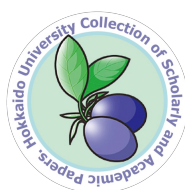




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# A New Estimate for the Ground State Energy of Schrödinger Operators

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## Abstract

A new estimate for the ground state energy of Schrödinger operators on  $L^2(\mathbf{R}^n)$  ( $n \geq 1$ ) is presented. As a corollary, it is shown that the ground state energy of the Schrödinger operator with a scalar potential  $V$  is more than the classical lower bound  $\text{ess.inf}_{x \in \mathbf{R}^n} V(x)$  if  $V$  is essentially bounded from below in a certain manner (enhancement of the ground state energy due to quantization). As an application, it is proven that the ground state energy of the Hamiltonian of the hydrogen-like atom is enhanced under a class of perturbations given by scalar potentials (vanishing at infinity).

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## 1 Introduction and the main results

In this Letter we present a new estimate for the ground state energy of Schrödinger operators and its applications. The method is based on a remarkable structure of a class of symmetric operators on an  $L^2$  space [1] (see Theorem 2.2 in Section 2 below).

Let  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$  ( $n \in \mathbf{N}$ ) and  $a(x) = (a_{ij}(x))_{i,j=1,\dots,n}$  be an  $n \times n$  real, positive semi-definite matrix of the form

$$a(x) = b(x)^* b(x), \quad (1.1)$$

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where  $b(x) = (b_{ij}(x))_{i,j=1,\dots,n}$  is an  $n \times n$  complex matrix. We assume that each function  $b_{ij} : x \rightarrow b_{ij}(x)$  is continuously differentiable on  $\mathbf{R}^n$ .

We denote by  $D_j$  the generalized partial differential operator in the  $j$ -th variable  $x_j$ . We define a symmetric operator  $H_0$  on  $L^2(\mathbf{R}^n)$  by

$$H_0 := - \sum_{j,k=1}^n D_j a_{jk} D_k \quad (1.2)$$

with  $D(H_0) := \{f \in L^2(\mathbf{R}^n) | f \in D(D_j), a_{jk} D_k f \in D(D_j), j, k = 1, \dots, n\}$ , where  $D(\cdot)$  denotes operator domain. It follows that  $H_0 \geq 0$ .

For  $p \geq 1$  (resp.  $p = \infty$ ), we denote by  $L_{\text{loc}}^p(\mathbf{R}^n)$  the set of functions  $f$  on  $\mathbf{R}^n$  such that for all compact sets  $K$  of  $\mathbf{R}^n$ ,  $\int_K |f(x)|^p dx < \infty$  (resp.  $\text{ess.sup}_{x \in \mathbf{R}^n} |f(x)| < \infty$ ), where  $\text{ess.sup}$  denotes essential supremum. Let  $V$  be a real-valued measurable function on  $\mathbf{R}^n$  such that

$$V \in L_{\text{loc}}^2(\mathbf{R}^n). \quad (1.3)$$

We denote the multiplication operator defined by the function  $V$  on  $L^2(\mathbf{R}^n)$  by the same symbol  $V$ . It follows that  $V$  is self-adjoint with  $D(V) \supset C_0^\infty(\mathbf{R}^n)$ .

We consider a symmetric operator of the form

$$H_{a,V} := H_0 + V \quad (1.4)$$

on  $L^2(\mathbf{R}^n)$  with  $D(H_{a,V}) = C_0^\infty(\mathbf{R}^n)$ . The ground state energy (the lower bound) of  $H_{a,V}$  is defined by

$$E_0(H_{a,V}) := \inf_{u \in C_0^\infty(\mathbf{R}^n), \|u\|_{L^2(\mathbf{R}^n)}=1} (u, H_{a,V} u)_{L^2(\mathbf{R}^n)}, \quad (1.5)$$

provided that  $H_{a,V}$  is bounded from below, where  $(\cdot, \cdot)_{L^2(\mathbf{R}^n)}$  and  $\|\cdot\|_{L^2(\mathbf{R}^n)}$  denote the inner product and the norm of  $L^2(\mathbf{R}^n)$  respectively.

If  $H_{a,V}$  is essentially self-adjoint and bounded from below, then  $E_0(H_{a,V}) = \inf \sigma(\overline{H}_{a,V})$ , the infimum of the spectrum of  $\overline{H}_{a,V}$  (the closure of  $H_{a,V}$ ). For the main theorems stated below, however, we do not need assume that  $H_{a,V}$  is essentially self-adjoint.

