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Weak Coupling Limit Removing an Ultraviolet Cut-off for a Hamiltonian of Particles Interacting with a Scalar Field

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Abstract

A functional integral representation of a heat semigroup acting on a Hilbert space is constructed. Its generator describes an interaction between particles and a quantized field. By using the functional integral representation, for the Nelson model with a **nonnegative mass** we investigate a weak coupling limit which removes an ultraviolet cut-off. As a result, it is shown that a many-body Schrödinger Hamiltonian with the many-body Coulomb potential (or the many-body Yukawa potential) appears in the limit.

1 INTRODUCTION

The aim of this paper is to derive a many-body Schrödinger Hamiltonian with the Coulomb potential (or the Yukawa potential) from taking a scaling limit of a Hamiltonian which describes an interaction of an arbitrary but conserved number of nonrelativistic particles and a quantized neutral scalar field. In particular, we pursue the study of "the Nelson model"([1,2,4,8,9,11]) with a **nonnegative mass**. The model, for example, explains an interaction of nonrelativistic nucleons and mesons. This paper is inspired by [4] in which, for the Nelson model with a **positive mass**, with a scalar potential and with the number of the nonrelativistic particles fixed, introducing a renormalization, the author elaborates taking a weak coupling limit and **at the same time** removing an ultraviolet cut-off. The main problem we consider in this paper is to extend the results in [4] to the Nelson model with the **nonnegative mass**, with a much more general scalar potential and with the number of nonrelativistic particles unfixed.

A mathematical formulation of physical descriptions of interactions between particles and quantized fields is, in this paper, reduced to a theory of self-adjoint operators acting in the tensor product of two Hilbert spaces. We shall assume that the number of the space dimensions in which the particles move and interact is three. Although the statistics of the nonrelativistic particle does not play any role, in this paper, it is considered to be a fermion. (Of course all the results can be extended to the case where the particle is a boson.) Let $\mathcal{F}_b$ and $\mathcal{F}_a$ be the Boson Fock space and the Fermion Fock space over $L^2(\mathbb{R}^3)$, respectively,

$$\mathcal{F}_b \equiv \bigoplus_{N=0}^{\infty} [\otimes_s^N L^2(\mathbb{R}^3)],$$
$$\mathcal{F}_a \equiv \bigoplus_{N=0}^{\infty} [\otimes_{as}^N L^2(\mathbb{R}^3)],$$

where $\otimes_s^N L^2(\mathbb{R}^3)$ (resp. $\otimes_{as}^N L^2(\mathbb{R}^3)$) $N \geq 1$, denotes the symmetric (resp. anti-symmetric) N-fold tensor product of $L^2(\mathbb{R}^3)$ and $\otimes_{as}^0 L^2(\mathbb{R}^3) = \otimes_a^0 L^2(\mathbb{R}^3) \equiv \mathbb{C}$. We denote the tensor product Hilbert space by

$$\mathcal{L} \equiv \mathcal{F}_a \otimes \mathcal{F}_b \cong \bigoplus_{Z=0}^{\infty} [L^2_{as}(\mathbb{R}^{3Z}) \otimes \mathcal{F}_b],$$

where $L^2_{as}(\mathbb{R}^{3Z}), Z \geq 1$, denotes the set of square integrable anti-symmetric measurable functions over $\mathbb{R}^{3Z}$ and $L^2_{as}(\mathbb{R}^0) \equiv \mathbb{C}$. The number Z counts the number of bare nonrelativistic particles and is called "charge". Moreover $L^2_{as}(\mathbb{R}^{3Z}) \otimes \mathcal{F}_b$ is called "sector". Let $a(k)$,$a^\dagger(k)$,$b(k)$ and $b^\dagger(k)$ be the formal annihilation and creation operators in $\mathcal{F}_b$ and the formal annihilation and creation operators in $\mathcal{F}_a$, respectively, (which are defined by bilinear forms) which satisfy the following canonical commutation or anti-commutation relations

$$\begin{aligned}
&[a(k), a^\dagger(k')] = \delta(k-k'), \\
&[a^\sharp(k), a^\sharp(k')] = 0, \\
&\{b(k), b^\dagger(k')\} = \delta(k-k'), \\
&\{b^\sharp(k), b^\sharp(k')\} = 0,
\end{aligned}$$

where $a^\sharp$ and $b^\sharp$ denote a or $a^\dagger$ and b or $b^\dagger$, respectively, $[A,B] = AB - BA$, $\{A,B\} = AB + BA$, and $\delta(x)$ is the delta function with mass $x \in \mathbb{R}^3$. Formally we put

$$\begin{aligned}
\Psi(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int b(k) e^{ikx} dk, \\
\Psi^\dagger(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int b^\dagger(k) e^{-ikx} dk.
\end{aligned}$$

The Nelson Hamiltonian H is explained as an operator acting in the Hilbert space $\mathcal{L}$ and, formally, have the following form (a mathematically rigorous definition of the Nelson Hamiltonian is given in (4.1))

$$\begin{aligned}
H &= \int \Psi^\dagger(x) \left(-\frac{1}{2m}\Delta\right) \Psi(x) dx \otimes I - g \int \Psi^\dagger(x) \phi_F(x) \Psi(x) dx + I \otimes \int \omega(k) a^\dagger(k) a(k) dk \\
&\equiv H_a \otimes I - gH_I + I \otimes H_b^s,
\end{aligned}$$

where Δ is the Laplacian in $L^2(\mathbb{R}^3)$, $g \in \mathbb{R}$, $m > 0$, a coupling constant, a mass of the non-relativistic particles, respectively, $\omega(k) = \sqrt{k^2+\mu^2}$, $\mu \geq 0$ and $\phi_F(x)$ a time-zero quantized neutral scalar field in $\mathcal{F}_b$. "The weak coupling Hamiltonian"([2]) is defined by

$$H_w = H_a \otimes I - \Lambda g H_I + \Lambda^2 I \otimes H_b^s.$$

Let V_Z, $Z = 0, 1, 2, ..$ be multiplication operators in $L^2(\mathbb{R}^{3Z})$ with some conditions and put

$$V = \bigoplus_{Z=0}^{\infty} [V_Z \otimes I]. \tag{1. 1}$$

The restriction of $H_w + V$ to the sector $L^2_{as}(\mathbb{R}^{3Z}) \otimes \mathcal{F}_b$ has the form

$$\begin{aligned} H_w + V|_{L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b} &= \left(\frac{1}{2m}\mathbf{p}_Z^2 + V_Z\right) \otimes I - \Lambda g \sum_{j=1}^{Z} \phi(x_j) + \Lambda^2 I \otimes H_b^s \Big|_{L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b} \\ &\equiv \left(\frac{1}{2m}\mathbf{p}_Z^2 + V_Z\right) \otimes I - \Lambda g H_I^Z + \Lambda^2 I \otimes H_b^s \Big|_{L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b} \\ &\equiv H_w^Z + V_Z \otimes I \Big|_{L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b}. \end{aligned}$$

Here $\mathbf{p}_Z = (\mathbf{p}^1, ..., \mathbf{p}^Z) = (\mathbf{p}_1^1, \mathbf{p}_2^1, ..., \mathbf{p}_3^Z) = (-i\frac{\partial}{\partial x_1^1}, -i\frac{\partial}{\partial x_2^1}, ..., -i\frac{\partial}{\partial x_3^Z})$. In [1,2], introducing an ultraviolet cut-off and defining $H_w^Z(\Lambda) + V_Z \otimes I$ as a self-adjoint operator in

$$\mathcal{L}_Z \equiv L^2(\mathbb{R}^{3Z}) \otimes \mathcal{F}_b,$$

the authors show that, for all $t \in \mathbb{R}$ and for each fixed $Z \geq 1$, the strong limit of $e^{-it(H_w^Z + V_Z\otimes I)}$ as $\Lambda \to \infty$ exists. On the other hand, E.Nelson puts an ultraviolet cut-off parameterized by $\Lambda > 0$ on the interaction term H_I^Z ([11]); he obtains a self-adjoint operator in $\mathcal{L}_Z$;

$$H^Z(\Lambda) = \frac{1}{2m}\mathbf{p}_Z^2 \otimes I + g H_I^Z(\Lambda) + I \otimes H_b^s.$$

Defining a renormalization term $E(\Lambda)$ which goes to minus infinity as $\Lambda \to \infty$, he shows the existence of a self-adjoint operator H_∞^Z so that

$$s - \lim_{\Lambda\to\infty} e^{-it(H^Z(\Lambda) - g^2 Z E(\Lambda))} = e^{-itH_\infty^Z}.$$

However an explicit form of H_∞^Z is unknown. In [4], for each fixed $Z \geq 1$, the author defines a scaling Hamiltonian in $\mathcal{L}_Z$ by

$$\begin{aligned} &H_w^Z(\Lambda) + V_Z \otimes I - g^2 Z E_w(\Lambda^\alpha) \\ &= \left(\frac{1}{2m}\mathbf{p}_Z^2 + V_Z\right) \otimes I + \Lambda g H_I^Z(\Lambda^\alpha) + \Lambda^2 I \otimes H_b^s - g^2 Z E_w(\Lambda^\alpha), \end{aligned}$$

where the region of $\alpha > 0$ is restricted to an interval in $\mathbb{R}$, a scalar potential V_Z is infinitesimally small with respect to $\mathbf{p}_Z^2$ and $E_w(\Lambda^\alpha)$ a renormalization which goes to minus infinity as $\Lambda \to \infty$. Then, only in the case of **positive mass** $\mu > 0$, he considers the strong limit of $e^{-it\left(H_w^Z(\Lambda)+V_Z\otimes I-g^2 Z E_w(\Lambda^\alpha)\right)}$ as $\Lambda \to \infty$, the so called "weak coupling limit removing the ultraviolet cut-off simultaneously (WCLUV)", from the viewpoint of the pure operator theory. He obtains

$$s - \lim_{\Lambda\to\infty} e^{-it(H_w^Z(\Lambda)+V_Z\otimes I-g^2 Z E_w(\Lambda^\alpha))} = e^{-it(H_{eff}^Z+V_Z)} \otimes P_b,$$

where P_b is a projection operator on a closed subspace in $\mathcal{F}_b$. Moreover he shows that the self-adjoint operator H_{eff}^Z in $L^2(\mathbb{R}^{3Z})$ is the Z-body Schrödinger Hamiltonian with the Yukawa potential;

$$H_{eff}^Z = \frac{1}{2m}\mathbf{p}_Z^2 - \frac{g^2}{4\pi}\sum_{i<j}^{Z} \frac{e^{-\mu|x^i - x^j|}}{|x^i - x^j|}, \quad \mu > 0.$$

In this paper, as an extension of the previous paper [4], we consider WCLUV of the Nelson model with the **nonnegative mass** $\mu \geq 0$, with a much more general scalar potentials $V_Z, Z = 0, 1, 2, \ldots$ and with the number of the nonrelativistic particles Z unfixed. In this situation, renormalizations should be

$$g^2 E_w(\Lambda^\alpha) \int \Psi^\dagger(x)\Psi(x)dx,$$

where $\int \Psi^\dagger(x)\Psi(x)dx$ is a formal description of the number operator in $\mathcal{F}_a$. We define a time-zero quantized neutral scalar field with an ultraviolet cut-off $\phi_F(\Xi^1_{\Lambda^\alpha}(x))$ (for the definition, see section 4) and put

$$H_w(\Lambda)_V = H_a \otimes I - \Lambda g \int \Psi^\dagger(x)\phi_F(\widetilde{\Xi}^1_{\Lambda^\alpha}(x))\Psi(x)dx + \Lambda^2 I \otimes H^s_b + V. \tag{1. 2}$$

The main interest in this paper is to be concerned with the strong limit of

$$\exp\left\{-it\left(H_w(\Lambda)_V - \left(g^2 E_w(\Lambda^\alpha)\int \Psi^\dagger(x)\Psi(x)dx\right)\otimes I\right)\right\}$$

as $\Lambda \to \infty$ and to show explicit forms of the asymptotic limits.

Since the number of the non-relativistic particles is conserved, this is essentially the same problem as the one fixing the number of the nonrelativistic particles. However the situation is of course even much more complicated in the nonnegative mass case. Before going into details, it is helpful to give an overview of our discussion. The basic method presented here is as follows: In the massive Nelson model with the number of the nonrelativistic particles Z fixed, the author in [4] introduces a unitary operator which separates a renormalization term and transforms the Hamiltonian $H^Z_w(\Lambda)$ to a Hamiltonian $\widetilde{H}^Z_w(\Lambda)$ which is easy to analyze WCLUV from the viewpoint of the pure operator theory. Nevertheless, in the massless case, $\mu = 0$, we can not define such a unitary operator. To avoid this difficulty, we introduce, in this paper, a well defined unitary operator with an infrared cut-off $K > 0$, which transforms the Hamiltonian $H^Z_w(\Lambda)$ to a Hamiltonian $\widetilde{H}^Z_w(K\Lambda)$. The transformed Hamiltonian $\widetilde{H}^Z_w(K\Lambda)$ is, unfortunately, much more complicated than $\widetilde{H}^Z_w(\Lambda)$. We shall, however, prove that, for sufficiently large $\Lambda > 0$, we can find it possible to construct a functional integral representation of the heat semi-group generated by $\widetilde{H}^Z_w(K\Lambda)$. We shall also show that the functional integral representation and the unitary transformation give a good way to analyze WCLUV of the Nelson model with nonnegative mass and with a much more general scalar potential. Accomplishing several steps of simple arguments, we reach a desired result for each fixed Z. Once we establish the existence and a concrete form of the strong limit of $e^{-it(H^Z_w(\Lambda)+V_Z\otimes I - ZE_w(\Lambda^\alpha))}$ as $\Lambda \to \infty$ on $\mathcal{L}_Z$, we can easily extend the results to the case of the strong limit of $e^{-it(H_w^Z(\Lambda)+V_Z\otimes I-ZE_w(\Lambda^\alpha))}\Big|_{L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b}$ as $\Lambda\to\infty$ on each sector $L^2_{as}(\mathbb{R}^{3Z})\otimes\mathcal{F}_b$. Hence one can derive the strong limit of $e^{-it\left(H_w(\Lambda)_V-\left(g^2E_w(\Lambda^\alpha)\int\Psi^\dagger(x)\Psi(x)dx\right)\otimes I\right)}$ as $\Lambda\to\infty$ on $\mathcal{L}$.

We organize this paper as follows. In section 2, we present basic notations and facts known in the constructive quantum field theory [3,13 and see references therein]. Section 3 is devoted to constructing a functional integral representation of a heat semi-group whose generator has the form ([7])

$$H_{formal}=\frac{1}{2m}\left(\mathbf{p}_Z\otimes I-A\right)^2+U+V_Z\otimes I+I\otimes H_b^s. \tag{1. 3}$$

Here $A=(A_1,...,A_{3Z})$ and U are symmetric operators in $\mathcal{L}_Z$ holds satisfying some conditions. The operator H_{formal} is an abstract version of $\widetilde{H}_w^Z(K\Lambda)$. Section 4 is the main section of this paper. In this section, we give a definition of the Nelson Hamiltonian in the Schrödinger representation in a rigorous manner and introduce a unitary operator with an infrared cut-off $K>0$. Moreover we analyze WCLUV of the Nelson Hamiltonian by using the functional integral representation established in section 3 and the unitary transformation with the infrared cut-off $K>0$. In section 5, we give a remark on WCLUV of the Nelson model in the space dimensions $d=1,2$.

