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Extension of the Drasin–Shea–Jordan theorem

N. H. Bingham and A. Inoue

§1. Introduction

In [BI1, BI2], we proved Mercerian results for Hankel transforms. In particular, in [BI2] we introduced a method, a type of localization, which seems useful for other problems, too. Here we apply the method to general integral transforms, and thereby extend the theorems of Drasin–Shea [DS] and Jordan [J]. As an application, we give a Mercerian result for Hankel transforms of non-monotone functions.

We recall the setting. Given a measurable kernel $k : (0, \infty) \rightarrow \mathbf{R}$, let

$$\check{k}(z) := \int_0^\infty t^{-z} k(t) \frac{dt}{t}$$

be its Mellin transform for $z \in \mathbf{C}$ such that the integral converges absolutely. For suitable functions $f, g : (0, \infty) \rightarrow \mathbf{R}$, the Mellin convolution is the function $f * g$ given by

$$(f * g)(x) := \int_0^\infty f(x/t) g(t) \frac{dt}{t}$$

for those $x > 0$ for which the integral converges absolutely. For $\rho \in \mathbf{R}$, we write R_ρ for the class of functions f *regularly varying* (at infinity) with *index* ρ : f is measurable, positive for large enough x , and

$$f(\lambda x)/f(x) \rightarrow \lambda^\rho \quad (x \rightarrow \infty) \quad \forall \lambda > 0;$$

see [BGT] for background. We are concerned here with comparisons between the asymptotic behaviour of the function f and that of its transform $k * f$. The simplest such results of this type are *Abelian*, and state that under suitable conditions

$$f(x) \sim x^\rho \ell(x) \quad (x \rightarrow \infty) \tag{1.1}$$

implies

$$(k * f)(x) \sim c x^\rho \ell(x) \quad (x \rightarrow \infty), \tag{1.2}$$

where

$$c = \check{k}(\rho). \tag{1.3}$$

Tauberian results supply a partial converse under suitable side-conditions (*Tauberian conditions*); see e.g. [BGT, Ch. 4] for a detailed treatment. In these circumstances, one has

$$(k * f)(x)/f(x) \rightarrow c \quad (x \rightarrow \infty). \quad (1.4)$$

The question arises of whether one can obtain (1.1) - and so (1.2) - from (1.4), with ρ as in (1.3). Such results are *Mercerian* in character; for a textbook treatment, see e.g. [BGT, Ch. 5]. The prototypes are due to Drasin and Shea ([DS, Th. 6.2]; [BGT, Th. 5.2.1]), with k non-negative, and Jordan ([J, Th. 1, 1a]; [BGT, Th. 5.3.1]), where k can change sign.

This area of Mercerian theorems of Drasin-Shea-Jordan type has long suffered from several outstanding problems. First, the proofs are long, complicated and highly technical, and one seeks to simplify them as much as possible. Next, until recently no such results were available for the important case where the integrals defining $\check{k}$ and $k * f$ are only conditionally rather than absolutely convergent, as is the case for Fourier and Hankel transforms, for example. In two recent works [BI1, BI2] we succeeded both in simplifying the proofs of Drasin-Shea-Jordan theorems and in extending them to Fourier and Hankel transforms.

Here we focus on a third long-standing problem: the need to rid Jordan's theorems [J] of various technical conditions which complicate their statements and reduce their applicability. Our approach here was suggested by the observation [BI1, §8.1] that in the Hankel case, for ρ small enough to give $k * f$ absolutely convergent, one of Jordan's theorems [J, Th. 1] nevertheless did not apply because its conditions exclude kernels such as the Fourier and Hankel ones, which oscillate infinitely often in sign. (However, Jordan's second theorem [J, Th. 1a] applies to some but not all Hankel cases; see §5 for details.) Our results succeed in solving the problem raised there. The main specific contributions are:

- (i) freeing Jordan's theorem from its non-essential technical conditions;
- (ii) eliminating the need to use Pólya peaks (see below for details);
- (iii) extending the results of [BI1, BI2] in the Hankel case with absolute convergence (the motivating situation, as mentioned above) to non-monotone functions. This last is to be expected: monotonicity was used to ensure convergence in the conditionally convergent case.

