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BV-Solutions of a Hyperbolic-Elliptic System for a Radiating Gas

Kazuo ITO *

Abstract

This paper is concerned with the initial value problem of a system owning a hyperbolic equation with respect to unknown function $u(x,t)$ and an elliptic one with respect to unknown function $q(x,t)$ in one space dimension. This system originates in the dynamics of a radiating gas.

The purpose in the present paper is to give the results on global existence and asymptotic behaviour of BV-solutions to the present system for the two cases: the first case is that initial data $u_0(x)$ decay as $|x| \to \infty$ and the second one is that initial data $u_0(x)$ tend to two given constants $u_\pm$ with $u_- < u_+$ as $x \to \pm\infty$.

In the first case, we prove that the present problem is well-posed in BV, that is, for any BV-initial datum, there exists a unique BV-solution. We also show that if initial data u_0 are small in a certain sence, then the solutions $u(\cdot,t)$ decay in the order $O(t^{-(1-1/p)/2})$ as $t \to \infty$ in $L^p(R)$ with $p \in [1,\infty]$.

Furthermore, in the second case, we prove that the present problem is well-posed in r_0+BV, where $r_0(x)$ is u_- when $x<0$ and u_+ when $x>0$. Finally, we prove the main result of the present paper that if both $|u_+ - u_-|$ and initial BV-perturbations to r_0 are small in a certain sense, then the rarefaction waves of the inviscid Burgers equation are stable for the present system and the convergence rates of $u(\cdot,t)$ to the rarefaction waves as $t \to \infty$ are $O(t^{-(1-1/p)/2})$ in $L^p(R)$ with $p \in [1,\infty]$.

1 Introduction

1.1 The background

The propagation of disturbances in a radiating gas has widely been studied by many physicists and mathematicians, see the references in [1]. In particular, K. Hamer [1] first observed the balance of two effects of nonlinearity and dissipation. Applying an asymptotic analysis to the system of equations of continuity, momentum, energy, state and a certain relation of heat flux and temparature, he derived the following simple model equations

$$\begin{cases} u_t + uu_x + q_x = 0, \\ -q_{xx} + q + u_x = 0. \end{cases} \tag{1}$$

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Here $u(x,t)$ and $q(x,t)$ roughly mean the velocity and the heat flux of the radiating gas depending on space variable $x \in R$ and time $t \geq 0$ respectively.

We consider (1) as the initial value problem with initial condition

$$u(x,0) = u_0(x). \tag{2}$$

We wish to treat the data $u_0(x)$ which may have discontinuities and tend to two given constants $u_\pm$ with the constraint $u_- \leq u_+$ as $x \to \pm\infty$.

The purpose in the present paper is to study global existence and asymptotic behaviour of solutions to (1)-(2) as $t \to \infty$.

1.2 Notation

Here we list the notations which will be often used throughout this paper.

Notation. We often denote by C positive universal constants without any notice. The function sign : $R \to R$ is defined by

$$\text{sign}x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & 0 < x. \end{cases}$$

Let $j(\lambda)$ be a nonnegative smooth function for $\lambda \in R$ with compact support in $[-1,1]$ and with $\int_R j(\lambda)d\lambda = 1$. For $\delta > 0$, we define $j_\delta(\lambda) = \delta^{-1}j(\delta^{-1}\lambda)$. The convolution operator $f \mapsto j_\delta * f$ for $f = f(x)$ is often called *mollifier* of f.

Let Ω be a region in R^N. For $1 \leq p \leq \infty$, $L^p(\Omega)$ denote the usual Lebesgue spaces on Ω. For nonnegative integers m, $W^{m,p}(\Omega)$ denote the usual Sobolev spaces on Ω. If $p = 2$, we simply write $H^m(\Omega) = W^{m,p}(\Omega)$. $BV(\Omega)$ is the space of all functions $f \in L^1(\Omega)$ whose total variations

$$\begin{aligned} \|Df\|(\Omega) &= \sup\{\int_\Omega f \text{div} g dx; g = (g_1, \cdots, g_N) \in C_0^\infty(\Omega; R^N), \\ &\quad |g(x)| \leq 1 \quad \text{for } x \in \Omega\} \end{aligned}$$

are finite. The $BV(\Omega)$-norms of f are given by

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \|Df\|(\Omega).$$

In particular, if $f \in W^{1,1}(\Omega)$, then

$$\|f\|_{BV(\Omega)} = \|f\|_{W^{1,1}(\Omega)}.$$

Let I be intervals in R and let X be Banach spaces. $Lip(I;X)$ are the spaces of all functions $f : I \to X$ such that

$$\sup_{t,t' \in I, t \neq t'} \frac{\|f(t) - f(t')\|_X}{|t - t'|}$$

are finite.

1.3 Survey on previous and recent results

We start to note that the first equation of (1) is a Burgers-type equation. So it is natural to suspect that even if initial data are smooth, solutions can not be necessarily smooth.

As might be suspected, Hamer [1] had suggested that if $|u_- - u_+|$ was larger than a certain positive constant, then travelling wave solutions of (1) tending to two constants $u_\pm$ with $u_- > u_+$ as $x \to \pm\infty$ might develop singularities.

The development of singularities was first proved by Kawashima [4] quite rigorously. His result shows that there are a negative constant α_1 and a $x_0 \in R$ such that if $u_0'(x_0) < \alpha_1$, then even if $u_0(x)$ are smooth, solutions $u(x,t)$ must have discontinuities in a finite time. From the above facts, we realize that we can not expect the global existence of smooth solutions, even if initial data are sufficiently smooth.

On the other hand, Kawashima [4] also showed that there exists a negative constant α_2 larger than α_1 such that if $u_0 \in C^1(R)$ and

$$u_0'(x) \geq \alpha_2 \quad \text{for all } x \in R, \tag{3}$$

then the initial value problem (1)-(2) has global C^1-solutions.

The first study on the well-posedness of the initial value problem (1)-(2) was started by Kawashima and Tanaka [6]. The class of initial data in which they worked are a subset of a smooth functions space, more precisely, a small neighborhood of 0 in $H^m(R)$ for $m \geq 2$ (say B) and also in $r_0 + B$, where $r_0(x)$ is defined by

$$r_0(x) = \begin{cases} u_-, & \text{for } x < 0, \\ u_+, & \text{for } x > 0. \end{cases} \tag{4}$$

Here they assumed that $u_- < u_+$ and $|u_+ - u_-|$ is sufficiently small. It should be noted that (3) is satisfied in the former case, for the above condition on $u_0(x)$ implies the smallnesses of u_0' in $L^\infty(R)$. For any u_0 belonging to the above classes, they obtained global smooth solutions. In particular, for u_0 belonging to the latter class, they showed the stability of the rarefaction waves of the inviscid Burgers equation.

Recently, a study on the stability of the shock waves, which should be considered for the case $u_- > u_+$, was started by Kawashima and Nishibata [5].

Various studies on the present problem are steadily expanding even today.

1.4 Our problem

The following lemma will be often utilized throughout this paper.

Lemma 1 *Let $1 \leq p \leq \infty$. We define linear operators $K : L^p(R) \to L^p(R)$ and $K_1 : L^p(R) \to L^p(R)$ by*

$$(Kf)(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy, \tag{5}$$

$$(K_1 f)(x) = \frac{1}{2}\Big(\int_x^{\infty} e^{x-y} f(y)dy - \int_{-\infty}^{x} e^{-(x-y)} f(y)dy\Big) \tag{6}$$

for $f \in L^p(R)$. Then K and K_1 have the following properties:

$$-Kf_{xx} + Kf = f, \tag{7}$$

$$
\begin{aligned}
&K(f_x) = K_1 f, && (8)\\
&(K_1 f)_x = Kf - f, && (9)\\
&\|Kf\|_{L^p(R)} \le \|f\|_{L^p(R)}, && (10)\\
&\|K_1 f\|_{L^p(R)} = \|K(f_x)\|_{L^p(R)} \le \|f\|_{L^p(R)}, && (11)\\
&\|f - Kf\|_{L^p(R)} \le \|f_x\|_{L^p(R)} && (12)
\end{aligned}
$$

for $f \in L^p(R)$ with suitable regularities.

The proof is easy and omitted.

We formally rewrite the problem (1)-(2). To do this, we write $q(x,t)$ in terms of K_1:

$$q(x,t) = -(K_1 u)(x,t).$$

Substituting the above into the first equation of (1) and using (7), we get

$$u_t + uu_x + u - Ku = 0.$$

Thus, our problem to be considered is

$$\begin{cases} u_t + uu_x + u - Ku = 0, \\ q = -K_1 u, \\ u(x,0) = u_0(x). \end{cases} \tag{13}$$

We easily realize that all the properties of $q(x,t)$ are always determined uniquely by $u(x,t)$. So, our studies are mainly concentrated on $u(x,t)$.

1.5 Our purpose

Our main purpose is to extend Kawashima and Tanaka's result [6] up to initial data which not only may be large but also may have discontinuities, as throughly as possible. The inevitability to have to comprehend solutions of (1)-(2) in a weak sense is now clear.

This paper is organized by two parts.

The first part, Section 2, studies the case when initial data belong to $BV(R)$. After establishing a definition of weak solutions (Definition 2), we give a series of results on uniqueness (Theorem 3), ordering principle (Corollary 4) and global existence (Theorem 7) of weak solutions of (13). Finally, for $p \in [1,\infty]$, L^p-decays (Theorem 16) of the weak solutions as $t \to \infty$ are given when the amplitudes of initial data are small but the total variations of them are not necessarily small.

Furthermore, the second part, Section 3, treats initial data belonging to $\{r_0\} + BV(R)$. After a definition of weak solutions (Definition 17) is stated, a series of results on uniqueness (Theorem 18), ordering principle (Corollary 19) and global existence (Theorem 24) of weak solutions of (13) is likewise given. Finally, utilizing the result on the stability of the rarefaction waves established by Kawashima and Tanaka [6] as well as our results mentioned above, we prove the main result (Theorem 29) of the present paper that the rarefaction waves are stable for (13) by certain of the $BV(R)$-perturbations to $r_0(x)$. More precisely, if the amplitudes of $u_0(x)$ are close to $r_0(x)$ but the total variations of $u_0(x)$ are not necessarily small in a certain sense, then the weak solutions $u(\cdot,t)$ are asymptotically close to the rarefaction waves in the order $O(t^{-(1-1/p)/2})$ as $t \to \infty$ in $L^p(R)$ with $p \in [1,\infty]$.

2 Weak solutions for the case $u_0(\pm\infty) = 0$

In this section, we consider the problem (13) for the case "$u_0(\pm\infty) = 0$".

2.1 Definition of weak solutions

First of all, following Kružkov [7], we introduce the definition of weak solutions of (13).

Definition 2 A function $(u,q)(x,t)$ is called a *weak solution* of (13) with the initial datum $u_0(x)$, if $(u,q)(x,t)$ satisfies the following conditions:

(i) $u(x,t)$ belongs to

$$C^0([0,\infty); L^1(R)) \cap L^\infty(R \times (0,\infty)). \tag{14}$$

(ii) $u(x,t)$ satisfies an *entropy inequality*

$$\begin{aligned} 0 \ \leq \ & \iint_{R\times(0,\infty)} \{|u-a|\phi_t + \frac{1}{2}|u-a|(u+a)\phi_x \\ & -\mathrm{sign}(u-a)(u-Ku)\phi\} dxdt, \end{aligned} \tag{15}$$

for any $a \in R$ and $0 \leq \phi \in C_0^\infty(R \times (0,\infty))$.

(iii) $u(x,0) = u_0(x)$ a.e.$x \in R$.

(iv) $q(x,t)$ satisfies the second equation of (13) with the above $u(x,t)$.

Remark. Classical solutions are always weak solutions.

2.2 Uniqueness of weak solutions

We show that weak solutions $(u,q)(x,t)$ defined above are unique.

Theorem 3 *(Uniqueness) Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (13) with initial data $u_0 \in L^1(R)$ and $v_0 \in L^1(R)$ respectively. Then,*

$$\|u(t) - v(t)\|_{L^1(R)} \leq \|u_0 - v_0\|_{L^1(R)} \quad a.e\ t \geq 0. \tag{16}$$

As a consequence, $q(x,t)$ are unique.