2 FUNDAMENTAL FACTS

In this section we define basic notations and prepare some concepts on the Schrödinger representation of a symmetric Boson Fock space and a functional integral representation of a heat semi-group. For a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, we denote the scalar product by $<f,g>_{\mathcal{H}}$ and the associated norm by $||f||_{\mathcal{H}}$, where the scalar product is linear in g and antilinear in f. For tempered distributions f and g, a symbol $\bar{f}$ denotes the complex conjugate of f and $\hat{f}$ (resp. $\check{g}$) the Fourier transform of f (resp. the inverse Fourier transform of g). We denote the domain of an operator A by $D(A)$. The notations $C_b^n(\mathbb{R}^d;\mathcal{H}), n=1,2,...$ denote the set of n-times strongly continuously differentiable functions, together with bounded up to n-times derivative, from $\mathbb{R}^d$ to the Hilbert space $\mathcal{H}$ and $C_b(\mathbb{R}^d)$ the set of the bounded continuous functions on $\mathbb{R}^d$. Let $\omega = \omega(k) = \sqrt{k^2+\mu^2}$, $k \in \mathbb{R}^3, \mu \geq 0$, and $\mathcal{S}_r'(\mathbb{R}^m)$ the set of real tempered distributions on $\mathbb{R}^m$. Let us define two Hilbert spaces as follows

$$\begin{aligned} \mathcal{H}_{-\frac{1}{2}} &= \left\{ f \in \mathcal{S}_r'(\mathbb{R}^3) \middle| \|f\|^2_{\mathcal{H}_{-\frac{1}{2}}} \equiv \int_{\mathbb{R}^3} \frac{|\hat{f}(k)|^2}{\omega(k)} dk < \infty \right\}, \\ \mathcal{W} &= \left\{ f \in \mathcal{S}_r'(\mathbb{R}^{3+1}) \middle| \|f\|^2_{\mathcal{W}} \equiv 2\int_{\mathbb{R}^{3+1}} \frac{|\hat{f}(k,k_0)|^2}{\omega(k)^2+k_0^2} dk dk_0 < \infty \right\}. \end{aligned}$$

For simplicity, we put $\|f\|_{\mathcal{H}_{-\frac{1}{2}}} = \|f\|_{-\frac{1}{2}}$. Let $\{\phi(f) | f \in \mathcal{H}_{-\frac{1}{2}}\}$ be the Gaussian mean zero random process indexed by $\mathcal{H}_{-\frac{1}{2}}$ so that the characteristic functions are given by

$$\int_Q e^{i\phi(f)} d\mu = e^{-\frac{1}{4}\|f\|^2_{-\frac{1}{2}}}, \quad f \in \mathcal{H}_{-\frac{1}{2}},$$

where $(Q, d\mu)$ denotes an underlying probability measure space. If we need, we regard ϕ as the variable of the Gaussian random processes $\phi(f)$. Similarly, let $\{\Phi(f) | f \in \mathcal{W}\}$ be the Gaussian mean zero random process indexed by $\mathcal{W}$ on an underlying probability measure space $(Q_E, d\mu_E)$ with the characteristic functions

$$\int_{Q_E} e^{i\Phi(f)} d\mu_E = e^{-\frac{1}{4}\|f\|^2_{\mathcal{W}}}, \quad f \in \mathcal{W}.$$

A Boson Fock space $\mathcal{F}_b$ in the Schrödinger representation and a Euclidean Boson Fock space $\mathcal{E}$ are defined by $\mathcal{F} = L^2(Q, d\mu)$ and $\mathcal{E} = L^2(Q_E, d\mu_E)$,respectively, which are equipped with scalar products

$$\begin{aligned} < \phi(f), \phi(g) >_{\mathcal{F}} &= \frac{1}{2} < f, g >_{-\frac{1}{2}}, \\ < \Phi(f), \Phi(g) >_{\mathcal{E}} &= \frac{1}{2} < f, g >_{\mathcal{W}}, \end{aligned}$$

respectively. " the Wick product" $: \phi(f_1)...\phi(f_n) :$ in $\mathcal{F}_b$ is defined by recurrences as follows:

$$: \phi(f) := \phi(f),$$

$$: \phi(f_1)..\phi(f_n) := \phi(f_1) : \phi(f_2)..\phi(f_n) : - \sum_{j=2}^{n} \langle \phi(f_1), \phi(f_j) \rangle_{\mathcal{F}} : \phi(f_2)..\widehat{\phi(f_j)}..\phi(f_n) :, n \geq 2,$$

where $\widehat{\phi(f)}$ means to the omission of the term $\phi(f)$. Set

$$\begin{aligned}\Gamma_0(\mathcal{F}) &= \mathbb{C},\\ \Gamma_n(\mathcal{F}) &= \mathbf{L}\{: \phi(f_1)...\phi(f_n) : | f_j \in \mathcal{H}_{-\frac{1}{2}}, j = 1, ..., n\}, \quad n \geq 1.\end{aligned}$$

Here $\mathbf{L}\{...\}$ denotes the linear hull of the vectors in $\{...\}$. Then it is well known that $\mathcal{F}$ has the following orthogonal decomposition

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \Gamma_N(\mathcal{F}).$$

"The finite particle subspace" is defined by

$$\mathcal{F}_0 = \bigcup_{N=0}^{\infty} \left[\bigoplus_{n=0}^{N} \Gamma_N(\mathcal{F}) \bigoplus_{n>N+1} \{0\} \right],$$

which is dense in $\mathcal{F}$. Let T be a contraction linear operator from $\mathcal{H}_{-\frac{1}{2}}$ to $\mathcal{W}$ and h a non-negative self-adjoint operator in $\mathcal{H}_{-\frac{1}{2}}$. Then a linear operator $\Gamma(T)$ from $\mathcal{F}$ to $\mathcal{E}$ and a linear operator $d\Gamma(h)$ in $\mathcal{F}$ are defined by

$$\begin{aligned}\Gamma(T)\Omega_{\mathcal{F}} &= \Omega_{\mathcal{E}},\\ \Gamma(T) : \phi(f_1)...\phi(f_n) : &= : \Phi(Tf_1)...\Phi(Tf_n) :, f_1, ..., f_n \in \mathcal{H}_{-\frac{1}{2}}, n \geq 1,\\ d\Gamma(h)\Omega_{\mathcal{F}} &= 0,\\ d\Gamma(h) : \phi(f_1)...\phi(f_n) : &= \sum_{j=1}^{n} : \phi(f_1)...\phi(hf_j)...\phi(f_n) :, f_1, ..., f_n \in D(h), n \geq 1.\end{aligned}$$

Here $\Omega_{\mathcal{F}} \equiv 1 \in \mathcal{F}$, $\Omega_{\mathcal{E}} \equiv 1 \in \mathcal{E}$. It is easily checked that $\Gamma(T)$ can be uniquely extended to a contraction linear operator from $\mathcal{F}$ to $\mathcal{E}$ and we denote its extension by the same symbol. Moreover since it is easily seen that $d\Gamma(h)$ is essentially self-adjoint on $\mathcal{F}_0$, we also denote its self-adjoint extension by the same symbol. We define a non-negative self-adjoint operator $\tilde{\omega}$ in $\mathcal{H}_{-\frac{1}{2}}$ by

$$\widehat{\tilde{\omega} f}(k) = \omega(k)\hat{f}(k), \quad f \in \mathcal{H}_{-\frac{1}{2}},$$

with the maximal domain,i.e., $D(\tilde{\omega}) = \left\{ f \in L^2(\mathbb{R}^3) | \omega \hat{f} \in L^2(\mathbb{R}^3) \right\}$. Let a family of operators $j_t : \mathcal{H}_{-\frac{1}{2}} \to \mathcal{W}, t \geq 0$, be defined by

$$j_t f = \delta_t \otimes f, \quad f \in \mathcal{H}_{-\frac{1}{2}},$$

where δ_t is the delta function with mass at $t \in \mathbb{R}$. It is well known that j_t is isometry. Choosing the special operators j_t and $\tilde{\omega}$, we define a family of isometries J_t and a non-negative self-adjoint operator H_b by

$$\begin{aligned} J_t &= \Gamma(j_t), \quad t \geq 0, \\ H_b &= d\Gamma(\tilde{\omega}). \end{aligned}$$

The operator H_b is called the free Hamiltonian in $\mathcal{F}$. It is also known that J_t is a positivity preserving operator,i.e., for $F \geq 0$,

$$J_t F \geq 0. \tag{2. 1}$$

The following fundamental relation between J_t and H_b is crucial in the next section.

$$J_t^* J_s = e^{-|t-s|H_b}. \tag{2. 2}$$

We introduce basic subspaces in $\mathcal{F}$. We denote by $\mathcal{S}(\mathbb{R}^m)$ the set of rapidly decreasing infinitely many times differentiable functions on $\mathbb{R}^m$. Then we define

$$\begin{aligned} \mathcal{F}^\infty &= \mathbf{L}\left\{ F(\phi(f_1), ..., \phi(f_m)), \Omega_{\mathcal{F}} \middle| F \in \mathcal{S}(\mathbb{R}^m), f_j \in \mathcal{H}_{-\frac{1}{2}}, j = 1, ..., m, m \geq 1 \right\}, \\ \mathcal{F}_+^\infty &= \mathbf{L}\left\{ F \in \mathcal{F}^\infty \left| F(\phi) \geq 0, a.s. \phi \in Q \right. \right\}. \end{aligned}$$

Then $\mathcal{F}^\infty$ is dense in $\mathcal{F}$ and note that, for any $F \in \mathcal{F}$ with $F \geq 0$, there exists $F_n \in \mathcal{F}_+^\infty$ so that $||F - F_n||_{\mathcal{F}} \to 0$ as $n \to \infty$. Hence if $< G, F >_{\mathcal{F}} \geq 0$ for any $F \in \mathcal{F}_+^\infty$, it holds that $G(\phi) \geq 0, a.s. \phi \in Q$. For proceeding our discussion explicitly, we give a unitary equivalence of $\mathcal{F}$ and $\mathcal{F}_b$. Define $\Omega = \{1, 0, 0, 0, ..\}$. Let the annihilation operator and the creation operator denote by $a(f), f \in L^2(\mathbb{R}^3)$ and $a^\dagger(g), g \in L^2(\mathbb{R}^3)$, respectively, which satisfy the following canonical commutation relations on a dense domain:

$$\begin{aligned}[a(f), a^\dagger(g)] &= <\bar{f}, g>_{L^2(\mathbb{R}^3)}, \\ [a^\sharp(f), a^\sharp(g)] &= 0.\end{aligned}$$

Using the formal annihilation and creation operator $a(k)$ and $a^\dagger(k)$, we formally write

$$a^\sharp(f) = \int a^\sharp(k) f(k) dk.$$

For a real-valued function f, we define

$$\phi_F(f) = \frac{1}{\sqrt{2}}\left\{a^\dagger(\frac{\bar{\hat{f}}}{\sqrt{\omega}}) + a(\frac{\hat{f}}{\sqrt{\omega}})\right\}, \quad \frac{\hat{f}}{\sqrt{\omega}} \in L^2(\mathbb{R}^3),$$

$$\pi_F(f) = \frac{i}{\sqrt{2}}\left\{a^\dagger(\sqrt{\omega}\bar{\hat{f}}) - a(\sqrt{\omega}\hat{f})\right\}, \quad \sqrt{\omega}\hat{f} \in L^2(\mathbb{R}^3).$$

Moreover we define, for $f = \Re f + i\Im f$,

$$\begin{aligned}\phi_F(f) &= \phi_F(\Re f) + i\phi_F(\Im f), \\ \pi_F(f) &= \pi_F(\Re f) + i\pi_F(\Im f).\end{aligned}$$

Define a map $\mathcal{T}$ from $\mathcal{F}_b$ to $\mathcal{F}$ by

$$\mathcal{T}\Omega = \Omega_{\mathcal{F}},$$

$$\mathcal{T} : \phi_F(f_1)...\phi_F(f_n) : \Omega =: \phi(f_1)...\phi(f_n) :, f_1, ..., f_n \in \mathcal{H}_{-\frac{1}{2}},$$

and extend $\mathcal{T}$ by linearity. Here "the Wick product" $: \phi_F(f_1)...\phi_F(f_n) :$ in $\mathcal{F}_b$ is defined by removing all the creation operators to the left side and all the annihilation operators to the right side without the commutation relations. One can easily extend $\mathcal{T}$ to the unitary operator from $\mathcal{F}_b$ to $\mathcal{F}$. We denote its extension by the same symbol $\mathcal{T}$. Then it can be easily checked that

$$\mathcal{T}^{-1}\phi(f)\mathcal{T} = \phi_F(f), \quad f \in \mathcal{H}_{-\frac{1}{2}},$$

$$\mathcal{T}^{-1}H_b^s\mathcal{T} = H_b$$

hold as operator equality in $\mathcal{F}_b$. The unitary operator $\mathcal{T}$ implements the following unitary equivalence

$$\mathcal{L}(=\mathcal{F}_a \otimes \mathcal{F}_b) \cong \mathcal{F}_a \otimes \mathcal{F} \cong \bigoplus_{Z=0}^{\infty}[L^2_{as}(\mathbb{R}^{3Z}) \otimes \mathcal{F}] \equiv \bigoplus_{Z=0}^{\infty} \mathcal{H}_Z. \tag{2. 3}$$

In what follows, we use the equivalence (2.3) whenever no confusion is to be feared and we also call $\mathcal{H}_Z$ "sector".

