tau_a(s_0)-\kappa_a'(s_0)}\kappa_a(s_0)(\tau_a(s_0)-3\kappa_a'(s_0))-\kappa_a(s_0)$. The last condition is equivalent to the condition that $\tau_a''(s_0)-\kappa_a'''(s_0)=0$. This completes the proof. □

3-2) Affine height functions. Let $S^2=\{(x_1,x_2,x_3)|x_1^2+x_2^2+x_3^2=1\}$ be the unit sphere in $\mathbb{R}^3$. We also define a family of smooth functions on γ parameterized by S^2

$$H:I\times S^2\to\mathbb{R}$$

by

$$H(s,u)=|\gamma'(s)\ \gamma''(s)\ u|.$$

We call H *an affine height function on* γ.

Differentiating $H(s,u)$ with respect to s and applying Frenet-Serret type formula (2.1), we have the followings:

$$h_u'(s)=N_{13}(s), \tag{3.6}$$
$$h_u''(s)=N_{23}(s)-\kappa_a(s)N_{12}(s), \tag{3.7}$$
$$h_u'''(s)=(\tau_a(s)-\kappa_a'(s))N_{12}(s)-\kappa_a(s)N_{13}(s), \tag{3.8}$$
$$\begin{aligned}h^{(4)}(s)=&(\tau_a'(s)-\kappa_a''(s)+\kappa^2(s))N_{12}(s)\\&+(\tau_a(s)-2\kappa_a'(s))N_{13}(s)-\kappa_a(s)N_{23}(s),\end{aligned} \tag{3.9}$$
$$\begin{aligned}h^{(5)}(s)=&(\tau_a''(s)-\kappa_a'''(s)+4\kappa_a(s)\kappa_a'(s)-2\kappa_a(s))N_{12}(s)\\&+(2\tau_a'(s)-3\kappa_a''(s)+\kappa_a^2(s))N_{13}(s)+(\tau_a(s)-3\kappa_a'(s))N_{23}(s),\end{aligned} \tag{3.10}$$

where $h_u(s)=H(s,u)$ for any $u\in S^2$ and $N_{ij}(s)=|\gamma^{(i)}(s)\ \gamma^{(j)}(s)\ u|$.

It follows from these formulae that we have the following proposition.

Proposition 3.2. *Let $\gamma : I \to \mathbb{R}^2$ be a space curve with $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$. Then*
(1) $h'_u(s_0) = 0$ *if and only if there exist* $\lambda, \mu \in \mathbb{R}$ *such that* $u = \lambda\gamma'(s_0) + \mu\gamma'''(s_0)$.
(2) $h'_u(s_0) = h''_u(s_0) = 0$ *if and only if there exists* $\mu \in \mathbb{R}$ *such that* $\mu \neq 0$ *and* $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$.
(3) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ *if and only if there exists*$\mu \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ *and* $\tau_a(s_0) - \kappa'_a(s_0) = 0$.
(4) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h^{(4)}_u = 0$ *if and only if there exists*$\lambda \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ *and* $\tau_a(s_0) - \kappa'_a(s_0) = \tau'_a(s_0) - \kappa''_a(s_0) = 0$.
((5) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h^{(4)}_u = h^{(5)}_u = 0$ *if and only if there exists*$\lambda \in \mathbb{R}$ *such that* $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0)+\gamma'''(s_0))$ *and* $\tau_a(s_0)-\kappa'_a(s_0) = \tau'_a(s_0)-\kappa''_a(s_0) = \tau''_a(s_0)-\kappa'''_a(s_0) = 0$.

Proof. (1) By the formula (3.6), $h'_u(s_0) = 0$ if and only if $u\ \gamma'(s_0)$, $\gamma'''(s_0)\ u$ are linearly dependent and $\gamma'(s_0)$, $\gamma'''(s_0)$ are linearly independent.
(2) It follows from (1) and the formula (4.7) that $h'_u(s_0) = h''_u(s_0) = 0$ if and only if there exist $\lambda, \mu \in \mathbb{R}$ such that $u = \lambda\gamma''(s_0) + \mu\gamma'''(s_0)$, $\lambda^2 + \mu^2 \neq 0$ and $0 = N_{23}(s_0) - \kappa_a(s_0)N_{12}(s_0) = \lambda - \mu\kappa_a(s_0)$.
(3) By (2) and (3.8), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ if and only if there exists$\mu \in \mathbb{R}$ such that $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau_a(s_0) - \kappa'_a(s_0))N_{12}(s_0) - \kappa_a(s_0)N_{13}(s_0) = (\tau_a(s_0) - \kappa'_a(s_0))\mu$. Because $\mu \neq 0$, we have $\tau_a(s_0) - \kappa'_a(s_0) = 0$.
(4) By (3) and (3.9), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h^{(4)}_u = 0$ if and only if there exists$\lambda \in \mathbb{R}$ such that $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau'_a(s) - \kappa''_a(s) + \kappa^2(s))N_{12}(s) + (\tau_a(s) - 2\kappa'_a(s))N_{13}(s) - \kappa_a(s)N_{23}(s) = (\tau'_a(s_0) - \kappa''_a(s_0))\mu$. By the fact that $\mu \neq 0$, we have $\tau'_a(s_0) - \kappa''_a(s_0) = 0$.
(5) By (4) and (3.10), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = h^{(4)}_u = h^{(5)}_u = 0$ if and only if there exists$\lambda \in \mathbb{R}$ such that $\mu \neq 0$, $u = \mu(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$ and $0 = (\tau''_a(s) - \kappa'''_a(s) + 4\kappa_a(s)\kappa'_a(s) - 2\kappa_a(s))N_{12}(s) + (2\tau'_a(s) - 3\kappa''_a(s) + \kappa^2_a(s))N_{13}(s) + (\tau_a(s) - 3\kappa'_a(s))N_{23}(s) = \tau''_a(s_0) - \kappa'''_a(s_0) - \kappa_a(s_0)(\tau_a(s_0) - \kappa'_a(s_0))$. Since $\tau_a(s_0) - \kappa'_a(s_0) = 0$, we have $\tau''_a(s_0) - \kappa'''_a(s_0) = 0$. This completes the proof. □

4. Equi-affine invariants of space curves

By the propositions in the last section, we can recognize that the function $\tau_a(s) - \kappa_a'(s)$ and the direction $\kappa_a(s)\gamma'(s) + \gamma'''(s)$ have special meanings. In fact, we define a new special equi-affine frame as follows:

$$\tilde{T}(s) = \gamma'(s),$$
$$\tilde{N}(s) = \gamma''(s),$$
$$\tilde{B}(s) = \kappa_a(s)\gamma'(s) + \gamma'''(s).$$

Differentiating each vector with respect to s, we have

$$\tilde{T}'(s) = \gamma''(s) = \tilde{N}(s) \tag{4.1}$$

$$\begin{aligned}\tilde{N}'(s) &= \gamma'''(s) = \tilde{B}(s) - \kappa_a(s)\gamma'(s) \\ &= -\kappa_a(s)\tilde{T}(s) + \tilde{B}(s)\end{aligned} \tag{3.2}$$

$$\begin{aligned}\tilde{B}'(s) &= \kappa_a'(s)\gamma'(s) + \kappa_a(s)\gamma''(s) + \gamma^{(4)}(s) \\ &= \kappa_a'(s)\gamma'(s) - \tau_a(s)\gamma'(s) \\ &= -(\tau_a(s) - \kappa_a'(s))\gamma'(s) = -(\tau_a(s) - \kappa_a'(s))\tilde{T}(s).\end{aligned} \tag{4.3}$$

Hence, if we put $\sigma_a(s) = \tau_a(s) - \kappa_a'(s)$, then we have the following new Frenet-Serret type formula:

$$\begin{cases} \tilde{T}'(s) = \tilde{N}(s) \\ \tilde{N}'(s) = -\kappa_a(s)\tilde{T}(s) + \tilde{B}(s) \\ \tilde{B}'(s) = -\sigma_a(s)\tilde{T}(s). \end{cases}$$

We remark that if $\sigma_a(s)$ is constantly equal to zero, then $\tilde{B}(s)$ is a constant vector along $\gamma(s)$. We may call $\tilde{B}(s)$ *an intrinsic affine binormal* and $\sigma_a(s)$ *an intrinsic affine torsion* of the curve $\gamma(s)$. It is clear that the frame $(\tilde{T}(s), \tilde{N}(s), \tilde{B}(s))$ is an equi-affine frame, so that we call it *an intrinsic special affine frame* (briefly, *an IS-frame*). Accordingly the assertion (2) of Proposition 3.1, we define a ruled surface generated by the intrinsic affine binormal of $\gamma(s)$:

$$RS(\gamma) = \{x \in \mathbb{R}^3 | x = \gamma(s) + \eta\tilde{B}(s)\ \eta \in \mathbb{R}\}.$$

On the other hand, we define a one-parameter family of functions $R : I \times \mathbb{R}^3 \to \mathbb{R}$ by

$$R(s, u) = |\gamma'(s)\ \gamma'''(s)\ \gamma(s) - u| = f_u'(s).$$

For a fixed point $s_0 \in I$, the plane $P_{s_0} = \{u \in \mathbb{R}^3 | R(s_0, u) = 0\}$ is *the affine rectifying plane* of $\gamma(s)$ at $\gamma(s_0)$,which is the affine plane generated by the intrinsic affine tangent $\tilde{T}(s_0)$ and the intrinsic affine binormal $\tilde{B}(s_0)$.

