HOKKAIDO UNIVERSITY

Title	The commutator ideal in Toeplitz algebras for uniform algebras and the analytic structure
Author(s)	Nakazi, T.; Sawada, H.
Citation	Hokkaido University Preprint Series in Mathematics, 371, 1-9
Issue Date	1997-02-01
DOI	https://doi.org/10.14943/83517
Doc URL	https://hdl.handle.net/2115/69121
Type	departmental bulletin paper
File Information	re371.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

The Commutator Ideal In Toeplitz Algebras For Uniform Algebras And The Analytic Structure

T. Nakazi and H. Sawada

Series #371. February 1997

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#347 T. Nakazi and T. Yamamoto, Weighted Norm Inequalities For Some Singular Integral Operators, 17 pages. 1996.

#348 Y. Ito and I. Nakamura, Hilbert schemes and simple singularities A_n and D_n, 22 pages. 1996

#349 R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, 42 pages. 1996

#350 F. Hiroshima, Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field, 21 pages. 1996.

#351 T. Nakazi and Y. Watatani, Invariant subspace theorems for subdiagonal algebras, 22 pages. 1996.

#352 Y. Nishiura and H. Suzuki, Nonexistence of stable turing patterns with smooth limiting interfacial configurations in higher dimensional spaces, 21 pages. 1996.

#353 Y.-G. Chen, Y. Giga and K. Sato, On instant extinction for very fast diffusion equations, 9 pages. 1996.

#354 A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free U(n)-action on Harish-Chandra modules, 25 pages. 1996.

#355 G. Ishikawa, Topology of plane trigonometric curves and the strangeness of plane curves derived from real pseudo-line arrangements, 18 pages. 1996.

#356 N.H. Bingham and A. Inoue, The Drasin-Shea-Jordan theorem for Hankel transforms of arbitrarily large order, 13 pages. 1996.

#357 S. Izumiya, Singularities of solutions for first order partial differential equations, 20 pages. 1996.

#358 N. Hayashi, P.I. Naumkin and T. Ozawa, Scattering theory for the Hartree equation, 14 pages. 1996.

#359 I. Tsuda and K. Tadaki, A logic-based dynamical theory for a genesis of biological threshold, 49 pages. 1996.

#360 I. Tsuda and A. Yamaguchi, Singular-continuous nowhere-differentiable attractors in neural systems, 40 pages. 1996.

#361 M. Nakamura and T. Ozawa, Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces, 17 pages. 1996.

#362 I. Nakamura, Hilbert schemes and simple singularities E_6, E_7 and E_8, 21 pages. 1996.

#363 T. Mikami, Equivalent conditions on the central limit theorem for a sequence of probability measures on R, 7 pages. 1996.

#364 S. Izumiya and T. Sano, Generic affine differential geometry of space curves, 23 pages. 1996.

#365 T. Tsukada, Stability of reticular optical caustics, 12 pages. 1996.

#366 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, 40 pages. 1996.

#367 A. Arai, A class of representations of the *-algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, 12 pages. 1996.

#368 K. Ito, BV-solutions of a hyperbolic-elliptic system for a radiating gas, 33 pages. 1997.

#369 M. Nakamura and T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, 20 pages. 1997.

#370 N.H. Bingham and A. Inoue, An Abel-Tauber theorem for Hankel transforms, 8 pages. 1997.

The Commutator Ideal In Toeplitz Algebras For Uniform Algebras And The Analytic Structure

by

Takahiko Nakazi *

and

Hironari Sawada

Department of Mathematics

Hokkaido University

Sapporo 060, Japan

and

Higashiyama Gakuen

Sakyoku Eikando Cho

Kyoto 606, Japan

AMS Subject Classification(1991) ; 46 J 15, 47 B 35
Key Words And Phrases ; Toeplitz algebra, commutator ideal, compact operator, uniform algebra, Gleason part.

