HOKKAIDO UNIVERSITY

Title	Norms of some singular integral operators and their inverse operators
Author(s)	Nakazi, T.; Yamamoto, T.
Citation	Hokkaido University Preprint Series in Mathematics, 387, 1-28
Issue Date	1997-09-01
DOI	https://doi.org/10.14943/83533
Doc URL	https://hdl.handle.net/2115/69137
Type	departmental bulletin paper
File Information	re387.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

NORMS OF SOME SINGULAR INTEGRAL OPERATORS AND THEIR INVERSE OPERATORS

T. Nakazi and T. Yamamoto

Series #387. September 1997

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#361 M. Nakamura and T. Ozawa, Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces, 17 pages. 1996.

#362 I. Nakamura, Hilbert schemes and simple singularities E_6, E_7 and E_8, 21 pages. 1996.

#363 T. Mikami, Equivalent conditions on the central limit theorem for a sequence of probability measures on R, 7 pages. 1996.

#364 S. Izumiya and T. Sano, Generic affine differential geometry of space curves, 23 pages. 1996.

#365 T. Tsukada, Stability of reticular optical caustics, 12 pages. 1996.

#366 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, 40 pages. 1996.

#367 A. Arai, A class of representations of the *-algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, 12 pages. 1996.

#368 K. Ito, BV-solutions of a hyperbolic-elliptic system for a radiating gas, 33 pages. 1997.

#369 M. Nakamura and T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, 20 pages. 1997.

#370 N.H. Bingham and A. Inoue, An Abel-Tauber theorem for Hankel transforms, 8 pages. 1997.

#371 T. Nakazi and H. Sawada, The commutator ideal in Toeplitz algebras for uniform algebras and the analytic structure, 9 pages. 1997.

#372 M.-H. Giga and Y. Giga, Stability for evolving graphs by nonlocal weighted curvature, 70 pages. 1997.

#373 T. Nakazi, Brown-Halmos type theorems of weighted Toeplitz operators, 14 pages. 1997.

#374 J. Inoue and S.-E. Takahashi, On characterizations of the image of Gelfand transform of commutative Banach algebras, 30 pages. 1997.

#375 L. Solomon and H. Terao, The double coxeter arrangement, 21 pages. 1997.

#376 G. Ishikawa and T. Morimoto, Solution surfaces of Monge-Ampère equations, 15 pages. 1997.

#377 G. Ishikawa, A relative transversality theorem and its applications, 16 pages. 1997.

#378 J. Inoue and T. Nakazi, On the zeroes of solutions of an extremal problem in H^1, 14 pages. 1997.

#379 Y. Giga and K. Ito, On pinching of curves moved by surface diffusion, 12 pages. 1997.

#380 F. Hiroshima, Weak coupling limit removing an ultraviolet cut-off for a Hamiltonian of particles interacting with a scalar field, 39 pages. 1997.

#381 Y. Giga, S. Matsui and Y. Shimizu, On estimates in Hardy spaces for the Stokes flow in a half space, 13 pages. 1997.

#382 A. Arai, A new estimate for the ground state energy of Schrödinger operators, 12 pages. 1997.

#383 M. Nakamura and T. Ozawa, The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order, 24 pages. 1997.

#384 K. Ito, Asymptotic stability of planar rarefaction wave for scalar viscous conservation law, 8 pages. 1997.

#385 A. Arai, Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems, 32 pages. 1997.

#386 P. Aviles and Y. Giga, On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields, 21 pages. 1997.

NORMS OF SOME SINGULAR INTEGRAL OPERATORS AND THEIR INVERSE OPERATORS

by

TAKAHIKO NAKAZI
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
and
TAKANORI YAMAMOTO
Department of Mathematics
Hokkai-Gakuen University
Sapporo 062, Japan

Abstract. Let α and β be bounded measurable functions on the unit circle T. Then the singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, (f \in L^2(T))$ where P_+ is an analytic projection and P_- is a co-analytic projection. In this paper, the norms of $S_{\alpha,\beta}$ and its inverse operator on the Hilbert space $L^2(T)$ are calculated in general, using α, β and $\alpha\bar{\beta} + H^\infty$. Moreover, the relations between these and the norms of Hankel operators are established. As an application, in some special case in which α and β are nonconstant functions, the norm of $S_{\alpha,\beta}$ is calculated in a completely explicit form. If α and β are constant functions, then it is well known that the norm of $S_{\alpha,\beta}$ on $L^2(T)$ is equal to $\max\{|\alpha|, |\beta|\}$. If α and β are nonzero constant functions, then it is also known that $S_{\alpha,\beta}$ on $L^2(T)$ has an inverse operator $S_{\alpha^{-1},\beta^{-1}}$ whose norm is equal to $\max\left\{|\alpha|^{-1}, |\beta|^{-1}\right\}$.

Keywords: Singular integral operators, Hardy spaces, Hankel operators, Toeplitz operators.

AMS Subject Classification: Primary 45E10, 47B35; Secondary 46J15.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $T = \{\zeta; |\zeta| = 1\}$. That is, $dm(\zeta) = d\theta/2\pi$ for $\zeta = e^{i\theta}$. The inner product in the Hilbert space $L^2 = L^2(T)$ is given by

$$(f, g) = \int_T f(\zeta)\overline{g(\zeta)}dm(\zeta).$$

The norm in L^2 is given by $\|f\|_2 = \sqrt{(f, f)}$. By $L^\infty = L^\infty(T)$ we denote the space of essentially bounded Lebesgue measurable functions. The norm in L^∞ is given by $\|f\|_\infty = \operatorname{ess\,sup}_T |f|$. Let H^2 (resp. H^∞) be the Hardy space of functions $f \in L^2$ (resp. $f \in L^\infty$) whose negative Fourier coefficients are zero. Let $\overline{H_0^2}$ be the space of functions $\overline{f} \in L^2$ whose nonnegative Fourier coefficients are zero. Then $L^2 = H^2 \oplus \overline{H_0^2}$. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi i}\int_T \frac{f(\eta)}{\eta - \zeta}d\eta \qquad (a.e.\zeta \in T),$$

where the integral is understood in the sence of Cauchy's principal value (cf. [4, p.12]). If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and Sf becomes a Lebesgue measurable function on T. Let an analytic projection and a co-analytic projection be

$$P_+ = (I + S)/2, \quad \text{and} \quad P_- = (I - S)/2,$$

where I denotes the identity operator. If $\alpha, \beta \in L^\infty$, then the singular integral operator $S_{\alpha,\beta}$ on L^2 is defined by

$$S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, \quad (f \in L^2).$$

Then $S_{1,1} = I$, $S_{1,-1} = S$, $S_{1,0} = P_+$ and $S_{0,1} = P_-$. The norm of $S_{\alpha,\beta}$ is defined by

$$\|S_{\alpha,\beta}\| = \sup_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta}f\|_2.$$