By the assertion (2) of Proposition 3.1, $R(s, u) = \frac{\partial R}{\partial s}(s, u) = 0$ if and only if $u \in RS(\gamma)$. It follows that the surface $RS(\gamma)$ is the envelope of the family of affine rectifying planes along $\gamma(s)$, so that it is called *affine rectifying surface.* Since the affine rectifying surface is the envelope of the one-parameter family of planes in $\mathbb{R}^3$, it is classically known that such a surface is a developable. By this reason, we also call it *an intrinsic affine binormal developable.* By definition and the Frenet-Serret type formula, the intrinsic affine developable is a cylindrical surface if and only if $\sigma_a(s) \equiv 0$.

By the assertion (1) of Proposition 3.2, the point $u = \lambda\gamma'(s) + \mu\gamma'''(s)$ seems to have special meanings for the curve $\gamma(s)$. If $\mu = 0$, then u is on the tangent line of $\gamma(s)$. so that this case is rather a critical case. Assume that $\mu \neq 0$, then we have $\frac{u}{\mu} = \frac{\lambda}{\mu}\gamma'(s) + \gamma'''(s)$, so that $\frac{u}{\mu}$ is on the following surface associated with the curve $\gamma(s)$.

$$GS(\gamma) = \{\nu\gamma'(s) + \gamma'''(s) | \nu \in \mathbb{R}, s \in I\}.$$

We call $GS(\gamma)$ *an affine rectifying Gaussian surface.* We do not know that these surfaces have classically been studied or not. We only describe the generic singularities appearing these surfaces.

In order to formulate the main results, we consider that a curve is an immersion $\gamma : S^1 \to \mathbb{R}^3$. Let $\mathrm{Imm}(S^1, \mathbb{R}^3)$ be the space of immersions equipped with the Whitney C^∞-topology. We consider an open subset

$$\mathrm{Imm}_a^+(S^1, \mathbb{R}^3) = \{\gamma \in \mathrm{Imm}(S^1, \mathbb{R}^3) | |\dot{\gamma}(t)\ \ddot{\gamma}(t)\ \dddot{\gamma}(t)| > 0\}.$$

Theorem 4.1. *There exists a dense subset $\mathcal{O} \subset \mathrm{Imm}_a^+(S^1, \mathbb{R}^3)$ such that for any $\gamma \in \mathcal{O}$ we have the following:*
(1) Let p be a point of the intrinsic affine binormal developable of γ at s_0, then, locally at p, the intrinsic affine binormal developable is
(i) *diffeomorphic to the plane in $\mathbb{R}^3$ if $\sigma_a'(s_0) \neq 0$*
(ii) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ if $\sigma_a'(s_0) = 0$ and $\sigma_a''(s_0) \neq 0$.*
(iii) *diffeomorphic to the swallow tail in $\mathbb{R}^3$ if $\sigma_a'(s_0) = \sigma_a''(s_0) = 0$ and $\sigma_a'''(s_0) \neq 0$.*
(2) Let p be a point on the affine rectifying Gaussian surface of γ at s_0, then locally at p, the affine rectifying Gaussian surface is
(i) *diffeomorphic to the plane in $\mathbb{R}^3$ if $\sigma_a(s_0) \neq 0$.*
(ii) *diffeomorphic to the cuspidal edge in $\mathbb{R}^3$ if $\sigma_a(s_0) = 0$ and $\sigma_a'(s_0) \neq 0$.*
(iii) *diffeomorphic to the swallow tail in $\mathbb{R}^3$ if $\sigma_a(s_0) = \sigma_a'(s_0) = 0$ and $\sigma_a''(s_0) \neq 0$. Here, the the cuspidal edge is given as $\mathbb{R} \times \{(x_1, x_2) | x_1^2 = x_2^3\}$ and the swallow tail is given as $\{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.*

5. Unfoldings of functions of one-variable

In this section we give the main part of the proof of Theorem 4.1. For the purpose, we review some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [5]. Let $F : (\mathbb{R}\times\mathbb{R}^r, (t_0, x_0)) \to \mathbb{R}$ be a function germ. We call f *an r-parameter unfolding* of $f(t) = F(t, x_0)$. The crucial notion is universal unfoldings, however, we only use some features of it, so we do not need to give the original definition of universal unfoldings (cf., [5]). We say that $f(t)$ has *an A_k singularity at* t_0 if $f^{(p)}(t_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(t_0) \neq 0$. In this case we adopt the following definition. Let $j^{k-1}(\frac{\partial F}{\partial x_i}(t, x_0))(t_0) = \sum_{j=1}^{k-1} \alpha_{j,i}t^j$ for $i = 1, \ldots, r$, where j^{k-1} denotes the $(k-1)$-jet. We say that F is *infinitesimally (p)versal* (respectively, *infinitesimally versal*) if the $(k-1)\times r$ (respectively, $k\times r$) matrix of coefficients (α_{ji}) (respectively, $(\alpha_{0i}, \alpha_{ji})$) has rank $k-1$ (respectively, k). There are two important sets for concerning the unfoldings relative to the above notions. *The bifurcation set* of F is the set

$$\mathcal{B}_F = \{x \in \mathbb{R}^r | \text{there exists } t \text{ with } \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0 \text{ at } (t, x)\}$$

and *the discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{there exists } t \text{ with } F = \frac{\partial F}{\partial t} = 0 \text{ at } (t, x)\}.$$

The proof of Theorem 4.1 is based on the following result.

Proposition 5.1 (cf., [5]). *Let $F : (\mathbb{R}\times\mathbb{R}^r, (t_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(t)$ which has the A_k singularity at t_0.*

(1) *Suppose that F is an infinitesimally (p)versal unfolding.*
  (a) *If $k = 2$, then $\mathcal{B}_F$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$.*
  (b) *If $k = 3$, then $\mathcal{B}_F$ is diffeomorphic to $C\times\mathbb{R}^{r-2}$.*
  (c) *If $k = 4$, then $\mathcal{B}_F$ is diffeomorphic to $S\times\mathbb{R}^{r-3}$.*

(2) *Suppose that F is an infinitesimally versal unfolding.*
  (a) *If $k = 1$, then $\mathcal{D}_F$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$.*
  (b) *If $k = 2$, then $\mathcal{D}_F$ is diffeomorphic to $C\times\mathbb{R}^{r-2}$.*
  (c) *If $k = 3$, then $\mathcal{D}_F$ is diffeomorphic to $S\times\mathbb{R}^{r-3}$.*

Here, $C = \{(x_1, x_2) | x_1^2 = x_2^3\}$ is the ordinary cusp and $S = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallow tail.

We now have the following key propositions for the proof of Theorem 4.1. Let $F : I\times\mathbb{R}^3 \to \mathbb{R}$ be the affine distance function on $\gamma : I \to \mathbb{R}^3$ with $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$.

Proposition 5.2. *If f_{u_0} has the A_k-singularity ($k = 2, 3, 4$) at s_0, then F is the infinitesimally (p)-versal unfolding of f_{u_0}.*

Proof. By definition, we have

$$\begin{aligned} F(s, u) &= |\gamma_a'(s)\ \gamma_a''(s)\ \gamma_a(s) - u\ | \\ &= \begin{vmatrix} x_1'(s) & x_1''(s) & x_1(s) - u_1 \\ x_2'(s) & x_2''(s) & x_2(s) - u_2 \\ x_3'(s) & x_3''(s) & x_3(s) - u_3 \end{vmatrix} \\ &= (x_1(s) - u_1)E_{23} + (x_2(s) - u_2)E_{31} + (x_3(s) - u_3)E_{12}, \end{aligned}$$

where $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, $u = (u_1, u_2, u_3)$ and $E_{ij} = x_i'(s)x_j''(s) - x_j'(s)x_i''(s)$. Then we have

$$
\begin{aligned}
F_{ij} &:= \frac{dE_{ij}}{ds} \\
&= x_i'(s)x_j'''(s) - x_j'(s)x_i'''(s), \\
\tilde{G}_{ij} &:= \frac{d^2E_{ij}}{ds^2} \\
&= x_i''(s)x_j'''(s) + x_i'(s)x_j^{(4)}(s) - x_j''(s)x_i'''(s) - x_j'(s)x_i^{(4)}(s) \\
&= x_i''(s)x_j'''(s) + x_i'(s)(-\kappa_a(s)x_j''(s) - \tau_a(s)x_j'(s)) \\
&\quad - x_j''(s)x_i'''(s) - x_j'(s)(-\kappa_a(s)x_i''(s) - \tau_a(s)x_i'(s)) \\
&= x_i''(s)x_j'''(s) - x_j''(s)x_i'''(s) - \kappa_a(s)(x_i'(s)x_j''(s) - x_j'(s)x_i''(s)) \\
&= G_{ij} - \kappa_a(s)E_{ij},
\end{aligned}
$$

where $G_{ij} = x_i''(s)x_j'''(s) - x_j''(s)x_i'''(s)$. We also have

$$
\begin{aligned}
\frac{dG_{ij}}{ds} &= x_i'''(s)x_j'''(s) + x_i''(s)x_j^{(4)}(s) - x_j'''(s)x_i'''(s) - x_j''(s)x_i^{(4)}(s) \\
&= x_i''(s)(-\kappa_a(s)x_j''(s) - \tau_a(s)x_j'(s)) - x_j''(s)(-\kappa_a(s)x_i''(s) - \tau_a(s)x_i'(s)) \\
&= \tau_a(s)(x_i'(s)x_j''(s) - x_j'(s)x_i''(s)) \\
&= \tau_a(s)E_{ij}.
\end{aligned}
$$