The operator  $H_{a,V}$  with  $a(x) = (\delta_{ij})_{i,j=1,\dots,n}$  is the Schrödinger operator with the scalar potential  $V$ :

$$H_V := -\Delta + V, \quad (1.6)$$

where  $\Delta := \sum_{j=1}^n D_j^2$  is the Laplacian on  $L^2(\mathbf{R}^n)$ .

The function space

$$W(\mathbf{R}^n) := \{f \in L_{\text{loc}}^\infty(\mathbf{R}^n) \mid f \text{ is real-valued, } D_j f \in L_{\text{loc}}^2(\mathbf{R}^n), \\ D_j D_k f \in L_{\text{loc}}^1(\mathbf{R}^n), j, k = 1, \dots, n\} \quad (1.7)$$

plays an important role in our theory. For each  $f \in W(\mathbf{R}^n)$ , we define a real-valued measurable function

$$\Phi_{a,V}(x; f) := \sum_{j,k=1}^n [D_j(a_{jk}(x) D_k f(x)) - (D_j f(x)) a_{jk}(x) D_k f(x)] + V(x). \quad (1.8)$$

We introduce

$$\mathcal{W}_{a,V} := \{f \in W(\mathbf{R}^n) \mid \text{ess.inf}_{x \in \mathbf{R}^n} \Phi_{a,V}(x; f) > -\infty\}, \quad (1.9)$$

where  $\text{ess.inf}$  denotes essential infimum. In the case where  $\mathcal{W}_{a,V} \neq \emptyset$ , we define for  $f \in \mathcal{W}_{a,V}$

$$E_{a,V}(f) := \text{ess.inf}_{x \in \mathbf{R}^n} \Phi_{a,V}(x; f). \quad (1.10)$$

The fundamental result of this Letter is the following theorem, which gives a criterion of the boundedness from below of  $H_{a,V}$  and an estimate for  $E_0(H_{a,V})$  simultaneously.

**THEOREM 1.1.** *Suppose that  $\mathcal{W}_{a,V} \neq \emptyset$ . Then  $H_{a,V}$  is bounded from below and*

$$E_0(H_{a,V}) \geq \sup_{f \in \mathcal{W}_{a,V}} E_{a,V}(f). \quad (1.11)$$

*Remark 1.1.* We emphasize that, in Theorem 1.1,  $V$  is not necessarily essentially bounded from below.

*Remark 1.2.* One can consider  $H_{a,V}$  as a quadratic form with form domain  $C_0^\infty(\mathbf{R}^n)$ . In this case, the basic assumption (1.3) for  $V$  can be weakened to the condition  $V \in L_{\text{loc}}^1(\mathbf{R}^n)$ .

*Remark 1.3.* Under the hypothesis of Theorem 1.1,  $H_{a,V}$  is bounded from below. Hence it has a self-adjoint extension  $\widetilde{H}_{a,V}$ , the Friedrichs extension. It follows that  $E_0(\widetilde{H}_{a,V}) \geq \sup_{f \in \mathcal{W}_{a,V}} E_{a,V}(f)$ .

*Remark 1.4.* Consider the case where  $V$  is essentially bounded from below. Then  $\mathcal{W}_{a,V} \neq \emptyset$ , since  $0 \in \mathcal{W}_{a,V}$ . We have  $E_{a,V}(0) = \text{ess.inf } V(x)$ . Hence Theorem 1.1 implies that

$$E_0(H_{a,V}) \geq \sup_{f \in \mathcal{W}_{a,V}} E_{a,V}(f) \geq \text{ess.inf } V(x).$$

Thus Theorem 1.1 may improve the classical lower bound  $\text{ess.inf } V(x)$ . As Theorem 1.3 below shows, under an additional condition, (1.11) indeed gives an improved lower bound.

The following theorem shows that inequality (1.11) is best possible in a sense.

**THEOREM 1.2.** *Suppose that  $\overline{H}_{a,V}$  has a real eigenvalue  $\lambda$  with an eigenvector  $\Omega$  (i.e.,  $\Omega \in D(\overline{H}_{a,V})$  and  $\overline{H}_{a,V}\Omega = \lambda\Omega$ ) having the following properties: (i)  $\Omega > 0$ , a.e.; (ii)  $\Omega \in D(H_0) \cap D(V)$  and  $\overline{H}_{a,V}\Omega = (H_0 + V)\Omega$ ; (iii)  $f_0 := -\log \Omega \in W(\mathbf{R}^n)$ . Then,  $f_0 \in \mathcal{W}_{a,V}$  and  $H_{a,V}$  is bounded from below with*

$$\lambda = E_0(H_{a,V}) = \sup_{f \in \mathcal{W}_{a,V}} E_{a,V}(f) = E_{a,V}(f_0). \quad (1.12)$$

*Remark 1.5.* Theorem 1.2, which is a variant of [1, Theorem 1.2], shows that any positive eigenfunction of  $\bar{H}_{a,V}$  with suitable regularities is a ground state of  $\bar{H}_{a,V}$ . We note that there is a similarity between Theorem 1.2 and the Allegretto-Piepenbrink theorem [2, Theorem 2.12] (cf. also Theorem 2.2 in Section 2 below).