Since $\|P_+\| = \|P_-\| = 1$, we have

$$\|S_{\alpha,\beta}\| \leq \|\alpha\|_\infty + \|\beta\|_\infty < \infty.$$

Hence, $S_{\alpha,\beta}$ is a bounded operator on L^2. Furthermore, it is well known that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \le \|S_{\alpha,\beta}\| \le \left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty .$$

If α and β are constant functions, then it is well known and not difficult to establish that

$$\|S_{\alpha,\beta}\| = \max\{|\alpha|, |\beta|\}$$

(cf. [3]). If α and β are nonconstant functions, then we will show in Section 2 that the formula of $\|S_{\alpha,\beta}\|$ is more complicated.

If $\phi \in L^\infty$, then the Toeplitz operator T_ϕ is defined by $T_\phi f = P_+(\phi f)$ for $f \in H^2$. This operator is bounded, and it's norm is equal to $\|\phi\|_\infty$ (cf. [2, p.179]). The Hankel operator H_ϕ is defined by $H_\phi f = P_-(\phi f)$ for $f \in H^2$. This operator is bounded, and by the Nehari theorem [8] (cf. [9, p.181]), it's norm is equal to $\inf\{\|\phi - k\|_\infty; k \in H^\infty\}$. Hence,

$$\|H_\phi\| \le \|T_\phi\| \le \|S_{\phi,1}\|.$$

Though the norm of T_ϕ or H_ϕ is known, the norm of $S_{\phi,1}$ is not known.

In this paper, we consider the operator $S_{\alpha,\beta}$ on L^2 for functions $\alpha, \beta \in L^\infty$. In Section 2, we give the formula of the norm of $S_{\alpha,\beta}$ on L^2 for $\alpha, \beta \in L^\infty$, which involve lower bounds over the algebra H^∞. It is a little surprising that the norm of the singular integral operator $S_{\alpha,\beta}$ is related to the norm of the Hankel operator $H_{\alpha\bar{\beta}}$ for some special α and β. In Section 3, we also give the formula of the norm of $S^{-1}_{\alpha,\beta}$ on L^2 for $\alpha, \beta \in L^\infty$, which involve upper bounds over the algebra H^∞. If $S_{\alpha,\beta}$ is invertible, then $\operatorname{ess\,inf}_T(\min\{|\alpha|, |\beta|\}) > 0$. When $\phi = \alpha/\beta$ and $\operatorname{ess\,inf}_T(\min\{|\alpha|, |\beta|\}) > 0$,

$$S_{\alpha,\beta} = \beta S_{\phi,1} = \beta(P_+\phi P_+ + P_-)(I + P_-\phi P_+).$$

Then $I+P_-\phi P_+$ is invertible and $(I+P_-\phi P_+)^{-1} = I-P_-\phi P_+$ (cf. [9, p.393]). Hence, $S_{\alpha,\beta}$ is invertible if and only if $\min\{\operatorname{ess\,inf}_T|\alpha|, \operatorname{ess\,inf}_T|\beta|\} > 0$ and $T_{\alpha/\beta}$ is invertible.

The first author (cf. [7]) calculated essentially the norm of the operator inverse to T_ϕ. We will show in Section 3 that the formula of the norm of the operator inverse to T_ϕ is similar to the fomula of the norm of the operator inverse to $S_{\alpha,\beta}$. The second author (cf. [11]) considered the norm of $S_{\alpha,\beta}$ on the weighted space $L^2(T, W)$ with a weight function W on T, and proved the Feldman-Krupnik-Markus theorem [3] using the Cotlar-Sadosky lifting theorem [1] when α and β are constant functions.

2. NORM OF THE OPERATOR $S_{\alpha,\beta}$

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish.

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \le \|S_{\alpha,\beta}\| \le \left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty .$$

We should mention that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{0+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty^{1/2},$$

and

$$\left\|\sqrt{|\alpha|^2+|\beta|^2}\right\|_\infty = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-0|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty^{1/2}.$$

It is not difficult to establish that

$$\|S_{\alpha,\beta}\|^2 \le \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty,$$

for any $k \in H^\infty$. The formula of the norm of $S_{\alpha,\beta}$ is as follows.

Theorem 1. Let $\alpha, \beta \in L^\infty$. Then

$$\|S_{\alpha,\beta}\|^2 = \inf_{k\in H^\infty}\left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty .$$

The infimum is attained.

Proof. For any $k \in H^\infty$, we define the quantity M_k according to

$$M_k = \left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty .$$

We prove that $\|S_{\alpha,\beta}\|^2 \geq \inf\{M_k; k \in H^\infty\}$. Let $\gamma = \|S_{\alpha,\beta}\|$. Then

$$\|S_{\alpha,\beta} f\|_2 \leq \gamma \|f\|_2, \quad (f \in L^2).$$

Let $W_1 = \gamma^2 - |\alpha|^2, W_2 = \gamma^2 - |\beta|^2$ and $W_3 = \gamma^2 - \alpha\bar{\beta}$. Then

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2) \geq 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \leq W_1 W_2.$$

Hence, $\gamma \geq \max\{|\alpha|, |\beta|\}$ and

$$|\gamma^2 - \alpha\bar{\beta} - g|^2 \leq (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2).$$

Let $k_0 = \gamma^2 - g$. Then $k_0 \in H^\infty$ and $|\gamma^2 - \alpha\bar{\beta} - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$\begin{aligned} 0 &\leq (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2) - |\alpha\bar{\beta} - k_0|^2 \\ &= \gamma^4 - (|\alpha|^2 + |\beta|^2)\gamma^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k_0|^2. \end{aligned}$$