We now denote that

$$
\begin{aligned}
H_{ij} = \frac{d^3E_{ij}}{ds^3} &= \frac{dG_{ij}}{ds} - \kappa_a'(s)E_{ij} - \kappa_a(s)\frac{dE_{ij}}{ds} \\
&= \tau_a(s)E_{ij} - \kappa_a'(s)E_{ij} - \kappa_a(s)F_{ij} \\
&= (\tau_a(s) - \kappa_a'(s))E_{ij} - \kappa_a(s)F_{ij}
\end{aligned}
$$

Let $j^{k-1}(\frac{\partial F}{\partial u_i}(s, u_0))(s_0)$ be the $(k-1)$-jet of $\frac{\partial F}{\partial u_i}(s, u_0)$ at s_0. Since

$$
\frac{\partial F}{\partial u_1}(s, u) = -E_{23}, \frac{\partial F}{\partial u_2}(s, u) = -E_{32}, \frac{\partial F}{\partial u_3}(s, u) = -E_{12},
$$

we have

$$
\begin{aligned}
j^3(\frac{\partial F}{\partial u_1}(s, u_0))(s_0) &= -F_{23}s - \frac{1}{2}\tilde{G}_{23}s^2 - \frac{1}{6}H_{23}s^3 \\
j^3(\frac{\partial F}{\partial u_2}(s, u_0))(s_0) &= -F_{31}s - \frac{1}{2}\tilde{G}_{31}s^2 - \frac{1}{6}H_{31}s^3 \\
j^3(\frac{\partial F}{\partial u_3}(s, u_0))(s_0) &= -F_{12}s - \frac{1}{2}\tilde{G}_{12}s^2 - \frac{1}{6}H_{12}s^3.
\end{aligned}
$$

We distinguish into three cases.

The case (1) When f_u has the A_2-singularity at s_0, by the assumption $\gamma(s)$ satisfies the relation $|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)| = 1$, so that we have

$$-x_1''(s)F_{23} - x_2''(s)F_{31} - x_3''(s)F_{12} = 1.$$

Thus $(F_{23}, F_{31}, F_{12}) \neq (0,0,0)$. This means that

$$\operatorname{rank}(-F_{23} \quad -F_{31} \quad -F_{12}) = 1.$$

The case (2) When f_u has the A_3-singularity at s_0, we require the 2×3 matrix

$$A = \begin{pmatrix} -F_{23} & -F_{31} & -F_{12} \\ -\frac{1}{2}\tilde{G}_{23} & -\frac{1}{2}\tilde{G}_{31} & -\frac{1}{2}\tilde{G}_{12} \end{pmatrix}$$

to be non-singular.

For the purpose, we set $F = (-F_{23}, -F_{31}, -F_{12})$ and $G = (-\frac{1}{2}\tilde{G}_{23}, -\frac{1}{2}\tilde{G}_{31}, -\frac{1}{2}\tilde{G}_{12})$. Then it is enough to show that $\det(M) \neq 0$, where M is the Gramm matrix of F, G. We need the following lemma.

Lemma 5.2.1. *If $f_{u_0}(s)$ has the A_3-singularity at s_0, then we have the following:*
(1) $x_1'(s_0)F_{23} + x_2'(s_0)F_{31} + x_3'(s_0)F_{12} = 0$
(2) $x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12} = 0$
(3) $x_1''(s_0)F_{23} + x_2''(s_0)F_{31} + x_3''(s_0)F_{12} = -1$
(4) $\kappa_a(s_0)x_i'(s_0) + x_i'''(s_0) = -(\tau_a(s_0) - \kappa_a'(s_0))(x_i(s_0) - u_{0,i}), i = 1,2,3$

Proof of Lemma 5.2.1. The assertion (1) and (2) from the direct calculations.
(3) Since $|\gamma'(s_0)\ \gamma''(s_0)\ \gamma'''(s_0)| = 1$, we have $-x_1''(s_0)F_{23} - x_2''(s_0)F_{31} - x_3''(s_0)F_{12} = 1$. Therefore, we have $x_1''(s_0)F_{23} + x_2''(s_0)F_{31} + x_3''(s_0)F_{12} = -1$.
(4) By Proposition 3.1, we have $u = 0 = \gamma(s_0) + \frac{1}{\tau_a(s_0) - \kappa_a'(s_0)}(\kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0))$, so that we have $-(\tau_a(s_0) - \kappa_a'(s_0))(\gamma(s_0) - u_0) = \kappa_a(s_0)\gamma'(s_0) + \gamma'''(s_0)$. $\square$

Proof of the case (2) in Proposition 5.2. We now calculate the Gramm matrix M of F, G. Firstly we have

$$\begin{aligned} F \cdot F &= F_{23}^2 + F_{31}^2 + F_{12}^2 \\ G \cdot G &= \frac{1}{4}(\tilde{G}_{23}^2 + \tilde{G}_{31}^2 + \tilde{G}_{12}^2) \\ F \cdot G = G \cdot F &= \frac{1}{2}(F_{23}\tilde{G}_{23} + F_{31}\tilde{G}_{31} + F_{12}\tilde{G}_{12}), \end{aligned}$$

where $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$. It follows that the Gramm matrix M of F, G is

$$\begin{pmatrix} F \cdot F & F \cdot G \\ G \cdot F & G \cdot G \end{pmatrix} = \begin{pmatrix} F_{23}^2 + F_{31}^2 + F_{12}^2 & \frac{1}{2}(F_{23}\tilde{G}_{23} + F_{31}\tilde{G}_{31} + F_{12}\tilde{G}_{12}) \\ \frac{1}{2}(F_{23}\tilde{G}_{23} + F_{31}\tilde{G}_{31} + F_{12}\tilde{G}_{12}) & \frac{1}{4}(\tilde{G}_{23}^2 + \tilde{G}_{31}^2 + \tilde{G}_{12}^2) \end{pmatrix}.$$

We now calculate the determinant of M as follows:

$$
\begin{aligned}
\det(M) &= \frac{1}{4}(F_{23}^2 + F_{31}^2 + F_{12}^2)(\tilde{G}_{23}^2 + \tilde{G}_{31}^2 + \tilde{G}_{12}^2) - \frac{1}{4}(F_{23}\tilde{G}_{23} + F_{31}\tilde{G}_{31} + F_{12}\tilde{G}_{12})^2 \\
&= \frac{1}{4}(F_{23}^2\tilde{G}_{23}^2 + F_{31}^2\tilde{G}_{23}^2 + F_{12}^2\tilde{G}_{23}^2 + F_{23}^2\tilde{G}_{31}^2 + F_{31}^2\tilde{G}_{31}^2 + F_{12}^2\tilde{G}_{31}^2 + F_{23}^2\tilde{G}_{12}^2 + F_{31}^2\tilde{G}_{12}^2 \\
&+ F_{12}^2\tilde{G}_{12}^2 - F_{23}^2\tilde{G}_{23}^2 - F_{31}^2\tilde{G}_{31}^2 \\
&- F_{12}^2\tilde{G}_{12}^2 - 2F_{23}F_{31}\tilde{G}_{23}\tilde{G}_{31} - 2F_{23}F_{12}\tilde{G}_{23}\tilde{G}_{12} - 2F_{31}F_{12}\tilde{G}_{31}\tilde{G}_{12}) \\
&= \frac{1}{4}(F_{31}^2\tilde{G}_{23}^2 + F_{12}^2\tilde{G}_{23}^2 + F_{23}^2\tilde{G}_{31}^2 + F_{12}^2\tilde{G}_{31}^2 + F_{23}^2\tilde{G}_{12}^2 + F_{31}^2\tilde{G}_{12}^2 \\
&- 2F_{23}F_{31}\tilde{G}_{23}\tilde{G}_{31} - 2F_{23}F_{12}\tilde{G}_{23}\tilde{G}_{12} - 2F_{31}F_{12}\tilde{G}_{31}\tilde{G}_{12}) \\
&= \frac{1}{4}\{(F_{31}\tilde{G}_{23} - F_{23}\tilde{G}_{31})^2 + (F_{23}\tilde{G}_{12} - F_{12}\tilde{G}_{23})^2 + (F_{12}\tilde{G}_{31} - F_{31}\tilde{G}_{12})^2\} \\
&= \frac{1}{4}\{(F_{31}G_{23} - \kappa_a(s_0)E_{23}F_{31} - F_{23}G_{31} + \kappa_a(s_0)E_{31}F_{23})^2 \\
&+ (F_{23}G_{12} - \kappa_a(s_0)E_{12}F_{23} - F_{12}G_{23} + \kappa_a(s_0)E_{23}F_{12})^2 \\
&+ (F_{12}G_{31} - \kappa_a(s_0)E_{31}F_{12} - F_{31}G_{12} - \kappa_a(s_0)E_{12}F_{31})^2\} \\
&= \frac{1}{4}[\{F_{31}G_{23} - F_{23}G_{31} + \kappa_a(s_0)(E_{31}F_{23} - E_{23}F_{31})\}^2 \\
&+ \{F_{23}G_{12} - F_{12}G_{23} + \kappa_a(s_0)(E_{23}F_{12} - E_{12}F_{23})\}^2 \\
&+ \{F_{12}G_{31} - F_{31}G_{12} + \kappa_a(s_0)(E_{12}F_{31} - E_{31}F_{12})\}^2] \\
&= \frac{1}{4}[\{(x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0))F_{31} - (x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))F_{23} \\
&+ \kappa_a(s_0)((x_3'(s_0)x_1''(s_0) - x_1'(s_0)x_3''(s_0))F_{23} - (x_2'(s_0)x_3''(s_0) - x_3'(s_0)x_2''(s_0))F_{31})\}^2 \\
&+ \{(x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0))F_{23} - (x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0))F_{12} \\
&+ \kappa_a(s_0)((x_2'(s_0)x_3''(s_0) - x_3'(s_0)x_2''(s_0))F_{12} - (x_1'(s_0)x_2''(s_0) - x_2'(s_0)x_1''(s_0))F_{23})\}^2 \\
&+ \{(x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))F_{12} - (x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0))F_{31} \\
&+ \kappa_a(s_0)((x_1'(s_0)x_2''(s_0) - x_2'(s_0)x_1''(s_0))F_{31} - (x_3'(s_0)x_1''(s_0) - x_1'(s_0)x_3''(s_0))F_{12})\}^2] \\
&= \frac{1}{4}[\{x_2''(s_0)x_3'''(s_0)F_{31} - x_3''(s_0)x_2'''(s_0)F_{31} - x_3''(s_0)x_1'''(s_0)F_{23} + x_1''(s_0)x_3'''(s_0)F_{23} \\
&+ \kappa_a(s_0)(x_3'(s_0)x_1''(s_0)F_{23} - x_1'(s_0)x_3''(s_0)F_{23} \\
&- x_2'(s_0)x_3''(s_0)F_{31} + x_3'(s_0)x_2''(s_0)F_{31})\}^2 \\
&+ \{x_1''(s_0)x_2'''(s_0)F_{23} - x_2''(s_0)x_1'''(s_0)F_{23} - x_2''(s_0)x_3'''(s_0)F_{12} + x_3''(s_0)x_2'''(s_0)F_{12} \\
&+ \kappa_a(s_0)(x_2'(s_0)x_3''(s_0)F_{12} - x_3'(s_0)x_2''(s_0)F_{12} \\
&- x_1'(s_0)x_2''(s_0)F_{23} + x_2'(s_0)x_1''(s_0)F_{23})\}^2 \\
&+ \{x_3''(s_0)x_1'''(s_0)F_{12} - x_1''(s_0)x_3'''(s_0)F_{12} - x_1''(s_0)x_2'''(s_0)F_{31} + x_2''(s_0)x_1'''(s_0)F_{31} \\
&+ \kappa_a(s_0)(x_1'(s_0)x_2''(s_0)F_{31} - x_2'(s_0)x_1''(s_0)F_{31} \\
&- x_3'(s_0)x_1''(s_0)F_{12} + x_1'(s_0)x_3''(s_0)F_{12})\}^2]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}[\{x_3'''(s_0)(x_1''(s_0)F_{23} + x_2''(s_0)F_{31}) - x_3''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31}) \\
&+ \kappa_a(s_0)(x_3'(s_0)(x_1''(s_0)F_{23} + x_2''(s_0)F_{31}) - x_3''(s_0)(x_1'(s_0)F_{23} + x_2'(s_0)F_{31}))\}^2 \\
&\{x_2'''(s_0)(x_1''(s_0)F_{23} + x_3''(s_0)F_{12}) - x_2''(s_0)(x_1'''(s_0)F_{23} + x_3'''(s_0)F_{12}) \\
&+ \kappa_a(s_0)(x_2'(s_0)(x_1''(s_0)F_{23} + x_3''(s_0)F_{12}) - x_2''(s_0)(x_1'(s_0)F_{23} + x_3'(s_0)F_{12}))\}^2 \\
&\{x_1'''(s_0)(x_2''(s_0)F_{31} + x_3''(s_0)F_{12}) - x_1''(s_0)(x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12}) \\
&+ \kappa_a(s_0)(x_1'(s_0)(x_2''(s_0)F_{31} + x_3''(s_0)F_{12}) - x_1''(s_0)(x_2'(s_0)F_{31} + x_3'(s_0)F_{12}))\}^2] \\
&= \frac{1}{4}[\{x_3'''(s_0)(-1 - x_3''(s_0)F_{12}) - x_3''(s_0)(-x_3'''(s_0)F_{12}) \\
&+ \kappa_a(s_0)(x_3'(s_0)(-1 - x_3''(s_0)F_{12}) - x_3''(s_0)(-x_3'(s_0)F_{12}))\}^2 \\
&+ \{x_2'''(s_0)(-1 - x_2''(s_0)F_{31}) - x_2''(s_0)(-x_2'''(s_0)F_{31}) \\
&+ \kappa_a(s_0)(x_2'(s_0)(-1 - x_2''(s_0)F_{31}) - x_2''(s_0)(-x_2'(s_0)F_{31}))\}^2 \\
&+ \{x_1'''(s_0)(-1 - x_1''(s_0)F_{23}) - x_1''(s_0)(-x_1'''(s_0)F_{23}) \\
&+ \kappa_a(s_0)(x_1'(s_0)(-1 - x_1''(s_0)F_{23}) - x_1''(s_0)(-x_1'(s_0)F_{23}))\}^2] \\
&= \frac{1}{4}[\{-x_3'''(s_0) - x_3''(s_0)x_3'''(s_0)F_{12} + x_3''(s_0)x_3'''(s_0)F_{12} \\
&+ \kappa_a(s_0)(-x_3'(s_0) - x_3'(s_0)x_3''(s_0)F_{12} + x_3'(s_0)x_3''(s_0)F_{12})\}^2 \\
&+ \{-x_2'''(s_0) - x_2''(s_0)x_2'''(s_0)F_{31} + x_2''(s_0)x_2'''(s_0)F_{31} \\
&+ \kappa_a(s_0)(-x_2'(s_0) - x_2'(s_0)x_2''(s_0)F_{31} + x_2'(s_0)x_2''(s_0)F_{31})\}^2 \\
&+ \{-x_1'''(s_0) - x_1''(s_0)x_1'''(s_0)F_{23} + x_1''(s_0)x_1'''(s_0)F_{23} \\
&+ \kappa_a(s_0)(-x_1'(s_0) - x_1'(s_0)x_1''(s_0)F_{23} + x_1'(s_0)x_1''(s_0)F_{23})\}^2] \\
&= \frac{1}{4}\{(-x_3'''(s_0) - \kappa_a(s_0)x_3'(s_0))^2 \\
&+ (-x_2'''(s_0) - \kappa_a(s_0)x_2'(s_0))^2 + (-x_1'''(s_0) - \kappa_a(s_0)x_1'(s_0))^2\} \\
&= \frac{1}{4}(\tau_a(s_0) - \kappa_a'(s_0))^2(\sum_{i=1}^{3} (x_i(s_0) - u_{0,i})^2).
\end{aligned}
$$

If $\det(M) = 0$, then we have $\frac{1}{4}(\tau_a(s_0) - \kappa_a'(s_0))^2(\sum_{i=1}^{3} (x_i(s_0) - u_i)^2) = 0$. By the assumption and Propostion 3.1, we have $\tau_a(s_0) - \kappa_a'(s_0) \neq 0$, so that $\sum_{i=1}^{3} (x_i(s_0) - u_{0,i})^2 = 0$. This means that $x_i(s_0) - u_{0,i} = 0.(i = 1, 2, 3)$

On the other hand, by the assumption, we have

$$
x_i(s_0) - u_{0,i} = -\frac{1}{\tau_a(s_0) - \kappa_a'(s_0)}(\kappa_a(s_0)x_i'(s_0) + x_i'''(s_0)).
$$

Thus we have $\kappa_a(s_0)x_i'(s_0) + x_i'''(s_0) = 0$ which contradicts the fact that

$$
|\gamma'(s)\ \gamma''(s)\ \gamma'''(s)\ | = 1.
$$

So we have $\det(M) \neq 0$, then $\operatorname{rank} A = 2$.
(3) When f_u has the A_4-singularity at s_0, we require the 3×3 matrix

$$B = \begin{pmatrix} -F_{23} & -F_{31} & -F_{12} \\ -\frac{1}{2}\tilde{G}_{23} & -\frac{1}{2}\tilde{G}_{31} & -\frac{1}{2}\tilde{G}_{12} \\ -\frac{1}{6}H_{23} & -\frac{1}{6}H_{31} & -\frac{1}{6}H_{12} \end{pmatrix}$$

to be non-singular. It is enough to show that $\det(B) \neq 0$. We also need the following lemma.

Lemma 5.2.2. *If $f_{u_0}(s)$ has the A_4-singularity at s_0, then we have the following:*
(1) $x_1''(s_0)E_{23} + x_2''(s_0)E_{31} + x_3''(s_0)E_{12} = 0$.
(2) $x_1'''(s_0)E_{23} + x_2'''(s_0)E_{31} + x_3'''(s_0)E_{12} = 1$.