Let

$$v_k(x) := \sum_{j=1}^n D_j a_{jk}(x), \quad k = 1, \dots, n. \quad (1.13)$$

and set

$$v(x) := (v_1(x), \dots, v_n(x)), \quad x \in \mathbf{R}^n. \quad (1.14)$$

**THEOREM 1.3.** *Consider the case where  $V$  is essentially bounded from below and set*

$$V_0 := \text{ess.inf}_{x \in \mathbf{R}^n} V(x). \quad (1.15)$$

*Suppose that  $a(x)$  and  $v(x)$  are bounded on  $\mathbf{R}^n$  and that there exists a point  $x_0 \in \mathbf{R}^n$  such that  $V_0 = V(x_0)$  and, for some  $R > 0$ ,*

$$\text{ess.inf}_{|x-x_0|>R} V(x) > V_0, \quad (1.16)$$

$$\inf_{|x-x_0| \leq R} \text{tr } a(x) > (R^2 + 1)R \sup_{|x-x_0| \leq R} |v(x)|, \quad (1.17)$$

*where  $\text{tr}$  denotes trace. Then  $E_0(H_{a,V}) > V_0$ .*

*Remark 1.6.* We can compute a constant  $\gamma$  such that  $E_0(H_{a,V}) \geq \gamma > V_0$ . See (3.11).

*Remark 1.7.* Obviously (1.17) holds for  $a(x) = (\delta_{ij})_{i,j=1,\dots,n}$ . Hence Theorem 1.3 can be applied to the Schrödinger operator  $H_V$ . In the context of quantum mechanics, Theorem 1.3 establishes a mathematically rigorous basis for the folklore that quantization should raise the ground state energy for a wide class of scalar potentials bounded from below (this traditional picture comes from the Heisenberg uncertainty principle).

This Letter is organized as follows. In Section 2 (resp. 3) we prove Theorems 1.1 and 1.2 (resp. Theorem 1.3). In Section 4 we apply Theorem 1.1 to the Hamiltonian of the hydrogen-like atom perturbed by a scalar potential and show that the ground state energy is enhanced under the perturbation.

## 2 Proof of Theorems 1.1 and 1.2

Theorem 1.1 is proven by applying an extension of [1, Corollary 3.4], which, for the reader's convenience, we prove it first.

We introduce an operator

$$L_j := \sum_{k=1}^n b_{jk}(x) D_k, \quad j = 1, \dots, n. \quad (2.1)$$

on  $L^2(\mathbf{R}^n)$  with  $D(L_j) = C_0^\infty(\mathbf{R}^n)$ . Then we have

$$H_0 = \sum_{j=1}^n L_j^* L_j \quad \text{on } C_0^\infty(\mathbf{R}^n). \quad (2.2)$$

For each  $f \in W(\mathbf{R}^n)$ , we define

$$\widehat{H}_0 f = - \sum_{j,k=1}^n D_j(a_{jk}(x) D_k f), \quad (2.3)$$

$$\widehat{L}_j f = \sum_{k=1}^n b_{jk}(x) D_k f. \quad (2.4)$$

We denote by  $\langle \cdot \rangle$  the expectation value with respect to the  $n$ -dimensional Lebesgue measure:

$$\langle f \rangle := \int_{\mathbf{R}^n} f(x) dx, \quad f \in L^1(\mathbf{R}^n). \quad (2.5)$$

LEMMA 2.1 [1, cf. Lemma 3.3]. *Let  $\Omega$  be an a.e. strictly positive function  $\Omega$  on  $\mathbf{R}^n$  such that  $\Omega \in W(\mathbf{R}^n)$  and  $\Omega^{-1} \in L_{\text{loc}}^\infty(\mathbf{R}^n)$ . Let*

$$\Omega_j := \Omega^{-1} \widehat{L}_j \Omega. \quad (2.6)$$

Then, for all  $u \in C_0^\infty(\mathbf{R}^n)$ ,

$$\langle |u|^2 \Omega^{-1} \widehat{H}_0 \Omega \rangle = \sum_{j=1}^n \left\{ 2 \operatorname{Re} \langle (L_j u)^* u \Omega_j \rangle - \langle |u|^2 |\Omega_j|^2 \rangle \right\}. \quad (2.7)$$