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

Abstract. Let A be a uniform algebra and H^2 the abstract Hardy space defined by A. Under some conditions on A, we show that its maximal ideal space has some analytic structure when the commutator ideal of a Toeplitz algebra on H^2 is just a set of all compact operators on H^2.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Fix a nonzero complex homomorphism τ on A and a representing measure m for τ whose support is X. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak star closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$. $M(A)$ denotes the maximal ideal space of A and X is regarded as a closed subset of $M(A)$. The Shilov boundary ∂A of A then becomes a closed subset of X. The τ fixed above belongs to $M(A)$. We set $A_\tau = \{f \in A \,;\, \tau(f) = 0\}$ and $H^p_\tau = \{f \in H^p \,;\, \int f dm = 0\}$.

For ϕ in L^∞, the Toeplitz operator T_ϕ is defined by

$$T_\phi(f) = P(\phi f) \qquad (f \in H^2)$$

where P is the orthogonal projection onto H^2, and the Hankel operator H_ϕ is defined by

$$H_\phi(f) = (I - P)(\phi f) \qquad (f \in H^2).$$

Let $\mathcal{T}(C(X))$ be the C^*-algebra generated by the operators $\{T_\phi \,;\, \phi \in C(X)\}$ and $\mathcal{C}(C(X))$ be the commutator ideal of $\mathcal{T}(C(X))$. $\mathcal{LC}(H^2)$ denotes the set of all compact operators on H^2. In this paper, we study the analytic structure on $M(A)$ when $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Let $\mathcal{A}$ be the set of all continuous functions on the closed unit disc $\partial\triangle \cup \triangle$ which are analytic on the open unit disc $\triangle$. If $A = \mathcal{A}|\partial\triangle, X = \partial\triangle$ and m is the normalized Lebesgue measure on $\partial\triangle$, then H^2 is the classical Hardy space. If $A = \mathcal{A}, X = \partial\triangle \cup \triangle$ and m is the normalized area measure on $\partial\triangle\cup\triangle$, then H^2 is called the Bergman space. In both examples, it is known ($cf.$[5, Theorem 7.23] and [1, Theorem 4.12]) that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Let z be a coordinate function in $\mathbf{C}$. If $A = \mathcal{A}|\partial\triangle$, then T_z is an isometry and if $A = \mathcal{A}$, then T_z is an essential isometry. In this paper, these two uniform algebras are typical examples such that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$.

We call A a uniform algebra with generalized approximation in modulus property when the set of all finite sums $\sum_i \mid f_i \mid^2$, $f_i \in A$, is dense in the cone of positive and continuous functions on X($cf.$ [7]). A lot of uniform algebras have such modulus property. If a function q in A is unimodular on X, we call q an inner function. T is called isometry when $T^*T = 1$ and T is called essential isometry when $1 - T^*T$ is compact. $G(\tau)$ denotes the Gleason part of τ, that is, $G(\tau) = \{\sigma \in M(A) \,;\, \|\tau - \sigma\| < 2\}$. $[E]_2$ denotes the closure of the subset of E in L^2.

In this paper, we assume that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. In §2, if there exists a nonconstant function q in A such that T_q is an isometry and not a unitary, we show that $G(\tau)$ is nontrivial and under some conditions on A and q, it is contained in a subset of $M(A)$ given an analytic structure which is determined by q. In §3, we consider the same problem when T_q is an essential isometry. In §4, we study it without the existence of T_q.

§2. Isometric Toeplitz operator

The typical example in this section is $A = \mathcal{A} \mid \partial\Delta$ and then T_z is an isometry.

Theorem 1. If $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an isometry and not a unitary, then $G(\tau) \neq \{\tau\}$.