Proof of Lemma 5.2.2. The assertion (1) from the direct calculation.
(2) By the assumption, we have $|\,\gamma'(s_0)\ \gamma''(s_0)\ \gamma'''(s_0)\,| = 1$, so that we also have $x_1'''(s_0)E_{23} - x_2'''(s_0)E_{31} - x_3'''(s_0)E_{12} = 1$. □

Proof of the case (3) in Proposition 5.2. We now calculate the determinant of B as follows:

$$\begin{aligned}
\det(B) &= -\frac{1}{12}\{(\tilde{G}_{31}H_{12} - \tilde{G}_{12}H_{31})F_{23} + (\tilde{G}_{12}H_{23} - \tilde{G}_{23}H_{12})F_{31} + (\tilde{G}_{23}H_{31} - \tilde{G}_{31}H_{23})F_{12}\} \\
&= -\frac{1}{12}[\{(G_{31} - \kappa_a(s_0)E_{31})((\tau_a(s_0) - \kappa_a'(s_0))E_{12} - \kappa_a(s_0)F_{12}) \\
&- (G_{12} - \kappa_a(s_0)E_{12})((\tau_a(s_0) - \kappa_a'(s_0))E_{31} - \kappa_a(s_0)F_{31})\}F_{23} \\
&+ \{(G_{12} - \kappa_a(s_0)E_{12})((\tau_a(s_0) - \kappa_a'(s_0))E_{23} - \kappa_a(s_0)F_{23}) \\
&- (G_{23} - \kappa_a(s_0)E_{23})((\tau_a(s_0) - \kappa_a'(s_0))E_{12} - \kappa_a(s_0)F_{12})\}F_{31} \\
&+ \{(G_{23} - \kappa_a(s_0)E_{23})((\tau_a(s_0) - \kappa_a'(s_0))E_{31} - \kappa_a(s_0)F_{31}) \\
&- (G_{31} - \kappa_a(s_0)E_{31})((\tau_a(s_0) - \kappa_a'(s_0))E_{23} - \kappa_a(s_0)F_{23})\}F_{12}] \\
&= -\frac{1}{12}[\{(\tau_a(s_0) - \kappa_a'(s_0))E_{12}G_{31} - \kappa_a(s_0)F_{12}G_{31} + \kappa_a^2(s_0)E_{31}F_{12} \\
&- (\tau_a(s_0) - \kappa_a'(s_0))E_{31}G_{12} + \kappa_a(s_0)F_{31}G_{12} - \kappa_a^2(s_0)E_{12}F_{31}\}F_{23} \\
&+ \{(\tau_a(s_0) - \kappa_a'(s_0))E_{23}G_{12} - \kappa_a(s_0)F_{23}G_{12} + \kappa_a^2(s_0)E_{12}F_{23} \\
&- (\tau_a(s_0) - \kappa_a'(s_0))E_{12}G_{23} + \kappa_a(s_0)F_{12}G_{23} - \kappa_a^2(s_0)E_{23}F_{12}\}F_{31} \\
&+ \{(\tau_a(s_0) - \kappa_a'(s_0))E_{31}G_{23} - \kappa_a(s_0)F_{31}G_{23} + \kappa_a^2(s_0)E_{23}F_{31} \\
&- (\tau_a(s_0) - \kappa_a'(s_0))E_{23}G_{31} + \kappa_a(s_0)F_{23}G_{31} - \kappa_a^2(s_0)E_{31}F_{23}\}F_{12}] \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))(E_{12}F_{23}G_{31} - E_{31}F_{23}G_{12} \\
&+ E_{23}F_{31}G_{12} - E_{12}F_{31}G_{23} + E_{31}F_{12}G_{23} - E_{23}F_{12}G_{31}) \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))\{(E_{12}G_{31} - E_{31}G_{12})F_{23} + (E_{23}G_{12} \\
&- E_{12}G_{23})F_{31} + (E_{31}G_{23} - E_{23}G_{31})F_{12}\}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0)) \\
&[\{(x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))E_{12} - (x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0))E_{31}\}F_{23} \\
&+ \{(x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0))E_{23} - (x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0))E_{12}\}F_{31} \\
&+ \{(x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0))E_{31} - (x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))E_{23}\}F_{12}\}] \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))[\{x_1'''(s_0)(x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) \\
&- x_1''(s_0)(x_2'''(s_0)E_{31} + x_3'''(s_0)E_{12})\}F_{23} \\
&+ \{x_2'''(s_0)(x_1''(s_0)E_{23} + x_3''(s_0)E_{12}) - x_2''(s_0)(x_1'''(s_0)E_{23} + x_3'''(s_0)E_{12})\}F_{31} \\
&+ \{x_3'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31}) - x_3''(s_0)(x_1'''(s_0)E_{23} + x_2'''(s_0)E_{31})\}F_{12}] \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))[\{x_1'''(s_0)(x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) - x_1''(s_0)(1 - x_1'''(s_0)E_{23})\}F_{23} \\
&+ \{x_2'''(s_0)(x_1''(s_0)E_{23} + x_3''(s_0)E_{12}) - x_2''(s_0)(1 - x_2'''(s_0)E_{31})\}F_{31} \\
&+ \{x_3'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31}) - x_3''(s_0)(1 - x_3'''(s_0)E_{12})\}F_{12}] \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))\{(x_1'''(s_0)(x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) - x_1''(s_0) + x_1''(s_0)x_1'''(s_0)E_{23})F_{23} \\
&+ (x_2'''(s_0)(x_1''(s_0)E_{23} + x_3''(s_0)E_{12}) - x_2''(s_0) + x_2''(s_0)x_2'''(s_0)E_{31})F_{31} \\
&+ (x_3'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31}) - x_3''(s_0) + x_3''(s_0)x_3'''(s_0)E_{12})F_{12}\} \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))\{(x_1'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) - x_1''(s_0))F_{23} \\
&+ (x_2'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) - x_2''(s_0))F_{31} \\
&+ (x_3'''(s_0)(x_1''(s_0)E_{23} + x_2''(s_0)E_{31} + x_3''(s_0)E_{12}) - x_3''(s_0))F_{12}\} \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))(-x_1''(s_0)F_{23} - x_2''(s_0)F_{31} - x_3''(s_0)F_{12}) \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0))\{-(x_1''(s_0)F_{23} + x_2''(s_0)F_{31} + x_3''(s_0)F_{12})\} \\
&= -\frac{1}{12}(\tau_a(s_0) - \kappa_a'(s_0)) \neq 0.
\end{aligned}
$$

This completes the proof of Proposition 5.2. □

We define a function $\tilde{H} : I \times S^2 \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{H}(s,u,v) = |\ \gamma'(s)\ \gamma''(s)\ u\ | - v$$

We also define $h_{u,v} : I \to \mathbb{R}$ by

$$h_{u,v}(s) = \tilde{H}(s,u,v)$$

for any $(u,v) \in S^2 \times \mathbb{R}$.

Proposition 5.3. *If h_{u_0,v_0} has an A_k - singularity (k=1,2,3) at the point s_0 then $\tilde{H}$ is the ifinitesimally versal unfolding of h_{u_0,v_0}.*

Proof. We now consider the local representative $\bar{H} : I \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ of $\tilde{H}$ which is given by

$$
\begin{aligned}
\bar{H}(s,y) &= \begin{vmatrix} x_1'(s) & x_1''(s) & y_1 \\ x_2'(s) & x_2''(s) & y_2 \\ x_3'(s) & x_3''(s) & \sqrt{1-y_1^2-y_2^2} \end{vmatrix} - y_3 \\
&= (x_2'(s)x_3''(s) - x_3'(s)x_2''(s))y_1 \\
&\quad + (x_3'(s)x_1''(s) - x_1'(s)x_3''(s))y_2 \\
&\qquad + (x_1'(s)x_2''(s) - x_2'(s)x_1''(s))\sqrt{1-y_1^2-y_2^2} - y_3 \\
&= y_1E_{23} + y_2E_{31} + \sqrt{1-y_1^2-y_2^2}\, E_{12} - y_3,
\end{aligned}
$$

where $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, $E_{ij} = x_i'(s)x_j''(s) - x_j'(s)x_i''(s)$, $y = (y_1, y_2, y_3)$ and $(u,v) = (y_1, y_2, \sqrt{1-y_1^2-y_2^2}, y_3)$. Then we have

$$
\begin{aligned}
F_{ij} &:= \frac{dE_{ij}}{ds} \\
&= x_i''(s)x_j''(s) + x_i'(s)x_j'''(s) - x_j''(s)x_i''(s) - x_j'(s)x_i'''(s) \\
&= x_i'(s)x_j'''(s) - x_j'(s)x_i'''(s) \\
\tilde{G}_{ij} &:= \frac{d^2E_{ij}}{ds^2} \\
&= x_i''(s)x_j'''(s) + x_i'(s)x_j^{IV}(s) - x_j''(s)x_i'''(s) - x_j'(s)x_i^{IV}(s) \\
&= x_i''(s)x_j'''(s) + x_i'(s)(-\kappa_a(s)x_j''(s) - \tau_a(s)x_j'(s)) \\
&\qquad - x_j''(s)x_i'''(s) - x_j'(s)(-\kappa_a(s)x_i''(s) - \tau_a(s)x_i'(s)) \\
&= x_i''(s)x_j'''(s) - \kappa_a(s)x_i'(s)x_j''(s) - x_j''(s)x_i'''(s) + \kappa_a(s)x_j'(s)x_i''(s) \\
&= x_i''(s)x_j'''(s) - x_j''(s)x_i'''(s) - \kappa_a(s)(x_i'(s)x_j''(s) - x_j'(s)x_i''(s)) \\
&= G_{ij} - \kappa_a(s)E_{ij},
\end{aligned}
$$

where $G_{ij} = x_i''(s)x_j'''(s) - x_j''(s)x_i'''(s)$.