*Proof.* Let  $J_\varepsilon$  ( $\varepsilon > 0$ ) be the Friedrichs mollifier and  $\Omega_m := J_{1/m} \Omega$ ,  $m = 1, 2, \dots$ . Then  $\Omega_m \in C^\infty(\mathbf{R}^n)$ . Moreover, for all compact sets  $K$  of  $\mathbf{R}^n$ ,  $p \in [1, \infty)$ , and  $j, k = 1, \dots, n$ ,

$$\|\Omega_m - \Omega\|_{L^p(K)} \rightarrow 0, \quad (2.8)$$

$$\|D_j \Omega_m - D_j \Omega\|_{L^2(K)} \rightarrow 0, \quad (2.9)$$

$$\|D_j D_k \Omega_m - D_j D_k \Omega\|_{L^1(K)} \rightarrow 0, \quad (2.10)$$

as  $m \rightarrow \infty$ . We have also for all  $r > 0$

$$\inf_{|x| < r} \Omega_m(x) \geq \text{ess. inf}_{|x| < r+m-1} \Omega(x) > 0, \quad (2.11)$$

$$\sup_{|x| < r} \Omega_m(x) \leq \text{ess. sup}_{|x| < r+m-1} \Omega(x) < \infty. \quad (2.12)$$

By integration by parts, one can easily show that (2.7) holds with  $\Omega$  replaced by  $\Omega_m$ . Then a simple limiting argument using (2.8)–(2.12) gives (2.7) (we need to take a subsequence  $\{\Omega_{m_i}\}_{i=1}^\infty$  of  $\{\Omega_m\}_m$  such that  $\Omega_{m_i} \rightarrow \Omega$  a.e. as  $i \rightarrow \infty$  on the support of  $u$ ).  $\square$

**THEOREM 2.2.** *Suppose that there exists an a.e. strictly positive function  $\Omega$  on  $\mathbf{R}^n$  such that  $\Omega \in W(\mathbf{R}^n)$ ,  $\Omega^{-1} \in L_{\text{loc}}^\infty(\mathbf{R}^n)$ , and*

$$\widehat{H}_0 \Omega(x) + V(x) \Omega(x) \geq \lambda \Omega(x), \quad \text{a.e. } x, \quad (2.13)$$

*with a real constant  $\lambda$ . Then  $E_0(H_{a,V}) \geq \lambda$ .*

*Proof.* This follows from Lemma 2.1 and a simple application of [1, Theorem 1.1] (take  $\Psi = \Omega^{-1} \widehat{H}_0 \Omega$  as the vector  $\Psi$  in [1, Theorem 1.1]).  $\square$

**LEMMA 2.3.** *Let  $f \in W(\mathbf{R}^n)$  and set*

$$\Omega_f = e^{-f}. \quad (2.14)$$

*Then  $\Omega_f \in W(\mathbf{R}^n)$ ,  $\Omega_f^{-1} \in L_{\text{loc}}^\infty(\mathbf{R}^n)$  and*

$$D_j \Omega_f = -(D_j f) \Omega_f, \quad D_k D_j \Omega_f = [-D_k D_j f + (D_j f)(D_k f)] \Omega_f. \quad (2.15)$$

*Proof.* Let  $f_m = J_{1/m} f$ . Then  $f_m \in C^\infty(\mathbf{R}^n)$  and, for all  $r > 0$ ,

$$\sup_{|x| < r} |f_m(x)| \leq \sup_{|x| < r+m-1} |f(x)| < \infty.$$

Moreover, (2.8)–(2.10) with  $\Omega_m$  and  $\Omega$  replaced by  $f_m$  and  $f$  respectively hold. Let  $u \in C_0^\infty(\mathbf{R}^n)$  and  $\text{supp } u \subset \{x \in \mathbf{R}^n \mid |x| \leq r\}$ . By integration by parts, we have  $\langle \Omega_{f_m} D_j u \rangle = \langle (D_j f_m) \Omega_{f_m} u \rangle$ . Using the elementary inequality

$$|e^s - e^t| \leq |s - t| e^{\max\{s, t\}}, \quad s, t \in \mathbf{R},$$

we have

$$|\langle (\Omega_{f_m} - \Omega_f) u \rangle| \leq e^{\text{ess. sup}_{|x| \leq r+1} |f(x)|} \sup_{x \in \mathbf{R}^n} |u(x)| \left( \int_{|x| \leq r} |f_m(x) - f(x)| dx \right)$$