Corollary 4 *(Ordering principle) Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (13) with initial data $u_0 \in L^1(R)$ and $v_0 \in L^1(R)$ respectively. Then, if $u_0(x) \leq v_0(x)$ a.e. $x \in R$, then,*

$$u(x,t) \leq v(x,t) \quad a.e.\ (x,t) \in R \times (0,\infty). \tag{17}$$

Throughout the present subsection, we write for simplicity

$$R_+^2 = R \times (0,\infty) \tag{18}$$

To prove Theorem 3, we closely follows the argument of Kružkov [7]. We begin with the following lemma.

Lemma 5 *There holds*

$$
\begin{aligned}
0 \;\le\; & \iint_{R^2_+} \{ |u(x,t) - v(x,t)| \psi_t(x,t) \\
& + \frac{1}{2} |u(x,t) - v(x,t)| (u(x,t) + v(x,t)) \psi_x(x,t) \\
& - |u(x,t) - v(x,t)| \psi(x,t) \\
& + |(Ku)(x,t) - (Kv)(x,t)| \psi(x,t) \} dxdt
\end{aligned} \tag{19}
$$

for any $\psi \in C_0^\infty(R^2_+)$ *with* $\psi \ge 0$.

Proof. In (15), we put $a = v(y,s)$ and

$$\phi = \phi(x,t;y,s) \in C_0^\infty(R^2_+ \times R^2_+)$$

for $(y,s) \in R^2_+$. Integrating it over R^2_+ with respect to (y,s), we have

$$
\begin{aligned}
0 \;\le\; & \iiiint_{(R^2_+)^2} \{ |u(x,t) - v(y,s)| \phi_t(x,t;y,s) \\
& + \frac{1}{2} |u(x,t) - v(y,s)| (u(x,t) + v(y,s)) \phi_x(x,t;y,s) \\
& - \mathrm{sign}(u(x,t) - v(y,s))(u(x,t) - (Ku)(x,t)) \cdot \\
& \cdot \phi(x,t;y,s) \} dxdtdyds,
\end{aligned} \tag{20}
$$

Similarly, changing the role of $u(x,t)$ to that of $v(y,s)$,

$$
\begin{aligned}
0 \;\le\; & \iiiint_{(R^2_+)^2} \{ |v(y,s) - u(x,t)| \phi_s(x,t;y,s) \\
& + \frac{1}{2} |v(y,s) - u(x,t)| (v(y,s) + u(x,t)) \phi_y(x,t;y,s) \\
& - \mathrm{sign}(v(y,s) - u(x,t))(v(y,s) - (Kv)(y,s)) \cdot \\
& \cdot \phi(x,t;y,s) \} dxdtdyds.
\end{aligned} \tag{21}
$$

Adding (20) and (21), we have

$$
\begin{aligned}
0 \;\le\; & \iiiint_{(R^2_+)^2} \{ |u(x,t) - v(y,s)| (\phi_t + \phi_s)(x,t;y,s) \\
& + \frac{1}{2} |u(x,t) - v(y,s)| (u(x,t) + v(y,s)) (\phi_x + \phi_y)(x,t;y,s) \\
& - \mathrm{sign}(u(x,t) - v(y,s))(u(x,t) - v(y,s) \\
& - (Ku)(x,t) + (Kv)(y,s)) \phi(x,t;y,s) \} dxdtdyds.
\end{aligned} \tag{22}
$$

Define

$$J_\delta(x-y, t-s) = j_\delta(x-y) j_\delta(t-s),$$

where j_δ is defined in Subsection 1.2. Since $\psi(x,t)$ has a compact support in R^2_+, there are constants $\delta_0 > 0$ and $T > 0$ such that

$$\psi(x,t) = 0 \quad \text{for } t \notin [\delta_0, T]. \tag{23}$$

For later convenience, we only treat δ satisfying $0 < \delta \le \delta_0/2$. Put

$$\phi(x,t;y,s) = J_\delta(x-y,t-s)\psi(x,t).$$

Then it is easy to see

$$(\phi_t + \phi_s)(x,t;y,s) = J_\delta(x-y,t-s)\psi_t(x,t),$$

and

$$(\phi_x + \phi_y)(x,t;y,s) = J_\delta(x-y,t-s)\psi_x(x,t).$$

From the above, we have

$$\begin{aligned}
0 \le & \iiiint_{(R_+^2)^2} \{J_\delta(x-y,t-s)|u(x,t)-v(y,s)|\psi_t(x,t) \\
& +\frac{1}{2}J_\delta(x-y,t-s)|u(x,t)-v(y,s)|(u(x,t)+v(y,s))\psi_x(x,t) \\
& -J_\delta(x-y,t-s)|u(x,t)-v(y,s)|\psi(x,t) \\
& +J_\delta(x-y,t-s)\mathrm{sign}(u(x,t)-v(y,s)) \cdot \\
& \cdot((Ku)(x,t)-(Kv)(y,s))\psi(x,t)\} dxdtdyds \\
\equiv & I_1(\delta)+I_2(\delta)+I_3(\delta)+I_4(\delta).
\end{aligned}$$

Claim.

$$\lim_{\delta\to 0} I_1(\delta) = \iint_{R_+^2} |u(x,t)-v(x,t)|\psi_t(x,t)dxdt, \tag{24}$$

$$\lim_{\delta\to 0} I_2(\delta) = \frac{1}{2}\iint_{R_+^2} |u(x,t)-v(x,t)|(u(x,t)+v(x,t))\psi_x(x,t)dxdt, \tag{25}$$

$$\lim_{\delta\to 0} I_3(\delta) = -\iint_{R_+^2} |u(x,t)-v(x,t)|\psi(x,t)dxdt, \tag{26}$$

and

$$\limsup_{\delta\to 0} I_4(\delta) \le \iint_{R_+^2} |(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdt. \tag{27}$$

If *Claim* is verified, then the proof of Lemma 5 is complete. In the remaining part, we prove (24)-(27). The limits (24)-(27) can be shown essentialy by KruŽkov's method [7]. Here we only show (27). We estimate $I_4(\delta)$ as follows:

$$\begin{aligned}
|I_4(\delta)| \le & \iiiint_{(R_+^2)^2} J_\delta(x-y,t-s)|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdtdyds \\
& + \iiiint_{(R_+^2)^2} J_\delta(x-y,t-s)|(Kv)(x,t)-(Kv)(y,s)|\psi(x,t)dxdtdyds \\
\equiv & I_{4,1}(\delta)+I_{4,2}(\delta).
\end{aligned}$$

$I_{4,1}(\delta)$ is computed as follows:

$$I_{4,1}(\delta) = \iint_{R_+^2}\left\{\iint_{R_+^2} J_\delta(x-y,t-s)dyds\right\}\cdot$$

$$
\begin{aligned}
&\cdot|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdt \\
= &\iint_{R^2_+}\left\{\int_R j_\delta(y)dy\int_{-\infty}^t j_\delta(s)ds\right\}\cdot \\
&\cdot|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdt \\
= &\iint_{R^2_+}\left\{\int_{-\infty}^t j_\delta(s)ds\right\}|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdt. \qquad (28)
\end{aligned}
$$

We denote by $\Psi_\delta(x,t)$ the integrand of (28). Then,

$$
\begin{aligned}
&\Psi_\delta(x,t)\to|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t) \quad \text{as } \delta\to 0, \\
&\Psi_\delta(x,t)\le|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t) \quad \text{for } 0<\delta\le\delta_0/2
\end{aligned}
$$

for a.e. $(x,t)\in R^2_+$. The right hand side is integrable in R^2_+. Then, the Lebesgue dominated convergence theorem implies

$$
\lim_{\delta\to 0} I_{4,1}(\delta)=\iint_{R^2_+}|(Ku)(x,t)-(Kv)(x,t)|\psi(x,t)dxdt.
$$

Secondly, we show $I_{4,2}(\delta)\to 0$ as $\delta\to 0$. To do this, we note that the integral region in $I_{4,2}(\delta)$ is actually limited to

$$
\{(x,t;y,s);|x-y|\le\delta,\quad |t-s|\le\delta\}.
$$

Changing the variables $x\to\xi$ and $t\to\tau$ by

$$
x-y=\xi,\quad t-s=\tau,
$$

we compute:

$$
\begin{aligned}
I_{4,2}(\delta) = &\iint_{|\xi|\le\delta,|\tau|\le\delta}\iint_{(y,s)\in R^2_+} J_\delta(\xi,\tau)|(Kv)(y+\xi,s+\tau)-(Kv)(y,s)|\cdot \\
&\cdot\psi(y+\xi,s+\tau)d\xi d\tau dyds \\
\le &\|\psi\|_{L^\infty(R^2_+)}\iint_{|\xi|\le\delta,|\tau|\le\delta} J_\delta(\xi,\tau)d\xi d\tau\cdot \\
&\cdot\sup_{|\xi|\le\delta,|\tau|\le\delta}\iint_{(y,s)\in R^2_+}|(Kv)(y+\xi,s+\tau)-(Kv)(y,s)|dyds \\
= &\|\psi\|_{L^\infty(R^2_+)}\cdot \\
&\cdot\sup_{|\xi|\le\delta,|\tau|\le\delta}\iint_{(y,s)\in R^2_+}|(Kv)(y+\xi,s+\tau)-(Kv)(y,s)|dyds.
\end{aligned}
$$

The last integral tends to 0 as $\delta\to 0$ by virtue of the Lebesgue theorem. Thus (27) is proved. The proof of Lemma 5 is complete. □

Proof of Theorem 3. Let $\chi_\varepsilon(x)$ $(\varepsilon>0)$ be a function in $C_0^\infty(R)$ satisfying

$$
0\le\chi_\varepsilon(x)\le 1,
$$

and

$$
\chi_\varepsilon(x)\to 1,\quad \chi'_\varepsilon(x)\to 0 \qquad (29)
$$

as $\varepsilon \to 0$ for any $x \in R$. Define

$$\psi_{\varepsilon,\delta}(x,t) = \int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma\chi_\varepsilon(x),$$

for any t_1 and t_2 with $0 < t_1 < t_2$ and $\delta > 0$ sufficiently small. Then $\psi_{\varepsilon,\delta} \in C_0^\infty(R_+^2)$ and $0 \leq \psi_{\varepsilon,\delta}(x,t) \leq 1$.

We consider (19) with $\psi = \psi_{\varepsilon,\delta}$. Since

$$\begin{aligned}\partial_t\psi_{\varepsilon,\delta} + \frac{u+v}{2}\partial_x\psi_{\varepsilon,\delta} &= (j_\delta(t-t_1) - j_\delta(t-t_2))\chi_\varepsilon(x) \\ &\quad + \int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma\frac{u+v}{2}\chi_\varepsilon'(x),\end{aligned}$$

(19) becomes

$$\begin{aligned}0 \leq &\iint_{R_+^2}\{(j_\delta(t-t_1) - j_\delta(t-t_2))\chi_\varepsilon(x) \\ &+ \int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma\frac{u+v}{2}\chi_\varepsilon'(x)\}|u(x,t) - v(x,t)|dxdt \\ &- \iint_{R_+^2}|u(x,t) - v(x,t)|\psi_{\varepsilon,\delta}(x,t)dxdt \\ &+ \iint_{R_+^2}|(Ku)(x,t) - (Kv)(x,t)|\psi_{\varepsilon,\delta}(x,t)dxdt.\end{aligned}$$

Note that

$$\psi_{\varepsilon,\delta}(x,t) \to \int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma \quad \text{as } \varepsilon \to 0$$

for any $\delta > 0$ and $(x,t) \in R_+^2$. Using the above fact, we let $\varepsilon \to 0$ and we have

$$\begin{aligned}&\iint_{R_+^2} j_\delta(t-t_2)|u(x,t) - v(x,t)|dxdt \\ &\leq \iint_{R_+^2} j_\delta(t-t_1)|u(x,t) - v(x,t)|dxdt \\ &\quad - \iint_{R_+^2}|u(x,t) - v(x,t)|\int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma dxdt \\ &\quad + \iint_{R_+^2}|(Ku)(x,t) - (Kv)(x,t)|\int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma dxdt.\end{aligned}$$

Note

$$\int_{t-t_2}^{t-t_1} j_\delta(\sigma)d\sigma \to \begin{cases} 0, & \text{for } t \leq t_1 < t_2 \text{ or } t_1 < t_2 \leq t, \\ 1, & \text{for } t_1 < t < t_2, \end{cases} \tag{30}$$

as $\delta \to 0$. Using (30), we let $\delta \to 0$, then

$$\begin{aligned}&\|u(t_2) - v(t_2)\|_{L^1(R)} \leq \|u(t_1) - v(t_1)\|_{L^1(R)} \\ &- \int_{t_1}^{t_2}\|u(t) - v(t)\|_{L^1(R)}dt + \int_{t_1}^{t_2}\|(Ku)(t) - (Kv)(t)\|_{L^1(R)}dt.\end{aligned}$$

Then, by virtue of (10),

$$\|u(t_2) - v(t_2)\|_{L^1(R)} \leq \|u(t_1) - v(t_1)\|_{L^1(R)}.$$

Using the continuity of $u(t)$ and $v(t)$ from $[0,\infty)$ to $L^1(R)$, we let $t_1 \to 0$ to get

$$\|u(t_2) - v(t_2)\|_{L^1(R)} \le \|u_0 - v_0\|_{L^1(R)}.$$

Since $t_2 > 0$ is arbitrary, we find that this is the desired estimate. We complete the proof of Theorem 3. $\square$

Outline of the proof of Corollary 4. Put

$$\Phi(u) = |u| + u \quad \text{for } u \in R.$$

Then, $\Phi(u)$ satisfies

$$\Phi(u) > 0 \quad \text{for } u > 0, \quad \Phi(u) = 0 \quad \text{for } u \le 0.$$

Claim: There holds

$$\begin{aligned} 0 \ &\le \int_{-\infty}^{\infty} \Phi(u(x,t) - v(x,t))dx \\ &\le \int_{-\infty}^{\infty} \Phi(u_0(x) - v_0(x))dx \quad \text{for } t \ge 0. \end{aligned} \tag{31}$$

If *Claim* is verified, then the first integral must be 0 because of $u_0(x) \le v_0(x)$. Therefore we obtain the conclusion $u(x,t) \le v(x,t)$ a.e. (x,t) in $R \times (0,\infty)$.