We denote $\frac{\partial \bar{H}}{\partial y_i}(s_0, y_0) + j^{k-1}(\frac{\partial \bar{H}}{\partial y_i}(s, y_0))(s_0)$ the $(k-1)$-jet of $\frac{\partial \bar{H}}{\partial y_i}(s,y)$ with constant term. We have

$$
\begin{aligned}
\frac{\partial \bar{H}}{\partial y_1}(s,y) &= -\frac{y_1}{\sqrt{1-y_1^2-y_2^2}}E_{12} + E_{23} = -\alpha y_1 E_{12} + E_{23} \\
\frac{\partial \bar{H}}{\partial y_2}(s,y) &= -\frac{y_2}{\sqrt{1-y_1^2-y_2^2}}E_{12} + E_{31} = -\alpha y_2 E_{12} + E_{31} \\
\frac{\partial \bar{H}}{\partial y_3}(s,y) &= -1,
\end{aligned}
$$

where $\alpha = \frac{1}{\sqrt{1-y_1^2-y_2^2}}$. It follows that

$$\begin{aligned}
\frac{\partial \bar{H}}{\partial y_1}(s_0, y_0) + j^2\left(\frac{\partial \bar{H}}{\partial y_1}(s, y_0)\right)(s_0) &= \frac{\partial \bar{H}}{\partial y_1}(s_0, y_0) + \frac{\partial^2 \bar{H}}{\partial s \partial y_1}(s_0, y_0)s + \frac{1}{2}\frac{\partial^3 \bar{H}}{\partial s^2 \partial y_1}(s_0, y_0)s^2 \\
&= (-\alpha y_1 E_{12} + E_{23}) + (-\alpha y_1 F_{12} + F_{23})s + \frac{1}{2}(-\alpha y_1 \tilde{G}_{12} + \tilde{G}_{23})\, s^2 \\
\frac{\partial \bar{H}}{\partial y_2}(s_0, y_0) + j^2\left(\frac{\partial \bar{H}}{\partial y_2}(s, y_0)\right)(s_0) &= \frac{\partial \bar{H}}{\partial y_2}(s_0, y_0) + \frac{\partial^2 \bar{H}}{\partial s \partial y_2}(s_0, y_0)s + \frac{1}{2}\frac{\partial^3 \bar{H}}{\partial s^2 \partial y_2}(s_0, y_0)\, s^2 \\
&= (-\alpha y_2 E_{12} + E_{31}) + (-\alpha y_2 F_{12} + F_{31})\, s + \frac{1}{2}(-\alpha y_2 \tilde{G}_{12} + \tilde{G}_{31})s^2 \\
\frac{\partial \bar{H}}{\partial y_3}(s_0, y_0) + j^2\left(\frac{\partial \bar{H}}{\partial y_3}(s, y_0)\right)(s_0) &= \frac{\partial \bar{H}}{\partial y_3}(s_0, y_0) + \frac{\partial^2 \bar{H}}{\partial s \partial y_3}(s_0, y_0)s + \frac{1}{2}\frac{\partial^3 \bar{H}}{\partial s^2 \partial y_3}(s_0, y_0)\, s^2 \\
&= -1.
\end{aligned}$$

(1) When h_{u_0,v_0} has the A_1-singularity at s_0, since

$$\operatorname{rank}(-\alpha y_1 E_{12} + E_{23} \quad -\alpha y_2 E_{12} + E_{31} \quad -1) = 1,$$

it is clear that $\bar{H}$ is the versal unfolding of h_{u_0,v_0}.
(2) When h_{u_0,v_0} has the A_2-singularity at s_0, we require 2×3 matrix

$$A = \begin{pmatrix} -\alpha y_1 E_{12} + E_{23} & -\alpha y_2 E_{12} + E_{31} & -1 \\ -\alpha y_1 F_{12} + F_{23} & -\alpha y_2 F_{12} + F_{31} & 0 \end{pmatrix}$$

to be non-singular. It is enough to show that

$$(-\alpha y_1 F_{12} + F_{23})^2 + (-\alpha y_2 F_{12} + F_{31})^2 \neq 0.$$

For the purpose, we need the following lemma.

Lemma 5.3.1. *If h_{u_0,v_0} has the A_2-singularity at s_0, then we have*

$$y_{0,1}F_{23} + y_{0,2}F_{31} = -F_{12}\sqrt{1 - y_1^2 - y_{0,2}^2} = -\frac{1}{\alpha}F_{12}$$

Proof of Lemma 5.3.1. Since $h'_{u_0,v_0}(s_0) = 0$, we have $|\gamma'(s_0)\ \gamma'''(s_0)\ u_0| = 0$. It follows that

$$\begin{aligned}
0 &= \begin{vmatrix} x_1'(s_0) & x_1'''(s_0) & y_{0,1} \\ x_2'(s_0) & x_2'''(s_0) & y_{0,2} \\ x_3'(s_0) & x_3'''(s_0) & \sqrt{1 - y_{0,1}^2 - y_{0,2}^2} \end{vmatrix} \\
&= (x_2'(s_0)x_3'''(s_0) - x_3'(s_0)x_2'''(s_0))y_{0,1} \\
&\quad + (x_3'(s_0)x_1'''(s_0) - x_1'(s_0)x_3'''(s_0))y_{0,2} + (x_1'(s_0)x_2'''(s_0) - x_2'(s_0)x_1'''(s_0))\sqrt{1 - y_{0,1}^2 - y_{0,2}^2} \\
&= y_1F_{23} + y_2F_{31} + \sqrt{1 - y_{0,1}^2 - y_{0,2}^2}\ F_{12}.
\end{aligned}$$

Therefore we have $y_{0,1}F_{23} + y_{0,2}F_{31} = -F_{12}\sqrt{1 - y_{0,1}^2 - y_{0,2}^2} = -\frac{1}{\alpha}F_{12}$. $\square$

Proof of the case (2) in Theorem 5.3. By Lemma 5.3.1, we have

$$\begin{aligned}
&(-\alpha y_{0,1}F_{12} + F_{23})^2 + (-\alpha y_{0,2}F_{12} + F_{31})^2 \\
&= \alpha^2 y_{0,1}^2 F_{12}^2 - 2\alpha y_{0,1}F_{12}F_{23} + F_{23}^2 + \alpha^2 y_{0,2}^2 F_{12}^2 - 2\alpha y_{0,2}F_{12}F_{31} + F_{31}^2 \\
&= F_{23}^2 + F_{31}^2 - 2\alpha(y_{0,1}F_{23} + y_{0,2}F_{31})F_{12} + \alpha^2(y_{0,1}^2 + y_{0,2}^2)F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 - 2\alpha(-\frac{1}{\alpha}F_{12})F_{12} + \alpha^2(y_{0,1}^2 + y_{0,2}^2)F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 + 2F_{12}^2 + \alpha^2(y_1^2 + y_2^2)F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 + \{2 + \alpha^2(y_{0,1}^2 + y_{0,2}^2)\}F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 + (1 + \frac{1 - y_{0,1}^2 - y_{0,2}^2}{1 - y_{0,1}^2 - y_{0,2}^2} + \frac{y_1^2 + y_2^2}{1 - y_1^2 - y_2^2})F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 + (1 + \frac{1}{1 - y_{0,1}^2 - y_{0,2}^2})F_{12}^2 \\
&= F_{23}^2 + F_{31}^2 + (1 + \alpha^2)F_{12}^2.
\end{aligned}$$

Since $1 + \alpha^2 > 0$ and $(F_{23}, F_{31}, F_{12}) \neq (0,0,0)$,

$$(-\alpha y_{0,1}F_{12} + F_{23})^2 + (-\alpha y_{0,2}F_{12} + F_{31})^2 \neq 0$$

holds, so that we have $rank\ A = 2$.

(3) When h_{u_0,v_0} has the A_3-singularity at s_0, then we require the 3×3 matrix

$$B = \begin{pmatrix} \alpha y_{0,1}E_{12} + E_{23} & -\alpha y_{0,2}E_{12} + E_{31} & -1 \\ -\alpha y_{0,1}F_{12} + F_{23} & -\alpha y_{0,2}F_{12} + F_{31} & 0 \\ \frac{1}{2}(-\alpha y_{0,1}\tilde{G}_{12} + \tilde{G}_{23}) & \frac{1}{2}(-\alpha y_{0,2}\tilde{G}_{12} + \tilde{G}_{31}) & 0 \end{pmatrix}$$

to be non-singular. It is enough to show that $\det(B) \neq 0$. We also need the following lemma.

Lemma 5.3.2. *If h_{u_0,v_0} has the A_3-singularity at s_0 then we have the following:*
(1) $-(x_1''(s_0)F_{23} + x_2''(s_0)F_{31} + x_3''(s_0)F_{12}) = 1$
(2) $x_1'(s_0)F_{23} + x_2'(s_0)F_{31} + x_3'(s_0)F_{12} = 0$
(3) $x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12} = 0$
(4) $\frac{y_i}{\mu} = \kappa_a(s_0)x_i'(s_0) + x_i'''(s_0)$ $(i = 1,2)$
$\frac{1}{\alpha\mu} = \kappa_a(s_0)x_3'(s_0) + x_3'''(s_0)$, *for some* $\mu \in \mathbb{R}, \mu \neq 0$.

Proof of Lemma 5.3.2. (1) By the assumptions, we have $|\gamma'(s_0)\ \gamma''(s_0)\ \gamma'''(s_0)| = 1$, so that $-x_1''(s_0)F_{23} - x_2''(s_0)F_{31} - x_3''(s_0)F_{12} = 1$. Therefore we have $-(x_2''(s_0)F_{23} + x_2''(s_0)F_{31} + x_3''(s_0)F_{12}) = 1$.
The cases (2), (3) are from the direct calculations.