Suppose

$$\gamma^2 \leq \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on some measurable subset E of T. Since

$$\gamma^2 \geq \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right|$$

on T, we have

$$\left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leq 0$$

on E. This implies $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\gamma^2 \geq \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\gamma^2 \geq \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on T. Hence $M_{k_0} \leq \gamma^2$. Since $\gamma = \|S_{\alpha,\beta}\|$, we have

$$\inf_{k\in H^\infty} M_k \leq M_{k_0} \leq \|S_{\alpha,\beta}\|^2.$$

We prove that $\|S_{\alpha,\beta}\|^2 \leq \inf\{M_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \leq M_k,$$

we have

$$\begin{aligned}
&(M_k - |\alpha|^2)(M_k - |\beta|^2) \\
&\geq \left(\sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} - \frac{|\alpha|^2-|\beta|^2}{2}\right) \\
&\times \left(\sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} + \frac{|\alpha|^2-|\beta|^2}{2}\right) \\
&= |\alpha\bar{\beta}-k|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&M_k\|f_1+f_2\|_2^2 - \|\alpha f_1 + \beta f_2\|_2^2 \\
&= \left\|\sqrt{M_k-|\alpha|^2}f_1\right\|_2^2 + \left\|\sqrt{M_k-|\beta|^2}f_2\right\|_2^2 - 2Re(\alpha\bar{\beta}f_1, f_2) \\
&\geq 2\left\|\sqrt{M_k-|\alpha|^2}f_1\right\|_2 \left\|\sqrt{M_k-|\beta|^2}f_2\right\|_2 - 2Re((\alpha\bar{\beta}-k)f_1, f_2) \\
&\geq 2\int_T \left(\sqrt{M_k-|\alpha|^2}\sqrt{M_k-|\beta|^2} - |\alpha\bar{\beta}-k|\right)|f_1 f_2|dm \geq 0.
\end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta} f\|_2^2 \le M_k \|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\inf_{k \in H^\infty} M_k \le M_{k_0} \le \|S_{\alpha,\beta}\|^2 \le \inf_{k \in H^\infty} M_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Remark 1. Let $\alpha, \beta \in L^\infty$, let $\phi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \sqrt{|\phi|^2 + \psi^2} + \sqrt{|\phi - k|^2 + \psi^2} \right\|_\infty .$$

The infimum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \| |\phi| + |\phi - k| \|_\infty .$$

Corollary 1. If $|\alpha|$ and $|\beta|$ are constant functions, then

$$\|S_{\alpha,\beta}\|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left\|H_{\alpha\bar{\beta}}\right\|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

Proof. It follows from Theorem 1 that

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$

$$= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left(\inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty\right)^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 2. Let $\alpha, \beta \in L^\infty$. Then

$$\|S_{\alpha,\beta}\|^2 \le \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \|H_{\alpha\bar{\beta}}\|.$$

Proof. It follows from the easy direction of Theorem 1 that

$$\|S_{\alpha,\beta}\|^2 \le \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right\|_\infty$$

$$\le \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2+|\beta|^2}{2} + |\alpha\bar{\beta}-k| + \left|\frac{|\alpha|^2-|\beta|^2}{2}\right| \right\|_\infty$$

$$= \inf_{k\in H^\infty} \left\| \max\left\{|\alpha|^2, |\beta|^2\right\} + |\alpha\bar{\beta}-k| \right\|_\infty$$

$$\le \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \inf_{k\in H^\infty} \|\alpha\bar{\beta}-k\|_\infty.$$

By the Nehari theorem [8], this proves the corollary.

Remark 2. If $\alpha\bar{\beta} \in H^\infty$, then the infimum in Theorem 1 is attained by $k = \alpha\bar{\beta}$. In this case,

$$\|S_{\alpha,\beta}\| = \|\max\{|\alpha|, |\beta|\}\|_\infty = \max\left\{\|\alpha\|_\infty, \|\beta\|_\infty\right\}.$$

Hence the equality holds in Corollary 2 because $\|H_{\alpha\bar{\beta}}\| = 0$. By Corollary 1, if $|\alpha| = |\beta| = constant$, then the equality holds in Corollary 2.

By Corollary 2, if $\phi \in L^\infty$ and $\|\phi\|_\infty \le 1$, then $\|S_{\phi,1}\|^2 \le 1 + \|H_\phi\|$. By Corollary 1, if $|\phi| = 1$, then $\|S_{\phi,1}\|^2 = 1 + \|H_\phi\|$. When ϕ is a real function whose range consists of only two points, the norm of $S_{\phi,1}$ will be calculated in a completely explicit form in Corollary 3. For example, by Corollary 3, if E is a measurable subset of the unit circle T satisfying $0 < m(E) < 1$, then $\|S_{\chi_E,1}\| = 2/\sqrt{3}$. It is well known that $\|H_{\chi_E}\| = 1/2$. In this case, $\|S_{\chi_E,1}\|^2 < 1 + \|H_{\chi_E}\|$. Hence the equality does not hold in Corollary 2. We need Lemma 1 to prove Corollary 3.

Lemma 1. Let a and b be real numbers satisfying $a \neq b$. Then the equality

$$\frac{a^2+1}{2}+\sqrt{(a-x)^2+\left(\frac{a^2-1}{2}\right)^2}=\frac{b^2+1}{2}+\sqrt{(b-x)^2+\left(\frac{b^2-1}{2}\right)^2}$$

holds for some real number x if and only if $|a+b|<2$. Then the x is unique and it is given by

$$x=x_0=\frac{2(a+b)(1-ab)}{4-(a+b)^2}.$$

Proof. Suppose the equality holds for $x=0$. Then $a^2=b^2$. Since $a \neq b$, we have $a=-b$. Hence $|a+b|=0<2$. Suppose the equality holds for some real number x satisfying $x \neq 0$. Then

$$\left\{\frac{b^2-a^2}{2}+\sqrt{(b-x)^2+\left(\frac{b^2-1}{2}\right)^2}\right\}^2=(a-x)^2+\left(\frac{a^2-1}{2}\right)^2.$$

It follows by direct computation that

$$2(b^2-a^2)\sqrt{(x-b)^2+\left(\frac{b^2-1}{2}\right)^2}=(b-a)\left\{4x-(a+b)(b^2+1)\right\}.$$