We show (31). Since the functions $u(x,t)$ and $v(x,t)$ are weak solutions of the first equation of (13), we have

$$\begin{aligned} 0 \ = \ &\iiiint_{(R^2_+)^2} [(u(x,t) - v(y,s))(\phi_t + \phi_s)(x,t;y,s) \\ &+\frac{1}{2}(u(x,t) - v(y,s))(u(x,t) + v(y,s))(\phi_x + \phi_y)(x,t;y,s) \\ &-\{(u(x,t) - v(y,s)) - ((Ku)(x,t) - (Kv)(y,s))\}\phi(x,t;y,s)] \\ &dxdtdyds \end{aligned} \tag{32}$$

for $\phi \in C_0^\infty(R^2_+ \times R^2_+)$. Adding (22) and (32), we have

$$\begin{aligned} 0 \ \le \ &\iiiint_{(R^2_+)^2} [\Phi(u(x,t) - v(y,s))(\phi_t + \phi_s)(x,t;y,s) \\ &+\frac{1}{2}\Phi(u(x,t) - v(y,s))(u(x,t) + v(y,s))(\phi_x + \phi_y)(x,t;y,s) \\ &-\{\Phi(u(x,t) - v(y,s)) - (\Phi(Ku)(x,t) - (Kv)(y,s))\}\phi(x,t;y,s)] \\ &dxdtdyds. \end{aligned}$$

Making use of the procedure used in the proof of Theorem 3, we arrive at

$$\begin{aligned} &\int_{-\infty}^{\infty} \Phi(u(x,t_2) - v(x,t_2))dx \\ &\le \int_{-\infty}^{\infty} \Phi(u(x,t_1) - v(x,t_1))dx \\ &- \int_{t_1}^{t_2}\int_{-\infty}^{\infty} \Phi(u(x,t) - v(x,t))dxdt \\ &+ \int_{t_1}^{t_2}\int_{-\infty}^{\infty} \Phi((Ku)(x,t) - (Kv)(x,t))dxdt \end{aligned}$$

for any t_1 and t_2 with $0 < t_1 < t_2$. It is easy to check that for $f = f(x)$,

$$\int_{-\infty}^{\infty} \Phi((Kf)(x))dx \le \int_{-\infty}^{\infty} \Phi(f(x))dx.$$

Thus, we obtain

$$\int_{-\infty}^{\infty} \Phi(u(x,t_2) - v(x,t_2))dx \le \int_{-\infty}^{\infty} \Phi(u(x,t_1) - v(x,t_1))dx.$$

Now, letting $t_2 \to t$ and $t_1 \to 0$, we conclude (31). The outline of the proof of Corollary 4 is complete. $\square$

2.3 Global existence of weak solutions

In this subsection, we consider global existence of weak solutions for initial data in $BV(R)$.

2.3.1 Known result

Kawashima and Tanaka [6] proved the global existence of smooth small solutions of (1)-(2).

To state their result, we define for $m = 2, 3, \cdots$, and for $\delta > 0$

$$E^m(\delta) = \{f \in H^m(R); \|f\|_{H^m(R)} \le \delta\}.$$

Then, their result is stated as follows.

Theorem 6 [6] *(Global existence of smooth small solutions) There are a $\delta_1 > 0$ and a $C > 0$ such that for any initial datum $u_0 \in E^m(\delta_1)$, there exists a solution $(u,q)(x,t)$ of (1)-(2) in*

$$\begin{aligned} &[C([0,\infty); H^m(R)) \cap C^1([0,\infty); H^{m-1}(R))], \\ &\times C([0,\infty); H^{m+1}(R)) \end{aligned} \tag{33}$$

satisfying

$$\begin{aligned} &\|u(t)\|^2_{H^m(R)} + \int_0^t (\|\partial_x u(\tau)\|^2_{H^{m-1}(R)} + \|q(\tau)\|^2_{H^{m+1}(R)})d\tau \\ &\le C\delta_1^2 \end{aligned} \tag{34}$$

for $t \ge 0$.

2.3.2 Global existence result

Our result is an extension of Theorem 6.

Theorem 7 *(Global existence of weak solutions) For any u_0 in $BV(R)$, there exists a weak global solution $(u,q)(x,t)$ of (13) satisfying*

$$u, q_x \in \bigcap_{T>0} BV(R \times (0,T)), \tag{35}$$

$$u \in Lip([0,\infty); L^1(R)), \tag{36}$$

$$q \in Lip([0,\infty); W^{1,1}(R)), \tag{37}$$

$$\|u(t) - u(t')\|_{L^1(R)} \leq (\|Du_0\|_{L^1(R)} + \|Du_0\|^2_{L^1(R)})|t - t'| \quad \textit{for } t, t' \geq 0, \tag{38}$$

$$\|u(t)\|_{L^1(R)} \leq \|u_0\|_{L^1(R)} \quad \textit{for } t \geq 0, \tag{39}$$

$$\|u\|_{L^1(R\times(0,T))} \leq T\|u_0\|_{L^1(R)}, \tag{40}$$

$$\|Du\|(R \times (0,T)) \leq T(2\|Du_0\|(R) + \|Du_0\|(R)^2), \tag{41}$$

$$\|u\|_{L^\infty(R\times(0,T))} \leq \|Du_0\|(R), \tag{42}$$

$$\|q(t) - q(t')\|_{W^{1,1}(R)} \leq 3\|u(t) - u(t')\|_{L^1(R)} \quad \textit{for } t, t' \geq 0, \tag{43}$$

$$\|q(t)\|_{W^{1,1}(R)} \leq 3\|u(t)\|_{L^1(R)} \quad \textit{for } t \geq 0, \tag{44}$$

$$\|q\|_{L^1(R\times(0,T))} \leq \|u\|_{L^1(R\times(0,T))}, \tag{45}$$

$$\|Dq\|(R \times (0,T)) \leq \|Du\|(R \times (0,T)), \tag{46}$$

$$\|Dq_x\|(R \times (0,\infty)) \leq 2\|Du\|(R \times (0,T)), \tag{47}$$

$$\|q\|_{L^\infty(R\times(0,T))} \leq \|u\|_{L^\infty(R\times(0,T))} \tag{48}$$

for any $T > 0$. Here $\|Du\|(R \times (0,T))$, $\|Du_0\|(R)$, and so on, denote the total variations of $u(x,t)$ on $R \times (0,T)$ and of u_0 on R, respectively.

2.3.3 The regularized problem

To prove Theorem 7, we consider a regularized problem of the first equation of (13) with "artificial viscous term" for $\varepsilon > 0$:

$$\begin{cases} u^\varepsilon_t + u^\varepsilon u^\varepsilon_x + u^\varepsilon - Ku^\varepsilon = \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon(x,0) = u_0(x) \in W^{2,1}(R). \end{cases} \tag{49}$$

Our aim is to constract a classical global solution of (49) and to pass the limit as $\varepsilon \to 0$. First of all, we state the local existence theorem for (49).

Lemma 8 *(Local existence for (49)) Let $\varepsilon > 0$. Then, for any initial datum $u_0 \in W^{2,1}(R)$, there is a $T_0 > 0$ depending on ε and $\|u_0\|_{W^{2,1}(R)}$ such that there exists a solution $u^\varepsilon(x,t)$ of (49) belonging to*

$$C([0,T_0]; W^{2,1}(R)) \cap C^1([0,T_0]; L^1(R)).$$

The solutions are unique in the above space.

Lemma 8 can be proved by virtue of an abstract theory for semilinear parabolic type evolution equations. For details, we refer to the reader for the book of A. Lunardi [8].

In order to construct a global solution of (49), we need a priori estimates for (49). These estimates play also a crucial role in the proof of Theorem 7.

Lemma 9 *(A priori estimates for (49)) Let $\varepsilon > 0$ and let $T > 0$. Assume that $u^\varepsilon(x,t)$ is a solution of (49) in*

$$C([0,T];W^{2,1}(R)) \cap C^1([0,T];L^1(R))$$

with the initial datum $u_0 \in W^{2,1}(R)$. Then, the following inequalities hold:

$$\|u^\varepsilon(t)\|_{L^1(R)} \leq \|u_0\|_{L^1(R)}, \tag{50}$$

$$\|u_x^\varepsilon(t)\|_{L^1(R)} \leq \|u_0'\|_{L^1(R)}, \tag{51}$$

$$\|u_{xx}^\varepsilon(t)\|_{L^1(R)} \leq 2\varepsilon^{-1}\|u_0'\|_{L^1(R)} + 2\varepsilon^{-1}\|u_0'\|^2_{L^1(R)} + \|u_0''\|_{L^1(R)}, \tag{52}$$

$$\|u_t^\varepsilon(t)\|_{L^1(R)} \leq \|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)} + \varepsilon\|u_0''\|_{L^1(R)}, \tag{53}$$

for $0 \leq t \leq T$.

Proof of Lemma 9. Here we only show (53). Inequalities (50) and (51) can be shown more easily and (52) can be obtained by means of (49)-(51) and (53).