Proof. Since T_q is an isometry, $T_q^* T_q$ is the identity operator and $1 - T_q T_q^*$ is an orthogonal projection from H^2 onto $H^2 \ominus qH^2$. Because $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and the range of a compact operator contains no closed infinite-dimensional subspace, $1 - T_q T_q^*$ is a compact operator and $\dim(H^2 \ominus qH^2) < \infty$. Let $B = H^2 \cap L^\infty$ and $D = (qH^2) \cap L^\infty$. If $\dim(H^2 \ominus qH^2) = n$, then $\dim(B/D) \leq n$. In fact, if $\dim(B/D) > n$, there exists $\{f_j\}_{j=1}^{n+1}$ in B such that $\{f_j + D\}_{j=1}^{n+1}$ is linearly independent in B/D. By hypothesis, $\{f_j + qH^2\}_{j=1}^{n+1}$ is not linearly independent in H^2/qH^2 and so there exists $\{\alpha_j\}_{j=1}^{n+1}$ in $\mathbf{C}$ with $\langle \alpha_1, \cdots, \alpha_{n+1} \rangle \neq \langle 0, \cdots, 0 \rangle$ such that $\alpha_1 f_1 + \cdots + \alpha_{n+1} f_{n+1}$ belongs to $(qH^2) \cap L^\infty = D$. This contradiction implies that $\dim(B/D) \leq n$. If $D = B$, then $1 \in D$ and so $qH^2 = H^2$. This contradicts that T_q is not a unitary. This contradiction shows that D is a weak star closed proper ideal of B. Since $\dim(B/D) < \infty$, there exists ϕ in $M(B)$ such that $B_\phi = \{f \in B \ ; \ \phi(f) = 0\} \supseteq D$ and B_ϕ is weak star closed. Since $1 \notin B_\phi$, there exists a function h in L^1 such that

$$a = \int h dm \neq 0 \text{ and } \int hg dm = 0 \qquad (g \in B_\phi).$$

Hence $\phi(f) = \int fhdm/a$ for all f in B. Now there exists a (positive) representing measure for ϕ that is absolutely continuous with respect to hdm/a (*cf.* [2, Theorem 2.1.1]). Thus $\phi \mid A$ belongs to $G(\tau)$. If $\tau(q) \neq 0$, then $\phi \mid A \neq \tau$. and so $G(\tau) \neq \{\tau\}$. If $\tau(q) = 0$, put $Q = (q + a)/(1 + \bar{a}q)$ for some nonzero constant a with $\mid a \mid < \|q\|_\infty$, then Q belongs to A and T_Q is an isometry. Hence the argument above is valid for the new nonconstant function Q with $\tau(Q) \neq 0$. This completes the proof.

Theorem 2. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an isometry and not a unitary.

(1) Suppose $H^2 \cap C(X) = A$ and $M(A)$ has no isolated point. If $\mid q \mid = 1$ on X, then

$$\hat{q}(\partial A) = \partial\Delta, \hat{q}(M(A)) = \bar{\Delta} \text{ and } G(\tau) \subseteq \hat{q}^{-1}(\Delta).$$

Moreover $\hat{q}^{-1}(\Delta) \subseteq M(A)$ can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transforms of functions in A become analytic on $\hat{q}^{-1}(\Delta)$. $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(\Delta)$ over Δ where $1 \leq n \leq \dim\ker T_q^* < \infty$.

(2) If A is an algebra with generalized approximation in modulus property, then (1) is valid without assuming that $\mid q \mid = 1$ on X.

(3) If m is a unique representing measure for τ, then (1) is valid with $G(\tau) = \hat{q}^{-1}(\triangle)$ without any assumptions in (1).