Since $a \neq b$, we have

$$2(a+b)\sqrt{(x-b)^2+\left(\frac{b^2-1}{2}\right)^2}=4x-(a+b)(b^2+1).$$

Hence,

$$\left\{4-(a+b)^2\right\}x^2=2(a+b)(1-ab)x.$$

Since $x \neq 0$, we have

$$\left\{4-(a+b)^2\right\}x=2(a+b)(1-ab).$$

If $(a+b)^2=4$, then $ab=1$. Hence $a^2=b^2=1$. Since $a \neq b$, we have $a+b=0$. This contradiction implies $(a+b)^2 \neq 4$. Hence,

$$x=\frac{2(a+b)(1-ab)}{4-(a+b)^2}.$$

It follows by direct computation that

$$4x-(a+b)(b^2+1)=\frac{(a+b)\left\{(a-b)^2+(b^2+ab-2)^2\right\}}{4-(a+b)^2}.$$

If $a\neq b$, then

$$2\sqrt{(x-b)^2+\left(\frac{b^2-1}{2}\right)^2}=\frac{(a-b)^2+(b^2+ab-2)^2}{4-(a+b)^2}>0$$

because $(a-b)^2>0$. Hence, $|a+b|<2$. This proof is reversible.

Corollary 3. Let E be a measurable subset of T satisfying $0<m(E)<1$, let a and b be real numbers, and let $\phi=a\chi_E+b\chi_{E^c}$. Then

(1) If $\max\{a^2,b^2\}\geq 2-ab$, then

$$\|S_{\phi,1}\|=\max\{|a|,|b|\}.$$

(2) If $\max\{a^2,b^2\}<2-ab$, then

$$\|S_{\phi,1}\|^2=\frac{4(1-ab)}{4-(a+b)^2}.$$

Proof. We define the functions $f(x)$ and $g(x)$ according to

$$f(x)=\frac{a^2+1}{2}+\sqrt{(a-x)^2+\left(\frac{a^2-1}{2}\right)^2},$$

and

$$g(x)=\frac{b^2+1}{2}+\sqrt{(b-x)^2+\left(\frac{b^2-1}{2}\right)^2}.$$

For the real number x_0 defined in Lemma 1, it follows that

$$f(x_0)=g(x_0)$$

$$=\frac{a^2+1}{2}+\sqrt{\left(\frac{(a-b)(a^2+ab-2)}{4-(a+b)^2}\right)^2+\left(\frac{a^2-1}{2}\right)^2}$$

$$= \frac{b^2+1}{2} + \sqrt{\left(\frac{(a-b)(b^2+ab-2)}{4-(a+b)^2}\right)^2 + \left(\frac{b^2-1}{2}\right)^2}$$

$$= \frac{4(1-ab)}{4-(a+b)^2}.$$

We prove (1). Since $\max\{a^2, b^2\} \geq ab$ and $\max\{a^2, b^2\} \geq 2-ab$, we have $\max\{a^2, b^2\} \geq 1$. Hence, $\max\{a^2, b^2, 1\} = \max\{a^2, b^2\}$. Since $\|\phi\|_\infty = \max\{|a|, |b|\}$ and $\max\{\|\phi\|_\infty, 1\} \leq \|S_{\phi,1}\|$, we have

$$\max\left\{a^2, b^2\right\} = \max\left\{a^2, b^2, 1\right\} \leq \|S_{\phi,1}\|^2.$$

Suppose $a = b$. Then ϕ becomes a constant function. Hence, by Remark 2,

$$\|S_{\phi,1}\| = \max\{|\phi|, 1\} = \max\{|a|, |b|, 1\} = \max\{|a|, |b|\}.$$

Suppose $a \neq b$. Since $0 < m(E) < 1$, we have

$$\|S_{\phi,1}\|^2 \leq \inf_{x\in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{(\phi - x)^2 + \left(\frac{\phi^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{x\in R} \|f(x)\chi_E + g(x)\chi_{E^c}\|_\infty = \inf_{x\in R} \left(\max\{f(x), g(x)\}\right).$$

By Lemma 1, if $|a+b| \geq 2$, then the equality $f(x) = g(x)$ does not hold for any real number x. Hence, $f(x) < g(x)$ or $f(x) > g(x)$. Hence,

$$\inf_{x\in R} \left(\max\{f(x), g(x)\}\right) = \max\{f(a), g(b)\}.$$

Since $f(a) = \max\{a^2, 1\}$ and $g(b) = \max\{b^2, 1\}$, we have

$$\max\left\{a^2, b^2\right\} \leq \|S_{\phi,1}\|^2 \leq \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.$$

By Lemma 1, if $|a+b| < 2$, then the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 1. Hence, $f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. If $a \leq x_0 \leq b$, then $\max\{a^2, b^2\} \leq 2-ab$. Since $\max\{a^2, b^2\} \geq 2-ab$, we have $\max\{a^2, b^2\} = 2-ab$. By this equality,

$$\max\left\{a^2, b^2\right\} \leq \|S_{\phi,1}\|^2 \leq \inf_{x\in R} \left(\max\{f(x), g(x)\}\right)$$

$$= f(x_0) = g(x_0) = \frac{4(1-ab)}{4-(a+b)^2} = \max\left\{a^2, b^2\right\}.$$

If $x_0 \le a \le b$ or $a \le b \le x_0$, then

$$\max\left\{a^2, b^2\right\} \le \|S_{\phi,1}\|^2 \le \inf_{x\in R}(\max\{f(x), g(x)\})$$

$$= \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.$$

We prove (2). Suppose $a = b$. Then ϕ becomes a constant function and $\phi = a = b$. Since $\max\{a^2, b^2\} < 2 - ab$, we have $a^2 = b^2 < 1$. By Remark 2,

$$\|S_{\phi,1}\|^2 = \max\{\phi^2, 1\} = \max\{a^2, 1\} = \max\{b^2, 1\} = 1.$$

Suppose $a \neq b$. It is sufficient to prove that $\|S_{\phi,1}\|^2 = f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. Since $\max\{a^2, b^2\} < 2 - ab$, we have $a^2 + b^2 < 2(2 - ab)$. Hence $|a + b| < 2$. Let $f(x)$ and $g(x)$ be functions defined in the proof of (1). By Lemma 1, the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 1. Hence, $f(x_0) = g(x_0)$. It follows by direct calculation that

$$a - x_0 = \frac{(b-a)(a^2+ab-2)}{4-(a+b)^2} < 0,$$

$$b - x_0 = \frac{(a-b)(b^2+ab-2)}{4-(a+b)^2} > 0.$$

Hence, $a < x_0 < b$. By Theorem 1, there exists a $k_0 \in H^\infty$ such that

$$\|S_{\phi,1}\|^2 = \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{k\in H^\infty} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty.$$

$$\le \inf_{x\in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - x|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{x \in R} \|f(x)\chi_E + g(x)\chi_{E^c})\|_\infty \leq \inf_{x \in R} (\max\{f(x), g(x)\}) .$$