We differentiate (49) with respect to t. Then we obtain an equation of u_t^ε in the distribution sense:

$$\begin{cases} u_{tt}^\varepsilon + (u^\varepsilon u_x^\varepsilon)_t + u_t^\varepsilon - Ku_t^\varepsilon = \varepsilon u_{txx}^\varepsilon \quad \text{in } D'(R \times (0,T)), \\ u_t^\varepsilon(0,x) = (-u_0 u_0' - u_0 + Ku_0 + \varepsilon u_0'')(x). \end{cases} \tag{54}$$

Applying the mollifier $j_\delta *$ with respect to x to (54), we have

$$\begin{cases} u_{tt}^{\varepsilon,\delta} + (u^\varepsilon u_x^\varepsilon)_t^\delta + u_t^{\varepsilon,\delta} - Ku_t^{\varepsilon,\delta} = \varepsilon u_{txx}^{\varepsilon,\delta}, \\ u_t^{\varepsilon,\delta}(0,x) = (-u_0 u_0' - u_0 + Ku_0 + \varepsilon u_0'')^\delta(x), \end{cases} \tag{55}$$

where we write $f^\delta = j_\delta * f$. We note that $u_{tt}^{\varepsilon,\delta}$ is a function, for $(u^\varepsilon u_x^\varepsilon)_t^\delta = (u^\varepsilon u_t^\varepsilon)_x^\delta$ and $u_{txx}^{\varepsilon,\delta}$ are functions. Put

$$g_\sigma(\lambda) = \int_{-\infty}^{\infty} j_\sigma(\lambda - \mu)\text{sign}(\mu)d\mu \tag{56}$$

for $\lambda \in R$ and for $\sigma > 0$. It is easy to check that for $\lambda \in R$

$$0 \leq g_\sigma(\lambda) \leq 1, \tag{57}$$

$$g_\sigma(\lambda) \to \text{sign}(\lambda) \quad \text{as } \sigma \to 0, \tag{58}$$

$$g_\sigma'(\lambda) = 2j_\sigma(\lambda) = 2\sigma^{-1}j(\sigma^{-1}\lambda). \tag{59}$$

Multiplying (55) by $g_\sigma(u_t^{\varepsilon,\delta})$ and integrating it over R, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_0^{u_t^{\varepsilon,\delta}} g_\sigma(\lambda)d\lambda dx + \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})(u^\varepsilon u_x^\varepsilon)_t^\delta dx \\ &+ \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})u_t^{\varepsilon,\delta}dx \\ &= \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})Ku_t^{\varepsilon,\delta}dx + \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})u_{txx}^{\varepsilon,\delta}dx. \end{aligned} \tag{60}$$

An integration by parts and (57) give

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_0^{u_t^{\varepsilon,\delta}} g_\sigma(\lambda)d\lambda dx + \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})(u^\varepsilon u_x^\varepsilon)_t^\delta dx$$
$$+\int_{-\infty}^{\infty} g_\sigma'(u_t^{\varepsilon,\delta})(u_{tx}^{\varepsilon,\delta})^2 dx + \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})u_t^{\varepsilon,\delta}dx$$
$$\leq \int_{-\infty}^{\infty} |Ku_t^{\varepsilon,\delta}|dx.$$

We find that the third integral is nonnegative and that the last integral is less than or equal to $\|u_t^{\varepsilon,\delta}(t)\|_{L^1(R)}$ in view of (10). Thus, we arrive at

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_0^{u_t^{\varepsilon,\delta}} g_\sigma(\lambda)d\lambda dx + \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})(u^\varepsilon u_x^\varepsilon)_t^\delta dx$$
$$+\int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta})u_t^{\varepsilon,\delta}dx \leq \|u_t^{\varepsilon,\delta}(t)\|_{L^1(R)}. \tag{61}$$

We denote the second integral by $I_2(\delta,\sigma)$. For $I_2(\delta,\sigma)$, we compute:

$$\begin{aligned} I_2(\delta,\sigma) &= -\int_{-\infty}^{\infty} g_\sigma'(u_t^{\varepsilon,\delta})u_{tx}^{\varepsilon,\delta}(u^\varepsilon u_t^\varepsilon)^\delta dx \\ &= -\int_{-\infty}^{\infty} g_\sigma'(u_t^{\varepsilon,\delta})u_{tx}^{\varepsilon,\delta}u^\varepsilon u_t^{\varepsilon,\delta}dx \\ &\quad -\int_{-\infty}^{\infty} g_\sigma'(u_t^{\varepsilon,\delta})u_{tx}^{\varepsilon,\delta}[(u^\varepsilon u_t^\varepsilon)^\delta - u^\varepsilon u_t^{\varepsilon,\delta}]dx \\ &\equiv I_{2,1}(\delta,\sigma) + I_{2,2}(\delta,\sigma). \end{aligned}$$

Claim:

$$|I_{2,1}(\delta,\sigma)| \leq 2\sigma\|u_x^\varepsilon(t)\|_{L^1(R)}, \tag{62}$$

$$\begin{aligned} |I_{2,2}(\delta,\sigma)| \leq &\int_{-\infty}^{\infty} |\partial_x[((u^\varepsilon)^2u_x^\varepsilon)^\delta - (u^\varepsilon)^2u_x^{\varepsilon,\delta}]|dx \\ &+\int_{-\infty}^{\infty} |\partial_x(u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta - u^\varepsilon u_x^{\varepsilon,\delta}])|dx \\ &+\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon Ku_{xx}^\varepsilon)^\delta - u^\varepsilon(Ku_{xx}^\varepsilon)^\delta]|dx \\ &+\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon u_{xx}^\varepsilon)^\delta - u^\varepsilon(u_{xx}^\varepsilon)^\delta]|dx. \end{aligned} \tag{63}$$

We show (62). By means of an integration by parts, we compute:

$$\begin{aligned} I_{2,1}(\delta,\sigma) &= -\int_{-\infty}^{\infty}\partial_x(\int_0^{u_t^{\varepsilon,\delta}} g_\sigma'(\lambda)\lambda d\lambda)\cdot u^\varepsilon dx \\ &= \int_{-\infty}^{\infty}(\int_0^{u_t^{\varepsilon,\delta}} g_\sigma'(\lambda)\lambda d\lambda)\cdot u_x^\varepsilon dx. \end{aligned}$$

It then follows from (59) that

$$|\int_0^{u_t^{\varepsilon,\delta}} g_\sigma'(\lambda)\lambda d\lambda| = |2\int_0^{u_t^{\varepsilon,\delta}} \sigma^{-1}j(\sigma^{-1}\lambda)\lambda d\lambda|$$
$$= |2\sigma\int_0^{\sigma^{-1}u_t^{\varepsilon,\delta}} j(\lambda)\lambda d\lambda| \leq 2\sigma\int_{-1}^{1} j(\lambda)|\lambda|d\lambda \leq 2\sigma.$$

Thus, we arrive at

$$|I_{2,1}(\delta,\sigma)| \leq 2\sigma \int_{-\infty}^{\infty} |u_x^\varepsilon| dx,$$

which gives (62).

To show (63), making use of (49), we compute:

$$\begin{aligned}
&(u^\varepsilon u_t^\varepsilon)^\delta - u^\varepsilon u_t^{\varepsilon,\delta} \\
&= \{u^\varepsilon(-u^\varepsilon u_x^\varepsilon - u^\varepsilon + Ku^\varepsilon + u_{xx}^\varepsilon)\}^\delta \\
&\quad -u^\varepsilon(-u^\varepsilon u_x^\varepsilon - u^\varepsilon + Ku^\varepsilon + u_{xx}^\varepsilon)^\delta \\
&= -[((u^\varepsilon)^2 u_x^\varepsilon)^\delta - u^\varepsilon(u^\varepsilon u_x^\varepsilon)^\delta] - [((u^\varepsilon)^2)^\delta - u^\varepsilon u^{\varepsilon,\delta}] \\
&\quad +[(u^\varepsilon K u^\varepsilon)^\delta - u^\varepsilon(Ku^\varepsilon)^\delta] + [(u^\varepsilon u_{xx}^\varepsilon)^\delta - u^\varepsilon u_{xx}^{\varepsilon,\delta}].
\end{aligned}$$

Since

$$\begin{aligned}
u^\varepsilon(u^\varepsilon u_x^\varepsilon)^\delta &= u^\varepsilon(u^\varepsilon u_x^{\varepsilon,\delta} + [(u^\varepsilon u_x^\varepsilon)^\delta - u^\varepsilon u_x^{\varepsilon,\delta}]) \\
&= (u^\varepsilon)^2 u_x^{\varepsilon,\delta} + u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta - u^\varepsilon u_x^{\varepsilon,\delta}],
\end{aligned}$$

we get

$$\begin{aligned}
&(u^\varepsilon u_t^\varepsilon)^\delta - u^\varepsilon u_t^{\varepsilon,\delta} \\
&= -[((u^\varepsilon)^2 u_x^\varepsilon)^\delta - (u^\varepsilon)^2 u_x^{\varepsilon,\delta}] + u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta - u^\varepsilon u_x^{\varepsilon,\delta}] \\
&\quad -[(u^\varepsilon(u^\varepsilon - Ku^\varepsilon))^\delta - u^\varepsilon(u^{\varepsilon,\delta} - Ku^{\varepsilon,\delta})] + [(u^\varepsilon u_{xx}^\varepsilon)^\delta - u^\varepsilon u_{xx}^{\varepsilon,\delta}].
\end{aligned}$$

It follows from the above that

$$\begin{aligned}
&|I_{2,2}(\delta,\sigma)| \\
&= |-\int_{-\infty}^{\infty} \partial_x \int_0^{u_t^{\varepsilon,\delta}} g'_\sigma(\lambda) d\lambda \cdot [(u^\varepsilon u_t^\varepsilon)^\delta - u^\varepsilon u_t^{\varepsilon,\delta}] dx| \\
&= |\int_{-\infty}^{\infty} \int_0^{u_t^{\varepsilon,\delta}} g'_\sigma(\lambda) d\lambda \cdot \partial_x[(u^\varepsilon u_t^\varepsilon)^\delta - u^\varepsilon u_t^{\varepsilon,\delta}] dx| \\
&\leq \int_{-\infty}^{\infty} |\partial_x[((u^\varepsilon)^2 u_x^\varepsilon)^\delta - (u^\varepsilon)^2 u_x^{\varepsilon,\delta}]| dx \\
&+ \int_{-\infty}^{\infty} |\partial_x\{u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta - u^\varepsilon u_x^{\varepsilon,\delta}]\}| dx \\
&+ \int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon(u^\varepsilon - Ku^\varepsilon))^\delta - u^\varepsilon(u^{\varepsilon,\delta} - Ku^{\varepsilon,\delta})]| dx \\
&+ \int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon u_{xx}^\varepsilon)^\delta - u^\varepsilon u_{xx}^{\varepsilon,\delta}]| dx.
\end{aligned}$$

Finally, noting $u^\varepsilon - Ku^\varepsilon = -Ku_{xx}^\varepsilon$, we obtain (63). *Claim* has been verified.

Integrating (61) with respect to time variable over $(0,t)$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_0^{u_t^{\varepsilon,\delta}(t)} g_\sigma(\lambda) d\lambda dx + \int_0^t \int_{-\infty}^{\infty} g_\sigma(u_t^{\varepsilon,\delta}) u_t^{\varepsilon,\delta} dx d\tau \\
&\leq \int_{-\infty}^{\infty} \int_0^{u_t^{\varepsilon,\delta}(0)} g_\sigma(\lambda) d\lambda dx + \int_0^t \|u_t^{\varepsilon,\delta}(\tau)\|_{L^1(R)} d\tau
\end{aligned}$$

$$
\begin{aligned}
&+2\sigma\int_0^t \|u_x^\varepsilon(\tau)\|_{L^1(R)}d\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[((u^\varepsilon)^2u_x^\varepsilon)^\delta-(u^\varepsilon)^2u_x^{\varepsilon,\delta}]|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x(u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta-u^\varepsilon u_x^{\varepsilon,\delta}])|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon K u_{xx}^\varepsilon)^\delta-u^\varepsilon(K u_{xx}^\varepsilon)^\delta]|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon u_{xx}^\varepsilon)^\delta-u^\varepsilon(u_{xx}^\varepsilon)^\delta]|dxd\tau. \qquad (64)
\end{aligned}
$$

Letting $\sigma \to 0$ with (57)-(58) in mind, we obtain

$$
\begin{aligned}
&\|u_t^\varepsilon(t)\|_{L^1(R)}+\int_0^t \|u_t^\varepsilon(\tau)\|_{L^1(R)}d\tau \\
&\le \|u_t^\varepsilon(0)\|_{L^1(R)}+\int_0^t \|u_t^\varepsilon(\tau)\|_{L^1(R)}d\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[((u^\varepsilon)^2u_x^\varepsilon)^\delta-(u^\varepsilon)^2u_x^{\varepsilon,\delta}]|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x(u^\varepsilon[(u^\varepsilon u_x^\varepsilon)^\delta-u^\varepsilon u_x^{\varepsilon,\delta}])|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon K u_{xx}^\varepsilon)^\delta-u^\varepsilon(K u_{xx}^\varepsilon)^\delta]|dxd\tau \\
&+\int_0^t\int_{-\infty}^{\infty} |\partial_x[(u^\varepsilon u_{xx}^\varepsilon)^\delta-u^\varepsilon(u_{xx}^\varepsilon)^\delta]|dxd\tau. \qquad (65)
\end{aligned}
$$

Here we recall the following lemma.

Lemma 10 *(Friedrichs) Let $a \in B^1(R)$ and $v \in W^{1,1}(R)$, where $B^1(R)$ denotes the space of all $C^1(R)$-functions $f(x)$ with finite $\|f\|_{W^{1,\infty}(R)}$-norm. Define*

$$[j_\delta *, a\partial_x]v = j_\delta * (av_x) - a\cdot(j_\delta * v_x).$$

Then,

1) there is a $C>0$ such that

$$\|[j_\delta *, a\partial_x]v\|_{W^{1,1}(R)} \le C\|v\|_{W^{1,1}(R)},$$

2) $[j_\delta *, a\partial_x]v \to 0$ as $\delta \to 0$ in $W^{1,1}(R)$.