Hence  $\lim_{m \rightarrow \infty} \langle \Omega_{f_m} u \rangle = \langle \Omega_f u \rangle$ . Similarly  $\lim_{m \rightarrow \infty} \langle (D_j f_m) \Omega_{f_m} u \rangle = \langle (D_j f) \Omega_f u \rangle$ . Hence  $\langle (D_j f) \Omega_f u \rangle = \langle \Omega_f D_j u \rangle$ , which means that  $D_j \Omega_f = -(D_j f) \Omega_f$ . The condition  $f \in L_{\text{loc}}^\infty(\mathbf{R}^n)$  implies that  $\Omega_f, \Omega_f^{-1} \in L_{\text{loc}}^\infty(\mathbf{R}^n)$ . It follows that  $D_j \Omega_f \in L_{\text{loc}}^2(\mathbf{R}^n)$ .

In the same way, we can prove the second equation of (2.15) and  $D_k D_j \Omega_f \in L^1_{\text{loc}}(\mathbf{R}^n)$ .  
□

We are now ready to prove Theorems 1.1 and 1.2.

### Proof of Theorem 1.1

Let  $f \in \mathcal{W}_{a,V}$  and  $\Omega_f = e^{-f}$ . Then the conclusion of Lemma 2.3 holds. By direct computation using (2.15), we have

$$\begin{aligned}\widehat{H}_0 \Omega_f(x) + V(x) \Omega_f(x) &= \Phi_{a,V}(x; f) \Omega_f(x) \\ &\geq E_{a,V}(f) \Omega_f(x), \quad \text{a.e. } x.\end{aligned}$$

By this fact and Lemma 2.3,  $\Omega = \Omega_f$  satisfies the assumption of Theorem 2.2 with  $\lambda = E_{a,V}(f)$ . Hence  $E_0(H_{a,V}) \geq E_{a,V}(f)$ . Thus the assertion of Theorem 1.1 follows.  
□

### Proof of Theorem 1.2

We can write  $\Omega = e^{-f_0}$ . Then the conclusion of Lemma 2.3 holds with  $f = f_0$  and  $\Omega_f = \Omega$ . Hence  $H_0 \Omega = \widehat{H}_0 \Omega$  and  $\widehat{H}_0 \Omega(x) + V(x) \Omega(x) = \lambda \Omega(x)$ , a.e.  $x$ , which implies that  $\Phi_{a,V}(x; f_0) = \lambda$ , a.e.  $x$ . Hence  $f_0 \in \mathcal{W}_{a,V}$  and  $E_{a,V}(f_0) = \lambda$ . Thus, by Theorem 1.1, we obtain  $E_0(H_{a,V}) \geq \lambda$ . On the other hand, by a simple limiting argument, one can easily see that  $E_0(H_{a,V}) \leq \lambda$ . Hence  $E_0(H_{a,V}) = \lambda = E_{a,V}(f_0)$ . The last equality of (1.12) follows from this fact,  $\sup_{f \in \mathcal{W}_{a,V}} E_{a,V}(f) \geq E_{a,V}(f_0)$  and (1.11). □

## 3 Proof of Theorem 1.3

The idea of the proof is to find a “trial function”  $f$  in  $\mathcal{W}_{a,V}$  such that  $E_{a,V}(f) > V_0$ . Let  $t, \varrho > 0$  and

$$f_{t,\varrho}(x) = t \sqrt{(x - x_0)^2 + \varrho^2}.$$

Then  $f_{t,\varrho} \in W(\mathbf{R}^n)$ . By direct computation, we have for all  $x \in \mathbf{R}^n$

$$\begin{aligned}\Phi_{a,V}(x; f_{t,\varrho}) &= \frac{t}{[(x - x_0)^2 + \varrho^2]^{3/2}} \{ (x - x_0) [\text{tr } a(x) - a(x)] (x - x_0) + \varrho^2 \text{tr } a(x) \} \\ &\quad + \frac{tv(x)(x - x_0)}{\sqrt{(x - x_0)^2 + \varrho^2}} - \frac{(x - x_0)a(x)(x - x_0)}{(x - x_0)^2 + \varrho^2} t^2 + V(x).\end{aligned}$$

Noting that  $\text{tr } a(x) - a(x)$  is positive semi-definite, we obtain

$$\begin{aligned}\Phi_{a,V}(x; f_{t,\varrho}) &\geq \frac{\varrho^2 \text{tr } a(x)}{[(x - x_0)^2 + \varrho^2]^{3/2}} t + \frac{tv(x)(x - x_0)}{\sqrt{(x - x_0)^2 + \varrho^2}} \\ &\quad - \frac{(x - x_0)a(x)(x - x_0)}{(x - x_0)^2 + \varrho^2} t^2 + V(x).\end{aligned}\tag{3.1}$$