3 FUNCTIONAL INTEGRAL REPRESENTATIONS

In this section, by using methods developed in the constructive quantum field theory ([3,13]) and Hilbert space valued stochastic integral ([7]), we derive a functional integral representation of the heat semi-group generated by formally defined Hamiltonians as (1.1) acting in the Hilbert space $\mathcal{L}_Z$. Throughout this section, we suppose that $A_\mu(x)$'s have forms:

$$A_\mu(x) = P_\mu(\phi(f_{\mu,1}(x)), ..., \phi(f_{\mu,M_\mu}(x))) + A^0_\mu(x) \otimes I, \quad \mu = 1, ..., 3Z, \tag{3. 1}$$

where $P_\mu(y_1, ..., y_{M_\mu})$ is a real polynomial, $f_{\mu,j} \in C^2_b(\mathbb{R}^{3Z}; \mathcal{H}_{-\frac{1}{2}})$, and $A^0_\mu \in C^2_b(\mathbb{R}^{3Z}; \mathbb{R})$. Before constructing a functional integral representation of $e^{-tH_{formal}}$, we remark that there exist some differences between the quantized and the classical cases with respect to the vector potential A_μ. The expression H_{formal} is merely formal. The operator H_{formal} is well defined on the domain $\mathcal{L}^\infty_Z = C^\infty_0(\mathbb{R}^{3Z})\hat{\otimes}[\mathcal{F}_0 \cap D(H_b)]$, but it is not known whether $\mathcal{L}^\infty_Z$ is a core for H_{formal}. Worse still, no concrete cores for $H_{formall}$ are known. It may be possible for $H_{formal}|_{\mathcal{L}^\infty_Z}$ not to have a unique self-adjoint extension. Then we have to make it clear which self-adjoint extensions we choose in consideration. Then we take the following strategy: We construct one self-adjoint extension of $H_{formal}|_{\mathcal{L}^\infty_Z}$ via a strongly continuous symmetric one-parameter contraction semi-group. Concretely, at the beginning, we construct a family of contractive self-adjoint operators $Q_s, s \geq 0$, from which we derive a strongly continuous symmetric one parameter contraction semi-group $G_t, t \geq 0$, by using an $\mathcal{F}$-valued stochastic integral. Secondly,we shall show that its generator $H_{00}(A)$, which is a positive self-adjoint operator, has the same action on the domain $\mathcal{L}_Z^\infty$ as that of

$$\widetilde{H}(A) = \frac{1}{2m}(\mathbf{p}_Z \otimes I - A)^2.$$

Next we define $H_0(A)$ by a quadratic form sum

$$H_0(A) = H_{00}(A) \dot{+} I \otimes H_b. \tag{3. 2}$$

Finally, by using a diamagnetic inequality for $e^{-tH_0(A)}$ ((3.10),[7,9]), we shall show that the following self-adjoint operator can be defined;

$$H(A) = H_0(A) \dot{+} U \dot{+} V_+ \otimes I \dot{-} V_- \otimes I, \tag{3. 3}$$

where U is relatively form bounded with respect to

$$H_F^Z = \frac{1}{2m}\mathbf{p}_Z^2 \otimes I + I \otimes H_b$$

with relative bound < 1 so that $U(\cdot)$ is an $\mathcal{F}$-valued continuous function on $\mathbb{R}^{3Z}$ and $V_+ \in L^1_{loc}(\mathbb{R}^{3Z})$ and V_- is relatively $\mathbf{p}_Z^2$-form bounded. We adopt $H(A)$ defined in (3.3) as a mathematically rigorous definition of the formally defined Hamiltonian H_{formal} and give a functional integral representation of $e^{-tH(A)}$. Note that we do not know whether (3.3) is the Friedrichs extension of H_{formal} with a suitable domain or not.

In what follows, for simplicity, we put the mass of the nonrelativistic particles $m = 1$. For each $x, y \in \mathbb{R}^{3Z}$, $A(x) + A(y)$ is a self-adjoint multiplication operator in $\mathcal{F}$. Hence we can define a unitary operator on $\mathcal{F}$ by

$$U(x,y) = \exp\left\{\frac{i}{2}(A(x) + A(y))(x-y)\right\}.$$

Let $p_s(x)$ be the heat kernel function, i.e.,

$$p_s(x) = (2\pi s)^{-\frac{3}{2}} \exp\left(-\frac{1}{2s}|x|^2\right), \quad s > 0, \quad x \in \mathbb{R}^{3Z}.$$

We define a family of contractive self-adjoint operators $\{Q_s\}_{s\geq 0}$ by the Bochner integral

$$(Q_sF)(x) = \int_{\mathbb{R}^{3Z}} p_s(x-y)U(x,y)F(y)dy,$$
$$(Q_0F)(x) = F(x), \quad F \in \mathcal{L}_Z, \quad x \in \mathbb{R}^{3Z}.$$

Note that Q_sF is weakly right continuous at $s=0$,

$$\lim_{s\downarrow 0} \langle Q_sF, G\rangle_{\mathcal{L}_Z} = \langle F, G\rangle_{\mathcal{L}_Z}, \quad F, G \in \mathcal{L}_Z. \tag{3. 4}$$

Then we show the following key lemma.

Lemma 3.1 *Let $G \in \mathcal{L}_Z$ and $F \in \mathcal{L}_Z^\infty$. Then $\langle Q_sF, G\rangle$ is differentiable at $s>0$ and, in $s=0$, it is right differentiable with*

$$\lim_{s\downarrow 0} \left\langle \frac{Q_s - Q_0}{s} F, G\right\rangle_{\mathcal{L}_Z} = -\left\langle \widetilde{H}(A)F, G\right\rangle_{\mathcal{L}_Z}. \tag{3. 5}$$

Proof: We show an outline of the proof. Put

$$\begin{aligned} A_\mu^{(1)}(x,y) &= \sum_{n=1}^{3Z}\left(\frac{\partial A_n(y)}{\partial y_\mu}(x_n - y_n) - (A_\mu(x) + A_\mu(y))\right), \\ A_\mu^{(2)}(x,y) &= \sum_{n=1}^{3Z}\left(\frac{\partial^2 A_n(y)}{\partial y_\mu^2}(x_n - y_n) - 2\frac{\partial A_\mu(y)}{\partial y_\mu}\right), \mu = 1, ..., 3Z. \end{aligned}$$

By the Fubini theorem one can see that

$$\begin{aligned} \frac{d}{ds}\langle Q_sF, G\rangle_{\mathcal{L}_Z} &= \frac{1}{2}\sum_{\mu=1}^{3Z}\int_{\mathbb{R}^{3Z}\times\mathbb{R}^{3Z}} dxdy p_s(x-y)\left\{\left\langle U(x,y)\frac{\partial^2 F(y)}{\partial y_\mu^2}, G(x)\right\rangle_{\mathcal{F}}\right. \\ &\quad +2\left\langle U(x,y)A_\mu^{(1)}(x,y)\frac{\partial F(y)}{\partial y_\mu}, G(x)\right\rangle_{\mathcal{F}} \\ &\quad \left.+\left\langle U(x,y)\left\{A_\mu^{(1)}(x,y)^2 + A_\mu^{(2)}(x,y)\right\}F(y), G(x)\right\rangle_{\mathcal{F}}\right\} \\ &\equiv \int_{\mathbb{R}^{3Z}} p_s(X)dX \int_{\mathbb{R}^{3Z}} dx\Gamma(x, x-X). \end{aligned}$$

Applying the following well known inequality

$$||\phi(f)\Psi||_{\mathcal{F}} \leq \sqrt{2}\sqrt{N+1}\,||f||_{-\frac{1}{2}}\,||\Psi||_{\mathcal{F}}, \quad \Psi \in \Gamma_N(\mathcal{F}),$$

and noting (3.1), by the direct calculation, we show that

$$\left|\int_{\mathbb{R}^{3Z}} dx\Gamma(x, x-X)\right| \leq \epsilon_1 + \epsilon_2|X| + \epsilon_3|X|^2.$$

Here $\epsilon_1, \epsilon_2, \epsilon_3$ are positive constants. Hence we have

$$\begin{aligned}\lim_{s\to 0}\int_{\mathbb{R}^{3Z}} p_s(X)dX\int_{\mathbb{R}^{3Z}}\Gamma(x, x-X)dx &= \int_{\mathbb{R}^{3Z}}\Gamma(x,x)dx \\ &= -\left\langle \widetilde{H}(A)F, G\right\rangle_{\mathcal{L}_Z}.\end{aligned}$$

Then we have

$$\lim_{s\downarrow 0}\frac{d}{ds}\left\langle Q_s F, G\right\rangle_{\mathcal{L}_Z} = -\left\langle \widetilde{H}(A)F, G\right\rangle_{\mathcal{L}_Z},$$

which,together with (3.4), implies (3.5). The proof is complete. □

We fix probabilistic notations. Let (Ω, db) be a probability space for the $3Z$-dimensional Brownian motion $(b(t))_{t\geq 0} = (b_\mu(t))_{t\geq 0, \mu=1,\dots,3Z}$. Put $x_\mu + b_\mu(t) = \omega_\mu(t), \mu = 1, \dots, 3Z$ and $\omega(t) = (\omega_1(t), \dots, \omega_{3Z}(t))$. Unless any confusions arise, we write the variable of $b(t)$ by $b \in \Omega$. We define two measure spaces $(\widetilde{Q}, d\nu)$ and $(\widetilde{Q}_E, d\nu_E)$ by

$$\mathbb{R}^{3Z}\times\Omega\times Q = \widetilde{Q}, \quad dx\otimes db\otimes d\mu = d\nu,$$

$$\mathbb{R}^{3Z}\times\Omega\times Q_E = \widetilde{Q}_E, \quad dx\otimes db\otimes d\mu_E = d\nu_E.$$

Lemma 3.2 *For all $t \geq 0$, the strong limit*

$$s-\lim_{n\to\infty}\left(Q_{\frac{t}{2^n}}\right)^{2^n} \equiv G_t$$

exists. Moreover G_t is a strongly continuous symmetric one-parameter contraction semi-group on $\mathcal{L}_Z$ and has the following functional integral representation

$$\langle F, G_t G\rangle_{\mathcal{F}} = \int_{\widetilde{Q}} d\nu\overline{F(\omega(t))}G(\omega(0))e^{i\widetilde{K}(A,t)}, \quad F, G \in \mathcal{L}_Z, \tag{3. 6}$$

where

$$\widetilde{K}(A,t)=\sum_{\mu=1}^{3Z}\int_0^t A_\mu(\omega(s))db_\mu(s)+\frac{1}{2}\sum_{\mu=1}^{3Z}\int_0^t\frac{\partial A_\mu(\omega(s))}{\partial x_\mu}ds\equiv\int_0^t A(\omega(s))\circ db_(s),$$

where $\int \ldots db_\mu(s)$ *is an* $\mathcal{F}$*-valued stochastic integral defined as a vector in* $L^2(\Omega;\mathcal{F})$*, the set of* $\mathcal{F}$*-valued* L^2*-functions on* Ω*.*

Proof: By the definition of Q_t, one can see that

$$\left\langle F,\left(Q_{\frac{t}{2^n}}\right)^{2^n}\left(Q_{\frac{s}{2^m}}\right)^{2^m}G\right\rangle_{\mathcal{L}_Z}=\int_{\mathbb{R}^{3Z}}dx\left\langle F(\omega(t+s)),e^{i\widetilde{K}_{m,n}(A,t,s)}G(\omega(0))\right\rangle_{L^2(\Omega;\mathcal{F})},$$

where

$$\begin{aligned}2\widetilde{K}_{m,n}(A,t,s)\;=\;&\sum_{k=1}^{2^m}\left\{A\left(\omega\left(\frac{sk}{2^m}\right)\right)+A\left(\omega\left(\frac{s\,(k-1)}{2^m}\right)\right)\right\}\\&\times\left\{\omega\left(\frac{sk}{2^m}\right)-\omega\left(\frac{s\,(k-1)}{2^m}\right)\right\}\\&+\sum_{k=1}^{2^n}\left\{A\left(\omega\left(\frac{tk}{2^n}+s\right)\right)+A\left(\omega\left(\frac{t\,(k-1)}{2^n}+s\right)\right)\right\}\\&\times\left\{\omega\left(\frac{tk}{2^n}+s\right)-\omega\left(\frac{t\,(k-1)}{2^n}+s\right)\right\}.\end{aligned}$$

Since, by (3.1), $A_\mu\in C_b^1(\mathbb{R}^{3Z};\mathcal{F}),\mu=1,...,3Z$, it is easily seen that in $L^2(\Omega;\mathcal{F})$,

$$s-\lim_{m\to\infty}\lim_{n\to\infty}\widetilde{K}_{m,n}(A,t,s)=\widetilde{K}(A,t+s).$$

Then we have, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty}\lim_{n\to\infty}\left\langle F,\left(Q_{\frac{t}{2^n}}\right)^{2^n}\left(Q_{\frac{s}{2^m}}\right)^{2^m}G\right\rangle_{\mathcal{L}_Z}=\int_{\widetilde{Q}}d\nu\overline{F(\omega(t+s)}G(\omega(0))e^{i\widetilde{K}(A,t+s)}.$$

Hence it follows that $\left(Q_{\frac{t}{2^n}}\right)^{2^n}$ is a Cauchy sequence in $\mathcal{F}$ and $s-\lim_{n\to\infty}\left(Q_{\frac{t}{2^n}}\right)^{2^n}$ has the functional integral representation as in (3.6). The semi-group property and the strong continuity of G_t in t can be easily checked by (3.6). Thus the proof is complete. □

Lemma 3.2 and Stone's theorem yield that there exists a positive self-adjoint operator $H_{00}(A)$ so that

$$G_t=e^{-tH_{00}(A)}.$$

Lemma 3.3 *The self-adjoint operator $H_{00}(A)$ is a self-adjoint extension of $\widetilde{H}(A)|_{\mathcal{L}_Z^\infty}$.*

Proof: Let $F \in D(H_{00}(A))$ and $G \in \mathcal{L}_Z^\infty$. Then we have, by Lemma 3.1,

$$\begin{aligned}
\left\langle \frac{1}{t}\left(e^{-tH_{00}(A)} - I\right) G, F \right\rangle_{\mathcal{L}_Z} &= \lim_{n\to\infty} \left\langle \frac{1}{t}\left\{\left(Q_{\frac{t}{2^n}}\right)^{2^n} - I\right\} G, F \right\rangle_{\mathcal{L}_Z} \\
&= \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{1}{2^n} \left\langle \frac{2^n}{t} \left\{Q_{\frac{t}{2^n}} - I\right\} G, \left(Q_{\frac{t}{2^n}}\right)^j F \right\rangle_{\mathcal{L}_Z} \\
&= -\int_0^t \left\langle \widetilde{H}(A) G, e^{-tsH_{00}(A)} F \right\rangle_{\mathcal{L}_Z} ds.
\end{aligned}$$

As $t \to 0$ on the both sides, we get

$$\langle G, H_{00}(A) F\rangle_{\mathcal{L}_Z} = \left\langle \widetilde{H}(A) G, F \right\rangle_{\mathcal{L}_Z}.$$

Thus the proof is complete. □

Remark 3.4 *Since the operator H_{formal} may be defined on a larger domain than $\mathcal{L}_Z^\infty$, one can extend Lemma 3.3 to a larger domain. Actually, putting*

$$\begin{aligned}
&\mathcal{M}_Z^\infty = H^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b)], \\
&H^2 = \{u \in C_b^1(\mathbb{R}^{3Z}; \mathbb{R}) \,\big|\, ||\partial^k u||_{L^2(\mathbb{R}^{3Z})} < \infty, |k| \le 2, \},
\end{aligned}$$

we also show that Lemma 3.1 and Lemma 3.3 hold with $\mathcal{L}_Z^\infty$ replaced by $\mathcal{M}_Z^\infty$ ([7]).

Define $H_0(A)$ as (3.1) and put $H_0(A) = H_0$ for simplicity. We state the main theorem in this section.