For the proof, see [11] and [9] for instance.

Now, by applying Lemma 10, we let $\delta \to 0$ in (65). Thus we arrive at

$$\|u_t^\varepsilon(t)\|_{L^1(R)} \le \|u_t^\varepsilon(0)\|_{L^1(R)}.$$

Finally, using (12) and

$$u_t^\varepsilon(x,0) = (-u_0u_0' - u_0 + Ku_0 + \varepsilon u_0'')(x),$$

we get (53). We complete the proof of Lemma 9. □

By virtue of Lemmas 8 and 9, we arrive at

Proposition 11 *(Global existence for (49)) Let $\varepsilon > 0$. Then, for any $u_0 \in W^{2,1}(R)$, there exists a global solution $u^\varepsilon(x,t)$ of (49) belonging to*

$$C([0,\infty); W^{2,1}(R)) \cap C^1([0,\infty); L^1(R)).$$

The solutions are unique in the above space.

2.3.4 Proof of the global existence theorem

We are now in position to state the proof of Theorem 7. The proof needs two steps. The first step treats initial data belonging to $W^{2,1}(R)$ and the second step $BV(R)$.

Proof of Theorem 7. First step. Our first task is to prove

Proposition 12 *For any $u_0 \in W^{2,1}(R)$, there exists a weak solution $(u,q)(x,t)$ of (13). Furthermore, the weak solutions satisfy (35)-(48).*

To prove Proposition 12, we try to extract a subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$ which converges to the desired weak solution. This procedure is stated as in the following lemma.

Lemma 13 *There exist a subsequence $\{u^k\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ and a function $u(x,t)$ in*

$$\begin{aligned} &Lip([0,\infty); L^1(R)) \cap L^\infty(R \times (0,\infty)) \\ &\cap \bigcap_{T>0} BV(R \times (0,\infty)) \end{aligned} \tag{66}$$

such that

$$\begin{cases} u^k \to u & \text{in } L^1_{\mathrm{loc}}(R \times (0,\infty)), \\ u^k \to u & \text{a. e. } (x,t) \in R \times (0,\infty), \end{cases} \quad \text{as } k \to \infty \tag{67}$$

and $u(x,t)$ satisfies (38)-(42).

Proof of Lemma 13. First we show (67). Let $\{\Omega_n\}_{n=1}^\infty$ be a sequence of domains in $R \times (0,\infty)$ with smooth boundary $\partial\Omega_n$ satisfying

$$\Omega_n \subset (-n,n) \times (0,n) \quad \text{for } n = 1, 2, \cdots,$$

and

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots \subset \bigcup_{n=1}^\infty \Omega_n = R \times (0,\infty).$$

It follows from (50), (51) and (53) that

$$\|u^\varepsilon\|_{L^1(\Omega_n)} \le n\|u_0\|_{L^1(R)},$$

and

$$\|\nabla_{x,t} u^\varepsilon\|_{L^1(\Omega_n)} \le n(2\|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)} + \varepsilon\|u_0''\|_{L^1(R)}).$$

When $n = 1$, by virtue of the Rellich theorem, we can find a subsequence $\{u^{\varepsilon(1,k)}\}_{k=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon>0}$ and a function $v^1 \in L^1(\Omega_1)$ such that

$$\begin{aligned} &\varepsilon(1,k) \to 0, \\ &u^{\varepsilon(1,k)} \to v^1 \quad \text{in } L^1(\Omega_1), \end{aligned}$$

as $k \to \infty$.

Inductively, there exist a subsequence $\{u^{\varepsilon(n,k)}\}_{k=1}^{\infty}$ of $\{u^{\varepsilon(n-1,k)}\}_{k=1}^{\infty}$ and a function $v^n \in L^1(\Omega_n)$ such that

$$\varepsilon(n,k) \to 0,$$
$$u^{\varepsilon(n,k)} \to v^n \quad \text{in } L^1(\Omega_n)$$

as $k \to \infty$, and $v^n(t,x) = v^{n-1}(t,x)$ for a.e. $(x,t) \in \Omega_{n-1}$.

Define

$$u^k = u^{\varepsilon(k,k)} \quad \text{for } k = 1, 2, \cdots,$$
$$u(t,x) = v^n(t,x) \quad \text{for } (x,t) \in \Omega_n.$$

Then,

$$u^k \to u \quad \text{as } k \to \infty \text{ in } L^1_{\text{loc}}(R \times (0,\infty)).$$

Furthermore, it is possible to extract a subsequence of $\{u^k\}_{k=1}^{\infty}$ (we denote this sequence again by $\{u^k\}_{k=1}^{\infty}$) such that

$$u^k \to u \quad \text{as } k \to \infty \text{ for a. e. } (x,t) \in R \times (0,\infty).$$

Thus (67) is proved.

We secondly show that $u(x,t)$ is in (66) abd satisfies (38)-(42). From Sobolev inequality and (51), we get

$$|u^k(x,t)| \le \|u^k_x(t)\|_{L^1(R)} \le \|u_0'\|_{L^1(R)}. \tag{68}$$

Letting $k \to \infty$,

$$|u(x,t)| \le \|u_0'\|_{L^1(R)},$$

which shows that $u(x,t)$ is in $L^\infty(R \times (0,\infty))$ and satisfies (42).

In order to show the Lipschitz continuity of $u(t)$ in $L^1(R)$, we write

$$u^k(t) - u^k(t') = \int_{t'}^{t} u^k_t(s)ds$$

for t, $t' \ge 0$. Taking the $L^1(-\rho,\rho)$-norm for any $\rho > 0$ and making use of (53), we have

$$\|u^k(t) - u^k(t')\|_{L^1(-\rho,\rho)} \le \left| \int_{t'}^{t} \|u^k_t(s)\|_{L^1(R)} ds \right|$$
$$\le (\|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)} + \varepsilon(k,k)\|u_0''\|_{L^1(R)})|t - t'|.$$

With (68) in mind, we apply the Lebesgue dominated convergence theorem to the first term. Letting $k \to \infty$,

$$\|u(t) - u(t')\|_{L^1(-\rho,\rho)} \le (\|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)})|t - t'|.$$

Now, letting $\rho \to \infty$, we conclude (38) and consequently (36). Similarly, the inequalities (39) and (40) can be shown by (50).

Finally we show that u belongs to $BV(R \times (0,T))$ for any $T > 0$. It follows from (51) and (53) that

$$\|\nabla_{x,t} u^k\|_{L^1(R\times(0,T))} \le T(2\|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)} + \varepsilon(k,k)\|u_0''\|_{L^1(R)})$$

for any $T > 0$. Then, by virtue of (67) and [13, Th. 5.2.1], we take the inferior limit in the above inequality as $k \to \infty$ to obtain

$$\begin{aligned} \|Du\|(R \times (0,T)) &\le \liminf_{k\to\infty} \|\nabla_{x,t} u^k\|_{L^1(R\times(0,T))} \\ &\le T(2\|u_0'\|_{L^1(R)} + \|u_0'\|^2_{L^1(R)}), \end{aligned}$$

which shows that u has a finite total variation on $R \times (0,T)$. Together with (40), we have shown that u belongs to $BV(R \times (0,T))$ and also satisfies (41) for any $T > 0$. The proof of Lemma 13 is complete. □

By virtue of Lemma 13, we can now prove Proposition 12.

Proof of Proposition 12. We have to show that the function $u(x,t)$ satisfies the entropy inequality (15). To do this, we follow the procedure in [7]. Let a be any real number and be fixed. For $\delta > 0$, define

$$\Phi_\delta(\lambda) = \int_{-\infty}^{\infty} j_\delta(\lambda - \mu)|\mu - a| d\mu. \tag{69}$$

It is easy to see that $\Phi_\delta \in C^\infty(R)$, $\Phi_\delta''(\lambda) \ge 0$ and

$$\Phi_\delta(\lambda) \to |\lambda - a|, \quad \Phi_\delta'(\lambda) \to \operatorname{sign}(\lambda - a)$$

as $\delta \to 0$ for a.e. $\lambda \in R$.

We multiply the equation (49) with $u^\varepsilon = u^k$ by $\Phi_\delta'(u^k)\phi$, where $\phi(x,t) \ge 0$ is any test function on $R \times (0,\infty)$. Then,

$$\begin{aligned} 0 \le &\iint_{R\times(0,\infty)} \{\Phi_\delta(u^k)\phi_t + \int_a^{u^k} \Phi_\delta'(\lambda)\lambda d\lambda \phi_x - \Phi_\delta'(u^k)u^k + \Phi_\delta'(u^k)Ku^k \\ &+\varepsilon(k,k)\Phi_\delta(u^k)\phi_{xx} - \varepsilon(k,k)\Phi_\delta''(u^k)(u_x^k)^2\phi\} dxdt. \end{aligned} \tag{70}$$

Noting that the last term is nonpositive, we let $\delta \to 0$ to get

$$\begin{aligned} 0 \le &\iint_{R\times(0,\infty)} \{|u^k - a|\phi_t + \int_a^{u^k} \operatorname{sign}(\lambda - a)\lambda d\lambda \phi_x - \operatorname{sign}(u^k - a)u^k \\ &+\operatorname{sign}(u^k - a)Ku^k + \varepsilon(k,k)|u^k - a|\phi_{xx}\} dxdt. \end{aligned} \tag{71}$$

Letting $k \to \infty$, we obtain the entropy inequality (15) for $u(x,t)$. Thus, together with (66), we conclude that $u(x,t)$ is the desired weak solution.

It remains to show (35), (37), (43)-(48). We here only prove (46); others can be shown more easily.

Let $\varphi(x,t)$ be

$$\begin{aligned} &\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(R \times (0,T); R^2), \\ &|\varphi(x,t)| \le 1 \quad \text{for } (x,t) \in R \times (0,T) \end{aligned}$$

for any $T > 0$. For a set A, define

$$\chi_A(y) = \begin{cases} 1, & y \in A, \\ 0, & y \notin A. \end{cases} \tag{72}$$

Then, by virtue of Fubini's Theorem, we compute:

$$
\begin{aligned}
&\iint_{R\times(0,T)} q\mathrm{div}\varphi dxdt\\
&= -\iint_{R\times(0,T)} K_1 u\mathrm{div}\varphi dxdt\\
&= -\frac{1}{2}\iint_{R_x\times(0,T)}\int_{R_y}(\chi_{(x,+\infty)}(y)-\chi_{(-\infty,x)}(y))e^{-|x-y|}u(y,t)dy\mathrm{div}\varphi(x,t)dxdt\\
&= -\frac{1}{2}\int_{R_y}(\chi_{(-\infty,0)}(y)-\chi_{(0,+\infty)}(y))e^{-|y|}\iint_{R_x\times(0,T)} u(x-y,t)\mathrm{div}\varphi(x,t)dxdtdy\\
&\leq \frac{1}{2}\int_{R_y} e^{-|y|}dy\cdot\|Du\|(R\times(0,T))\\
&\leq \|Du\|(R\times(0,T)),
\end{aligned}
$$

which gives (46).

Now, all the assertions of Proposition 12 are verified and the proof is complete. □

Second step. We consider the general case $u_0 \in BV(R)$. To this end, we utilize the following fact [13, Th. 5.3.3]:

Lemma 14 *For any $f \in BV(R)$, there is a sequence $\{f_j\}_{j=1}^{\infty}$ in $C_0^{\infty}(R)$ such that*

$$
\begin{aligned}
&f_j \to f \quad \textit{in } L^1(R),\\
&\|f_j'\|_{L^1(R)} \to \|f'\|(R)
\end{aligned}
$$

as $j \to \infty$.