We recall ([BGT, §2.1.2]) the Matuszewska indices of a positive function f . The *upper Matuszewska index* $\alpha(f)$ is the infimum of those α for which there exists a constant

$C = C(\alpha)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \leq C(1 + o(1))\lambda^\alpha \quad (x \rightarrow \infty) \quad \text{uniformly in } \lambda \in [1, \Lambda],$$

the lower Matuszewska index $\beta(f)$ is the supremum of those β for which, for some constant $D = D(\beta)$ and all $\Lambda > 1$,

$$f(\lambda x)/f(x) \geq D(1 + o(1))\lambda^\beta \quad (x \rightarrow \infty) \quad \text{uniformly in } \lambda \in [1, \Lambda].$$

One says f has *bounded increase*, $f \in BI$, if $\alpha(f) < \infty$, *bounded decrease*, $f \in BD$, if $\beta(f) > -\infty$.

The upper order $\rho(f)$ of a positive function f is defined by

$$\rho(f) := \limsup_{x \rightarrow \infty} \frac{\log f(x)}{\log x}.$$

§2. Results

Theorem 1. Let k be a real kernel such that $\check{k}(z)$ converges absolutely for $a < \Re z < b$. Assume also

$$\check{k}(z) \neq \check{k}(\rho) \quad \text{for } \Re z = \rho \text{ and } z \neq \rho, \quad (2.1)$$

$$|\check{k}'(\rho)| + |\check{k}''(\rho)| > 0. \quad (2.2)$$

Let f be non-negative, measurable, and locally bounded on $[0, \infty)$, vanish in a neighbourhood of zero, have finite upper order $\rho \in (a, b)$, and $f \in BD \cup BI$. If

$$(k * f)(x)/f(x) \rightarrow c \neq 0 \quad (x \rightarrow \infty), \quad (2.3)$$

then $c = \check{k}(\rho)$ and $f \in R_\rho$.

For non-negative k , the theorem above coincides with that of Drasin and Shea [DS] (see also [BGT, Theorem 5.2.1]). The point of the theorem is that it dispenses with the following extra assumptions of Jordan [J] (see also [BGT, Theorem 5.3.1]):

[J, Theorem 1]: $\check{k}$ is monotone on $[\rho, b)$, and either

Case 1. $k(\cdot) = 0$ a. e. on $(0, 1)$

or

Case 2. $k(\cdot) \geq 0$ on some $(0, \delta)$, where $0 < \delta < 1$, with $k(\cdot) > 0$ on a subset of $(0, \delta)$ of positive measure, and also $|\check{k}(b-)| = \infty$ in case $b < \infty$.

[J, Theorem 1a]: $\check{k}$ is monotone on $(a, \rho]$, and *either*

Case 1. $k(1/\cdot) = 0$ a. e. on $(0, 1)$

or

Case 2. $k(1/\cdot) \geq 0$ on some $(0, \delta)$, where $0 < \delta < 1$, with $k(1/\cdot) > 0$ on a subset of $(0, \delta)$ of positive measure, and also $|\check{k}(a+)| = \infty$ in case $a > -\infty$.

The proof of Theorem 1 will be given in §3. As we indicated above, the key to the proof is the localization technique introduced in [BI2]. The idea is simple. There are two kinds of similar integral transforms $f \mapsto E_1 * f$ and $f \mapsto E_2 * f$ (see §3) which have been already used in, e.g., [DS], [J] and [BI1]. Instead of such separate use, we apply both transforms at the same time: $f \mapsto E_1 * E_2 * f$. Then this has the effect of localizing the problem completely, that is, it is enough to restrict to the narrow strip $\rho - \epsilon < \Re z < \rho + \epsilon$ for $\epsilon > 0$ small enough instead of the intermediate strips $\rho - \epsilon < \Re z < b$ or $a < \Re z < \rho + \epsilon$ used in the previous works. The usefulness of the idea is shown, for example, by the fact that, by virtue of it, we can avoid the use of the Pólya Peak Theorem of Drasin and Shea, which has been so essential before but does not work well enough in our problem (see [DS], [BGT, §§2.5, 5.2.3]).