(4) By the assumptions, there exists $\mu \in \mathbb{R}$ with $\mu \neq 0$ such that $u = \mu(\kappa_a(s_0)\gamma'(s_0)+\gamma'''(s_0))$. Therefore we have

$$\begin{aligned}
y_{0,i} &= \mu(\kappa_a(s_0)x_i'(s_0) + x_i'''(s_0)) \\
\frac{1}{\alpha} &= \sqrt{1 - y_{0,1}^2 - y_{0,2}^2} = \mu(\kappa_a(s_0)x_3'(s_0) + x_3'''(s_0)).
\end{aligned}$$

Thus we have $\frac{y_{0,i}}{\mu} = \kappa_a(s_0)x_i'(s_0) + x_i'''(s_0) \quad (i = 1,2)$, $\frac{1}{\alpha\mu} = \kappa_a(s_0)x_3'(s_0) + x_3'''(s_0)$. $\square$

Proof of the case (3) in Theorem 5.3. We have

$$\begin{aligned}
\det(B) &= -\{\frac{1}{2}(-\alpha y_1 F_{12} + F_{23})(-\alpha y_2 \tilde{G}_{12} + \tilde{G}_{31}) - \frac{1}{2}(-\alpha y_2 F_{12} + F_{31})(-\alpha y_1 \tilde{G}_{12} + \tilde{G}_{23})\} \\
&= -\frac{1}{2}\{(-\alpha y_1 F_{12} + F_{23})(-\alpha y_2 \tilde{G}_{12} + \tilde{G}_{31}) - (-\alpha y_2 F_{12} + F_{31})(-\alpha y_1 \tilde{G}_{12} + \tilde{G}_{23})\} \\
&= -\frac{1}{2}\{F_{23}\tilde{G}_{31} - F_{31}\tilde{G}_{23} + \alpha(y_{0,1}F_{31}\tilde{G}_{12} - y_{0,2}F_{23}\tilde{G}_{12} + y_{0,2}F_{12}\tilde{G}_{23} - y_1 F_{12}\tilde{G}_{31})\} \\
&= -\frac{1}{2}[F_{23}(G_{31} - \kappa_a(s_0)E_{31}) - F_{31}(G_{23} - \kappa_a(s_0)E_{23}) \\
&+ \alpha\{y_{0,1}F_{31}(G_{12} - \kappa_a(s_0)E_{12}) - y_{0,2}F_{23}(G_{12} - \kappa_a(s_0)E_{12}) \\
&+ y_{0,2}F_{12}(G_{23} - \kappa_a(s_0)E_{23}) - y_{0,1}F_{12}(G_{31} - \kappa_a(s_0)E_{31})\}] \\
&= -\frac{1}{2}[F_{23}G_{31} - F_{31}G_{23} + \kappa_a(s_0)(E_{23}F_{31} - E_{31}F_{23}) \\
&+ \alpha\{y_{0,1}(F_{31}G_{12} - F_{12}G_{31}) + y_{0,2}(F_{12}G_{23} - F_{23}G_{12}) \\
&+ \kappa_a(s_0)(y_{0,1}(E_{31}F_{12} - E_{12}F_{31}) + y_{0,2}(E_{12}F_{23} - E_{23}F_{12})\}] \\
&= -\frac{1}{2}[(x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))F_{23} - (x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0))F_{31} \\
&+ \kappa_a(s_0)\{(x_2'(s_0)x_3''(s_0) - x_3'(s_0)x_2''(s_0))F_{31} - (x_3'(s_0)x_1''(s_0) - x_1'(s_0)x_3''(s_0))F_{23}\} \\
&+ \alpha\{y_{0,1}(F_{31}(x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0)) - F_{12}(x_3''(s_0)x_1'''(s_0) - x_1''(s_0)x_3'''(s_0))) \\
&+ y_{0,2}(F_{12}(x_2''(s_0)x_3'''(s_0) - x_3''(s_0)x_2'''(s_0)) - F_{23}(x_1''(s_0)x_2'''(s_0) - x_2''(s_0)x_1'''(s_0))) \\
&+ \kappa_a(s_0)(y_{0,1}((x_3'(s_0)x_1''(s_0) - x_1'(s_0)x_3''(s_0))F_{12} - (x_1'(s_0)x_2''(s_0) - x_2'(s_0)x_1''(s_0))F_{31}) \\
&+ y_{0,2}((x_1'(s_0)x_2''(s_0) - x_2'(s_0)x_1''(s_0))F_{23} - (x_2'(s_0)x_3''(s_0) - x_3'(s_0)x_2''(s_0))F_{12})\}] \\
&= -\frac{1}{2}[-x_3'''(s_0)(x_1''(s_0)F_{23} + x_2''(s_0)F_{31}) + x_3''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31}) \\
&+ \kappa_a(s_0)\{-x_3'(s_0)(x_2''(s_0)F_{31} + x_1''(s_0)F_{23}) + x_3''(s_0)(x_2'(s_0)F_{31} + x_1'(s_0)F_{23})\} \\
&+ \alpha\{y_{0,1}(-x_1'''(s_0)(x_2''(s_0)F_{31} + x_3''(s_0)F_{12}) + x_1''(s_0)(x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12})) \\
&+ y_{0,2}(-x_2'''(s_0)(x_3''(s_0)F_{12} + x_1''(s_0)F_{23}) + x_2''(s_0)(x_3'''(s_0)F_{12} + x_1'''(s_0)F_{23})) \\
&+ \kappa_a(s_0)(y_{0,1}(-x_1'(s_0)(x_3''(s_0)F_{12} + x_2''(s_0)F_{31}) + x_1''(s_0)(x_3'(s_0)F_{12} + x_2'(s_0)F_{31})) \\
&+ y_{0,2}(-x_2'(s_0)(x_1''(s_0)F_{23} + x_3''(s_0)F_{12}) + x_2''(s_0)(x_1'(s_0)F_{23} + x_3'(s_0)F_{12}))\}]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}[x_3'''(s_0)(1 + x_3''(s_0)F_{12}) + x_3''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31}) \\
&+ \kappa_a(s_0)\{x_3'(s_0)(1 + x_3''(s_0)F_{12}) + x_3''(s_0)(x_2'(s_0)F_{31} + x_1'(s_0)F_{23})\} \\
&+ \alpha\{y_{0,1}(x_1'''(s_0)(1 + x_1''(s_0)F_{23}) + x_1''(s_0)(x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12})) \\
&+ y_{0,2}(x_2'''(s_0)(1 + x_2''(s_0)F_{31}) + x_2''(s_0)(x_3'''(s_0)F_{12} + x_1'''(s_0)F_{23})) \\
&+ \kappa_a(s_0)(y_{0,1}(x_1'(s_0)(1 + x_1''(s_0)F_{23}) + x_1''(s_0)(x_3'(s_0)F_{12} + x_2'(s_0)F_{31})) \\
&+ y_{0,2}(x_2'(s_0)(1 + x_2''(s_0)F_{31}) + x_2''(s_0)(x_1'(s_0)F_{23} + x_3'(s_0)F_{12}))\}] \\
&= -\frac{1}{2}[x_3'''(s_0) + x_3''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12}) \\
&+ \kappa_a(s_0)(x_3'(s_0) + x_3''(s_0)(x_1'(s_0)F_{23} + x_2'(s_0)F_{31} + x_3'(s_0)F_{12}) \\
&+ \alpha\{y_{0,1}(x_1'''(s_0) + x_1''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12})) \\
&+ y_{0,2}(x_2'''(s_0) + x_2''(s_0)(x_1'''(s_0)F_{23} + x_2'''(s_0)F_{31} + x_3'''(s_0)F_{12})) \\
&+ \kappa_a(s_0)(y_{0,1}(x_1'(s_0) + x_1''(s_0)(x_1'(s_0)F_{23} + x_2'(s_0)F_{31} + x_3'(s_0)F_{12})) \\
&+ y_{0,2}(x_2'(s_0) + x_2''(s_0)(x_1'(s_0)F_{23} + x_2'(s_0)F_{31} + x_3'(s_0)F_{12}))\}] \\
&= -\frac{1}{2}[x_3'''(s_0) + \kappa_a(s_0)x_3'(s_0) + \alpha\{y_{0,1}x_1'''(s_0) + y_{0,2}x_2'''(s_0) \\
&+ \kappa_a(s_0)(y_{0,1}x_1'(s_0) + y_{0,2}x_2'(s_0))\}] \\
&= -\frac{\alpha}{2}\{\frac{1}{\alpha}(\kappa_a(s_0)x_3'(s_0) + x_3'''(s_0)) + y_{0,1}x_1'''(s_0) + y_{0,2}x_2'''(s_0) \\
&+ \kappa_a(s_0)y_1x_1'(s_0) + \kappa_a(s_0)y_2x_2'(s_0)\} \\
&= -\frac{\alpha}{2}\{y_{0,1}(\kappa_a(s_0)x_1'(s_0) + x_1'''(s_0)) + y_{0,2}(\kappa_a(s_0)x_2'(s_0) + x_2'''(s_0)) \\
&+ \frac{1}{\alpha}(\kappa_a(s_0)x_3'(s_0) + x_3'''(s_0))\} \\
&= -\frac{\alpha}{2}\{\frac{y_{0,1}^2}{\mu} + \frac{y_{0,2}^2}{\mu} + \frac{1}{\alpha}(\frac{1}{\alpha\mu})\} \\
&= -\frac{\alpha}{2\mu}(y_{0,1}^2 + y_{0,2}^2 + \frac{1}{\alpha^2}) \\
&= -\frac{\alpha}{2\mu} \neq 0.
\end{aligned}
$$

It follows that we have rank $B = 3$. This completes the proof of Theorem 5.3 □

6. Generic properties

In oder to prove Theorem 4.1 we need transversality theorems which guaranty that the properties in Thoerem 4.1 are generic in $\mathrm{Imm}_a^+(S^1,\mathbb{R}^3)$. Let $\gamma : S^1 \to \mathbb{R}^3$ be a curve with $|\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)| > 0$. Since $\dot s(t) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{\frac{1}{6}}$, we have

$$\frac{dt}{ds} = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{6}},\ \gamma'(s(t)) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{6}}\dot\gamma(t)$$

and

$$\gamma''(s(t)) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{3}}\ddot\gamma(t) - \frac{1}{6}\dot\gamma(t)|\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{4}{3}}|\dot\gamma(t)\ \ddot\gamma(t)\ \gamma^{(4)}(t)|,$$

so that the affine distance function is given as follows:

$$\begin{aligned} F(t,u) = F(s(t),u) &= |\gamma'(s(t))\ \gamma''(s(t))\ \gamma(s(t)) - u| \\ &= ||\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{6}}\dot\gamma(t)\ |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{3}}\ddot\gamma(t)\ \gamma(t) - u| \\ &= |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{2}}|\dot\gamma(t)\ \ddot\gamma(t)\ \gamma(t) - u|. \end{aligned}$$