Theorem 3.5 *Let $A_\mu(x) \in D(H_b), \mu = 1, .., 3Z$, for all $x \in \mathbb{R}^{3Z}$, and*

$$\sup_{x\in\mathbb{R}^{3Z}} ||H_b A_\mu(x)||_{\mathcal{F}} < \infty. \tag{3. 7}$$

Furthermore we suppose that a multiplication operator U is a bounded operator on $\mathcal{L}_Z$ and $U(\cdot)$ is an $\mathcal{F}$-valued continuous function on $\mathbb{R}^{3Z}$. Then, for $F, G \in \mathcal{L}_Z$,

$$\left\langle F, e^{-t(H_0+U)} G \right\rangle_{\mathcal{L}_Z} = \int_{\widetilde{Q}_E} d\nu_E \overline{J_t F(\omega(t))} J_0 G(\omega(0)) e^{i\mathcal{K}(A,t)} e^{-E(U,t)}. \tag{3. 8}$$

where

$$\begin{aligned}
\mathcal{K}(A,t) &= \sum_{\mu=1}^{3Z}\int_0^t J_s A_\mu(\omega(s))db_\mu(s) + \sum_{\mu=1}^{3Z}\frac{1}{2}\int_0^t J_s \frac{\partial A_\mu(\omega(s))}{\partial x_\mu} ds \equiv \int_0^t J_s A(\omega(s)) \circ db(s), \\
E(U,t) &= \int_0^t J_s U(\omega(s))ds,
\end{aligned}$$

where $\int_0^t ...db_\mu(s)$ *is an* $\mathcal{E}$*-valued stochastic integral defined as a vector in* $L^2(\Omega;\mathcal{E})$*, the set of* $\mathcal{E}$*-valued* L^2*-functions on* Ω*.*

Note that (3.7) implies that $J_s A_\mu(x) \in C_b^1(\mathbb{R}\times\mathbb{R}^{3Z};\mathcal{E})$ as the variable $(s,x)\in\mathbb{R}\times\mathbb{R}^{3Z}$. Then $\mathcal{K}(A,t)$ is well defined as vectors in $L^2(\Omega;\mathcal{E})$.

Proof: By the strong Trotter product formula ([10]) we see that

$$\begin{aligned}
\left\langle F, e^{-t(H_0+U)}G\right\rangle_{\mathcal{L}_Z} &= \lim_{n\to\infty}\left\langle F, \left(e^{-\frac{t}{2^n}H_{00}(A)}e^{-\frac{t}{2^n}I\otimes H_b}e^{-\frac{t}{2^n}U}\right)^{2^n} G\right\rangle_{\mathcal{L}_Z} \\
&\equiv \lim_{n\to\infty} S_{2^n}.
\end{aligned}$$

Let $\frac{t}{2^n}=\epsilon$ for simplicity for the time being. From the definition of $H_{00}(A)$ and (2.2), it holds that

$$\begin{aligned}
S_{2^n} &= \lim_{k\to\infty}\Big\langle F, J_t^*\left(J_t\left(Q_{\frac{\epsilon}{2^k}}\right)^{2^k} J_t^*\right)\left(J_{t-\epsilon}e^{-\epsilon U}J_{t-\epsilon}^*\right)\left(J_{t-\epsilon}\left(Q_{\frac{\epsilon}{2^k}}\right)^{2^k} J_{t-\epsilon}^*\right)\left(J_{t-2\epsilon}e^{-\epsilon U}J_{t-2\epsilon}^*\right)\cdots \\
&\qquad \cdots\left(J_\epsilon e^{-\epsilon U}J_\epsilon^*\right)\left(J_\epsilon\left(Q_{\frac{\epsilon}{2^k}}\right)^{2^k} J_\epsilon^*\right)\left(J_0 e^{-\epsilon U}J_0^*\right)J_0 G\Big\rangle_{\mathcal{L}_Z} \\
&= \lim_{k\to\infty}\int_{\mathbb{R}^{3Z}\times\underbrace{\mathbb{R}^{3Z2^k}\times\ldots\times\mathbb{R}^{3Z2^k}}_{2^n}} dx d\vec{x}_1\ldots d\vec{x}_{2^n} P_\epsilon(\vec{x}_1)\ldots P_\epsilon(\vec{x}_{2^n}) \\
&\qquad \times\Big\langle F(x), J_t^*\left(E_t W_t(\vec{x}_1)E_t\right)\left(E_{t-\epsilon}U_{t-\epsilon}(x_1^{2^k})E_{t-\epsilon}\right)\left(E_{t-\epsilon}W_{t-\epsilon}(\vec{x}_2)E_{t-\epsilon}\right)\cdots \\
&\qquad \cdots\left(E_\epsilon U_\epsilon(x_{2^n-1}^{2^k})E_\epsilon\right)\left(E_\epsilon W_\epsilon(\vec{x}_{2^n})E_\epsilon\right)\left(E_0 U_0(x_{2^n}^{2^k})E_0\right)J_0 G(x_{2^n}^{2^k})\Big\rangle_{\mathcal{F}} \\
&\equiv \lim_{k\to\infty} S_{2^n,2^k},
\end{aligned} \tag{3. 9}$$

where $J_s J_s^* = E_s$ and $\vec{x}_j = (x_j^1,\ldots,x_j^{2^k})\in\mathbb{R}^{3Z2^k}, j=1,\ldots,2^n$,

$$P_\epsilon(\vec{x}_j) = p_\epsilon(x_{j-1}^{2^k}-x_j^1)p_\epsilon(x_j^1-x_j^2)\ldots p_\epsilon(x_j^{2^k-1}-x_j^{2^k}),$$

$$W_s(\vec{x_j}) = \exp\left\{\frac{i}{2}J_s\sum_{l=1}^{2^k}\left(A(x_j^{l-1}) + A(x_j^l)\right)(x_j^{l-1} - x_j^l)\right\}, \qquad x_j^0 = x_{j-1}^{2^k},\ x_0^{2^k} \equiv x,$$
$$U_s(x_j^i) = \exp\left(-\epsilon J_s U(x_j^i)\right).$$

By the Markov property of E_s's known in the theory of the constructive quantum field theory [13], one can neglect E_s's in (3.9). Then it holds that

$$S_{2^n,2^k} = \int_{\mathbb{R}^{3Z}} dx \left\langle J_t F(\omega(t)), e^{\left\{i\sum_{j=0}^{2^n-1} J_{\frac{jt}{2^n}}\mathcal{K}_k(\frac{jt}{2^n}) - \frac{t}{2^n}\sum_{j=1}^{2^n} J_{\frac{jt}{2^n}}U(\omega(\frac{jt}{2^n}))\right\}} J_0 G(x)\right\rangle_{L^2(\Omega;\mathcal{E})},$$

where

$$\begin{aligned}\mathcal{K}_k\left(\frac{jt}{2^n}\right) &= \frac{1}{2}\sum_{m=1}^{2^k}\left(A\left(\omega\left(\frac{m}{2^k}\frac{t}{2^n}+\frac{jt}{2^n}\right)\right) + A\left(\omega\left(\frac{(m-1)}{2^k}\frac{t}{2^n}+\frac{jt}{2^n}\right)\right)\right)\\ &\quad\times\left(\omega\left(\frac{m}{2^k}\frac{t}{2^n}+\frac{jt}{2^n}\right) - \omega\left(\frac{(m-1)}{2^k}\frac{t}{2^n}+\frac{jt}{2^n}\right)\right), \quad j = 0,\dots,2^n-1.\end{aligned}$$

As is seen in the proof of Lemma 3.2, since $A \in C_b^2(\mathbb{R}^{3Z},\mathcal{F})$, we have

$$\lim_{k\to\infty} S_{2^n,2^k} = \int_{\mathbb{R}^{3Z}} dx \left\langle J_t F(\omega(t)), e^{\left\{i\sum_{j=0}^{2^n-1} \mathcal{K}(\frac{jt}{2^n}) - \frac{t}{2^n}\sum_{j=1}^{2^n} J_{\frac{jt}{2^n}}U(\omega(\frac{jt}{2^n}))\right\}} J_0 G(x)\right\rangle_{L^2(\Omega;\mathcal{E})},$$

where

$$\mathcal{K}\left(\frac{jt}{2^n}\right) = \int_{\frac{jt}{2^n}}^{\frac{(j+1)t}{2^n}} J_{\frac{jt}{2^n}} A(\omega(s)) \circ db(s).$$

Since $J_s A_\mu(x) \in C_b^1(\mathbb{R}\times\mathbb{R}^{3Z};\mathcal{E})$, one can easily see that

$$s-\lim_{n\to\infty}\sum_{j=0}^{2^n-1}\mathcal{K}\left(\frac{jt}{2^n}\right) = \int_0^t J_s A(\omega(s)) \circ db(s)$$

in $L^2(\Omega;\mathcal{E})$. Since $J_s U(x)$ is $\mathcal{E}$-valued continuous function on $\mathbb{R}\times\mathbb{R}^{3Z}$, we have

$$\lim_{n\to\infty}\frac{t}{2^n}\sum_{j=0}^{2^n-1} J_{\frac{jt}{2^n}} U\left(\omega\left(\frac{jt}{2^n}\right)\right) = \int_0^t J_s U(\omega(s))ds,$$

in $L^2(\Omega;\mathcal{E})$. Hence, again by the Lebesgue dominated convergence theorem, we get the desired result. $\square$

Remark 3.6 *(1) For any particle's mass $m > 0$, the functional integral representation in Theorem 3.5 is replaced by*

$$\left\langle F, e^{-t(H_0+U)}G\right\rangle_{\mathcal{L}_Z} = \int_{\widetilde{Q}_E} d\nu_E \overline{J_t F(\omega(t/m))} J_0 G(\omega(0)) e^{i\mathcal{K}_m(A,t)} e^{-E_m(U,t)}.$$

where

$$\mathcal{K}_m(A,t) = \int_0^{\frac{t}{m}} J_{ms}A(\omega(s)) \circ db(s), \quad E_m(U,t) = E(U,t).$$

(2) Since it is enough to assume that $J_s A_\mu(x) \in C_b^1(\mathbb{R} \times \mathbb{R}^{3Z}; \mathcal{E}), \mu = 1, ..., 3Z$, to define the $\mathcal{E}$-valued stochastic integral $\int_0^t J_s A(\omega(s)) \circ db(s)$, for example, under the notations in (3.1), the following is a sufficient condition to define the right hand side of (3.8):

$$f_{\mu,j}(\cdot) \in C_b^1(\mathbb{R}^{3Z}; \mathcal{H}_{-\frac{1}{2}}), \quad f_{\mu,j}(x) \in D(\tilde{\omega}), \quad \sup_{x \in \mathbb{R}^{3Z}} \|\tilde{\omega} f_{\mu,j}(x)\|_{-\frac{1}{2}} < \infty.$$

However, it is difficult to prove (3.8) for such A's. One of the reason is that, since we do not know a concrete core for H_{formal}, we can not use a standard limiting argument. The following is a sufficient condition for (3.8)to hold

$$f_{\mu,j}(\cdot) \in C_b^2(\mathbb{R}^{3Z}; \mathcal{H}_{-\frac{1}{2}}), \quad f_{\mu,j}(x) \in D(\tilde{\omega}), \quad \sup_{x \in \mathbb{R}^{3Z}} \|\tilde{\omega} f_{\mu,j}(x)\|_{-\frac{1}{2}} < \infty.$$

We consider an extension of Theorem 3.5 to a much more general multiplication operator U. From Theorem 3.5 and (2.1), the following diamagnetic inequality immediately follows

$$\left|\left\langle F, e^{-t(H_0+U)}G\right\rangle_{\mathcal{L}_Z}\right| \le \left\langle |F|, e^{-t(H_F^Z+U)}|G|\right\rangle_{\mathcal{L}_Z}. \tag{3. 10}$$

Let us define for $G \in \mathcal{L}_Z$,

$$\mathrm{sgn}G(x) = \begin{cases} \frac{G(x)}{|G(x)|}, & |G(x)| \neq 0, \\ 0, & otherwise. \end{cases}$$

Lemma 3.7 *Let $|U|$ be a multiplication operator which is H_F^Z-form bounded with relative bound ϵ. Then $|U|$ is H_0-form bounded with relative bound $\le \epsilon$.*

Proof: Put $F = \mathrm{sgn}\left(e^{-tH_0}(\psi \otimes \Psi)\right)$, where $\psi \in C_0^\infty(\mathbb{R}^{3Z})$ with $\psi \geq 0$ and $\Psi \in \mathcal{F}_+^\infty$. By putting such F and $U = 0$ in (3.10), we have

$$\left\langle \psi \otimes \Psi, |e^{-tH_0}G| \right\rangle_{\mathcal{L}_Z} \leq \left\langle \psi \otimes \Psi, e^{-tH_F^Z}|G| \right\rangle_{\mathcal{L}_Z}.$$

Hence it follows that for a.e. $(x, \phi) \in \mathbb{R}^{3Z} \times Q$,

$$\left| e^{-tH_0}G \right|(x, \phi) \leq \left(e^{-tH_F^Z}|G| \right)(x, \phi).$$

Then a fundamental calculation shows that for $E > 0$

$$\left|(H_0 + E)^{-\frac{1}{2}} G\right|(x, \phi) \leq \left(\left(H_F^Z + E\right)^{-\frac{1}{2}} |G|\right)(x, \phi), \quad a.e. (x, \phi) \in \mathbb{R}^{3Z} \times Q.$$

Hence we have

$$\sup_{\|G\|_{\mathcal{L}_Z}=1} \left\| |U|^{\frac{1}{2}}(H_0 + E)^{-\frac{1}{2}}G \right\|_{\mathcal{L}_Z} \leq \sup_{\|G\|_{\mathcal{L}_Z}=1} \left\| |U|^{\frac{1}{2}} \left(H_F^Z + E\right)^{-\frac{1}{2}} G \right\|_{\mathcal{L}_Z} < \infty.$$

Thus the lemma follows. □

We define a class of multiplication operators in $L^2(\mathbb{R}^{3Z})$.

Definition 3.8 *We say that a multiplication operator $W \in \mathcal{M}_\pm(Z)$, $Z \geq 1$, if, and only if, with $W = W_+ - W_-$ (W_+ is the nonnegative part of W and $-W_-$ is the negative part of W), W_- is relatively form bounded with respect to p_Z^2 with relative bound < 1 and $W_+ \in L^1_{loc}(\mathbb{R}^{3Z})$. Moreover we say that $V_Z \in \mathcal{M}_\pm(Z)_r$ if, and only if, $V_Z \in \mathcal{M}_\pm(Z)$ and V_{Z+} and V_{Z-} are reduced by the closed subspace $L^2_{as}(\mathbb{R}^{3Z})$ in $L^2(\mathbb{R}^{3Z})$.*

Theorem 3.9 *Let U be an H_F^Z-form bounded multiplication operator with relative bound < 1 so that $U(\cdot)$ is an $\mathcal{F}$-valued continuous function on $\mathbb{R}^{3Z}$, and $V \in \mathcal{M}_\pm(Z)$. Then Theorem 3.5 holds with $H_0 + U$ replaced by $H_0 \dot{+} U \dot{+} V_+ \otimes I \dot{-} V_- \otimes I$.*

Proof: We show an outline of the proof. We assume, at first, $V \in L^\infty(\mathbb{R}^{3Z})$. Then, by virtue of Lemma 3.7, $H_0 \dot{+} U + V \otimes I$ is a well defined self-adjoint operator (KLMN Theorem [12]).