We apply Theorem 1 to Hankel transforms. There the kernel k to be considered is

$$k_\nu(x) := x^{-\frac{3}{2}} J_\nu(1/x), \quad (2.4)$$

where $\nu > -\frac{1}{2}$ and J_ν is the Bessel function. Here we treat only absolutely convergent integrals. The Mellin transform $\check{k}_\nu(z)$ of k_ν converges absolutely for $-\nu - \frac{3}{2} < \Re z < -1$ (so $a = -\nu - \frac{3}{2}$, $b = -1$), and is given by Weber's integral:

$$\check{k}_\nu(z) = 2^{z+\frac{1}{2}} \frac{\Gamma(\frac{3}{4} + \frac{\nu}{2} + \frac{z}{2})}{\Gamma(\frac{1}{4} + \frac{\nu}{2} - \frac{z}{2})}. \quad (W)$$

We write F_ν , or simply F , for the Hankel transform $k_\nu * f$ of f :

$$F_\nu(x) := \int_0^\infty k_\nu(x/t) f(t) \frac{dt}{t} \quad (0 < x < \infty). \quad (2.5)$$

We note that, in the usual terminology, it is not F_ν but $x^{-1} F_\nu(1/x)$ which is called the Hankel transform of f :

$$x^{-1} F_\nu(1/x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(t) dt \quad (0 < x < \infty).$$

Theorem 2. Let $-1/2 < \nu < \infty$. Let $t^{\nu+1/2}f(t) \in L^1_{\text{loc}}[0, \infty)$, f be eventually positive, have finite upper order $\rho := \rho(f) \in (-\nu - \frac{3}{2}, -1)$, $f \in BD \cup BI$, and let f have Hankel transform F_ν . If

$$F_\nu(x)/f(x) \rightarrow c \neq 0 \quad (x \rightarrow \infty), \quad (2.6)$$

then $c = \check{k}_\nu(\rho)$ and $f \in R_\rho$.

The theorem above is an analogue of the results of [BI1, BI2]. The difference is that in Theorem 2 the monotonicity assumption of [BI1, BI2] is weakened at the cost of absolute convergence of integrals. For example, if we consider Fourier transforms of radial functions on $\mathbf{R}^n$, then we naturally meet Hankel transforms of order $\nu = \frac{1}{2}(n-2)$, by Bochner's theorem (see e.g. [BC, II.7, Th. 40]). In such a situation, the monotonicity assumption of [BI1, BI2] will sometimes be too strong.

Since we consider only absolutely convergent integrals here, we cannot include the case $\nu = -\frac{1}{2}$, that is, the Fourier cosine transform. For, the Mellin transform of the cosine kernel $k_{-1/2}(x) = (2/\pi)^{1/2}x^{-1}\cos(1/x)$ has no strip of absolute convergence. To consider the cosine transform, it is indispensable to treat conditionally convergent integrals as in [BI1, BI2]. On the other hand, the Fourier sine transform has strips of both absolute and conditional convergence; our results here apply only to the first, those of [BI1, BI2] to both.

As we have shown in [BI1, BI2], for all $\nu > -\frac{1}{2}$, $\check{k}_\nu$ satisfies both the conditions (2.1) and (2.2). So for f as in Theorem 1, Theorem 2 is an immediate consequence of Theorem 1. On the other hand, the Drasin-Shea theorem (the case $k(\cdot) \geq 0$) cannot be applied to the kernel k_ν as it changes sign. Further, if ν is large enough, then neither Theorem 1 nor Theorem 1a of Jordan [J] can be applied to k_ν (see §5).