Since $a < x_0 < b$, we have

$$\inf_{x \in R} (\max\{f(x), g(x)\}) = f(x_0) = g(x_0).$$

Hence,

$$\|S_{\phi,1}\|^2 \leq f(x_0) = g(x_0).$$

Then

$$f(x_0) = \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2},$$

$$g(x_0) = \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2}.$$

Suppose there exists an $\varepsilon > 0$ such that $\|S_{\phi,1}\|^2 \leq f(x_0) - \varepsilon$. Then

$$\left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \leq \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2} - \varepsilon.$$

Since $f(x_0) = g(x_0)$, we have $\|S_{\phi,1}\|^2 \leq g(x_0) - \varepsilon$. Hence,

$$\left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - k_0|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty \leq \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2} - \varepsilon.$$

Hence, there exists an $\varepsilon' > 0$ such that

$$|a - k_0| \leq |a - x_0| - \varepsilon' \quad \text{on} \quad E,$$

$$|b - k_0| \leq |b - x_0| - \varepsilon' \quad \text{on} \quad E^c.$$

Since $a < x_0 < b$, we have $|a - x_0| + |b - x_0| = b - a$. If $|a - x_0| \geq |b - x_0|$, then

$$|2|a - x_0|\chi_{E^c} + a - k_0|$$

$$= |a\chi_E + (b + |a - x_0| - |b - x_0|)\chi_{E^c} - k_0|$$

$$\leq |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|a - x_0| - |b - x_0|)\chi_{E^c}$$

$$\leq |a - x_0| - \varepsilon'.$$

Hence, $\inf\{\|2\chi_{E^c} - k\|_\infty; k \in H^\infty\} < 1$. This is a contradiction (cf. [5, p.198]). If $|a - x_0| \leq |b - x_0|$, then

$$\begin{aligned} &| - 2|b - x_0|\chi_E + b - k_0| \\ &= |(a + |a - x_0| - |b - x_0|)\chi_E + b\chi_{E^c} - k_0| \\ &\leq |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|b - x_0| - |a - x_0|)\chi_E \\ &\leq |b - x_0| - \varepsilon'. \end{aligned}$$

Hence, $\inf\{\|2\chi_E - k\|_\infty; k \in H^\infty\} < 1$. This is a contradiction (cf. [5, p.198]). These two contradictions imply that ε' must be zero. This contradiction implies that ε must be zero. Hence,

$$\|S_{\phi,1}\|^2 = f(x_0) = g(x_0).$$

This completes the proof.

When a and b are complex numbers, we give Corollary 4. When $|\phi|$ is not constant, Corollary 4 does not contain the completely explicit form of the norm of $S_{\phi,1}$. We need Lemma 2 to prove Corollary 4.

Lemma 2. Let a and b be complex numbers, and let θ be a real number satisfying $Re\left(e^{i\theta}(a - b)\right) = 0$. If the equality

$$\frac{|a|^2 + 1}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2} = \frac{|b|^2 + 1}{2} + \sqrt{|e^{i\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2}$$

holds for some real number x, then $|a| = |b|$ and the equality holds for any real numbers x.

Proof. Suppose the equality holds for some real number x. Then

$$\left\{\frac{|a|^2 - |b|^2}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2}\right\}^2 = |e^{i\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2.$$

Since $Re\left(e^{i\theta}(a-b)\right)=0$, we have

$$(|a|^2-|b|^2)\sqrt{|e^{i\theta}a-x|^2+\left(\frac{|a|^2-1}{2}\right)^2}$$

$$=2Re\left(e^{i\theta}(a-b)\right)x+|b|^2-|a|^2+\frac{(|b|^2-|a|^2)(|a|^2-1)}{2}$$

$$=\frac{(|b|^2-|a|^2)(|a|^2+1)}{2}.$$

Hence, $|a|=|b|$ and the equality

$$|e^{i\theta}a-x|^2=|e^{i\theta}b-x|^2$$

holds for any x because $Re\left(e^{i\theta}(a-b)\right)=0$.

Corollary 4. Let E be a measurable subset of T satisfying $0<m(E)<1$, let a and b be complex numbers, and let $\phi=a\chi_E+b\chi_{E^c}$. Then

(1) If θ is a real number satisfying $Re\left(e^{i\theta}(a-b)\right)=0$, then

$$\|S_{\phi,1}\|^2\le\max_{z=a,b}\left\{\frac{|z|^2+1}{2}+\sqrt{(Im\,(e^{i\theta}z))^2+\left(\frac{|z|^2-1}{2}\right)^2}\right\}.$$

The equality does not hold in general.

(2) If $|a|=|b|$, then the equality holds for some θ in (1), and

$$\|S_{\phi,1}\|^2=\frac{|a|^2+1}{2}+\sqrt{\left(\frac{|a-b|}{2}\right)^2+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Proof. We prove (2). Suppose $a=b$. Then ϕ becomes a constant function. Hence, by Remark 2,

$$\|S_{\phi,1}\|^2=\max\left\{|\phi|^2,1\right\}=\max\left\{|a|^2,1\right\}=\frac{|a|^2+1}{2}+\sqrt{0+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Suppose $a \neq b$. Since $|\phi| = |a| = |b|$, it follows from Corollary 1 that

$$\|S_{\phi,1}\|^2 = \frac{|a|^2+1}{2} + \sqrt{\|H_\phi\|^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

Let

$$\psi = \frac{2\phi - (a+b)}{a-b}.$$

Then

$$\psi = \begin{cases} 1 & \text{on } E \\ -1 & \text{on } E^c \end{cases}$$

Hence, $\inf\{\|\psi - k\|_\infty; k \in H^\infty\} = 1$ (cf. [5,p.198]). By the Nehari theorem [8],

$$\|H_\phi\| = \inf_{k \in H^\infty} \|\phi - k\|_\infty = \frac{|a-b|}{2}.$$

Hence,

$$\|S_{\phi,1}\|^2 = \frac{|a|^2+1}{2} + \sqrt{\left(\frac{|a-b|}{2}\right)^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