Proof. (1) Since $H^2 \cap C(X) = A$, we can show $\dim(A/qA) < \infty$ as in the proof of Theorem 1. By hypothesis on q, $\hat{q}(\partial A) \subseteq \partial\triangle$. We will show that $\hat{q}(\partial A) = \partial\triangle$. If $\hat{q}(\partial A) \neq \partial\triangle$, then $W = \mathbf{C}\backslash\hat{q}(\partial A)$ is connected and unbounded. Since q is a nonconstant function because T_q is not a unitary, qA is a proper closed ideal in A and so there exists ϕ in $M(A)$ such that $\phi(q) = 0$. This implies $0 \in W$ and hence $(q-\lambda)A \subsetneq A$ for any $\lambda \in W$ by the index theory for Fredholm operators ($cf.$ [6, p157]). Since W is unbounded, there exists $\lambda \in W$ such that $(q-\lambda)^{-1} \in A$ and so $(q-\lambda)A = A$. This contradiction shows that $\hat{q}(\partial A) = \partial\triangle$ and $\mathbf{C}\backslash\hat{q}(\partial A) = \triangle \cup \bar{\triangle}^c$. Now $\hat{q}(M(A)) = \bar{\triangle}$. For if $\hat{q}(M(A)) \neq \bar{\triangle}$, then there exists $\lambda \in \triangle$ such that $(q-\lambda)A = A$ because $\hat{q}(M(A)) \supseteq \partial\triangle$. On the other hand, $qA \subsetneq A$ and so $(q-\lambda)A \subsetneq A$ for any $\lambda \in \triangle$ by the index theory for Fredholm operators. This contradiction shows that $\hat{q}(M(A)) = \bar{\triangle}$. By [6, Chapter VI, Theorem 6.3], $\hat{q}^{-1}(\triangle) \subseteq M(A)$ can be given the structure of a (possibly disconnected) open Riemann surface with discrete point identifications, and hence the Gelfand transform of functions in A become analytic on $\hat{q}^{-1}(\triangle)$. Since $\dim(A/qA) = \dim\ker T_q^*$, by [6, Chapter VI, Theorem 6.3] $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(\triangle)$ over $\triangle$ where $1 \leq n \leq \dim\ker T_q^* < \infty$. Now we will show that $G(\tau) \subseteq \hat{q}^{-1}(\triangle)$ and then the proof will be compleded. If $|\tau(q)| = 1$, then $\int |q| \, dm = |\int q dm|$ and so q is constant. This contradiction implies that $|\tau(q)| < 1$. Put $\tau(q) = a$ and $Q = (q-a)/(1-\bar{a}q)$, then $Q \in A$ and $|Q| = 1$. and $\tau(Q) = 0$. If $\phi \in G(\tau)$ and $\phi \notin \hat{q}^{-1}(\triangle)$, then $|\phi(q)| = 1$ and so $|\phi(Q)| = 1$. This and [6, Chapter VI, Theorem 2.1] imply that $\phi \notin G(\tau)$ and hence $G(\tau) \subseteq \hat{q}^{-1}(\triangle)$.

(2) Since T_q is an isometry, $\int |q|^2 |f|^2 \, dm = \int |f|^2 \, dm$ for all $f \in A$. By the hypothesis on A, $\int |q|^2 \, udm = \int udm$ for any positive function u in $C(X)$ and so $|q| = 1$ on X.

(3) Since T_q is an isometry, $\int |q|^2 |f|^2 \, dm = \int |f|^2 \, dm$ for all $f \in H^2$. If u is a nonnegative function in L^∞ and $\varepsilon > 0$, then there exists a function f_ε in H^∞ such that $u + \varepsilon = |f_\varepsilon|^2$ ($cf.$ [6, Chapter V, Theorem 5.4]). This implies that $|q| = 1$ on X. By Theorem 1, $G(\tau) \neq \{\tau\}$ and so there exists a unimodular function Z such that $H_\tau^\infty = ZH^\infty$ ($cf.$ [6, Chapter VI, Theorem 7.1]). By the proof of Theorem 1, $\dim H^\infty/qH^\infty < \infty$ and so $qH^\infty \subset H_\phi^\infty$ for some $\phi \in G(\tau)$ because $H^2 \cap L^\infty = H^\infty$($cf.$ [6, Chapter V, Theorem 6.1]). Put $a_1 = \phi(Z)$, then $H_\phi^\infty = \frac{Z - a_1}{1 - \bar{a}_1 Z} H^\infty$ and so

$$q = \frac{Z - a_1}{1 - \bar{a}_1 Z} q_1, \; q_1 \in H^\infty \text{ and } |q_1| = 1 \quad a.e..$$