The strong Trotter product formula for forms ([10]) and a limiting argument with respect to V yield (3.8) with $H_0 + U$ replaced by $H_0 \dot{+} U + V \otimes I$ in the same way as the proof of Theorem 3.5. Next, for any $V \in \mathcal{M}_\pm(Z)$, defining

$$V_{+n} = \begin{cases} V_+, & V_+ < n, \\ n, & otherwise, \end{cases} \quad V_{-m} \begin{cases} V_-, & V_- < m, \\ m, & otherwise, \end{cases}$$

as is mentioned above, one can see that (3.8) holds with $H_0 + U$ replaced by $H_0 \dot{+} U + V_{+n} \otimes I - V_{-m} \otimes I$. By using convergence theorems for forms as $n, m \to \infty$ ([7.Theorem 4.13, 14.Theorem 6.2]), we get the desired result. □

We conclude this section with giving a typical example of functional integral representations. Let $f = (f_1, ..., f_{3Z})$ satisfy

$$\frac{f_\mu}{\sqrt{\omega}} \in L^2(\mathbb{R}^3), \quad f_\mu(k) = f_\mu(-k), \quad \mu = 1, ..., 3Z. \tag{3. 11}$$

We introduce a notation $\tilde{f}_\mu(x)$ for $x = (x^1, ..., x^N) \in \mathbb{R}^{3Z}$ as follows;

$$\tilde{f}_\mu(x) = \frac{1}{\sqrt{(2\pi)^3}} \left(f_\mu(k) e^{-ikx^{[\frac{\mu}{3}+1]}} \right)^\vee, \mu = 1, ..., 3Z,$$

where $[s]$ is the integer part of s, the inverse Fourier transformation $\vee$ is taken with respect to the variable $k \in \mathbb{R}^3$. Because of (3.11), $\tilde{f}_\mu(x)$ is a real valued function for each $x \in \mathbb{R}^{3Z}$. Then it is rather obvious that, for each $x \in \mathbb{R}^{3Z}$, $\tilde{f}_\mu(x) \in \mathcal{H}_{-\frac{1}{2}}$. Thus multiplication operators

$$A_\mu(f) = \int_{\mathbb{R}^{3Z}}^{\oplus} \phi(\tilde{f}_\mu(x)) dx \equiv \int_{\mathbb{R}^{3Z}}^{\oplus} A_\mu(f(x)) dx,$$

can be well defined in $\mathcal{L}_Z$. Put $A(f) = (A_1(f), ..., A_{3Z}(f))$. Let $\hat{\varrho} = (\hat{\varrho}_1, ..., \hat{\varrho}_{3Z})$, $\hat{\tau} = (\hat{\tau}_1, ..., \hat{\tau}_{3Z})$ and $\hat{\eta} = (\hat{\eta}_1, ..., \hat{\eta}_{3Z})$ satisfy that $\hat{\varrho}_\mu(k) = \hat{\varrho}_\mu(-k), \hat{\tau}_\mu(k) = \hat{\tau}_\mu(-k)$ and $\hat{\eta}_\mu(k) = \hat{\eta}_\mu(-k)$, and

$$\frac{\hat{\varrho}_\mu}{(\sqrt{\omega})^n}, \frac{\hat{\eta}_\mu}{(\sqrt{\omega})^m}, \frac{\hat{\tau}_\mu}{(\sqrt{\omega})^l} \in L^2(\mathbb{R}^3), \quad n, m = -1, 0, 1, 2, \ l = 1, 2, \ \mu = 1, ..., 3Z. \tag{3. 12}$$

We fix $\hat{\varrho}, \hat{\tau}$ and $\hat{\eta}$ in what follows. Define

$$H_{\hat{\varrho},\hat{\tau},\hat{\eta}} = \frac{1}{2m}\left(\mathrm{p}_Z \otimes I - \gamma A(\hat{\varrho})\right)^2 + \alpha \sum_{\mu=1}^{3Z} A_\mu(\hat{\tau}) + \beta \sum_{\mu=1}^{3Z} A_\mu(\hat{\eta})^2 + I \otimes H_b,$$

where α, β and γ are real constants.

Theorem 3.10 *Suppose that the absolute values of coupling constants γ and β are sufficiently small. Then $H_{\hat{\varrho},\hat{\tau},\hat{\eta}}$ is self-adjoint on $D(H_F^Z)$ and essentially self-adjoint on any core for H_F^Z. In particular, $\mathcal{L}_Z^\infty$ is a core for $H_{\hat{\varrho},\hat{\tau},\hat{\eta}}$. Moreover we have, for $V \in \mathcal{M}_\pm(Z)$,*

$$\left\langle F, e^{-t(H_{\hat{\varrho},\hat{\tau},\hat{\eta}} \dot{+} V_+ \otimes I \dot{-} V_- \otimes I)} G \right\rangle_{\mathcal{L}_Z} = \int_{\tilde{Q}_E} d\nu_E \overline{J_t F(\omega(t/m))} J_0 G(\omega(0)) e^{\mathcal{K}_{\hat{\varrho},\hat{\tau},\hat{\eta}}(t)} e^{-E_V(t)\otimes I}. \quad (3.\ 13)$$

where

$$\begin{aligned}\mathcal{K}_{\hat{\varrho},\hat{\tau},\hat{\eta}}(t) = \sum_{\mu=1}^{3Z} \Bigg\{ & i\gamma\Phi\left(\int_0^{t/m} j_{ms}\tilde{\hat{\varrho}}_\mu(\omega(s)) \circ db_\mu(s)\right) \\ & -\alpha\Phi\left(\int_0^t j_s \tilde{\hat{\tau}}_\mu(\omega(s))ds\right) - \beta\int_0^t J_s\phi\left(\tilde{\hat{\eta}}_\mu(\omega(s))\right)^2 ds\Bigg\}, \\ E_V(t) = \int_0^t V(\omega(s))ds. &\end{aligned}$$

Proof: For simplicity, we put $||\cdot||_{L^2(\mathbb{R}^3)} = ||\cdot||_2$ and $H_{\hat{\varrho},\hat{\tau},\hat{\eta}} = H_F^Z + H_R + \alpha H_I + \beta H_{II}$ in this proof. Suppose that $V \in C_b(\mathbb{R}^{3Z})$. Note that

$$\begin{aligned} ||\phi(f)\Psi||_{\mathcal{F}} &\le \frac{1}{\sqrt{2}}\left(2\left\|\frac{\hat{f}}{\omega}\right\|_2 \left\|H_b^{\frac{1}{2}}\Psi\right\|_{\mathcal{F}} + \left\|\frac{\hat{f}}{\sqrt{\omega}}\right\|_2 ||\Psi||_{\mathcal{F}}\right), \quad \Psi \in D(H_b^{\frac{1}{2}}), \\ \left\|\left[(H_b+I)^{\frac{1}{2}}, \phi(f)\right]\Psi\right\|_{\mathcal{F}} &\le \frac{1}{\sqrt{2}}\left(\frac{1}{2\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(\lambda+1)^2}d\lambda\right) \\ &\quad \times \left(\left\|\hat{f}\right\|_2 \left\|H_b^{\frac{1}{2}}\Psi\right\|_{\mathcal{F}} + \left\|\sqrt{\omega}\hat{f}\right\|_2 ||\Psi||_{\mathcal{F}}\right), \quad \Psi \in D(H_b). \end{aligned}$$

By assumptions $\frac{\hat{\tau}_\mu}{\omega}, \frac{\hat{\tau}_\mu}{\sqrt{\omega}} \in L^2(\mathbb{R}^3)$, one can see that H_I is infinitesimally small with respect to $I \otimes H_b$. Moreover we see that, by the above inequality, for $\Psi \in D(H_F^Z)$,

$$||H_R\Psi||_{\mathcal{L}_Z} \le \sum_{\mu=1}^{3Z}\left\{\gamma^2\left(\left\|\frac{\hat{\varrho}_\mu}{\omega}\right\|_2 \sum_{n=-1}^{2}\left\|\frac{\hat{\varrho}_\mu}{(\sqrt{\omega})^n}\right\|_2 + \left\|\frac{\hat{\varrho}_\mu}{\sqrt{\omega}}\right\|_2 \sum_{n=1}^{2}\left\|\frac{\hat{\varrho}_\mu}{(\sqrt{\omega})^n}\right\|_2\right)\right.$$

$$
\begin{aligned}
&\qquad +|\gamma| \sum_{n=-1}^{2} \left\| \frac{\hat{\varrho}_\mu}{(\sqrt{\omega})^n} \right\|_2 \Bigg\} \left\| (H_F^Z + I)\Psi \right\|_{\mathcal{L}_Z} \times const., \\
||H_{II}\Psi||_{\mathcal{L}_Z} &\leq \sum_{\mu=1}^{3Z} |\beta| \left(\left\| \frac{\hat{\eta}_\mu}{\omega} \right\|_2 \sum_{n=-1}^{2} \left\| \frac{\hat{\eta}_\mu}{(\sqrt{\omega})^n} \right\|_2 + \left\| \frac{\hat{\eta}_\mu}{\sqrt{\omega}} \right\|_2 \sum_{n=1}^{2} \left\| \frac{\hat{\eta}_\mu}{(\sqrt{\omega})^n} \right\|_2 \right) \\
&\qquad \times ||(H_F^Z + I)\Psi||_{\mathcal{L}_Z} \times const..
\end{aligned}
$$

As a result, since γ and β are sufficiently small, it holds that $H_R + \alpha H_I + \beta H_{II}$ is relatively bounded with respect to H_F^Z with relative bound < 1. Hence, the Kato-Rellich theorem ([12]) yields that $H_{\hat{\varrho},\hat{\tau},\hat{\eta}}$ is self-adjoint on $D(H_F^Z)$ and essentially self-adjoint on any core for H_F^Z. Next we prove (3.13). In addition to (3.12), we suppose that

$$\omega\sqrt{\omega}\hat{\varrho}_j \in L^2(\mathbb{R}^3).$$

Then $\frac{\hat{\varrho}_j}{\sqrt{\omega}}, \sqrt{\omega}\hat{\varrho}_j, \omega\sqrt{\omega}\hat{\varrho}_j \in L^2(\mathbb{R}^3)$ implies that $A_\mu(\hat{\varrho}) \in C_b^2(\mathbb{R}^{3Z}, \mathcal{F})$, and $A_\mu(\hat{\varrho}(x)) \in D(H_b)$ for $x \in \mathbb{R}^{3Z}$ with

$$\sup_{x\in\mathbb{R}^{3Z}} ||H_b A_\mu(\hat{\varrho}(x))||_{\mathcal{F}} < \infty, \quad \mu = 1, 2, ..., 3Z.$$

Hence, noting that H_I and H_{II} are regarded as $\mathcal{F}$-valued continuous function on $\mathbb{R}^{3Z}$, by Theorem 3.5, one can see that the right hand side of (3.13) is surely

$$\left\langle F, e^{-t\left(H_0(\gamma A(\hat{\varrho})) \dot{+} (\alpha H_I + \beta H_{II}) + V\otimes I\right)} G \right\rangle_{\mathcal{L}_Z}.$$

Since $\mathcal{L}_Z^\infty$ is a core for $H_{\hat{\varrho},\hat{\tau},\hat{\eta}}$, by Lemma 3.3, we can see that

$$H_0(\gamma A(\hat{\varrho})) \dot{+} (\alpha H_I + \beta H_{II}) + V \otimes I = H_{\hat{\varrho},\hat{\tau},\hat{\eta}} + V \otimes I$$

as self-adjoint operators in $\mathcal{L}_Z$. Hence (3.13) follows for such $\hat{\varrho}$'s. Next we suppose that $\hat{\varrho}$ satisfies (3.12). Then the right hand side of (3.13) is well defined for such $\hat{\varrho}$'s (see Remark 3.6 (2)). One can easily find sequences $\hat{\varrho}(n) = (\hat{\varrho}(n)_1, ..., \hat{\varrho}(n)_{3Z})$, so that

$$\lim_{n\to\infty} \left\| \frac{\hat{\varrho}_\mu}{(\sqrt{\omega})^m} - \frac{\hat{\varrho}(n)_\mu}{(\sqrt{\omega})^m} \right\|_2 = 0, \quad \mu = 1, 2, ..., 3Z, m = 2, 1, 0, -1.$$

Then, for sufficiently small β, γ, $\mathcal{L}_Z^\infty$ is a common core for $H_{\hat{\varrho}(n),\hat{\tau},\hat{\eta}} + V \otimes I \equiv H(n), n \in \mathbb{N}$. Moreover

$$s - \lim_{n\to\infty} H(n)\Psi = (H_{\hat{\varrho},\hat{\tau},\hat{\eta}} + V \otimes I)\Psi, \quad \Psi \in \mathcal{L}_Z^\infty,$$

which implies that, on $\mathcal{L}_Z$,

$$s - \lim_{n\to\infty} e^{-tH(n)} = e^{-t(H_{\hat{\varrho},\hat{\tau},\hat{\eta}}+V\otimes I)}. \tag{3. 14}$$

On the other hand, (3.13) holds with $\hat{\varrho}$ replaced by $\hat{\varrho}(n)$ and, then, put the right hand side of (3.13) by $I(n)$. Since one can see that

$$\left\| \Phi\left(\int_0^t j_s \widetilde{\hat{\varrho}}_\mu(\omega(s)) db_\mu(s) - \int_0^t j_s \widetilde{\hat{\varrho}(n)}_\mu(\omega(s)) db_\mu(s)\right)\right\|^2_{L^2(\Omega;\mathcal{E})}$$
$$\leq const. \times \int_\Omega d\mu \int_0^t \left\| \frac{\hat{\varrho}_\mu}{\sqrt{\omega}} - \frac{\hat{\varrho}(n)_\mu}{\sqrt{\omega}}\right\|_2^2 ds \to 0,$$

as $n \to \infty$, and, similarly,

$$\left\| \Phi\left(\int_0^t j_s \frac{\partial \widetilde{\hat{\varrho}}_\mu(\omega(s))}{\partial x_\mu} ds - \int_0^t j_s \frac{\partial \widetilde{\hat{\varrho}(n)}_\mu(\omega(s))}{\partial x_\mu} ds\right)\right\|_{L^2(\Omega;\mathcal{E})}$$
$$\leq const. \times \int_\Omega d\mu \int_0^t \left\| \sqrt{\omega}\hat{\varrho}_\mu - \sqrt{\omega}\hat{\varrho}(n)_\mu \right\|_2 ds \to 0$$

as $n \to \infty$. Hence we have

$$s - \lim_{n\to\infty} \mathcal{K}_{\hat{\varrho}(n),\hat{\tau},\hat{\eta}}(t) = \mathcal{K}_{\hat{\varrho},\hat{\tau},\hat{\eta}}(t)$$

in $L^2(\Omega;\mathcal{E})$. A simple limiting argument shows that $I(n)$ converges to the right hand side of (3.13). Combining (3.14), we get (3.13) with $V \in C_b(\mathbb{R}^{3Z})$. For any $V \in \mathcal{M}_\pm(Z)$, by the same limiting argument as in Theorem 3.10, we get the desired result. □

4 WCLUV OF THE NELSON MODEL

In this section, our first task is to define the Nelson Hamiltonian $H_w(\Lambda)_V$ with a scalar potential V, the nonnegative mass $\mu \geq 0$ and a scaling parameter $\Lambda > 0$ in the Schrödinger representation in a rigorous manner as a self-adjoint operator in $\mathcal{L} = \bigoplus_{Z=0}^{\infty} \mathcal{H}_Z$. Moreover we investigate WCLUV of the Nelson Hamiltonian $H_w(\Lambda)_V$ by using a unitary transformation $\mathcal{U}_K(\Lambda)$ with an infrared cut-off $K > 0$ and a functional integral representation derived in the preceding section. In what follows, we denote the $3Z$-dimensional Brownian motion by $(b(t))_{t\geq 0} = (b_\mu^j(t))_{t\geq 0, \mu=1,2,3, j=1,..,Z}$. Set

$$\Xi_\Lambda(k) = \begin{cases} 1, & |k| < \Lambda, \\ 0, & \textit{otherwise}. \end{cases}$$