Since $a \neq b, |a| = |b|$ and $Re\left(e^{i\theta}(a-b)\right) = 0$, it follows that there exist real numbers u and v such that

$$e^{i\theta}a = u + iv, \quad \text{and} \quad e^{i\theta}b = u - iv.$$

Since $|a| = |b|$, we have

$$\left(\frac{|a-b|}{2}\right)^2 = \frac{|a|^2 - Re(a\bar{b})}{2} = \frac{u^2+v^2-(u^2-v^2)}{2}$$

$$= v^2 = \left(Im\left(e^{i\theta}a\right)\right)^2 = \left(Im\left(e^{i\theta}b\right)\right)^2.$$

We prove (1). By (2), it is sufficient to prove (1) when $|a| \neq |b|$. We define the functions $f(x)$ and $g(x)$ according to

$$f(x) = \frac{|a|^2+1}{2} + \sqrt{|e^{i\theta}a - x|^2 + \left(\frac{|a|^2-1}{2}\right)^2},$$

and

$$g(x) = \frac{|b|^2+1}{2} + \sqrt{|e^{i\theta}b - x|^2 + \left(\frac{|b|^2-1}{2}\right)^2}.$$

Since $|a| \neq |b|$, it follows from Lemma 2 that the equality $f(x) = g(x)$ does not hold for any real number x. If $|a| < |b|$, then $f(x) < g(x)$ because $Re\left(e^{i\theta}(a-b)\right) = 0$. Hence, by Theorem 1,

$$\|S_{\phi,1}\|^2 = \inf_{k\in H^\infty} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi-k|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$\leq \inf_{x\in R} \left\| \frac{|\phi|^2+1}{2} + \sqrt{|\phi - e^{-i\theta}x|^2 + \left(\frac{|\phi|^2-1}{2}\right)^2} \right\|_\infty$$

$$= \inf_{x\in R} \|f(x)\chi_E + g(x)\chi_{E^c})\|_\infty \leq \inf_{x\in R} g(x) = g\left(Re\left(e^{i\theta}b\right)\right)$$

$$= \frac{|b|^2+1}{2} + \sqrt{\left(Im\left(e^{i\theta}b\right)\right)^2 + \left(\frac{|b|^2-1}{2}\right)^2}.$$

Similarly, if $|a| > |b|$, then

$$\|S_{\phi,1}\|^2 \leq \frac{|a|^2+1}{2} + \sqrt{\left(Im\left(e^{i\theta}a\right)\right)^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

This completes the proof.

3. NORM OF THE OPERATOR INVERSE TO $S_{\alpha,\beta}$

The first author essentially gave Theorem 2 (cf. [7, Corollary 3]). It is not difficult to establish that

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 \geq \operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right),$$

for any $k \in H^\infty$. The formula of the norm of T_ϕ is as follows.

Theorem 2. Let $\phi \in L^\infty$. Then

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 = \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right).$$

The supremum is attained. When T_ϕ is invertible, this is equal to $\|T_\phi^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity J_k according to

$$J_k = \operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right).$$

We prove that $\inf\{\|T_\phi f\|_2^2; f \in H^2, \|f\|_2 = 1\} \leq \sup\{J_k; k \in H^\infty\}$. Let $\varepsilon = \inf\{\|T_\phi f\|_2; f \in H^2, \|f\|_2 = 1\}$. Then

$$\varepsilon \|f_1\|_2 \leq \|T_\phi f_1\|_2, \quad (f_1 \in H^2).$$

Since $\|T_\phi f_1\|_2^2 + \|H_\phi f_1\|_2^2 = \|\phi f_1\|_2^2$, we have

$$\|H_\phi f_1\|_2^2 \leq \left(\left(|\phi|^2 - \varepsilon^2\right) f_1, f_1\right).$$

Let $W_1 = |\phi|^2 - \varepsilon^2, W_2 = 1, W_3 = \phi$. Since $(\phi f_1, f_2) = (H_\phi f_1, f_2)$, for any $f_1 \in H^2, f_2 \in \overline{H_0^2}$, we have

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2)$$

$$\geq \|H_\phi f_1\|_2^2 + \|f_2\|_2^2 + 2Re\,(H_\phi f_1, f_2) = \|H_\phi f_1 + f_2\|_2^2 \geq 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a $k_0 \in H^\infty$ such that

$$|W_3 - k_0|^2 \leq W_1 W_2.$$

Hence,

$$|\phi - k_0|^2 \leq |\phi|^2 - \varepsilon^2.$$

Since $\varepsilon = \inf\{\|T_\phi f\|_2; f \in H^2, \|f\|_2 = 1\}$, we have

$$\inf_{f \in H^2, \|f\|_2 = 1} \|T_\phi f\|_2^2 \leq J_{k_0} \leq \sup_{k \in H^\infty} J_k.$$

We prove that $\inf\{\|T_\phi f\|_2^2; f \in H^2, \|f\|_2 = 1\} \geq \sup\{J_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, $|\phi|^2 - J_k \geq |\phi - k|^2$ and $H_{\phi - k} = H_\phi$. Hence, for any $f_1 \in H^2$,

$$\|T_\phi f_1\|_2^2 - J_k \|f_1\|_2^2 = \|\phi f_1\|_2^2 - \|H_\phi f_1\|_2^2 - J_k \|\phi f_1\|_2^2$$

$$= \left((|\phi|^2 - J_k) f_1, f_1\right) - \|H_\phi f_1\|_2^2 \geq \left(|\phi - k|^2 f_1, f_1\right) - \|H_{\phi - k} f_1\|_2^2$$

$$= \|(\phi - k) f_1\|_2^2 - \|H_{\phi - k} f_1\|_2^2 = \|T_{\phi - k} f_1\|_2^2 \geq 0.$$