The vanishing of f near zero as in Theorem 1 will be too restrictive, and actually is not assumed in Theorem 2. Since this is the point of the proof of Theorem 2 in §4, we discuss it briefly. Clearly, we could drop the vanishing of f near zero if we were able to show

$$\left| \int_0^1 f(u)k_\nu(x/u)\frac{du}{u} \right| = o(f(x)) \quad (x \rightarrow \infty). \quad (2.7)$$

However, it seems difficult to prove this directly. In fact, we can show

$$\left| \int_0^1 f(u) k_\nu(x/u) \frac{du}{u} \right| \leq c_1 x^{-(\nu + \frac{3}{2})}$$

for some $c_1 > 0$, and the consequence $f \in R_\rho$ of Theorem 2 certainly implies $x^{-(\nu + \frac{3}{2})} = o(f(x))$ as $x \rightarrow \infty$. Unfortunately, we do not know how to prove the last assertion directly, that is, without Theorem 2. For this reason, in the proof of Theorem 2, we follow the line of [BI1] and use Theorem 1 to reduce its complicated arguments rather than use the theorem directly.

§3. Proof of Theorem 1

For brevity, we will as far as possible keep, step by step, to the proof of [BGT, Theorem 5.2.1] (the Drasin–Shea theorem). By (2.3), f is eventually positive. We note a crude but useful bound. For $\gamma \in \mathbf{R}$, $G : (0, \infty) \rightarrow \mathbf{R}$ measurable such that $\check{G}(\gamma)$ converges absolutely, and f non-negative and measurable with bound $f(x) \leq d(\gamma)x^\gamma$ ($x > 0$), the Mellin convolution $G * f$ exists and satisfies the bound

$$|G * f(x)| \leq d(\gamma)x^\gamma \int_0^\infty t^{-\gamma} |G(t)| dt/t = d(\gamma) |\check{G}|^\sim(\gamma) x^\gamma \quad (0 < x < \infty).$$

Steps 1 and 2 are exactly as those of the proof of [BGT, Theorem 5.3.1] (Jordan’s theorem), so $\check{k}(\rho) = c$. Here we can assume $f \in BI$ instead of $f \in BD$; see [BGT, §5.2.4].

Steps 3-5. By the Riemann–Lebesgue Lemma and Vitali’s theorem in complex analysis [T1, §5.2], we may take $\epsilon > 0$ so small that $[\rho - 2\epsilon, \rho + 2\epsilon] \subset (a, b)$ and that $\check{k}(z)$ takes the value $\check{k}(\rho)$ only at $z = \rho$ in the strip $\rho - 2\epsilon \leq \Re z \leq \rho + 2\epsilon$ (see [BGT, §5.1.3], Jordan [J, p. 191]). Write p_1, p_2 for $\rho - \epsilon, \rho + \epsilon$, and consider

$$E_1(x) := I_{[1, \infty)}(x) x^{p_1}, \quad E_2(x) := I_{(0, 1]}(x) x^{p_2} \quad (0 < x < \infty).$$

The convergence strips of $\check{E}_1, \check{E}_2$ are $\{\Re z > p_1\}, \{\Re z < p_2\}$. Take $\gamma \in (\rho, p_2)$. Then, by definition of ρ ,

$$f(x) \leq d(\gamma)x^\gamma \quad (0 < x < \infty)$$

for some $d(\gamma) > 0$. Write

$$F(x) := (k * f)(x) \quad (0 < x < \infty).$$

It is important to consider, instead of f and F themselves, the following regularized versions:

$$h(x) := (E_2 * E_1 * f)(x), \quad H(x) := (E_2 * E_1 * F)(x) \quad (0 < x < \infty).$$