We proceed with the proof as follows. From the above lemma, we can obtain a sequence $\{u_{0j}\}_{j=1}^{\infty}$ in $C_0^{\infty}(R)\subset W^{2,1}(R)$ which approximates u_0 in the above sense. For initial data u_{0j}, it follows from Proposition 12 that there exist weak solutions $u_j(x,t)$ of (13) satisfying (35), (36), (38)-(42). On the other hand, it follows from Theorem 3 that

$$\|u_j(t)-u_l(t)\|_{L^1(R)} \leq \|u_{0j}-u_{0l}\|_{L^1(R)}$$

for $j,l = 1,2,\cdots$. This inequality shows that $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^1(R\times(0,T))$ for any $T>0$. Therefore, for any $T>0$, there is a function u^T in $L^1(R\times(0,T))$ such that

$$u_j \to u^T \quad \text{as } j\to\infty \text{ in } L^1(R\times(0,T)).$$

We note that $u^{T_1}(x,t) = u^{T_2}(x,t)$ for a.e. $(x,t)\in R\times(0,T_1)$ when $T_1 < T_2$. From this fact, we can define $u(x,t) = u^T(x,t)$ for a.e. $(x,t)\in(0,T)$. Then, for any $T>0$,

$$u_j \to u \quad \text{as } j\to\infty \text{ in } L^1(R\times(0,T)).$$

Furthermore, it is possible to extract a subsequence of $\{u_j\}_{j=1}^{\infty}$ (denoting this subsequence again by $\{u_j\}_{j=1}^{\infty}$) such that

$$u_j(x,t) \to u(x,t) \quad \text{as } j\to\infty,$$

for a. e. $(x,t)\in R\times(0,\infty)$.

Now, passing the limit as $j \to \infty$ in (38)-(42) for u_j and u_{0j} as in the proof of Lemma 13, we obtain (38)-(42) for u and u_0. Furthermore, letting $j \to \infty$ in

$$
\begin{aligned}
0 \ \le \ & \iint_{R\times(0,\infty)} \{|u_j - a|\phi_t + \frac{1}{2}|u_j - a|(u_j + a)\phi_x \\
& -\text{sign}(u_j - a)(u_j - Ku_j)\phi\} dxdt,
\end{aligned}
$$

with any $a \in R$ and $0 \le \phi \le C_0^\infty(R \times (0,\infty))$, we find that $u(x,t)$ satisfies the entropy inequality (15). Thus, we have proved that $u(x,t)$ is the desired weak solution of (13).

It is easy to check that $q = -K_1 u$ is also the desired weak solution of (13). This completes the proof of Theorem 7. □

2.4 L^p-decays of weak solutions

In this subsection we consider the stability of the 0-solution for small discontinuous perturbations. In other words, we study L^p-decays of the weak solutions $u(\cdot,t)$ of (13) constructed in Theorem 7 when BV-data are small in a certain sense.

2.4.1 Known result

Kawashima and Tanaka [6] showed that the 0-solution is stable when purturbations are smooth and small.

To state their result, we define for $m = 2, 3, \cdots$ and for $\delta > 0$

$$E_1^m(\delta) = \{f \in (H^m \cap L^1)(R); \|f\|_{(H^m\cap L^1)(R)} \le \delta\}.$$

Then, their result is stated as follows.

Theorem 15 [6] *(Decays of smooth small solutions) Let $m \ge 2$ be integers. Then, there are a $\delta_2 \in (0, \delta_1)$ (here δ_1 is in Theorem 6) and positive constants C such that for any initial data $u_0 \in E_1^m(\delta_2)$, the solutions $(u,q)(x,t)$ obtained in Theorem 6 satisfy the following decay estimates*

$$\|\partial_x^k u(t)\|_{H^{m-1-2k}(R)} \le C\delta_2(1+t)^{-(\frac{1}{4}+\frac{k}{2})}, \tag{73}$$

$$\|\partial_x^k q(t)\|_{H^{m-1-2k}(R)} \le C\delta_2(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \tag{74}$$

for $t \ge 0$ and $k = 0, 1, \cdots, [(m-1)/2]$. Here, $[\alpha]$ denotes the maximal integer less than α.

2.4.2 L^p-decays of weak solutions

Our result is an extension of Theorem 15. To state it, we define for $\delta > 0$,

$$
\begin{aligned}
S(\delta) \ = \ & \{\psi \in BV(R); \exists \psi^i \in E_1^3(\delta), i = 1, 2, \quad \text{s.t.} \\
& \psi^1(x) \le \psi(x) \le \psi^2(x) \quad \text{a.e. } x \in R\}.
\end{aligned}
$$

Then, our result is stated as follows.

Theorem 16 *(L^p-decays of weak solutions) Let $p \in [1, \infty]$ and let δ_2 be the positive number established in Theorem 15 with $m = 3$. Then, there is a $C > 0$ such that for any initial datum $u_0 \in S(\delta_2)$, the weak solutions $u(x,t)$ of (13) constructed in Theorem 7 satisfy*

$$\|u(t)\|_{L^p(R)} \leq C\delta_2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \tag{75}$$

for $t \geq 0$.

Remark. We can also obtain $L^p(R)$-decays of $q(\cdot, t)$:

$$\|q(t)\|_{L^p(R)} \leq C\delta_2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

But it does not seem that this is optimal.

Proof of Theorem 16. Since $u_0 \in S(\delta_2)$, there are functions $u_0^i \in E_1^3(\delta_2)$, $i = 1, 2$, such that

$$u_0^1(x) \leq u_0(x) \leq u_0^2(x). \tag{76}$$

It then follows from Theorems 6 and 15 that for initial data $u_0^i(x)$, the classical solutions $u^i(x,t)$ of the first equation of (13) exist and, in particular, satisfy

$$\|u^i(t)\|_{L^2(R)} \leq C\delta_2(1+t)^{-1/4},$$

$$\|u_x^i(t)\|_{L^2(R)} \leq C\delta_2(1+t)^{-3/4}.$$

From the above,

$$\begin{aligned} \|u^i(t)\|_{L^\infty(R)} &\leq \sqrt{2}\|u^i(t)\|_{L^2(R)}^{1/2}\|u_x^i(t)\|_{L^2(R)}^{1/2} \\ &\leq C\delta_2(1+t)^{-1/2}, \end{aligned} \tag{77}$$

which shows that $u^i(x,t)$ are in $L^\infty(R \times (0,\infty))$. Furthermore, we can also prove that u^i are in $C([0,\infty); L^1(R))$. This is shown by a standard calculations utilized in the proof of local existence theorem and by a procedure as in the proof of Lemma 9. We omit the details.

On the other hand, it follows from Theorem 7 that for the initial datum $u_0 \in S(\delta_2)$, there exists a weak solution $u(x,t)$ of the first equation (13).

Now, we can apply Corollary 4 to (76). Thus, we arrive at

$$u^1(x,t) \leq u(x,t) \leq u^2(x,t) \quad \text{for a.e. } (x,t) \in R \times (0,\infty). \tag{78}$$

From (78), we immediately have

$$|u(x,t)| \leq \max_{i=1,2} |u^i(x,t)|.$$

Combining the above with (77), we conclude that the inequality (75) with $p = \infty$ holds. Finally, making use of (39) and an inequality

$$\|f\|_{L^p(R)} \leq \|f\|_{L^1(R)}^{1/p}\|f\|_{L^\infty(R)}^{1-1/p} \tag{79}$$

for $f = f(x)$, we arrive at (75) with $p \in [1,\infty)$. The proof of Theorem 16 is complete. □

3 Weak solutions for the case $u_0(\pm\infty) = u_\pm$

In this section, we study uniqueness, global existence and asymptotic behaviour of weak solutions of (13) for the case "$u_0(\pm\infty) = u_\pm$", where u_- and u_+ are given two constant with the constraint $u_- < u_+$.

3.1 Definition of weak solutions

First of all, we introduce the rarefaction waves. For any u_-, u_+ with $u_- < u_+$, the initial value problem

$$\begin{cases} \tilde{r}_t + \tilde{r}\tilde{r}_x = 0 & \text{in } R \times (-1, \infty), \\ \tilde{r}(x, -1) = u_\pm & \text{in } R_\pm, \end{cases} \tag{80}$$

where

$$R_- = (-\infty, 0), \quad R_+ = (0, +\infty), \tag{81}$$

gives a unique solution $\tilde{r}(x,t)$, which is explicitly written as

$$\tilde{r}(x,t) = \begin{cases} u_-, & x < u_-(t+1), \\ x/(t+1), & u_-(t+1) \le x \le u_+(t+1), \\ u_+, & u_+(t+1) < x. \end{cases} \tag{82}$$

Following Nishikawa and Nishihara [12], we define

$$r(x,t) = \tilde{r}|_{R\times[0,\infty)}(x,t) \quad \text{for } (x,t) \in R \times [0, \infty). \tag{83}$$

We call $r(x,t)$ the *rarefaction waves.*

From now on, we treat the initial data $u_0(x)$ belonging to

$$\{r_0\} + BV(R). \tag{84}$$

We often write $u_0(x)$ as

$$u_0(x) = r_0(x) + v_0(x), \quad v_0 \in BV(R). \tag{85}$$

Now we state the definition of weak solutions.

Definition 17 A function $(u,q)(x,t)$ is called a *weak solution* of (13) with initial datum $u_0(x)$ belonging to $\{r_0\} + L^1(R)$, if $(u,q)(x,t)$ satisfies the following conditions:

(i) $u(x,t)$ belongs to

$$\{r\} + C^0([0,\infty); L^1(R)) \cap L^\infty(R \times (0,\infty)). \tag{86}$$

(ii) $u(x,t)$ satisfies an *entropy inequality*

$$\begin{aligned} 0 \ \le \ & \iint_{R\times(0,\infty)} \{|u-a|\phi_t + \frac{1}{2}|u-a|(u+a)\phi_x \\ & -\mathrm{sign}(u-a)(u-Ku)\phi\}dxdt, \end{aligned} \tag{87}$$

for any $a \in R$ and $0 \le \phi \in C_0^\infty(R \times (0,\infty))$.

(iii) $u(x,0) = u_0(x)$ a.e.$x \in R$.

(iv) $q(x,t)$ satisfies the second equation of (13) with the above $u(x,t)$.

Remark. Classical solutions are always weak solutions.

3.2 Uniqueness of weak solutions

As in Subsection 2.2, we have the following theorem.

Theorem 18 *(Uniqueness) Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (13) with initial data u_0 and v_0 in $\{r_0\} + L^1(R)$, respectively. Then,*

$$\|u(t) - v(t)\|_{L^1(R)} \leq \|u_0 - v_0\|_{L^1(R)} \quad a.e\ t \geq 0. \tag{88}$$

As a consequence, $q(x,t)$ are unique.

Corollary 19 *(Ordering principle) Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (13) as in Theorem 18. Then, if $u_0(x) \leq v_0(x)$ a.e. $x \in R$, then,*

$$u(x,t) \leq v(x,t) \quad a.e.\ (x,t) \in R \times (0,\infty). \tag{89}$$

Theorem 18 and Corollary 19 can be proved as in Subsection 2.2, so we omit the proof.

3.3 Global existence of weak solutions

Throughout the remaining subsections, we write

$$d = |u_+ - u_-| = u_+ - u_- > 0, \tag{90}$$

$$b = \max(u_-, u_+). \tag{91}$$

3.3.1 Smooth rarefaction waves

We introduce smooth rarefaction waves $w(x,t)$ proposed by Hattori and Nishihara [2]. Let us consider the initial value problem:

$$\begin{cases} \tilde{w}_t + \tilde{w}\tilde{w}_x = \tilde{w}_{xx} & \text{in } R \times (-1,\infty), \\ \tilde{w}(x,-1) = u_\pm & \text{in } R_\pm, \end{cases} \tag{92}$$

where $R_\pm$ are in (81). For any u_-, u_+ with $u_- < u_+$, the problem (92) admits a unique solution $\tilde{w}(x,t)$. Following Nishikawa and Nishihara [12], we define

$$w(x,t) = \tilde{w}|_{R\times[0,\infty)}(x,t) \quad \text{for } (x,t) \in R \times [0,\infty). \tag{93}$$

We call $w(x,t)$ *smooth rarefaction waves.* If we put

$$w_0(x) = w(x,0), \tag{94}$$

then $w(x,t)$ satisfy

$$\begin{cases} w_t + ww_x = w_{xx} & \text{in } R \times (0,\infty), \\ w(x,0) = w_0(x) & \text{in } R. \end{cases} \tag{95}$$

We wish to mention here the properties of the smooth rarefaction waves in the following lemmas, which will be often used in the remaining part of this paper.

Lemma 20 [2] *The smooth rarefaction waves $w(x,t)$ given by (95) satisfy the following properties on $R \times [0,\infty)$: for any $p \in [1,\infty]$, there are constants $C > 0$ such that*

$$w \in C^\infty(R \times [0,\infty)), \tag{96}$$

$$u_- < w(x,t) < u_+ \quad \textit{and} \quad |w(x,t) - u_\pm| \le C_d \exp(-c|x|^2), \tag{97}$$

$$w_x(x,t) > 0, \tag{98}$$

$$\|w_x(t)\|_{L^p(R)} \le C d^{1/p}(1+t)^{-(1-1/p)} \tag{99}$$

for $t \ge 0$.