Hence, for any $k \in H^\infty$,

$$\|T_\phi f_1\|_2^2 \geq J_k \|f_1\|_2^2, \quad (f_1 \in H^2).$$

Therefore,

$$\sup_{k \in H^\infty} J_k \leq \inf_{f \in H^2, \|f\|_2 = 1} \|T_\phi f\|_2^2 \leq J_{k_0} \leq \sup_{k \in H^\infty} J_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Corollary 5. If $|\phi|$ is a constant function, then

$$\inf_{f \in H^2, \|f\|_2 = 1} \|T_\phi f\|_2^2 = |\phi|^2 - \|H_\phi\|^2.$$

Proof. It follows from Theorem 2 that

$$\inf_{f \in H^2, \|f\|_2 = 1} \|T_\phi f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right)$$

$$= |\phi|^2 + \sup_{k \in H^\infty} \left(\operatorname{ess\,inf}_T \left(-|\phi - k|^2\right)\right) = |\phi|^2 - \inf_{k \in H^\infty} \|\phi - k\|_\infty^2.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 6. Let $\phi \in L^\infty$. Then

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 \geq \operatorname{ess\,inf}_T |\phi|^2 - \|H_\phi\|^2.$$

Proof. It follows from the easy direction of Theorem 2 that

$$\begin{aligned}\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 &\geq \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(|\phi|^2 - |\phi - k|^2\right)\right) \\ &\geq \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T |\phi|^2 + \operatorname{ess\,inf}_T \left(-|\phi - k|^2\right)\right) \\ &= \operatorname{ess\,inf}_T |\phi|^2 + \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(-|\phi - k|^2\right)\right) \\ &= \operatorname{ess\,inf}_T |\phi|^2 - \inf_{k\in H^\infty} \|\phi - k\|_\infty^2.\end{aligned}$$

By the Nehari theorem [8], this proves the corollary.

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish.

$$\begin{aligned}\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2 &\leq \operatorname{ess\,inf}_T \left(\min\left\{|\alpha|^2, |\beta|^2\right\}\right) \\ &= \operatorname{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{0 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}\right).\end{aligned}$$

It is not difficult to establish that

$$\begin{aligned}&\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2 \\ &\geq \operatorname{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}\right),\end{aligned}$$

for any $k \in H^\infty$. The formula of the norm of $S_{\alpha,\beta}^{-1}$ is as follows. The proof of Theorem 3 is essentially the same as the proof of Theorem 1.

Theorem 3. Let $\alpha, \beta \in L^\infty$. Then

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2$$

$$= \sup_{k \in H^\infty} \left(\operatorname{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right) \right).$$

The supremum is attained. When $S_{\alpha,\beta}$ is invertible, the supremum is equal to $\|S_{\alpha,\beta}^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity N_k according to

$$N_k = \operatorname{ess\,inf}_T \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right).$$

We prove that $\inf\{\|S_{\alpha,\beta} f\|_2^2; f \in L^2, \|f\|_2 = 1\} \leq \sup\{N_k; k \in H^\infty\}$. Let $\varepsilon = \inf\{\|S_{\alpha,\beta} f\|_2; f \in L^2, \|f\|_2 = 1\}$. Then

$$\varepsilon \|f\|_2 \leq \|S_{\alpha,\beta} f\|_2, \quad (f \in L^2).$$

Let $W_1 = |\alpha|^2 - \varepsilon^2, W_2 = |\beta|^2 - \varepsilon^2$ and $W_3 = \alpha\bar{\beta} - \varepsilon^2$. Then

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2Re\,(W_3 f_1, f_2) \geq 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem [1], $W_1 \geq 0, W_2 \geq 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \leq W_1 W_2.$$

Hence, $\varepsilon \leq \min\{|\alpha|, |\beta|\}$ and there exists a $g \in H^\infty$ such that

$$|\alpha\bar{\beta} - \varepsilon^2 - g|^2 \leq \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right).$$

Let $k_0 = \varepsilon^2 + g$. Then $k_0 \in H^\infty$ and $|\alpha\bar{\beta} - \varepsilon^2 - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$0 \leq \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right) - |\alpha\bar{\beta} - k_0|^2$$

$$= \varepsilon^4 - \left(|\alpha|^2 + |\beta|^2\right)\varepsilon^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k|^2.$$

Suppose

$$\varepsilon^2 \geq \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on some measurable subset E of T. Since

$$\varepsilon^2 \leq \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2+|\beta|^2}{2} - \left|\frac{|\alpha|^2-|\beta|^2}{2}\right|$$

on T, we have

$$\left|\frac{|\alpha|^2-|\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \leq 0$$

on E. This implies $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\varepsilon^2 \leq \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\varepsilon^2 \leq \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on T. Hence $\varepsilon^2 \leq N_{k_0}$. Since $\varepsilon = \inf\{\|S_{\alpha,\beta}f\|_2; f \in L^2, \|f\|_2 = 1\}$, we have

$$\inf_{f\in L^2, \|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 \leq N_{k_0} \leq \sup_{k\in H^\infty} N_k.$$

We prove that $\inf\{\|S_{\alpha,\beta}f\|_2^2; f \in L^2, \|f\|_2 = 1\} \geq \sup\{N_k; k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \geq N_k,$$

we have

$$(|\alpha|^2 - N_k)(|\beta|^2 - N_k)$$

$$\geq \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} + \frac{|\alpha|^2 - |\beta|^2}{2} \right)$$
$$\times \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} - \frac{|\alpha|^2 - |\beta|^2}{2} \right)$$
$$= |\alpha\bar{\beta} - k|^2.$$

Hence,

$$\|\alpha f_1 + \beta f_2\|_2^2 - N_k \|f_1 + f_2\|_2^2$$
$$= \left\| \sqrt{|\alpha|^2 - N_k} f_1 \right\|_2^2 + \left\| \sqrt{|\beta|^2 - N_k} f_2 \right\|_2^2 + 2Re(\alpha\bar{\beta} f_1, f_2)$$
$$\geq 2 \left\| \sqrt{|\alpha|^2 - N_k} f_1 \right\|_2 \left\| \sqrt{|\beta|^2 - N_k} f_2 \right\|_2 + 2Re((\alpha\bar{\beta} - k) f_1, f_2)$$
$$\geq 2 \int_T \left(\sqrt{|\alpha|^2 - N_k} \sqrt{|\beta|^2 - N_k} - |\alpha\bar{\beta} - k| \right) |f_1 f_2| dm \geq 0.$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta} f\|_2^2 \geq N_k \|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\sup_{k \in H^\infty} N_k \leq \inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 \leq N_{k_0} \leq \sup_{k \in H^\infty} N_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof.