If $q_1 H^\infty \neq H^\infty$, then by the same argument

$$q_1 = \frac{Z - a_2}{1 - \bar{a}_2 Z} q_2, \; q_2 \in H^\infty \text{ and } |q_2| = 1 \quad a.e.$$

where $|a_2| < 1$. Repeating the above process, we can show that for some finite integer n,

$$q = \prod_{j=1}^{n} \frac{Z - a_j}{1 - \bar{a}_j Z} \text{ and } | a_j | < 1 \quad (j = 1, \cdots, n).$$

By the same argument in (1), $\hat{q}(\partial A) = \partial\Delta$, $\hat{q}(M(A)) = \bar{\Delta}$ and $G(\tau) \subseteq \hat{q}^{-1}(\Delta)$. it is well known (*cf*. [6, Chapter VI, Theorem 7.2]) that the Gefand transforms of functions in A become analytic on $G(\tau)$. Hence $G(\tau) = \hat{q}^{-1}(\Delta)$. It is not difficult to see that $\hat{q}$ is an n-sheeted analytic cover of $G(\tau)$ over Δ where $1 \leq n = \dim\ker T_q^* < \infty$.

Even if A is a Dirichlet algebra and $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$, A need not be isometrically isomorphic to the disc algebra $\mathcal{A} \mid \partial\Delta$. In fact, suppose $A = H^\infty(\partial\Delta) \cap QC$, $C(X) = QC$ and m is the normalized Lebesgue measure on $\partial\Delta$, where $H^\infty(\partial\Delta)$ is the weak star closure of the disc algebra $\mathcal{A} \mid \partial\Delta$ in $L^\infty(m)$ and $QC = (H^\infty(\partial\Delta) + C(\partial\Delta)) \cap (\overline{H^\infty(\partial\Delta) + C(\partial\Delta)})$. Then $A \underset{\neq}{\supset} \mathcal{A} \mid \partial\Delta$, A is a Dirichlet algebra, A is not generated by inner functions in $H^\infty(\partial\Delta)$ and $\mathcal{C}(C(X)) = \mathcal{C}(C(\partial\Delta)) = \mathcal{LC}(H^2)$.

Let Γ be a discrete abelian group, let G be its dual group, and let Γ_+ be a subsemigroup of Γ such that $\Gamma_+ - \Gamma_+ = \Gamma$. Put

$$A = \{f \in C(G) \; ; \; \int \phi\bar{\chi} dm = 0 \quad (\chi \notin \Gamma_+)\}$$

and let m be the Haar measure on G. Then A is a uniform algebra on $X = G$. *G.J.* Murphy [8] showed that $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ if and only if $\Gamma = Z$ where Z is a set of all integers. (1) of the following theorem shows this.

Theorem 3. Suppose A is a nontrivial uniform algebra generated by inner functions $\{q_\alpha\}_{\alpha\in\Lambda}$.

(1) $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ if and only if $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$.

(2) When $A = H^2 \cap C(X)$, $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ if and only if $\dim(A/q_\alpha A) < \infty$ for all $\alpha \in \Lambda$.

Proof. (1) By the proof of Theorem 1, if $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$ then $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$. Conversely if $\dim(H^2 \ominus q_\alpha H^2) < \infty$ for all $\alpha \in \Lambda$, then $T_{q_\alpha}^* T_{q_\alpha} - T_{q_\alpha} T_{q_\alpha}^* = 1 - T_{q_\alpha} T_{q_\alpha}^*$ is in $\mathcal{LC}(H^2)$. Our situation is a little different from that in [9] but the proof of [9, Corollary 2.3] shows that $\mathcal{T}(C(X))$ is irreducible. By [5, Theorem 5.39], $\mathcal{LC}(H^2) \subseteq \mathcal{C}(C(X))$. Let ϕ be arbitrary function in $C(X)$ and $\alpha, \beta \in \Lambda$, then