Lemma 21 [6] *The smooth rarefaction waves $w(x,t)$ also satisfy the following properties for $p \in [1,\infty]$: there are constants $C > 0$ such that*

$$\|\partial_x^k w(t)\|_{L^p(R)} \le Cd(1+t)^{-\frac{1}{2}(k-\frac{1}{p})}, \quad k = 1,2,\cdots, \tag{100}$$

$$\|\partial_x^k w(t)\|_{L^p(R)} \le C(1+t)^{-\frac{1}{2}(k+1-\frac{1}{p})}, \quad k = 2,3,\cdots, \tag{101}$$

$$\|(w-r)(t)\|_{L^p(R)} \le \begin{cases} C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, & 1 < p \le \infty, \\ C\log(2+d\sqrt{t}), & p = 1, \end{cases} \tag{102}$$

$$\|(w-r)_x(t)\|_{L^p(R)} \le C(1+t)^{-\frac{1}{2}(2-\frac{1}{p})} \tag{103}$$

for $t \ge 0$.

Lemma 22 *The smooth rarefaction waves $w(x,t)$ also satisfy: for any $p \in [1,\infty]$, there are $C > 0$ such that*

$$\|w_t(t)\|_{L^p(R)} \le C(d^{1/p}+1)(1+t)^{-(1-1/p)} \quad \textit{for } t \ge 0, \tag{104}$$

$$\|(w-r)_t(t)\|_{L^p(R)} \le C(1+b)(1+t)^{-\frac{1}{2}(2-\frac{1}{p})} \quad \textit{for } t \ge 0, \tag{105}$$

$$\|w-r\|_{W^{1,1}(R\times(-\alpha,T))} \le C(1+b)T\log(2+d\sqrt{T}) \tag{106}$$

for any $\alpha > 0$ and $T > 0$.

Proof. The inequality (104) can be obtained from Lemmas 20 and 21. Furthermore, it follows from Lemma 21 that

$$\begin{aligned} \|(w-r)_t(t)\|_{L^p(R)} &= \|\{-(w-r)w_x - r(w-r)_x + w_{xx}\}(t)\|_{L^p(R)} \\ &\le \|(w-r)(t)\|_{L^p(R)}\|w_x(t)\|_{L^\infty(R)} + \|r(t)\|_{L^\infty(R)}\|(w-r)_x(t)\|_{L^p(R)} \\ &\quad + \|w_{xx}(t)\|_{L^p(R)} \\ &\le C(1+b)(1+t)^{-\frac{1}{2}(2-\frac{1}{p})}, \end{aligned}$$

which gives (105). The estimate (106) can be shown in a similar way. □

3.3.2 Known result

Kawashima and Tanaka [6] proved that if initial perturbation to r_0 are smooth and small in a certain sense, then smooth global solutions of (1)-(2) exist. Furthermore, utilyzing the technique in [3], they showed that the solutions are asymptotically close to the rerefaction waves $r(x,t)$ as $t \to \infty$.

To state their result, we define a functions space

$$F = \{f \in \{r_0\} + (L^2 \cap L^1)(R); f' \in H^2(R)\}$$

endowed with the norm

$$\|f\|_F = \|f - r_0\|_{(L^2 \cap L^1)(R)} + \|f'\|_{H^2(R)}.$$

For $\delta > 0$, we also define

$$M(\delta) = \{f \in F; \|f\|_F \leq \delta\}. \tag{107}$$

Then,

Theorem 23 [6] *Let $p \in [1, \infty]$. Then, there are $\delta_0 > 0$, $d_0 > 0$ and $C > 0$; for any $u_0 \in M(\delta_0)$ and for any $d \leq d_0$, there exists a solution $(u,q)(x,t)$ of (1)-(2) in*

$$\begin{aligned} &(\{w\} + C([0,\infty); H^2(R)) \cap C^1([0,\infty); H^1(R))), \\ &\times C([0,\infty); H^3(R)), \end{aligned} \tag{108}$$

satisfying

$$\|u(t) - r(t)\|_{L^p(R)} \leq C\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \tag{109}$$

$$\|q(t) + r_x(t)\|_{L^p(R)} \leq C\delta_0(1+t)^{-(1-\frac{1}{2p})}, \tag{110}$$

for $t \geq 0$. The solutions $(u,q)(x,t)$ are unique in (108).

Remark. Note that $\|r_x(t)\|_{L^p(R)} = d^{1/p}t^{-(1-1/p)}$. This implies that (110) shows $-r_x(t)$ is the principal term of $q(t)$ in $L^p(R)$ as $t \to \infty$.

In the subsequent part, we shall extend Theorem 23 up to $BV(R)$-initial perturbations to $r_0(x)$.

3.3.3 Global existence result

Our global existence result is stated as follows.

Theorem 24 *(Global existence of weak solutions) For any u_-, u_+ with $u_- < u_+$ and for any $u_0(x)$ in $\{r_0\} + BV(R)$, there exists a weak solution $(u,q)(x,t)$ of (13) satisfying*

$$u - r, q_x \in \bigcap_{T>0} BV(R \times (0,T)) \tag{111}$$

$$u - r \in Lip([0,\infty); L^1(R)), \tag{112}$$

$$q \in Lip([0,\infty); W^{1,1}(R)). \tag{113}$$

Furthermore, for any $T > 0$, there are positive constants $C(d, \|u_0 - r_0\|_{BV(R)}, T)$ such that

$$\|u - r\|_{(BV\cap L^\infty)(R\times(0,T))} + \|q_x\|_{BV(R\times(0,T))} \leq C(d, \|u_0 - r_0\|_{BV(R)}, T), \tag{114}$$

$$\|u(t) - u(t')\|_{L^1(R)} + \|q(t) - q(t')\|_{W^{1,1}(R)} \leq C(d, \|u_0 - r_0\|_{BV(R)}, T)|t - t'| \tag{115}$$

for t, $t' \in [0, T]$.

In the subsequent subsections, we give a series of lemmas, which enable us to prove Theorem 24. Our strategy is to seek weak solutions of the form

$$u(x,t) = w(x,t) + V(x,t)$$

with the initial condition

$$\begin{aligned} u(x,0) = u_0(x) &= r_0(x) + v_0(x) \\ &\equiv w_0(x) + V_0(x), \end{aligned}$$

where $v_0(x)$ is in (85). In this situation, we easily see from (102) and (103) that $V_0(x)$ belong to $BV(R)$ and

$$\|V_0\|_{BV(R)} \leq \|u_0 - r_0\|_{BV(R)} + C. \tag{116}$$

As in Subsection 2.3.3, we first treat $V_0(x)$ belonging to $W^{2,1}(R)$ and secondly $BV(R)$. But we already know in view of the procedure in the proof of Theorem 7 that it is sufficient to treat $V_0(x)$ belonging to $W^{2,1}(R)$. So, in the following, we only study the case that $V_0 \in W^{2,1}(R)$.

3.3.4 The regularized problem

We consider a regularized problem for the first equation of (13):

$$\begin{cases} u^\varepsilon_t + u^\varepsilon u^\varepsilon_x + u^\varepsilon - Ku^\varepsilon = \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon(x,0) = w_0(x) + V_0(x) \quad \text{for } V_0 \in W^{2,1}(R). \end{cases} \tag{117}$$

Then, we have the following global existence theorem.

Theorem 25 *(Global existence for (117)) Let $\varepsilon > 0$. Then, for any initial datum $u_0(x)$ in $\{w_0\} + W^{2,1}(R)$, there exists a solution $u^\varepsilon(x,t)$ of (117) in*

$$\{w\} + (C([0,\infty); W^{2,1}(R)) \cap C^1([0,\infty); L^1(R))). \tag{118}$$

In particular, the solutions $u^\varepsilon(x,t)$ satisfy that for any $T > 0$, there are positive constants $C(d, \|u_0 - r_0\|_{BV(R)}, \varepsilon\|V_0''\|_{L^1(R)}, \varepsilon, T)$ and $C(d, \|u_0 - r_0\|_{BV(R)}, T)$ such that

$$\lim_{\varepsilon \searrow 0} C(d, \|u_0 - r_0\|_{BV(R)}, \varepsilon\|V_0''\|_{L^1(R)}, \varepsilon, T) = C(d, \|u_0 - r_0\|_{BV(R)}, T) \tag{119}$$

$$\|u^\varepsilon - r\|_{W^{1,1}(R\times(0,T))} \leq C(d, \|u_0 - r_0\|_{BV(R)}, \varepsilon\|V_0''\|_{L^1(R)}, \varepsilon, T). \tag{120}$$

The solutions are unique in (118).

To prove Theorem 25, we seek solutions $u^\varepsilon(x,t)$ of the form

$$u^\varepsilon(x,t) = w(x,t) + V^\varepsilon(x,t).$$

Then, the equation that $V^\varepsilon(x,t)$ should satisfy is

$$\begin{cases} V_t^\varepsilon + (wV^\varepsilon)_x + V^\varepsilon V_x^\varepsilon + V^\varepsilon - KV^\varepsilon = \varepsilon V_{xx}^\varepsilon + \varepsilon w_{xx} - w_{xx} + Kw_{xx}, \\ V^\varepsilon(x,0) = V_0(x) \in W^{2,1}(R). \end{cases} \tag{121}$$

As in Subsubsection 2.3.3, we wish to obtain global solutions of (121). To do this, we need a local existence theorem and a priori estimates, which are stated as follows.

Lemma 26 *(Local existence for (121)) Let $\varepsilon > 0$. Then, for any $V_0 \in W^{2,1}(R)$, there is a $T_0 > 0$ depending on ε and $\|V_0\|_{W^{2,1}(R)}$ such that there exists a solution $V^\varepsilon(x,t)$ of (121) in*

$$C([0,T_0]; W^{2,1}(R)) \cap C^1([0,T_0]; L^1(R)).$$

The solutions are unique in the above space.

The proof of Lemma 26 is omitted in the same reason as in Subsubsection 2.3.3.

Lemma 27 *(A priori estimates for (121)) Let $\varepsilon > 0$ and $T > 0$. Assume that $V^\varepsilon(x,t)$ is a solution of (121) in*

$$C([0,T]; W^{2,1}(R)) \cap C^1([0,T]; L^1(R))$$

with the initial datum $V_0 \in W^{2,1}(R)$. Then, the following inequalities hold: there are constants $C > 0$ depending on d and b such that

$$\|V^\varepsilon(t)\|_{L^1(R)} \le \|V_0\|_{L^1(R)} + C(1+\varepsilon)\log(2+t), \tag{122}$$

$$\|\nabla_{x,t} V^\varepsilon(t)\|_{L^1(R)} \le C\{\|V_0'\|_{L^1(R)} + \varepsilon\|V_0''\|_{L^1(R)} + 1 + \varepsilon\}(1+t)^C, \tag{123}$$

$$\|V_{xx}^\varepsilon(t)\|_{L^1(R)} \le C(\|V_0''\|_{L^1(R)} + 1) + C(\|V_0'\|_{L^1(R)}, t, \varepsilon^{-1}) \tag{124}$$

for $0 \le t \le T$.

Outline of the proof. Here we only show an outline of the proof of (123). Other estimates can be shown in a similar way. Furthermore, we only do formal computations, for all computations in the following can be justified as in the proof of Lemma 9.