The affine height function is also given as follows:

$$H(t,u) = H(s(t),u) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{2}}|\dot\gamma(t)\ \ddot\gamma(t)\ u|.$$

We now consider a smooth mapping $\Phi : (\mathbb{R}^9 - V)\times\mathbb{R}^3 \to \mathbb{R}$ defined by $\Phi((\mathbf{x},\mathbf{y},\mathbf{z}),\mathbf{w}) = |\mathbf{x}\ \mathbf{y}\ \mathbf{z}|^{-\frac{1}{2}}|\mathbf{x},\mathbf{y},\mathbf{w}|$, where $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^3$ and $V = \{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^9 | |\mathbf{x}\ \mathbf{y}\ \mathbf{z}| = 0\}$. It follows that Φ induces a smooth mapping

$$\Phi_*^\ell : J^\ell(S^1,(\mathbb{R}^9 - V)\times\mathbb{R}^3) \to J^\ell(S^1,\mathbb{R})$$

given by $\Phi_*^\ell(j^\ell\Gamma(t_0)) = j^\ell\Phi\circ\Gamma(t_0)$. We also consider an open set of the ℓ-jet space $J^\ell(S^1,\mathbb{R}^3)$ given by

$$J_{a+}^\ell(S^1,\mathbb{R}^3) = \{j^\ell\gamma(t_0) | |\dot\gamma(t_0)\ \ddot\gamma(t_0)\ \dddot\gamma(t_0)| > 0\}$$

and an immersion

$$I^\ell : J_{a+}^\ell(S^1,\mathbb{R}^3) \to J^\ell(S^1,(\mathbb{R}^9 - V)\times\mathbb{R}^3)$$

given by

$$I^\ell(j^\ell\gamma(t_0)) = j^\ell(\dot\gamma,\ddot\gamma,\dddot\gamma,\ \gamma)(t_0).$$

Let $W \subset J^\ell(S^1,\mathbb{R})$ be a subbundle of $\alpha\times\beta : J^\ell(S^1,\mathbb{R}) \to S^1\times\mathbb{R}$, We call W *a semi-algebraic subbundle* if the fibre $W_{(s,x)}$ is a semi-algebraic submanifold of $J^\ell(1,1)$. We now have the following simple lemmas.

Lemma 6.1. *Let W be a semi-algebraic subbundle of $J^\ell(S^1,\mathbb{R})$. Then $(\Phi_*^\ell\circ I^\ell)^{-1}(W)$ is a stratified set with codimension of each strata is not smaller than the codimesion of W.*

Lemma 6.2. *Let W be a submanifold in $J^\ell_{a+}(S^1, \mathbb{R}^3)$. Then the set*

$$\mathcal{T}_W = \{\gamma \in \mathrm{Imm}^+_a(S^1, \mathbb{R}^3) | j^\ell_1\Delta_\gamma \text{ transverse to } W\}$$

is a dense subset of $\mathrm{Imm}^+_a(S^1, \mathbb{R}^3)$. *Here, we deine $j^\ell_1\Delta_\gamma : S^1 \times \mathbb{R}^3 \to \mathbb{R}$ by $j^\ell_1\Delta_\gamma(t, u) = j^\ell(\gamma(\cdot) - u)(t)$.*

We can prove the assertion (1) of Theorem4.1.

Proof of (1) of Theorem 4.1. We now consider the A_k-orbit $\mathcal{A}_k(z)$ in $J^\ell(1, 1)$, then we have $\operatorname{codim} \mathcal{A}_k(z) = k$. Let $\bar{\mathcal{A}}_k(z)$ be the natural semi-algebraic subbundle of $J^\ell(S^1, \mathbb{R})$ corresponding to $\mathcal{A}_k(z)$. By Lemma 6.1, $(\Phi^\ell_* \circ I^\ell)^{-1}(\bar{\mathcal{A}}_k(z))$ is a stratified set with the codimension of each strata is not smaller than k.

By Lemma 6.2, there exists a residual subset $\mathcal{O} \subset \mathrm{Imm}^+_a(S^1, \mathbb{R}^3)$ such that for any $\gamma \in \mathcal{O}$, $j^\ell_1\Delta_\gamma$ cannot have the intersection with $(\Phi^\ell \circ I^\ell)^{-1}(\bar{\mathcal{A}}_k(z))$ for $k \geq 5$. This means that the affine distance function $F(t, u) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{2}}|\dot\gamma(t)\ \ddot\gamma(t)\ \gamma(t) - u|$ has not the A_k-singularity for $k \geq 5$. It follows from Proposition 5.2 and Theorem 5.1 that the bifurcation set $\mathcal{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^2$, $C \times \mathbb{R}$ or S. By Proposition 3.1, $\mathcal{B}_F$ is the intrinsic affine binormal developable. This completes the proof of (1) of Theorem 4.1. □

For the proof of the assertion (2) of Theorem 4.1, we consider a smooth mapping

$$\Psi^\ell : J^\ell_{a+}(S^1, \mathbb{R}^3) \times S^2 \to J^{\ell-1}(S^1, \mathbb{R})$$

defined by $\Psi^\ell(j^\ell\gamma(t), v) = j^{\ell-1}|\dot\gamma\ \ddot\gamma\ \dddot\gamma|^{-\frac{1}{2}}|\dot\gamma\ \ddot\gamma\ v|(t)$. We have the followign simple lemmas.

Lemma 6.3. *Let W be a semi-algebraic subbundle of $J^{\ell-1}(S^1, \mathbb{R})$. Then $(\Psi^\ell)^{-1}(W)$ is a stratified set with the codimension of each strata is not smaller than the codimension of W.*

Lemma 6.4. *Let W be a submanifold in $J^\ell_{a+}(S^1, \mathbb{R}) \times S^2$. Then the set*

$$\mathcal{T}_W = \{\gamma \in \mathrm{Imm}^+_a(S^1, \mathbb{R}^3) | j^\ell\gamma \times 1_{S^2}) \text{ transverse to } W\}$$

is a residual subset of $\mathrm{Imm}^+_a(S^1, \mathbb{R}^3)$.

We now prove the assertion (2) of Theorem 4.1.

Proof of (2) of Theorem 4.1. We now consider the A_k-orbit $\mathcal{A}_k(z)$ in $J^\ell(1, 1)$, then we have $\operatorname{codim} \mathcal{A}_k(z) = k$. We also have the natural semi-algebraic subbundle $\bar{\mathcal{A}}_k(z) \subset J^\ell(S^1, \mathbb{R})$. By Lemma 6.3, $(\Psi^\ell)^{-1}(\bar{\mathcal{A}}_k(z))$is a stratified set with the codimension of each strata is not smaller than k. By Lemma 6.4, there exists a residual subset $\mathcal{O} \subset \mathrm{Imm}^+_a(S^1, \mathbb{R}^3)$ such that for any $\gamma \in \mathcal{O}$, the image of $j^\ell(\gamma \times 1_{S^1})$ does not have the intersection with $(\Psi^\ell)^{-1}(\bar{\mathcal{A}}_k(z))$ for $k \geq 4$. This means that the afine height function $H(t, u) = |\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)|^{-\frac{1}{2}}|\dot\gamma(t)\ \ddot\gamma(t)\ u|$ has not the A_k-singularities for $k \geq 4$. It follows from Proposition 5.3 and Theorem 5.1 that the discriminant set $\mathcal{D}_{\tilde{H}}$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^2$, $C \times \mathbb{R}$ or S.

By Proposition 3.2, the discriminant set $\mathcal{D}_{\tilde{H}}$ of $\tilde{H}$ is given as follows:

$$\mathcal{D}_{\tilde{H}} = \{(\lambda\gamma'(s) + \mu\gamma'''(s), \mu) \in S^2 \times \mathbb{R} | \lambda \in \mathbb{R}, s \in S^1, \lambda^2 + \mu^2 \neq 0\}.$$

We consider $\mathcal{D}_{\tilde{H}}$ around the point $(\lambda\gamma'(s)+\mu\gamma'''(s),\mu)$ with $\mu \neq 0$ (i.e., $\lambda\gamma'(s)+\mu\gamma'''(s)$ is not tangent to γ). Since $\lambda\gamma'(s)+\mu\gamma'''(s) \in S^2$, we have

$$\mu = \frac{\pm 1}{\|\frac{\lambda}{\mu}\gamma'(s)+\gamma'''(s)\|},$$

where $\|(x_1,x_2,x_3)\| = \sqrt{x_1^2+x_2^2+x_3^2}$. We define a set

$$\mathcal{D}^+_{\tilde{H}} = \{(\lambda\gamma'(s)+\mu\gamma'''(s),\mu) \in \mathcal{D}_{\tilde{H}} | \mu > 0\}.$$

This set is an open subset of $\mathcal{D}_{\tilde{H}}$. Define a map

$$\Phi : \mathbb{R}^3 - 0 \to S^2 \times \mathbb{R}_+$$

by

$$\Phi(x_1,x_2,x_3) = (\frac{(x_1,x_2,x_3)}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}),$$

where $\mathbb{R}_+$ is the set of positive real numbers. It is clear that Φ is a diffeomorphism and $\Phi(GS(\gamma)) = \mathcal{D}^+_{\tilde{H}}$. This completes the proof of (2) of Theorem 4.1. □

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