(since $\gamma \in (p_1, \infty) \cap (-\infty, p_2) \cap (a, b)$, the crude bound above shows successively that the integrals converge absolutely, whence Fubini's theorem gives associativity of the convolutions). Again by Fubini's theorem,

$$H(x) = (E_2 * E_1 * (k * f))(x) = (k * h)(x) \quad (0 < x < \infty). \quad (3.1)$$

By the following integral representations

$$h(x) = x^{p_2} \int_x^\infty (E_1 * f)(t) dt / t^{1+p_2} = x^{p_1} \int_0^x (E_2 * f)(t) dt / t^{1+p_1},$$

$x^{-p_1}h(x)$ is increasing and $x^{-p_2}h(x)$ is decreasing. So

$$\frac{(xu)^{-p_1}h(xu)}{x^{-p_1}h(x)} \leq 1 \quad (0 < u \leq 1, x > 0),$$

$$\frac{(xu)^{-p_2}h(xu)}{x^{-p_2}h(x)} \leq 1 \quad (1 \leq u < \infty, x > 0).$$

Combining, we obtain the key estimate

$$\frac{h(xu)}{h(x)} \leq \max(u^{p_1}, u^{p_2}) \quad (0 < u < \infty, 0 < x < \infty). \quad (3.2)$$

As in (5.2.2') of [BGT], we have

$$\frac{(E_1 * F)(x)}{(E_1 * f)(x)} = \frac{\int_0^x F(t) dt / t^{1+p_1}}{\int_0^x f(t) dt / t^{1+p_1}} \rightarrow \check{k}(\rho) \quad (x \rightarrow \infty)$$

(here we may use $f \in BI$ instead of $f \in BD$ to assure $\int_0^\infty f(t) dt / t^{1+p_1} = \infty$). So

$$\frac{H(x)}{h(x)} = \frac{\int_x^\infty (E_1 * F)(t) dt / t^{1+p_2}}{\int_x^\infty (E_1 * f)(t) dt / t^{1+p_2}} \rightarrow \check{k}(\rho) \quad (x \rightarrow \infty). \quad (3.3)$$

Since $f \in R_\rho$ follows from $h \in R_\rho$ by standard Tauberian arguments (see below), (3.1) and (3.3) imply that the problem has been reduced to that of h from that of f . The advantage here is the useful bound (3.2) on h which we cannot expect the original function f to satisfy.

Now we can follow the standard arguments. Choose any sequence $x_n \uparrow \infty$. Consider $j_n(u) := h(x_n u)/h(x_n)$. The functions $u^{-p_1} j_n(u)$ are increasing on $(0, \infty)$ and by (3.2) uniformly bounded on compact u -sets in $(0, \infty)$. By Helly selection, we can find a sequence of integers $n' \rightarrow \infty$ such that $j_{n'}$ converges pointwise on $(0, \infty)$, to j , say. Then $u^{-p_1} j(u)$ is increasing, $j(1) = 1$, and by (3.2)

$$j(u) \leq \max(u^{p_1}, u^{p_2}) \quad (0 < u < \infty). \quad (3.4)$$

From (3.1), for $r > 0$,

$$\frac{H(rx_{n'})}{h(rx_{n'})} \cdot \frac{h(rx_{n'})}{h(x_{n'})} = \int_0^\infty \frac{h(rx_{n'}/t)}{h(x_{n'})} k(t) \frac{dt}{t}.$$

We now have suitably dominated convergence of the integrand by (3.2) (note that $p_1, p_2 \in (a, b)$), and $H(rx_{n'})/h(rx_{n'}) \rightarrow \check{k}(\rho)$ by (3.3). So we obtain the following integral equation for j :

$$\check{k}(\rho) j(r) = (k * j)(r) \quad (0 < r < \infty). \quad (3.5)$$

The equation (3.5) is the same type as those in [DS], [J].