Differentiating (121) with respect to x, multiplying it by $g_\sigma(V_x^\varepsilon)$ (where g_σ is in (56)), and integrating it over $R \times (0,t)$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{V_x^\varepsilon(t)} g_\sigma(\lambda)d\lambda dx - \int_{-\infty}^{\infty}\int_0^{V_0'} g_\sigma(\lambda)d\lambda dx \\ &+ \iint_{R\times(0,t)} g_\sigma(V_x^\varepsilon)(wV^\varepsilon)_{xx} dx d\tau \\ &+ \iint_{R\times(0,t)} g_\sigma(V_x^\varepsilon)(V^\varepsilon V_x^\varepsilon)_x dx d\tau + \iint_{R\times(0,t)} g_\sigma(V_x^\varepsilon)V_x^\varepsilon dx d\tau \\ &\le \int_0^t \|V_x^\varepsilon(\tau)\|_{L^1(R)} d\tau - \varepsilon \iint_{R\times(0,t)} g_\sigma'(V_x^\varepsilon)(V_{xx}^\varepsilon)^2 dx d\tau \\ &+(\varepsilon+2)\int_0^t \|w_{xxx}(\tau)\|_{L^1(R)} d\tau. \end{aligned} \tag{125}$$

We denote by $J(\sigma)$ the third integral in the left hand side. An integration by parts gives

$$\begin{aligned} J(\sigma) &= -\iint_{R\times(0,t)} g'_\sigma(V^\varepsilon_x)V^\varepsilon_{xx}wV^\varepsilon_x dxd\tau \\ &\quad -\iint_{R\times(0,t)} g'_\sigma(V^\varepsilon_x)V^\varepsilon_{xx}w_xV^\varepsilon dxd\tau \\ &\equiv J_1(\sigma)+J_2(\sigma). \end{aligned}$$

By virtue of (99) and the procedure utilized in the proof of Lemma 9, we obtain

$$|J_1(\sigma)| \le C\sigma\int_0^t \|w_x(\tau)\|_{L^1(R)}d\tau \le Cd\sigma t. \tag{126}$$

For $J_2(\sigma)$, we compute:

$$\begin{aligned} J_2(\sigma) &= -\iint_{R\times(0,t)} \partial_x\int_0^{V^\varepsilon_x} g'_\sigma(\lambda)d\lambda w_xV^\varepsilon dxd\tau \\ &= \iint_{R\times(0,t)}\int_0^{\sigma^{-1}V^\varepsilon_x} j(\lambda)d\lambda(w_{xx}V^\varepsilon+w_xV^\varepsilon_x)dxd\tau. \end{aligned}$$

Then, it follows from (101) that

$$\begin{aligned} |J_2(\sigma)| &\le 2\int_0^t \|w_{xx}(\tau)\|_{L^1(R)}\|V_x(\tau)\|_{L^1(R)}d\tau \\ &\le C\int_0^t(1+\tau)^{-1}\|V_x(\tau)\|_{L^1(R)}d\tau. \end{aligned}$$

Thus, letting $\sigma\to 0$ in (125), we arrive at

$$\begin{aligned} \|V^\varepsilon_x(t)\|_{L^1(R)} &\le \|V'_0\|_{L^1(R)}+C(1+\varepsilon) \\ &\quad +\int_0^t(1+\tau)^{-1}\|V^\varepsilon_x(\tau)\|_{L^1(R)}d\tau. \end{aligned} \tag{127}$$

Similarly, we can obtain

$$\begin{aligned} \|V^\varepsilon_t(t)\|_{L^1(R)} &\le \|V^\varepsilon_t(0)\|_{L^1(R)}+C(1+\varepsilon)(1+b) \\ &\quad +\int_0^t C\{(d+b)(1+\tau)^{-1}+(1+\tau)^{-3/2}\}\|V^\varepsilon_x(\tau)\|_{L^1(R)}d\tau. \end{aligned} \tag{128}$$

Adding (127) and (128), we have

$$\begin{aligned} &\|\nabla_{x,t}V^\varepsilon(t)\|_{L^1(R)} \\ &\le \|\nabla_{x,t}V^\varepsilon(0)\|_{L^1(R)}+C(1+\varepsilon)(1+b) \\ &+\int_0^t C\{(1+d+b)(1+\tau)^{-1}+(1+\tau)^{-3/2}\}\|\nabla_{x,t}V^\varepsilon(\tau)\|_{L^1(R)}d\tau. \end{aligned} \tag{129}$$

Applying the Gronwall lemma to (129), we obtain

$$\begin{aligned} &\|\nabla_{x,t}V^\varepsilon(t)\|_{L^1(R)} \\ &\le (\|\nabla_{x,t}V^\varepsilon(0)\|_{L^1(R)}+C(1+\varepsilon)(1+b))(1+t)^{C(1+d+b)}. \end{aligned}$$

After clearing the right hand side, we get (123). The proof of Lemma 27 is complete. □

Combining Lemmas 26 and 27, we obtain

Proposition 28 *(Global existence for (121)) Let $\varepsilon > 0$. Then, for any $V_0 \in W^{2,1}(R)$, there exists a solution $V^\varepsilon(x,t)$ of (121) in*

$$C([0,\infty); W^{2,1}(R)) \cap C^1([0,\infty); L^1(R)).$$

The solutions are unique in the above space.

Finally, we prove Theorem 25.

Proof of Theorem 25. For $V^\varepsilon(x,t)$ constructed in Proposition 28, $u^\varepsilon(x,t) = w(x,t) + V^\varepsilon(x,t)$ is a classical solution of (117). Furthermore, by virtue of Lemma 27, (106) and (116), we have for any $T > 0$

$$\begin{aligned}
\|u^\varepsilon - r\|_{W^{1,1}(R\times(0,T))} &\leq \|V^\varepsilon\|_{W^{1,1}(R\times(0,T))} + \|w - r\|_{W^{1,1}(R\times(0,T))} \\
&\leq C(\|V_0\|_{W^{1,1}(R)}, \varepsilon\|V_0''\|_{L^1(R)}, \varepsilon, T) + C(d,T) \\
&\leq C(d, \|u_0 - r_0\|_{BV(R)}, \varepsilon\|V_0''\|_{L^1(R)}, \varepsilon, T).
\end{aligned}$$

The above constant satisfies (119) and (120) for any $T > 0$. The proof of Theorem 25 is complete. □

3.3.5 Outline of the proof of the global existence theorem

Outline of the proof of Theorem 24. As in Lemma 13, we can find a subsequence $\{V^k\}_{k=1}^\infty$ of $\{V^\varepsilon\}_{\varepsilon>0}$ and a function $V(x,t)$ in

$$\begin{aligned}
&Lip([0,\infty); L^1(R)) \cap L^\infty(R\times(0,\infty)) \\
&\cap \bigcap_{T>0} BV(R\times(0,T))
\end{aligned}$$

such that

$$\begin{cases} V^k \longrightarrow V & \text{in } L^1_{\text{loc}}(R\times(0,\infty)), \\ V^k \longrightarrow V & \text{a. e. } (x,t) \in R\times(0,\infty), \end{cases} \tag{130}$$

as $k \longrightarrow \infty$. Furthermore, there are positive constants C depending on d and b such that

$$\|V(t) - V(t')\|_{L^1(R)} \leq C(\|V_0'\|_{L^1(R)} + 1)(1+T)^C|t - t'| \tag{131}$$

for t, $t' \in [0,T]$,

$$\|V(t)\|_{L^1(R)} \leq \|V_0\|_{L^1(R)} + C\log(2+t) \quad \text{for } t \geq 0, \tag{132}$$

$$\|V\|_{L^1(R\times(0,T))} \leq T(\|V_0\|_{L^1(R)} + C\log(2+T)), \tag{133}$$

$$\|DV\|(R\times(0,T)) \leq C(\|V_0'\|_{L^1(R)} + 1)(1+T)^C, \tag{134}$$

$$\|V\|_{L^\infty(R\times(0,T))} \leq C(\|V_0'\|_{L^1(R)} + 1)(1+T)^C \tag{135}$$

for any $T > 0$.

Now, if we let $k \longrightarrow \infty$, then the sequence

$$u^k(x,t) = w(x,t) + V^k(x,t)$$

has the limit. Define

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) \quad \text{for a.e. } (x,t) \in R\times(0,\infty).$$

Then, there holds:

$$u(x,t) = w(x,t) + V(x,t).$$

To verify (114) for $u-r$, we take the inferior limit as $k \to \infty$ in (120) along the sequence $\{u^k\}_{k=1}^{\infty}$. Then, we obtain (114) for $u-r$. The inequality (115) for u, is directly obtained from (104) with $p=1$ and (131). Finally, multiplying (117) by $\Phi_\delta(u^\varepsilon)\phi$, where Φ_δ is in (69) and $\phi(x,t) \geq 0$ is any test function on $R\times(0,\infty)$, we proceed as in (70)-(71). Thus we find that $u(x,t) = w(x,t) + V(x,t)$ satisfies the entropy inequality (87). Thus, the function $u(x,t)$ is the desired weak solution of (13). Furthermore, it is easy to check that $q = -K_1 u$ with the above u satisfies (114) and (115). The outline of the proof of Theorem 24 is complete. □

3.4 The stability of the rarefaction waves

For $\delta > 0$, we define

$$\begin{aligned} N(\delta) &= \{\psi \in \{r_0\} + BV(R); \exists \psi^i \in M(\delta), i=1,2 \ \text{ s.t.} \\ &\qquad \psi^1(x) \leq \psi(x) \leq \psi^2(x) \quad \text{a.e. } x \in R\}, \end{aligned}$$

where $M(\delta)$ is in (107).

Our main result of the present paper is the following.

Theorem 29 *(Stability of the rarefaction waves) Let $p \in [1,\infty]$ and let δ_0 and d_0 be the positive numbers established in Theorem 23. Then, for any $u_0 \in N(\delta_0)$ and for any $d \leq \delta_0$, there is a constant $C_{d_0} > 0$ such that the weak solution $u(x,t)$ obtained in Theorem 24 satisfies*

$$\|u(t) - r(t)\|_{L^p(R)} \leq C_{d_0}\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \tag{136}$$

for $t \geq 0$.

Remark. It does not seem to be able to obtain the optimal asymptotic behaviour of $q(t)$ at present.

Proof of Theorem 29. Since $u_0 \in N(\delta_0)$, there are functions $u_0^i \in M(\delta_0)$ such that

$$u_0^1(x) \leq u_0(x) \leq u_0^2(x) \quad \text{a.e. } x \in R. \tag{137}$$

It follows from Theorem 23 that for initial data u_0^i, $i=1,2$, there exist the solutions $u^i(x,t)$ in

$$\{w\} + (C([0,\infty); H^2(R)) \cap C^1([0,\infty); H^1(R)))$$

satisfying

$$\|u^i(t) - r(t)\|_{L^p(R)} \leq C_{d_0}\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \tag{138}$$

for $t \geq 0$. We note that u^i, $i=1,2$, are also in $L^\infty(R\times(0,\infty))$. We can also prove that

$$u^i \in \{r\} + C([0,\infty); L^1(R)).$$

This is shown by a tedious calculation used in the proof of the local existence theorem and by $L^1(R)$- a priori estimate. We omit the details. Furthermore, it follows from Theorem 24 that for initial data $u_0(x)$, there exists the weak solution $u(x,t)$.

Thus we can now apply Corollary 19 to (137) and we have

$$u^1(x,t) \leq u(x,t) \leq u^2(x,t).$$

Then

$$|u(x,t) - r(x,t)| \leq \max_{i=1,2} |u^i(x,t) - r(x,t)|.$$

Applying (138) to the above, we get

$$\begin{aligned} \|u(t) - r(t)\|_{L^p(R)} &\leq \max_{i=1,2} \|u^i(t) - r(t)\|_{L^p(R)} \\ &\leq C_{d_0}\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}. \end{aligned}$$

This completes the proof of Theorem 29. □

References

[1] K. Hamer, Nonlinear effects on the propagation of sound waves in a radiating gas, Quart. J. Mech. Appl. Math., 24(1971), 155–168.

[2] Y. Hattori and K. Nishihara, A note on the stability of the rarefaction wave of the Burgers equation, Japan I. Indust. Appl. Math., 8(1991), 85–96.

[3] K.Ito, Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimensions, Maththematical Models and Methods in Applied Sciences, 6(1996), 315–338.

[4] S. Kawashima, private communication.

[5] S. Kawashima and S. Nishibata, Shock waves for a model system of the radiating gas, preprint.

[6] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model equation of a radiating gas, in preparation.

[7] S. N. Kružkov, First order quasilinear equations in several independent variables, Math. USSR Sbornik, 10(1970), 217–243.

[8] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, 1995.

[9] A. Matsumura, Initial value problems for some quasilinear partial differential equations in mathematical physics, Doctoral thesis, Kyoto University, 1980.

[10] A. Matsumura and K. Nishihara, Asymptotic toward the rarefaction wave of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math., 3(1986), 1–13.

[11] S. Mizohata, Theory of partial differential equations, Cambridge Univ. Press, 1973.

[12] M. Nishikawa and K. Nishihara, Asymptotics toward the planar rarefaction wave for viscous conservation law in two space dimensions, preprint.

[13] W. P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Math., Springer-Verlag, New York, 1989.