$$T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi = (T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\alpha} + (T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}) + (T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}) T_\phi$$

because $T_{\bar{q}_\beta} T_\phi T_{q_\alpha} = T_{\bar{q}_\beta \phi} T_{q_\alpha}$ and $T_{q_\alpha} T_{\bar{q}_\beta \phi} = T_{q_\alpha} T_{\bar{q}_\beta} T_\phi$. Since $(T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi) T_{q_\beta} = 0$ and $\dim\ker T_{q_\beta}^* < \infty$ for arbitrary $\beta \in \Lambda$, both $T_\phi T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_\phi$, $T_{\bar{q}_\beta \phi} T_{q_\alpha} - T_{q_\alpha} T_{\bar{q}_\beta \phi}$ and $T_{q_\alpha} T_{\bar{q}_\beta} - T_{\bar{q}_\beta} T_{q_\alpha}$ are compact. Thus $T_\phi T_{\bar{q}_\beta q_\alpha} - T_{\bar{q}_\beta q_\alpha} T_\phi$ is compact. Since A is generated by $\{q_\alpha\}_{\alpha\in\Lambda}$, $C(X)$ is generated by $\{q_\alpha, \bar{q}_\alpha\}_{\alpha\in\Lambda}$. Therefore for any ϕ, ψ in $C(X)$ $T_\phi T_\psi - T_\psi T_\phi$ is compact and so $\mathcal{LC}(H^2) = \mathcal{C}(C(X))$.

(2) As in the proof Theorem 1, using $H^2 \cap C(X) = A$, we can show that $\dim(H^2 \ominus q_\alpha H^2) = \dim(A/q_\alpha A)$. Now (2) follows from (1).

§3. Essential isometric Toeplitz operator

The typical example in this section is $A = \mathcal{A}$. Then T_z is an essential isometry and bounded below.

Theorem 4. If $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A such that T_q is an essential isometry and does not have a dense range, then $G(\tau) \neq \{\tau\}$.

Proof. By the hypothesis, $1 - T_q^* T_q \in \mathcal{LC}(H^2)$ and $T_q^* T_q - T_q T_q^* \in \mathcal{LC}(H^2)$. Hence $1 - T_q T_q^*$ belongs to $\mathcal{LC}(H^2)$ and range $(1 - T_q T_q^*) \supseteq \ker T_q^*$. This implies that $\dim \ker T_q^* < \infty$ and so $\dim(H^2 \ominus [qH^2]) < \infty$, because the range of a compact operator contains no closed infinite-dimensional subspace. Now the proof of Theorem 1 shows that $G(\tau) \neq \{\tau\}$.

Theorem 5. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$ and there exists a function q in A_τ such that T_q is an essential isometry and bounded below. If $H^2 \cap C(X) = A$ and $(qH^2) \cap C(X) = qA$, then $1 \leq \dim A/qA < \infty$. Hence if $M(A)$ has no isorated point and W is the connected component of $\mathbf{C} \backslash \hat{q}(\partial A)$ which contains 0, then $\hat{q}^{-1}(W) \subseteq M(A)$ can be given the structure of a (possiblely disconnected) open Riemann surface, with discrete point identifications, so that the Gelfand transforms of functions in A become analytic on $\hat{q}^{-1}(W)$. $\hat{q}$ is an n-sheeted analytic cover of $\hat{q}^{-1}(W)$ over W where $1 \leq n \leq \dim \ker T_q^* < \infty$.