Put, for each $x \in \mathbb{R}^{3Z}$ and $K > 0$,

$$\tilde{\Xi}_K^Z(x) = \frac{1}{\sqrt{(2\pi)^3}} \sum_{j=1}^{Z} \left(\Xi_K(k) e^{-ikx^j}\right)^\vee,$$

$$(\Xi_{\Lambda^\alpha})_{K,\mu}^j(x) = \frac{1}{\sqrt{(2\pi)^3}} \left(\frac{\Xi_{\Lambda^\alpha}(k)(I - \Xi_K(k)) e^{-ikx^j} k_\mu}{\omega(k)} \right)^\vee, \quad \mu = 1,2,3, j = 1, ..., Z,$$

where the inverse Fourier transformation $\vee$ is taken with respect to the variable $k \in \mathbb{R}^3$. An operator $H_w^Z(\Lambda)$, $Z \geq 0$, acting in $\mathcal{L}_Z = L^2(\mathbb{R}^{3Z}) \otimes \mathcal{F}$ is defined by

$$\begin{aligned} H_w^Z(\Lambda) &= \frac{1}{2m} \mathbf{p}_Z^2 \otimes I - \Lambda g H_I^Z(\Lambda^\alpha) + \Lambda^2 I \otimes H_b, \quad Z \geq 1, \\ H_w^0(\Lambda) &= \Lambda^2 I \otimes H_b, \end{aligned}$$

where

$$H_I^Z(\Lambda^\alpha) = \int_{\mathbb{R}^{3Z}}^{\oplus} \phi(\tilde{\Xi}_{\Lambda^\alpha}^Z(x)) dx, \quad Z \geq 1, \alpha > 0.$$

Proposition 4.1 **([11])** *For any $\Lambda > 0$, the operator $H_w^Z(\Lambda)$ is self-adjoint on $D(H_F^Z)$ and bounded from below. Moreover it is essentially self-adjoint on any core for H_F^Z.*

Lemma 4.2 *Suppose that $V_Z \in \mathcal{M}_\pm(Z)_r$, $Z \geq 1$, and S denotes the projection operator from $\mathcal{L}_z$ onto $\mathcal{H}_Z$. Then, for any $t \geq 0$, S and the operator $e^{-t(H_w^Z(\Lambda) \dot{+} V_{Z+} \otimes I \dot{-} V_{Z-} \otimes I)}$ is commutative.*

Proof: Since $\mathbf{p}_Z^2$, V_{Z+} and V_{Z-} are reduced by $L_{as}^2(\mathbb{R}^{3Z})$, the operators $e^{-t\mathbf{P}_Z^2 \otimes I}$, $e^{-tV_{Z+} \otimes I}$ and $e^{-t(-V_{Z-} \otimes I)}$ are commutative with S. Moreover since, for any $\Psi \in \mathcal{H}_Z$,

$$\left(\exp\left(-t(-gH_I^Z(\Lambda^\alpha))\right) \Psi\right)(x) = \exp\left(-t(-g\phi(\tilde{\Xi}_{\Lambda^\alpha}^Z(x)))\right) \Psi(x),$$

and $\phi(\tilde{\Xi}^Z_{\Lambda^\alpha}(x))$ is symmetric in $x \in \mathbb{R}^{3Z}$, we see that

$$Se^{-t(-gH_I^Z)}\Psi = e^{-t(-gH_I^Z)}S\Psi.$$

Hence from the Trotter product formula it follows that

$$\begin{aligned}
&Se^{-t(H_w^Z(\Lambda)\dot{+}V_{Z+}\otimes I\dot{-}V_{Z-}\otimes I)}\Psi \\
&= s-\lim_{n\to\infty} S\left(e^{-\frac{t}{n}\mathrm{p}_Z^2\otimes I}e^{-\frac{t}{n}(-gH_I)}e^{-\frac{t}{n}I\otimes H_b}e^{-\frac{t}{n}V_{Z+}\otimes I}e^{-\frac{t}{n}(-V_{Z-}\otimes I)}\right)^n\Psi \\
&= s-\lim_{n\to\infty} \left(e^{-\frac{t}{n}\mathrm{p}_Z^2\otimes I}e^{-\frac{t}{n}(-gH_I)}e^{-\frac{t}{n}I\otimes H_b}e^{-\frac{t}{n}V_{Z+}\otimes I}e^{-\frac{t}{n}(-V_{Z-}\otimes I)}\right)^n S\Psi \\
&= e^{-t(H_w^Z(\Lambda)\dot{+}V_{Z+}\otimes I\dot{-}V_{Z-}\otimes I)}S\Psi.
\end{aligned}$$

Thus proof is complete. □

Lemma 4.2 yields that $H_w^Z(\Lambda)\dot{+}V_{Z+}\otimes I\dot{-}V_{Z-}\otimes I$ can be reduced by the closed subspace $\mathcal{H}_Z$ in $\mathcal{L}_Z$. Define

$$H_w^Z(\Lambda)_{V_Z} \equiv H_w^Z(\Lambda)\dot{+}V_{Z+}\otimes I\dot{-}V_{Z-}\otimes I\Big|_{\mathcal{H}_Z}.$$

Let $N_b \geq 0$ be the number operator in $\mathcal{F}_a$;

$$\begin{aligned}
&D(N_b) = \left\{\Psi = \{\Psi_Z\}_{Z=0}^\infty \in \mathcal{F}_a \Big| \sum_{Z=0}^\infty Z^2 \|\Psi_Z\|^2_{L^2_{as}(\mathbb{R}^{3Z})} < \infty\right\}, \\
&(N_b\Psi)_Z = Z\Psi_Z, \quad \Psi \in D(N_b).
\end{aligned}$$

For a set of scalar potentials $V = \{V_Z\}_{Z=0}^\infty$ with $V_Z \in \mathcal{M}_\pm(Z)_r$, $Z \geq 1$, and $V_0 = 0$, we define

$$H_w(\Lambda)_V\dot{+}\epsilon N_b\otimes I \equiv \bigoplus_{Z=0}^\infty H_w^Z(\Lambda)_{V_Z}\dot{+}\epsilon N_b\otimes I, \tag{4. 1}$$

where $\epsilon \geq 0$. Hence, putting V as (1.1), we may formally write $H_w(\Lambda)_V$ as (1.2). Set

$$E_w(\Lambda^\alpha) = -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{\Xi_{\Lambda^\alpha}(k)}{\omega^2(k)}dk.$$

At first, we investigate the strong limit of $e^{-t\left(H_w^Z(\Lambda)-g^2ZE_w(\Lambda^\alpha)\right)}$ as $\Lambda\to\infty$ on $\mathcal{L}_Z$ with fixed $Z\geq 1$. Put

$$\pi_K = \int_{\mathbb{R}^{3Z}}^\oplus \mathcal{T}\pi_F\left(\frac{1}{\sqrt{(2\pi)^3}}\sum_{j=1}^Z\frac{\Xi_{\Lambda^\alpha}(I-\Xi_K)e^{-ikx^j}}{\omega^2}\right)\mathcal{T}^{-1}dx.$$

We define a unitary operator on $\mathcal{L}_Z$ with an infrared cut-off $K > 0$ by

$$\mathcal{U}_K(g) = \exp\left(-ig\pi_K\right).$$

Here notations π_K and $\mathcal{T}$ are defined in section 2. Put $\mathcal{U}_K(\Lambda) = \mathcal{U}_K\left(\frac{g}{\Lambda}\right)$.

Theorem 4.3 *The unitary operator $\mathcal{U}_K(\Lambda)$ maps $\mathcal{L}_Z^\infty$ to $D(H_F^Z)$ with*

$$\begin{aligned}&\mathcal{U}_K(\Lambda)^{-1}\left(H_w^Z(\Lambda) - g^2 Z E_w(\Lambda^\alpha)\right)\mathcal{U}_K(\Lambda)\Big|_{\mathcal{L}_Z^\infty}\\ &= \frac{1}{2m}\left(\mathbf{p}_Z \otimes I - \frac{g}{\Lambda}A_K(\Lambda)\right)^2 + U_K(\Lambda) + \Lambda^2 I \otimes H_b\Big|_{\mathcal{L}_Z^\infty},\end{aligned} \qquad (4.\ 2)$$

where

$$\begin{aligned}A_K(\Lambda) &= \left(\int_{\mathbb{R}^{3Z}}^{\oplus} \phi\left((\Xi_{\Lambda^\alpha})_{K,1}^1(x)\right)dx, \int_{\mathbb{R}^{3Z}}^{\oplus} \phi\left((\Xi_{\Lambda^\alpha})_{K,2}^1(x)\right)dx, .., \int_{\mathbb{R}^{3Z}}^{\oplus} \phi\left((\Xi_{\Lambda^\alpha})_{K,3}^N(x)\right)dx\right),\\ U_K(\Lambda) &= -g\Lambda H_I^Z(K) + g^2 V_K(\Lambda) \otimes I - g^2 Z E_w(K),\\ V_K^Z(\Lambda) &= V_K^Z(\Lambda, x) = -\frac{1}{2(2\pi)^3}\sum_{i\neq j}^{Z}\int_{\mathbb{R}^3}\frac{\Xi_{\Lambda^\alpha}(k)(1-\Xi_K(k))e^{-ik(x^i-x^j)}}{\omega^2(k)}dk.\end{aligned}$$

Proof: Note that $[[\mathbf{p}_\mu^j \otimes I, \pi_K], \pi_K] = 0, \mu = 1, 2, 3, j = 1, .., N$. Then (4.2) follows from a direct calculation. Details are omitted. □

Remark 4.4 *If $\mu > 0$, we can put $K = 0$. Only in the case $\mu = 0$ we need to introduce the infrared cut-off $K > 0$. Because, if $\mu = 0$, we can not define the unitary operator $\mathcal{U}_0(\Lambda)$, since $\frac{\Xi_{\Lambda^\alpha}}{\omega\sqrt{\omega}} \notin L^2(\mathbb{R}^3)$. Furthermore, in the case $\mu > 0$, if we put $K = 0$, the term $\Lambda H_I^Z(K)$ in the right hand side of (4.2) disappears. Then, the right hand side of (4.2) becomes a simple form.*

We set the right hand side of (4.2) by $\widetilde{H}_w^Z(K\Lambda)$ and

$$H_F^Z(\Lambda) = \frac{1}{2m}\mathbf{p}_Z^2 \otimes I + \Lambda^2 I \otimes H_b$$

and define a symmetric operator $R(\Lambda, K)$ by

$$\widetilde{H}_w^Z(K\Lambda) = H_F^Z(\Lambda) + U_K(\Lambda) + R(\Lambda, K).$$

Let

$$\begin{aligned}
V_\mu^Z(x) &= -\frac{1}{4\pi}\sum_{i<j}^{Z}\frac{e^{-\mu|x^i-x^j|}}{|x^i-x^j|},\\
V_K^Z(x) &= -\frac{1}{2(2\pi)^3}\sum_{i\neq j}^{Z}\int_{\mathbb{R}^3}\frac{\Xi_K(k)e^{-ik(x^i-x^j)}}{\omega^2(k)}dk,\\
V_K^{Zc}(x) &= -\frac{1}{2(2\pi)^3}\sum_{i\neq j}^{Z}\int_{\mathbb{R}^3}\frac{(I-\Xi_K(k))e^{-ik(x^i-x^j)}}{\omega^2(k)}dk.
\end{aligned}$$

Note that $V_K^Z + V_K^{Zc} = V_\mu^Z$ and V_K^Z is bounded.

Lemma 4.5 *Let $0 < \alpha < \frac{1}{2}$. Then, for any $\epsilon > 0$, there exists Λ_0 and $b(\epsilon) > 0$ so that for all $\Lambda > \Lambda_0$, $D(-\Lambda g H_I^Z(K) + g^2 V_K(\Lambda)\otimes I + R(\Lambda, K)) \supset D(H_F^Z(\Lambda)) = D(H_F^Z)$ and for $\Psi \in D(H_F^Z)$*

$$\left|\left|\left(-g\Lambda H_I^Z(K) + g^2 V_K(\Lambda)\otimes I + R(\Lambda,K)\right)\Psi\right|\right|_{\mathcal{L}_Z} \le \epsilon\left|\left|H_F^Z(\Lambda)\Psi\right|\right|_{\mathcal{L}_Z} + b(\epsilon)\,||\Psi||_{\mathcal{L}_Z}.$$

Proof: From [4.Theorem 3.8 (1)] it follows that, with $b_1(\epsilon)\ge 0$,

$$\left|\left|\left(g^2 V_K(\Lambda)\otimes I + R(\Lambda,K)\right)\Psi\right|\right|_{\mathcal{L}_Z} \le \epsilon\left|\left|H_F^Z\Psi\right|\right|_{\mathcal{L}_Z} + b_1(\epsilon)\,||\Psi||_{\mathcal{L}_Z}.$$

Moreover it follows that, with $b_2(\epsilon)\ge 0$,

$$\left|\left|-g\Lambda H_I^Z(K)\Psi\right|\right|_{\mathcal{L}_Z} \le \epsilon\left|\left|\left(I\otimes\Lambda^2 H_b\right)\Psi\right|\right|_{\mathcal{L}_Z} + b_2(\epsilon)\,||\Psi||_{\mathcal{L}_Z}.$$

Combining above two inequalities, we get the desired result. □

It follows from Lemma 4.5 that, for sufficiently large $\Lambda > 0$ and $0 < \alpha < \frac{1}{2}$, $\widetilde{H}_w^Z(K\Lambda)$ is self-adjoint on $D(H_F^Z)$ and essentially self-adjoint on any core for $H_F^Z(\Lambda)$. Especially, $\mathcal{L}_Z^\infty$ is a core for $\widetilde{H}_w^Z(K\Lambda)$ with sufficiently large $\Lambda > 0$.

Corollary 4.6 *Suppose that $0 < \alpha < \frac{1}{2}$ and Λ is sufficiently large. Then $\mathcal{U}_K(\Lambda)$ maps $D(H_w^Z(\Lambda))$ onto $D(\widetilde{H}_w^Z(K\Lambda))$ with*

$$\mathcal{U}_K(\Lambda)^{-1}e^{-tH_w^Z(\Lambda)}\mathcal{U}_K(\Lambda) = e^{-t\widetilde{H}_w^Z(K\Lambda)}. \qquad (4.\ 3)$$

Proof: Since Λ is sufficiently large, $\mathcal{L}_Z^\infty$ is a core for $\widetilde{H}_w^Z(K\Lambda)$. Then the equality (4.2) can be extended to the equality on the domain $D(\widetilde{H}_w^Z(K\Lambda))$. Then (4.3) follows. □

Lemma 4.7 *Let $0 < \alpha < \frac{1}{2}$. Then, for sufficiently large Λ, $e^{-t\widetilde{H}_w^Z(K\Lambda)}$ has the following functional integral representation:*

$$\left\langle F, e^{-t\widetilde{H}_w^Z(K\Lambda)}G\right\rangle_{\mathcal{L}_Z} = \int_{\widetilde{Q}_E} d\nu_E \overline{J_{\Lambda^2 t}F(\omega(t/m))} J_0 G(\omega(0)) e^{\mathcal{K}_\Lambda(t)} e^{-E_\Lambda(t)\otimes I}. \tag{4. 4}$$

where

$$\begin{aligned} \mathcal{K}_\Lambda(t) = g & \left\{ \frac{i}{\Lambda} \sum_{j=1}^{Z} \sum_{\mu=1}^{3} \Phi\left(\int_0^{\frac{t}{m}} j_{m\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s)) \circ db^j_\mu(s) \right) \right. \\ & \left. -\Lambda\Phi\left(\int_0^t j_{\Lambda^2 s}\widetilde{\Xi}_K(\omega(s))ds \right) \right\}, \\ E_\Lambda(t) = g^2 & \int_0^t \left(V_K^Z(\Lambda, \omega(s)) - ZE_w(K) \right) ds. \end{aligned}$$

Proof: Since, for sufficiently large $\Lambda > 0$, $\mathcal{L}_Z^\infty$ is a core for $\widetilde{H}_w^Z(K\Lambda)$, (4.4) follows from Lemma 3.3 and Theorem 3.5. □

Remark 4.8 *Without the unitary transformation $\mathcal{U}_K(\Lambda)$, of course, we can get a functional integral representation of $e^{-t(H_w^Z(\Lambda)-g^2 ZE_w(\Lambda^\alpha))}$ directly. However, in that case, we meet a difficulty in which we cannot control divergent terms as $\Lambda \to \infty$ in the Hamiltonian. An essential effect of introducing the unitary transformation is to separate divergent terms in $H_w^Z(\Lambda)$ (see (4.2)).*

We state the main theorem in this section.