Remark 3. Let $\alpha, \beta \in L^\infty$, let $\phi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname{ess\,inf}_T \left(\sqrt{|\phi|^2 + \psi^2} - \sqrt{|\phi - k|^2 + \psi^2} \right) \right).$$

The supremum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\inf_{f \in L^2, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname{ess\,inf}_T \left(|\phi| - |\phi - k| \right) \right).$$

Corollary 7. If $|\alpha|$ and $|\beta|$ are constant functions, then

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 = \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{\|H_{\alpha\bar{\beta}}\|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}.$$

Proof. It follows from Theorem 3 that

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2$$

$$= \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right) \right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} + \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(-\sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right) \right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} + \sup_{k\in H^\infty} \left(-\left\| \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right\|_\infty \right)$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} - \inf_{k\in H^\infty} \left\| \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \right\|_\infty$$

$$= \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{\inf_{k\in H^\infty} \|\alpha\bar{\beta}-k\|_\infty^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}.$$

By the Nehari theorem [8], this proves the corollary.

Corollary 8. Let $\alpha, \beta \in L^\infty$. Then

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 \geq \operatorname{ess\,inf}_T \left(\min\{|\alpha|^2, |\beta|^2\}\right) - \|H_{\alpha\bar{\beta}}\|.$$

Proof. It follows from the easy direction of Theorem 3 that

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2$$

$$\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right)$$

$$\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-|\alpha\bar{\beta}-k|-\left|\frac{|\alpha|^2-|\beta|^2}{2}\right|\right)\right)$$

$$= \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}-|\alpha\bar{\beta}-k|\right)\right)$$

$$\geq \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\operatorname*{ess\,inf}_T\left(-|\alpha\bar{\beta}-k|\right)\right)$$

$$= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_T\left(-|\alpha\bar{\beta}-k|\right)\right)$$

$$= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(-\|\alpha\bar{\beta}-k\|_\infty\right)$$

$$= \operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)-\inf_{k\in H^\infty}\|\alpha\bar{\beta}-k\|_\infty.$$

By the Nehari theorem [8], this proves the corollary.

Remark 4. If $\alpha\bar{\beta}\in H^\infty$, then the supremum in Theorem 3 is attained by $k=\alpha\bar{\beta}$. In this case,

$$\inf_{f\in L^2,\|f\|_2=1}\|S_{\alpha,\beta}f\|_2=\operatorname*{ess\,inf}_T\left(\min\left\{|\alpha|,|\beta|\right\}\right).$$

For functions $\alpha,\beta\in L^\infty$, $S_{\alpha,\beta}$ is left invertible if and only if

$$\operatorname*{ess\,inf}_T\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar{\beta}-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)>0,$$

for some $k\in H^\infty$. By Corollary 5, for a function $\phi\in L^\infty$ satisfying $|\phi|=1$, T_ϕ is left invertible if and only if $S_{\phi,1}$ is left invertible if and only if $\|H_\phi\|<1$ (cf. [9, p.203]).

Corollary 9. Let $\phi\in L^\infty, |\phi|=1$. Then

$$\inf_{f\in H^2,\|f\|_2=1}\|T_\phi f\|_2=\|S_{\phi,1}\|\left(\inf_{f\in L^2,\|f\|_2=1}\|S_{\phi,1}f\|_2\right).$$

Proof. It follows from Corollary 5 that

$$\inf_{f\in H^2,\|f\|_2=1} \|T_\phi f\|_2^2 = 1 - \|H_\phi\|^2.$$

It follows from Corollary 1 that

$$\|S_{\phi,1}\|^2 = 1 + \|H_\phi\|.$$

It follows from Corollary 7 that

$$\inf_{f\in L^2,\|f\|_2=1} \|S_{\phi,1} f\|_2^2 = 1 - \|H_\phi\|.$$

These equalities proves the corollary.

Corollary 10. If a and b are invertible functions in H^∞, then

$$\|S_{a,\bar{b}} L_{1/\bar{b}}\|^{-2} = \sup_{k\in H^\infty} \left(\operatorname{ess\,inf}_T \left(\frac{|a|^2+|b|^2}{2|a|^2} - \sqrt{\left|\frac{\bar{b}}{a} - k\right|^2 + \left(\frac{|a|^2-|b|^2}{2|a|^2}\right)^2} \right) \right),$$

where $L_{1/\bar{b}}$ denotes the Laurent operator on L^2. The supremum is attained.

Proof. Since a and b are invertible functions H^∞, it follows that $S_{\bar{b}/a,1}$ is invertible, and

$$S_{\bar{b}/a,1}^{-1} = S_{a,\bar{b}} L_{1/\bar{b}},$$

(cf. [6, p.88], [10, Theorem 3]). Hence,

$$\|S_{a,\bar{b}} L_{1/\bar{b}}\|^{-2} = \|S_{\bar{b}/a,1}^{-1}\|^{-2} = \inf_{f\in L^2,\|f\|_2=1} \|S_{\bar{b}/a,1} f\|_2^2.$$

By Theorem 3, this proves the corollary.

Acknowledgements. This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

References

[1] M.Cotlar and C.Sadosky, On the Helson-Szegö theorem and a related class of modified Toeplitz kernels, Proc. Sym. Pure Math. 35 (1979), 383-407.

[2] R.G.Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

[3] I.Feldman, N.Krupnik and A.Markus, On the norm of two adjoint projections, Integral Equations and Operator Theory 14 (1991), 69-90.

[4] I.Gohberg and N.Krupnik, One-Dimensional Linear Singular Integral Equations, Vols. I,II, Birkhäuser Verlag, Basel, 1992.

[5] P.Koosis, Introduction to H_p Spaces, Cambridge U. Press, London, 1980.

[6] G.S.Litvinchuk and I.M.Spitkovskii, Factorization of Measurable Matrix Functions, Birkhäuser Verlag, Basel, 1987.

[7] T.Nakazi, Absolute values of Toeplitz operators and Hankel operators, Canad. Math. Bull. 34 (1991), 249-253.

[8] Z.Nehari, On bounded bilinear forms, Ann. of Math. 65 (1957), 153-162.

[9] N.K.Nikol'skii, Treatise on the Shift Operator, Springer Verlag, Berlin, 1986.

[10] T.Yamamoto, Invertibility of some singular integral operators and a lifting theorem, Hokkaido Math. J. 22 (1993), 181-198.

[11] T.Yamamoto, Boundedness of some singular integral operators in weighted L^2 spaces, J. Operator Theory 32 (1994), 243-254.