To estimate the right hand side of (3.1) from below, we first consider the case where  $|x - x_0| \leq R$ . We set

$$L_R(a) := \inf_{|x-x_0| \leq R} \operatorname{tr} a(x), \quad (3.2)$$

$$a_{R1} := \sup_{|x-x_0| \leq R} \|a(x)\|, \quad v_{R1} := \sup_{|x-x_0| \leq R} |v(x)|, \quad (3.3)$$

where  $\|a(x)\|$  is the norm of  $a(x)$  as a bounded linear operator on  $\mathbf{C}^n$ . Since  $L_R(a) > 0$  by (1.17), it follows that  $a_{R1} > 0$ . By the fact that, for  $|x - x_0| \leq R$ ,

$$\frac{(x - x_0)^2}{(x - x_0)^2 + \varrho^2} \leq \frac{R^2}{R^2 + \varrho^2},$$

we have

$$\Phi_{a,V}(x; f_{t,\varrho}) \geq C(t, \varrho),$$

where

$$C(t, \varrho) := \frac{1}{(R^2 + \varrho^2)^{3/2}} [L_R(a)\varrho^2 - v_{R1}R(R^2 + \varrho^2)] t - \frac{a_{R1}R^2}{R^2 + \varrho^2} t^2 + V_0. \quad (3.4)$$

Let  $\varrho > 1$ . Then, by (1.17), we can show that

$$L_R(a)\varrho^2 > v_{R1}(R^2 + \varrho^2)R.$$

Hence, putting

$$t_1 := \frac{1}{a_{R1}R^2\sqrt{R^2 + \varrho^2}} [L_R(a)\varrho^2 - v_{R1}R(R^2 + \varrho^2)] > 0, \quad (3.5)$$

we have for all  $t \in (0, t_1)$  and  $|x - x_0| \leq R$

$$\Phi_{a,V}(x; f_{t,\varrho}) \geq C(t, \varrho) > V_0. \quad (3.6)$$

We next consider the case  $|x - x_0| > R$ . Let

$$a_{R2} := \sup_{|x-x_0| > R} \|a(x)\|, \quad v_{R2} := \sup_{|x-x_0| > R} |v(x)|, \quad (3.7)$$

$$V_R := \operatorname{ess.inf}_{|x-x_0| > R} V(x) > V_0. \quad (3.8)$$

Then we have

$$\Phi_{a,V}(x; f_{t,\varrho}) \geq D(t)$$

where

$$D(t) := -v_{R2}t - a_{R2}t^2 + V_R. \quad (3.9)$$

We define a constant  $t_2 > 0$  as follows: in the case  $a_{R2} > 0$ ,

$$t_2 := \frac{-v_{R2} + \sqrt{v_{R2}^2 + 4a_{R2}(V_R - V_0)}}{2a_{R2}} > 0. \quad (3.10)$$

In the case  $a_{R2} = 0$ ,

$$t_2 := \frac{V_R - V_0}{v_{R2}}, \quad (3.11)$$

where  $t_2 := \infty$  if  $v_{R2} = 0$ . Then we have for all  $t \in (0, t_2)$  and  $|x - x_0| > R$

$$\Phi_{a,V}(x; f_{t,\varrho}) \geq D(t) > V_0. \quad (3.12)$$

By (3.6) and (3.12), we obtain

$$E_{a,V}(f_{t,\varrho}) \geq \gamma(t, \varrho) > V_0, \quad t \in (0, \min\{t_1, t_2\}), \quad \varrho > 1,$$

where

$$\gamma(t, \varrho) := \min\{C(t, \varrho), D(t)\}. \quad (3.13)$$

Hence, by Theorem 1.1, we have for all  $t \in (0, \min\{t_1, t_2\})$  and  $\varrho > 1$

$$E_0(H_{a,V}) \geq \gamma(t, \varrho) > V_0. \quad (3.14)$$

□

## 4 Application to the Hamiltonian of a perturbed hydrogen-like atom

The Hamiltonian of the hydrogen-like atom is given by

$$H_{\text{hyd}} = -\Delta - \frac{Z}{|x|} \quad (4.1)$$

on  $L^2(\mathbf{R}^n)$  with  $Z$  a positive constant (the physical case is the case  $n = 3$ ). A simple application of Theorem 1.2 shows that  $H_{\text{hyd}}$  is bounded from below with

$$E_0(H_{\text{hyd}}) = -\frac{Z^2}{(n-1)^2}, \quad (4.2)$$

see [1, §4.1].