Theorem 4.9 *Let $0 < \alpha < \frac{1}{2}$. Then*

$$s - \lim_{\Lambda\to\infty} e^{-t(H_w(\Lambda)-g^2 ZE_w(\Lambda^\alpha))} = e^{-t\left(\frac{1}{2m}\mathrm{p}_Z^2 + g^2 V_\mu^Z\right)} \otimes P_b,$$

where P_b denotes the projection operator onto the subspace $\{z\Omega_{\mathcal{F}} | z \in \mathbb{C}\} \subset \mathcal{F}$.

Before proving Theorem 4.9, we need some lemmas as follows.

Lemma 4.10 *Let $0 < \alpha < \frac{1}{2}$ and $F_l \in \mathcal{F}^\infty$ so that*

$$F_l = e^{i\phi(f_l)}, \quad f_l \in \mathcal{H}_{-\frac{1}{2}}, l = 1, .., M.$$

Suppose that $0 \le t_1 < t_2 < .... < t_M \le t$ and put $F_\Lambda^E(t) = e^{i\sum_{l=1}^M \Phi(j_{\Lambda^2 t_l} f_l)}$. Then for $(x, b) \in \mathbb{R}^{3Z} \times \Omega$,

$$\lim_{\Lambda\to\infty} \int_{Q_E} d\mu_E F_\Lambda^E(t) e^{\mathcal{K}_\Lambda(t)} = \Pi_{l=1}^M < \Omega_{\mathcal{F}}, F_l >_{\mathcal{F}} e^{-\int_0^t \left(V_K^Z(\omega(s)) + g^2 Z E_w(K)\right) ds}. \tag{4. 5}$$

Proof: For simplicity, we put the mass of the nonrelativistic particles $m = 1$ in this proof. Since the integral on the left hand side (L.H.S) of (4.5) is calculated as follows:

$$L.H.S.of\ (4.5) = \lim_{\Lambda\to\infty} \exp\left\{-\frac{1}{4}(2iIII_\Lambda + I_\Lambda - II_\Lambda)\right\},$$

where

$$\begin{aligned} III_\Lambda &= \left\langle \sum_{l=1}^M j_{\Lambda^2 t_l} f_l + \sum_{j=1}^Z \sum_{\mu=1}^3 \frac{g}{\Lambda} \int_0^t j_{\Lambda^2 s} (\Xi_{\Lambda^\alpha})_{K,\mu}^j (\omega(s)) \circ db_\mu^j(s), \right. \\ &\qquad \left. -\Lambda g \int_0^t j_{\Lambda^2 s} \tilde{\Xi}_K(\omega(s)) ds \right\rangle_{\mathcal{W}}, \\ I_\Lambda &= \left\| \sum_{l=1}^M j_{\Lambda^2 t_l} f_l + \sum_{j=1}^Z \sum_{\mu=1}^3 \frac{g}{\Lambda} \int_0^t j_{\Lambda^2 s} (\Xi_{\Lambda^\alpha})_{K,\mu}^j (\omega(s)) \circ db_\mu^j(s) \right\|_{\mathcal{W}}^2, \\ II_\Lambda &= \left\| \Lambda g \int_0^t j_{\Lambda^2 s} \tilde{\Xi}_K^Z(\omega(s)) ds \right\|_{\mathcal{W}}^2 \end{aligned}$$

We estimate $I_\Lambda, II_\Lambda, III_\Lambda$ separately.

$$\begin{aligned} I_\Lambda &= \left\| \sum_{l=1}^M j_{\Lambda^2 t_l} f_l \right\|_{\mathcal{W}}^2 + \left\| \sum_{j=1}^Z \sum_{\mu=1}^3 \frac{g}{\Lambda} \int_0^t j_{\Lambda^2 s} (\Xi_{\Lambda^\alpha})_{K,\mu}^j (\omega(s)) \circ db_\mu^j(s) \right\|_{\mathcal{W}}^2 \\ &\quad + 2\Re \left\langle \sum_{l=1}^M j_{\Lambda^2 t_l} f_l, \sum_{j=1}^Z \sum_{\mu=1}^3 \frac{g}{\Lambda} \int_0^t j_{\Lambda^2 s} (\Xi_{\Lambda^\alpha})_{K,\mu}^j (\omega(s)) \circ db_\mu^j(s) \right\rangle_{\mathcal{W}}^2 \\ &= I_\Lambda^1 + I_\Lambda^2 + 2\Re I_\Lambda^3. \end{aligned}$$

Then one can see that, by the definition of the operator j_t,

$$\lim_{\Lambda\to\infty} I_\Lambda^1 = \lim_{\Lambda\to\infty}\sum_{i,j=1}^{M}\left\langle f_i, e^{-\Lambda^2|t_i-t_j|\omega} f_j\right\rangle_{-\frac{1}{2}}$$
$$= \sum_{i=1}^{M} ||f_i||^2_{-\frac{1}{2}}. \tag{4. 6}$$

Noting that j_s is isometry from $\mathcal{H}_{-\frac{1}{2}}$ to $\mathcal{W}$, we have

$$\left|\left|j_{\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))\right|\right|^2_{\mathcal{W}} \le \frac{\mathbf{S}^2}{(2\pi)^3}\int_K^{\Lambda^\alpha} r dr,$$
$$\left|\left|j_{\Lambda^2 s}\frac{\partial(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))}{\partial x^j_\mu}\right|\right|^2_{\mathcal{W}} \le \frac{\mathbf{S}^2}{(2\pi)^3}\int_K^{\Lambda^\alpha} r^3 dr,$$

where $\mathbf{S}^2$ denotes the volume of the 2-dimensional sphere. Hence, noting that $0<\alpha<\frac{1}{2}$, it holds that

$$\lim_{\Lambda\to\infty}\int_\Omega d\mu\left|\left|\sum_{j=1}^{Z}\sum_{\mu=1}^{3}\frac{g}{\Lambda}\int_0^t j_{\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))db^j_\mu(s)\right|\right|^2_{\mathcal{W}}$$
$$= \lim_{\Lambda\to\infty}\int_\Omega d\mu\sum_{j=1}^{Z}\sum_{\mu=1}^{3}\frac{g^2}{\Lambda^2}\int_0^t ds\left|\left|j_{\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))\right|\right|^2_{\mathcal{W}}$$
$$\le \lim_{\Lambda\to\infty}\frac{3Ztg^2\mathbf{S}^2}{(2\pi)^3}\frac{1}{\Lambda^2}\int_K^{\Lambda^\alpha} r dr$$
$$= 0,$$

which implies that, since "$\frac{g}{\Lambda}\int_0^t j_{\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))db^j_\mu(s)$" is the $\mathcal{W}$-valued Martingale,

$$\lim_{\Lambda\to\infty}\left|\left|\sum_{j=1}^{Z}\sum_{\mu=1}^{3}\frac{g}{\Lambda}\int_0^t j_{\Lambda^2 s}(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))db^j_\mu(s)\right|\right|^2_{\mathcal{W}} = 0. \tag{4. 7}$$

Moreover

$$\lim_{\Lambda\to\infty}\left|\left|\sum_{j=1}^{Z}\sum_{\mu=1}^{3}\frac{g}{\Lambda}\int_0^t j_{\Lambda^2 s}\frac{\partial(\Xi_{\Lambda^\alpha})^j_{K,\mu}(\omega(s))}{\partial x^j_\mu}ds\right|\right|^2_{\mathcal{W}} \le \lim_{\Lambda\to\infty}\frac{3Ztg^2\mathbf{S}^2}{(2\pi)^3}\frac{1}{\Lambda^2}\int_K^{\Lambda^\alpha} r^3 dr$$
$$= 0.$$

Hence we have

$$\lim_{\Lambda\to\infty} I_\Lambda^2 = 0.$$

By (4.6) and (4.7), we have

$$\lim_{\Lambda\to\infty} I_\Lambda^3 = 0.$$

Consequently

$$\lim_{\Lambda\to\infty} I_\Lambda = \sum_{l=1}^{M} \|f_l\|_{-\frac{1}{2}}^2. \tag{4. 8}$$

Next we consider II_Λ, which is the most complicated and essential to derive the Z-body Coulomb (or the Z-body Yukawa) potential. Note that

$$\widehat{j_t f}(k,k_0) = \frac{1}{\sqrt{2\pi}} e^{-ik_0 t}\hat{f}(k).$$

Then we have

$$\begin{aligned} II_\Lambda &= \Lambda^2 \frac{2g^2}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{R}} \frac{dk\,dk_0}{\omega^2(k)+k_0^2}\left|\frac{1}{\sqrt{2\pi}}\int_0^t e^{-ik_0\Lambda^2 s}\sum_{j=1}^{Z}\Xi_K(k)e^{-ik\omega^j(s)}ds\right|^2 \\ &= \frac{2g^2}{(2\pi)^3}\int_{\mathbb{R}^3}\Xi_K(k)dk\int_{\mathbb{R}}\frac{dk_0}{\omega^2(k)+\frac{k_0^2}{\Lambda^4}}\left|\frac{1}{\sqrt{2\pi}}\int_0^t e^{-ik_0 s}\sum_{j=1}^{Z}e^{-ik\omega^j(s)}ds\right|^2. \end{aligned}$$

By the Lebesgue dominated convergence theorem, one can see that

$$\begin{aligned} \lim_{\Lambda\to\infty} II_\Lambda &= \frac{2g^2}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\Xi_K(k)dk}{\omega^2(k)}\int_{\mathbb{R}}dk_0\left|\frac{1}{\sqrt{2\pi}}\int_0^t e^{-ik_0 s}\sum_{j=1}^{Z}e^{-ik\omega^j(s)}ds\right|^2 \\ &= \frac{2g^2}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\Xi_K(k)dk}{\omega^2(k)}\int_0^t\left|\sum_{j=1}^{Z}e^{-ik\omega^j(s)}\right|^2 ds \\ &= 4\left\{\int_0^t ds\frac{g^2}{2(2\pi)^3}\sum_{i\neq j}^{Z}\int_{\mathbb{R}^3}\frac{\Xi_K(k)e^{-ik(\omega^i(s)-\omega^j(s))}}{\omega^2(k)}dk - g^2 Z\int_0^t ds E_w(K)\right\}. \\ &= -4\int_0^t\left(g^2 V_K^Z(\omega(s)) + g^2 Z E_w(K)\right)ds. \end{aligned} \tag{4. 9}$$

Here in the second equality in (4.9) we use the unitarity in $L^2(\mathbb{R}^3)$ of the Fourier transformation with respect to k_0. Note that II_Λ is monotonously increasing as $\Lambda\to\infty$. Finally we consider III_Λ. From (4.7) and (4.9) we have

$$\lim_{\Lambda\to\infty} III_\Lambda = \lim_{\Lambda\to\infty}\left\langle\sum_{l=1}^{M} j_{\Lambda^2 t_l}f_l, -\Lambda g\int_0^t j_{\Lambda^2 s}\tilde{\Xi}_K(\omega(s))ds\right\rangle_{\mathcal{W}}$$

$$= \lim_{\Lambda\to\infty} \frac{-2\Lambda g}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3\times\mathbb{R}} \frac{dkdk_0}{\sqrt{2\pi}} \sum_{l=1}^{M} \frac{e^{ik_0\Lambda^2 t_l}\bar{\hat{f}}_l(k)}{\omega^2(k)+k_0^2} \int_0^t \frac{ds}{\sqrt{2\pi}} e^{-i\Lambda^2 s k_0} \Xi_K(k) \sum_{j=1}^{Z} e^{-ik\omega^j(s)}$$

$$= \lim_{\Lambda\to\infty} \frac{-1}{\sqrt{(2\pi)^3}} \frac{g}{\Lambda} \int_{\mathbb{R}^3} dk \sum_{l=1}^{M} \frac{e^{-\Lambda^2|t_l-s|\omega(k)}\bar{\hat{f}}_l(k)\Xi_K(k)}{\omega(k)} \int_0^t \sum_{j=1}^{Z} e^{-ik\omega^j(s)} ds.$$

Hence by the Schwartz inequality

$$\lim_{\Lambda\to\infty} |III_\Lambda| \le \lim_{\Lambda\to\infty} \frac{Zt}{\sqrt{(2\pi)^3}} \sum_{l=1}^{M} ||f_l||_{-\frac{1}{2}} \frac{g}{\Lambda} \left(\int_{\mathbb{R}^3} e^{-2\Lambda^2|t_l-s|\omega(k)} \frac{\Xi_K(k)}{\omega(k)^2} dk \right)^{\frac{1}{2}}$$
$$= 0. \qquad (4.\ 10)$$

Then by (4.8),(4.9) and (4.10) it holds that

$$\lim_{\Lambda\to\infty} e^{-\frac{1}{4}(I_\Lambda - II_\Lambda + 2iIII_\Lambda)} = e^{-\int_0^t \left(g^2 V_K^Z(\omega(s)) + g^2 Z E_\omega(K)\right)ds} e^{-\frac{1}{4}\sum_{l=1}^{M} ||f_l||^2_{-\frac{1}{2}}}.$$

Thus the proof is complete. □

Lemma 4.11 *Let $0<\alpha<\frac{1}{2}$. Then, for $F,G\in\mathcal{F}$ and $(x,b)\in\mathbb{R}^{3Z}\times\Omega$,*

$$\lim_{\Lambda\to\infty} \int_{Q_E} d\mu_E \overline{J_{\Lambda^2 t}F} J_0 G e^{\mathcal{K}_\Lambda(t)} = \langle F, P_b G\rangle_{\mathcal{F}} e^{-\int_0^t \left(g^2 V_K^Z(\omega(s)) + g^2 Z E_\omega(K)\right)ds}. \qquad (4.\ 11)$$