Proof. Since T_q is bounded below, qH^2 is closed in H^2. By Theorem 4 and the proof, $\dim(H^2 \ominus qH^2) < \infty$ and by the proof of Theorem 1, $\dim\{H^2 \cap C(X)/(qH^2) \cap C(X)\} < \infty$. By the hypothesis, $1 \leq \dim(A/qA) < \infty$. The second statement is just [5, Chapter VI, Theorem 6.3]

If $1 - T_q^* T_q$ is finite rank or $\ker(1 - T_q^* T_q)$ is nontrivial, under some conditions on q and A, T_q is just an isometry. If T_q is an essential isometry and m is a unique representing measure for τ, then T_q is just an isometry. In genertal, when each compact Toepliz operator is zero, if T_q is an essential isometry then T_q is just isometry. In Theorem 5, if $|q| = 1$ on ∂A, then (1) in Theorem 2 is true but T_q may not be an isometry when $\partial A \neq X$.

§4. Hankel operator

In the previous sections, under the exstence of some special function q in A, we could find some analytic structure on $G(\tau)$ or some subset of $M(A)$. In this section, we do not put on such a hypothesis. If A is a polydisc algebra, that is, the set of all continuous functions on the closure $\overline{\triangle^n}$ of the unit polydisk $\triangle^n$ whose restriction to $\triangle^n$ is holomorplic there, then by Theorem 3 $\mathcal{C}(C(X)) \neq \mathcal{LC}(H^2)$. (1) of Theorem 6 also shows it because each compact Hankel operator is zero. (see [4]).

Theorem 6. Suppose $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$.

(1) There exists at least one nonzero compact Hankel operator whose symbol is in $C(X)$.

(2) If m is a unique representing measure for τ, then $G(\tau) \neq \{\tau\}$ and hence there is a one-to-one continuous map σ of $\triangle$ onto $G(\tau)$, such that $\hat{f}\circ\sigma$ is analytic on $\triangle$ for all f in A.

Proof. (1) Suppose each compact Hankel operator is zero. For any $f \in A$,

$$H_{\bar{f}}^* H_{\bar{f}} = T_{\bar{f}}^* T_f - T_f T_{\bar{f}}^*$$

is compact because $\mathcal{C}(C(X)) = \mathcal{LC}(H^2)$. Hence $H_{\bar{f}}$ is compact and so by the hypothesis, $H_{\bar{f}} = 0$. For any $g \in H^2$,

$$\bar{f}g = (M_{\bar{f}} - H_{\bar{f}})g = T_{\bar{f}}(g)$$

where $M_{\bar{f}}$ is multiplication operator by $\bar{f}$. Thus both f and $\bar{f}$ belong to H^2 and so

$$\int | f - \tau(f) |^2 \, dm = \int (f - \tau(f)) dm \int (\bar{f} - \tau(\bar{f})) dm = 0$$

Hence $f = \tau(f)$ *a.e.* and we get a contradiction. (2) If m is a unique representing measure and $G(\tau) = \{\tau\}$, then it is known [3] that each compact Hankel operator is zero.

References

1. S.Axler, Bergman spaces and their operators, Surveys of Some Recent Results in Operalor Theory Vol.1 (J.B.Conway and B.B.Morrel, editors), Pitman Research Notes in Math. 171, 1988, 1-50.

2. A.Browder, Introduction to Function Algebras, Benjamin, New York, 1969.

3. R.E.Curto, P.S.Muhly, T.Nakazi and J.Xia, Hankel operators and uniform algebras, Arch. Math. 43(1984), 440-447.

4. R.E.Curto, P.S.Muhly and T.Nakazi, Uniform algebras, Hankel operators and invariant subspaces, Operator Theory : Advances and Applications, 17(1986), 109-119.

5. R.G.Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York-London, 1972.

6. T.Gamelin, Uniform Algebras, Prentice-Hall, Englédwood Cliffs New Jersey, 1969.

7. J.Janas, Toeplitz spectral inclusion and generalized in modulus property, Proc. Amer. Math. Soc. Vol.104, 1988.

8. G.J.Murphy, Ordered groups and Toeplitz algebras, J.Operator Theory 18(1987), 303-326.

9. J.Tomiyama and K.Yabuta, Toeplitz operators for uniform algebras, Tôhoku Math. J. 30(1978), 117-129.