We consider a perturbation of  $H_{\text{hyd}}$  by a nonnegative potential  $U$  on  $\mathbf{R}^n$ :

$$H := H_{\text{hyd}} + U = -\Delta - \frac{Z}{|x|} + U \quad (4.3)$$

with  $D(H) = C_0^\infty(\mathbf{R}^n)$ . We want to show that the perturbation  $U$  with an additional condition enhances the ground state energy of the hydrogen-like atom even in the case  $\text{ess.inf}_{x \in \mathbf{R}^n} U(x) = 0$ .

**THEOREM 4.1.** *Let  $n \geq 2$  and  $U \geq 0$ . Suppose that there exists a constant  $R > n(n-1)/2Z$  such that*

$$\text{ess.inf}_{|x| \leq R} U(x) > 0. \quad (4.4)$$

Then

$$E_0(H) > E_0(H_{\text{hyd}}). \quad (4.5)$$

*Remark.* The conclusion of Theorem 4.1 is nontrivial only if  $\text{ess.inf}_{x \in \mathbf{R}^n} U(x) = 0$ . As the proof below shows, we can compute a constant  $\delta$  such that  $E_0(H) \geq \delta > E_0(H_{\text{hyd}})$ , see (4.25). This gives an estimate for  $E_0(H)$ .

*Proof.* We set  $r = |x|$ ,  $x \in \mathbf{R}^n$ , and

$$\Phi(x; f) := \Phi_{I,V}(x; f) = \Delta f(x) - \sum_{j=1}^n |D_j f(x)|^2 - \frac{Z}{r} + U(x), \quad (4.6)$$

where  $V(x) = -Zr^{-1} + U(x)$ . We take as a trial function

$$g_{s,\alpha}(x) = rs + \alpha \log(1+r), \quad x \in \mathbf{R}^n,$$

where  $s > 0$  and  $\alpha > 0$  are parameters to be chosen suitably. It is easy to see that  $g_{s,\alpha} \in W(\mathbf{R}^n)$  with

$$\begin{aligned} D_j g_{s,\alpha}(x) &= \frac{x_j}{r} \left( s + \frac{\alpha}{1+r} \right), \\ D_k D_j g_{s,\alpha}(x) &= \frac{\delta_{jk}}{r} \left( s + \frac{\alpha}{1+r} \right) - \frac{x_j x_k}{r^3} \left( s + \frac{\alpha}{1+r} + \frac{\alpha r}{(1+r)^2} \right). \end{aligned}$$

Hence

$$\Phi(x; g_{s,\alpha}) = \frac{(n-1)(s+\alpha) - Z}{r} - \frac{(n-1)\alpha}{1+r} - \frac{\alpha}{(1+r)^2} - \left( s + \frac{\alpha}{1+r} \right)^2 + U(x). \quad (4.7)$$

Let

$$U_R := \text{ess.inf}_{r \leq R} U(x) > 0. \quad (4.8)$$

Take  $s, \alpha > 0$  such that

$$(n-1)(s+\alpha) > Z. \quad (4.9)$$

Then, for  $r \leq R$ , we have

$$\Phi(x; g_{s,\alpha}) \geq \delta_1(s, \alpha, R), \quad (4.10)$$

where

$$\delta_1(s, \alpha, R) := \frac{(n-1)(s+\alpha) - Z}{R} - \alpha n - (s+\alpha)^2 + U_R. \quad (4.11)$$

Let

$$\alpha_1 = \frac{U_R}{n}, \quad \alpha_2 = \frac{-n + \sqrt{n^2 + 4 \left( U_R + \frac{Z^2}{(n-1)^2} \right)}}{2}, \quad (4.12)$$

For all  $\alpha \in (0, \min\{\alpha_1, \alpha_2\})$ , the interval

$$I_1(\alpha) := \left(0, \sqrt{U_R - \alpha n + \frac{Z^2}{(n-1)^2}} - \alpha\right) \quad (4.13)$$

is not empty. We define a set

$$S_1 := \{(s, \alpha) | 0 < \alpha < \min\{\alpha_1, \alpha_2\}, s \in I_1(\alpha), s \text{ and } \alpha \text{ satisfy (4.9)}\}. \quad (4.14)$$

This is a non-empty set. It follows that, for all  $(s, \alpha) \in S_1$ ,  $\delta_1(s, \alpha, R) > E_0(H_{\text{hyd}})$ . Hence

$$\text{ess.inf}_{r \leq R} \Phi(x; g_{s, \alpha}) \geq \delta_1(s, \alpha, R) > E_0(H_{\text{hyd}}), \quad (s, \alpha) \in S_1. \quad (4.15)$$