Proof: At first we assume that $F,G\in\mathcal{F}^\infty$ so that, with $f\in\mathcal{S}(\mathbb{R}^n), g\in\mathcal{S}(\mathbb{R}^m)$,

$$F = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} \check{f}(t_1,..,t_n) e^{i\sum_{j=1}^n \phi(f_j)t_j} dt_1...dt_n,$$
$$G = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \check{g}(s_1,..,s_m) e^{i\sum_{j=1}^m \phi(g_j)s_j} ds_1...ds_m.$$

Then from Lemma 4.10 it follows (4.11). Since $\mathcal{F}^\infty$ is dense in $\mathcal{F}$, one can easily see (4.11) for any $F,G\in\mathcal{F}$ by an approximation argument. □

The proof of Theorem 4.9

It is easily seen that

$$s - \lim_{\Lambda\to\infty} \mathcal{U}_K(\Lambda) = I.$$

Hence, by Corollary 4.6, and uniform boundedness of $e^{-t\widetilde{H}_w^Z(K\Lambda)}$ and $e^{-t\left(\frac{1}{2m}\mathrm{p}^2+g^2V_\mu^Z\right)}\otimes P_b$, it is enough to show that for any $\Theta, \Psi \in C_0^\infty(\mathbb{R}^{3Z})\hat{\otimes}\mathcal{F}^\infty$,

$$\lim_{\Lambda\to\infty}\left\langle\Theta, e^{-t\widetilde{H}_w^Z(K\Lambda)}\Psi\right\rangle_{\mathcal{L}_Z} = \left\langle\Theta, e^{-t\left(\frac{1}{2m}\mathrm{p}^2+V_\mu^Z\right)}\otimes P_b\Psi\right\rangle_{\mathcal{L}_Z}.$$

Let

$$\Theta = u\otimes F, \quad \Psi = v\otimes G,$$

where F and G are the same as those in the proof of Lemma 4.10 and $u, v \in C_0^\infty(\mathbb{R}^{3Z})$. Then

$$\left\langle\Theta, e^{-t\widetilde{H}_w^Z(K\Lambda)}\Psi\right\rangle_{\mathcal{L}_Z} = \int_{\mathbb{R}^{3Z}\times\Omega} dxdb\overline{u(\omega(t/m))}v(\omega(0))e^{-E_\Lambda(t)}\int_{Q_E} d\mu_E\overline{J_{\Lambda^2 t}F}J_0Ge^{\mathcal{K}_\Lambda(t)}.$$

We, at first, deal with $E_\Lambda(t) = g^2\int_0^t \left(V_K^Z(\Lambda,\omega(s)) - ZE_w(K)\right)ds$. Since

$$V_K^Z(\Lambda, x^1, ..., x^Z) = -\frac{1}{2\pi^2}\sum_{i<j}^{Z}\frac{1}{|x^i-x^j|}\int_K^{\Lambda^\alpha}\frac{r}{r^2+\mu^2}\sin(r|x^i-x^j|)dr,$$

using a contour integral, one can easily check that there exist $\delta_1 > 0$ and $\delta_2 > 0$, which are independent of $\Lambda > 0$, so that

$$|V_K^Z(\Lambda,\omega(s))| \le \delta_1|V_\mu^Z(\omega(s))| + \delta_2, \quad \mu \ge 0. \tag{4. 12}$$

Set

$$\Omega_0 = \left\{(x,b)\in\mathbb{R}^{3Z}\times\Omega \,\middle|\, \int_0^t\frac{e^{-\mu|\omega^i(s)-\omega^j(s)|}}{|\omega^i(s)-\omega^j(s)|}ds = \infty, i\ne j, i,j = 1,...,Z\right\}.$$

The measure of Ω_0 is zero. One can easily check that for $(x,b)\in[\mathbb{R}^{3Z}\times\Omega]\setminus\Omega_0$,

$$\lim_{\Lambda\to\infty} V_K^Z(\Lambda,\omega(s)) \;=\; V_K^{Zc}(\omega(s)) < \infty, \quad a.e. s\in[0,t]. \tag{4. 13}$$

Combining (4.12) and (4.13), one can see that,for $(x,b)\in[\mathbb{R}^{3Z}\times\Omega]\setminus\Omega_0$, by the Lebesgue dominated convergence theorem,

$$\lim_{\Lambda\to\infty}\int_0^t dsV_K^Z(\Lambda,\omega(s)) = \int_0^t dsV_K^{Zc}(\omega(s)).$$

Hence for almost everywhere $(x,b) \in \mathbb{R}^{3Z} \times \Omega$,

$$\lim_{\Lambda\to\infty} e^{-E_\Lambda(t)} = e^{-\int_0^t ds\left(g^2 V_K^{Z^c}(\omega(s)) - g^2 Z E_w(K)\right)ds}.$$

On the other hand, under the notations in the proof of Lemma 4.10, we see that

$$\left|e^{\mathcal{K}_\Lambda^Z(t)}\right| \le e^{\frac{1}{4}II_\Lambda}. \tag{4. 14}$$

As is seen in the proof of Lemma 4.10, the right hand side of (4.14) is monotonously increasing as $\Lambda \to \infty$. Hence we have, from the definition of F and G,

$$\begin{aligned}\left|\int_{Q_E} d\mu_E \overline{J_{\Lambda^2 t}F} J_0 G e^{\mathcal{K}_\Lambda^Z(t)}\right| &\le \left|\int_{\mathbb{R}^{n+m}} \overline{\check{f}(t)}\check{g}(s)dtds\right| e^{-\int_0^t\left(g^2 V_K^Z(\omega(s)) + g^2 Z E_w(K)\right)ds} \\ &\equiv C_{fg} e^{-\int_0^t\left(g^2 V_K^Z(\omega(s)) + g^2 Z E_w(K)\right)ds}.\end{aligned}$$

Then,by (4.12), we have for almost everywhere $(x,b) \in \mathbb{R}^{3Z} \times \Omega$,

$$\begin{aligned}&\left|\overline{u(\omega(t/m))}v(\omega(0))e^{-E_\Lambda(t)} \int_{Q_E} d\mu_E \overline{J_{\Lambda^2 t}F} J_0 G e^{\mathcal{K}_\Lambda^Z(t)}\right| \\ &\le |\overline{u(\omega(t/m))}v(\omega(0))| C_{fg} e^{-\int_0^t\left(g^2 V_K^Z(\omega(s)) + g^2 Z E_w(K) - \delta_1 |V_\mu^Z(\omega(s))| - \delta_2 - g^2 Z E_w(K)\right)ds}.\end{aligned} \tag{4. 15}$$

Since we see that

$$\left\langle |u|, e^{-t(-\frac{1}{2m}\mathrm{p}_Z^2 + \delta_1 V_\mu^Z - \delta_2 + g^2 V_K^Z)} |v| \right\rangle_{L^2(\mathbb{R}^{3Z})} < \infty,$$

the right hand side of (4.15) is integrable. By (4.11), we have

$$\begin{aligned}&\lim_{\Lambda\to\infty} \left\langle \Theta, e^{-t\widetilde{H}_w^Z(K\Lambda)}\Psi \right\rangle_{\mathcal{L}_Z} \\ &= \lim_{\Lambda\to\infty} \int_{\mathbb{R}^{3Z}\times\Omega} dx db \overline{u(\omega(t/m))}v(\omega(0))e^{-E_\Lambda(t)} \int_{Q_E} d\mu_E \overline{J_{\Lambda^2 t}F} J_0 G e^{\mathcal{K}_\Lambda^Z(t)} \\ &= \int_{\mathbb{R}^{3Z}\times\Omega} dx db \overline{u(\omega(t/m))}v(\omega(0)) e^{-\int_0^t\left(g^2 V_K^{Z^c}(\omega(s)) + g^2 V_K^Z(\omega(s))\right)ds} \langle F, P_b G\rangle_{\mathcal{F}}. \\ &= \left\langle u, e^{-t\left(\frac{1}{2m}\mathrm{p}^2 + g^2 V_\mu^Z\right)} v \right\rangle_{L^2(\mathbb{R}^{3Z})} \langle F, P_b G\rangle_{\mathcal{F}}.\end{aligned}$$

The proof is complete. □

From Theorem 4.9 it immediately follows the following corollary.

Corollary 4.12 *Let* $0 < \alpha < \frac{1}{2}$ *and* $V_Z \in \mathcal{M}_\pm(Z)$. *Then*

$$s - \lim_{\Lambda \to \infty} e^{-t(H_w^Z(\Lambda) \dot{+} V_{Z+} \otimes I \dot{-} V_{Z-} \otimes I - g^2 Z E_w(\Lambda^\alpha))} = e^{-t\left(\frac{1}{2m}\mathrm{p}_Z^2 \dot{+} V_{Z+} \dot{-} V_{Z-} + g^2 V_\mu^Z\right)} \otimes P_b. \qquad (4.\ 16)$$

Proof: At first, for $V_Z \in C_b(\mathbb{R}^{3Z})$, it can be easily seen that

$$\mathcal{U}_K(\Lambda)^{-1} e^{-t(H_w^Z(\Lambda) + V_Z \otimes I)} \mathcal{U}_K(\Lambda) = e^{-t(\widetilde{H}_w^Z(\Lambda) + V_Z \otimes I)}. \qquad (4.\ 17)$$

By a limiting argument as in Theorem 3.9, one can extend (4.17) with $+V \otimes I$ replaced by $\dot{+} V_{Z+} \otimes I \dot{-} V_{Z-} \otimes I$. Hence from Theorem 4.9 it follows (4.16). □

Since $V_\mu^Z(x)$ is symmetric in $x \in \mathbb{R}^{3Z}$, for $V_Z \in \mathcal{M}_\pm(Z)_r$, $\frac{1}{2m}\mathrm{p}_Z^2 \dot{+} V_{Z+} \dot{-} V_{Z-} + g^2 V_\mu^Z$ is reduced by the closed subspace $L_{as}^2(\mathbb{R}^{3Z})$ in $L^2(\mathbb{R}^{3Z})$. For a set of scalar potentials $V = \{V_Z\}_{Z=0}^\infty$ with $V_Z \in \mathcal{M}_\pm(Z)_r, Z \geq 1$, and $V_0 = 0$, put

$$\bigoplus_{Z=0}^{\infty} \left[\frac{1}{2m}\mathrm{p}_Z^2 \dot{+} V_{Z+} \dot{-} V_{Z-} + g^2 V_\mu^Z \Big|_{L_{as}^2(\mathbb{R}^{3Z})} \right] = H_{\infty, V, V_\mu}.$$

By Corollary 4.12 it immediately follows the following corollary.

Corollary 4.13 *Let* $0 < \alpha < \frac{1}{2}$ *and* $V = \{V_Z\}_{Z=0}^\infty$ *with* $V_Z \in \mathcal{M}_\pm(Z)_r$, $Z \geq 1$, *and* $V_0 = 0$. *Then*

$$s - \lim_{\Lambda \to \infty} e^{-t(H_w(\Lambda)_V - g^2 E_w(\Lambda^\alpha) N_b \otimes I)} = e^{-tH_{\infty, V, V_\mu}} \otimes P_b.$$

Proof: By Corollary 4.12, for $\Psi = (\Psi_0, \Psi_1, ...) \in \mathcal{L}$, $e^{-t(H_w^Z(\Lambda) \dot{+} V_{Z+} \otimes I \dot{-} V_{Z-} \otimes I - g^2 Z E_w(\Lambda^\alpha))} \Psi_Z$ strongly converges to $e^{-t\left(\frac{1}{2m}\mathrm{p}_Z^2 \dot{+} V_{Z+} \dot{-} V_{Z-} + g^2 V_\mu^Z\right)} \otimes P_b \Psi_Z$ as $\Lambda \to \infty$ with a uniform bound with respect to Z. Hence we see that, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} s - \lim_{\Lambda \to \infty} e^{-t(H_w^Z(\Lambda)_V - g^2 E_w(\Lambda^\alpha) N_b \otimes I)} \Psi &= s - \lim_{\Lambda \to \infty} \bigoplus_{Z=0}^{\infty} e^{-t(H_w^Z(\Lambda) \dot{+} V_{Z+} \otimes I \dot{-} V_{Z-} \otimes I - g^2 Z E_w(\Lambda^\alpha))} \Psi_Z \\ &= \bigoplus_{Z=0}^{\infty} \left[e^{-t\left(\frac{1}{2m}\mathrm{p}_Z^2 \dot{+} V_{Z+} \dot{-} V_{Z-} + g^2 V_\mu^Z\right)} \otimes P_b \right] \Psi_Z. \end{aligned}$$

Thus the proof is complete. □

5 CONCLUDING REMARKS

(1) Let $V_Z = 0, Z \geq 0$, in Corollary 4.13. Formally we may write

$$H_{\infty,0,V_\mu} = \int \Psi^\dagger(x)\left(-\frac{1}{2m}\Delta\right)\Psi(x)dx - \frac{g^2}{4\pi}\int \Psi^\dagger(x)\Psi^\dagger(y)\frac{e^{-\mu|x-y|}}{|x-y|}\Psi(y)\Psi(x)dxdy.$$

(2) For each $Z \geq 1$, define a self-adjoint operator in each sector $\mathcal{H}_Z$ by

$$R.H.S\ of\ (4.16) = e^{-tH_{eff}^Z} \otimes P_b.$$

We can also investigate WCLUV of the massive Nelson model with a set of scalar potentials $V = \{V_Z\}_{Z=0}^{\infty}$ with $V_Z \in \mathcal{M}_\pm(Z)_r$ in the space dimensions $d = 1, 2, 3$ by functional integral representations, restricting ultraviolet cut-off parameters $0 < \alpha < \frac{2}{d+1}$. Without the proofs, we only show the Hamiltonian H_{eff}^Z in the case where the space dimensions d are 1,2 and 3 in Fig.5.1. Since, in the case where $d = 1, 2$ and $\mu = 0$, each scalar potential in H_{eff}^Z does not converge, we can not hope to get nontrivial WCLUV of the massless Nelson model in the case $d = 1, 2$.

	H_{eff}^Z	Renormalization $E_w(\Lambda^\alpha)$	μ	α
$d=3$	$\frac{1}{2m}\mathbf{p}_Z^2 - \frac{g^2}{4\pi}\sum_{i<j}^Z \frac{e^{-\mu\lvert x^i-x^j\rvert}}{\lvert x^i-x^j\rvert} + V_Z$	$-\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{\Xi_{\Lambda^\alpha}(k)}{k^2+\mu^2}dk$	$\mu \geq 0$	$0 < \alpha < \frac{1}{2}$
$d=2$	$\frac{1}{2m}\mathbf{p}_Z^2 - \frac{g^2}{2\pi}\sum_{i\neq j}^Z \int_0^\infty \frac{rJ_0(r\lvert x^i-x^j\rvert)}{r^2+\mu^2}dr + V_Z$	$-\frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2}\frac{\Xi_{\Lambda^\alpha}(k)}{k^2+\mu^2}dk$	$\mu > 0$	$0 < \alpha < \frac{2}{3}$
$d=1$	$\frac{1}{2m}\mathbf{p}_Z^2 - \frac{g^2}{4}\sum_{i,j=1}^Z \frac{e^{-\mu\lvert x^i-x^j\rvert}}{\mu} + V_Z$	0	$\mu > 0$	$0 < \alpha < 1$

Fig.5.1

Here $V_Z \in \mathcal{M}_\pm(Z)_r$, and J_0 is the Bessel function with the order 0, i.e, $J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n}$.

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