We next consider the case  $r \geq R$ . We set

$$U'_R := \text{ess.inf}_{r \geq R} U(x) \geq 0. \quad (4.16)$$

Then we have

$$\Phi(x; g_{s, \alpha}) \geq \delta_2(s, \alpha, R), \quad (4.17)$$

where

$$\delta_2(s, \alpha, R) := -\frac{\alpha n}{1+R} - \left(s + \frac{\alpha}{1+R}\right)^2 + U'_R. \quad (4.18)$$

Let

$$\alpha_3 = \frac{(1+R)Z^2}{n(n-1)^2}, \quad (4.19)$$

$$\alpha_4 = \frac{\left(-n + \sqrt{n^2 + 4\left(U'_R + \frac{Z^2}{(n-1)^2}\right)}\right)(1+R)}{2}. \quad (4.20)$$

Then, for all  $\alpha \in (0, \min\{\alpha_3, \alpha_4\})$ , the interval

$$I_2(\alpha) := \left(0, \sqrt{U'_R - \frac{\alpha n}{1+R} + \frac{Z^2}{(n-1)^2}} - \frac{\alpha}{1+R}\right) \quad (4.21)$$

is not empty. By virtue of the condition  $R > n(n-1)/2Z$ , we have

$$\alpha_5 := \frac{2(1+R)Z}{R(n-1)} \left(1 - \frac{n(n-1)}{2ZR}\right) > 0. \quad (4.22)$$

Then the set

$$S_2 := \{(s, \alpha) | 0 < \alpha < \min\{\alpha_3, \alpha_4, \alpha_5\}, s \in I_2(\alpha), s \text{ and } \alpha \text{ satisfy (4.9)}\} \quad (4.23)$$

is not empty and, for all  $(s, \alpha) \in S_2$ , we have  $\delta_2(s, \alpha, R) > E_0(H_{\text{hyd}})$ . Hence

$$\text{ess.inf}_{r \geq R} \Phi(x; g_{s, \alpha}) \geq \delta_2(s, \alpha, R) > E_0(H_{\text{hyd}}), \quad (s, \alpha) \in S_2. \quad (4.24)$$

It follows from (4.15) and (4.24) that, for all  $(s, \alpha) \in S_1 \cap S_2$ ,

$$\text{ess.inf}_{x \in \mathbf{R}^n} \Phi(x; g_{s, \alpha}) \geq \delta(s, \alpha, R) > E_0(H_{\text{hyd}}),$$

where

$$\delta(s, \alpha, R) := \min\{\delta_1(s, \alpha, R), \delta_2(s, \alpha, R)\}. \quad (4.25)$$

(Note that  $S_1 \cap S_2 \neq \emptyset$ .) Thus, by Theorem 1.1, we obtain

$$E_0(H) \geq \delta(s, \alpha, R) > E_0(H_{\text{hyd}}), \quad (s, \alpha) \in S_1 \cap S_2. \quad (4.26)$$

□

In the following examples, we take  $n \geq 2$ .

EXAMPLE 4.1. *The Yukawa potential.* Let  $0 < \lambda \leq 1$ ,  $m > 0$  be constants and

$$U(x) = \frac{Z(1 - \lambda e^{-mr})}{r} > 0.$$

Then  $H$  becomes the Schrödinger operator with the Yukawa potential  $-Z\lambda e^{-mr}/r$ :

$$H_{\text{Yukawa}} = -\Delta - \frac{Z\lambda e^{-mr}}{r}.$$

It is obvious that, for all  $R > 0$ , (4.4) is satisfied. Hence we can apply Theorem 4.1 to conclude that

$$E_0(H_{\text{Yukawa}}) > E_0(H_{\text{hyd}}).$$

EXAMPLE 4.2. The Hamiltonian of a quantum system of a charged particle interacting with two fixed electric charges, one positive and the other negative, is given (up to a constant multiple) by

$$H_g = -\Delta - \frac{Z}{|x|} + \frac{g}{|x - x_0|},$$

where  $x_0 \in \mathbf{R}^n$  is a fixed point and  $g > 0$  is a constant. It is obvious that  $U(x) := g/|x - x_0|$  satisfies (4.4) for all  $R > 0$ . Hence we can apply Theorem 4.1 to conclude that

$$E_0(H_g) > E_0(H_{\text{hyd}}).$$

## References

- [1] Arai, A., A criterion for the boundedness from below with a class of symmetric operators and its applications, *J. Math. Anal. Appl.* **145**, 539-554 (1990).
- [2] Cycon, H. L., Froese, R. G., Kirsch, W. and Simon, B., *Schrödinger Operators*, Springer, Berlin, Heidelberg, 1987.