Step 6. Write

$$\phi(x) := j(e^x) e^{-\rho x}, \quad K(x) := k(e^x) e^{-\rho x} \quad (-\infty < x < \infty).$$

Choose c_1, c_2 such that $0 < \epsilon < c_1 < c_2 < 2\epsilon$. Then by (3.4),

$$\phi(x) \leq \max(e^{-\epsilon x}, e^{\epsilon x}) = e^{\epsilon |x|},$$

whence $e^{-c_1 |x|} \phi(x) \in L^2(\mathbf{R})$. On the other hand, $e^{-c_2 |x|} K(x) \in L^1(\mathbf{R})$ as

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-c_2 |x|} |K(x)| dx &\leq \int_{-\infty}^{\infty} e^{-c_2 x} |K(x)| dx \\ &= \int_0^\infty t^{-(c_2 + \rho)} |k(t)| \frac{dt}{t} = |k|^\sim(c_2 + \rho) < \infty \end{aligned}$$

by $c_2 + \rho \in (a, b)$.

We write the Fourier transform as

$$\hat{K}(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x) e^{-ixz} dx.$$

Then

$$\hat{K}(z) = \frac{1}{\sqrt{2\pi}} \check{k}(\rho + iz) \quad (|\Im z| \leq c_2). \quad (3.6)$$

The equation (3.5) becomes

$$\phi(x) = \frac{1}{\hat{K}(0)} \int_{-\infty}^{\infty} K(x-y)\phi(y)dy \quad (-\infty < x < \infty).$$

Here we note that

$$\hat{K}(0) = \check{k}(\rho) = c \neq 0.$$

By (3.6), the transcendental equation $\hat{K}(z) = \hat{K}(0)$ has a unique root $z = 0$ in the strip $|\Im z| \leq c_2$, which is at most double by (2.2). By Theorem 146 of Titchmarsh [T2],

$$\phi(x) = a_1 + a_2 x \quad (x \in \mathbf{R})$$

for some $a_1, a_2 \in \mathbf{C}$. That is,

$$j(x) = x^\rho (a_1 + a_2 \log x) \quad (0 < x < \infty).$$

Since j is real, so are a_1, a_2 . As $j(1) = 1$, we have $a_1 = 1$. Since $j(\cdot) \geq 0$, $a_2 = 0$. Thus $j(x) \equiv x^\rho$. Therefore the partial limit u^ρ of $h(ux_n)/h(x_n)$ does not depend on the sequence (x_n) chosen. Thus $h(ux)/h(x) \rightarrow u^\rho$ as $x \rightarrow \infty$, so $h \in R_\rho$. Since $x^{-p_1}(E_1 * f)(x)$ is increasing, $\log((E_1 * f)(x)/x^{1+p_2})$ is slowly decreasing. By the Monotone Density Theorem (cf. [BGT, §1.7]), we obtain $E_1 * f \in R_\rho$ from $h \in R_\rho$.

Steps 7-10 are proved as in the proof of [BGT, Theorem 5.2.1]. •

§4. Proof of Theorem 2

In this section, we write, for simplicity, F rather than F_ν . We write $f \asymp g$ if $f = O(g)$ and $g = O(f)$. We first show an analogue of [BGT, Proposition 2.10.3] (an O -version of the monotone density theorem) which we will need later.

Proposition 4.1. *Let $U(x) = \int_x^\infty u(t)dt$ ($x > 0$), where u is measurable, eventually positive, and satisfies the weak Tauberian condition $u \in BD \cup BI$. If $\beta(U) > -\infty$ and $\alpha(U) < 0$, then*

$$u(x) \asymp U(x)/x \quad (x \rightarrow \infty).$$

Though the proof below is quite similar to that of [BGT, Proposition 2.10.3], we show it for reader's convenience.

Proof of Proposition 4.1. We assume $u \in BD$, as the case $u \in BI$ is similar. Choose α such that $\alpha(U) < \alpha < 0$. By [BGT, Proposition 2.2.1], there exist positive constants C and X such that

$$U(y)/U(x) \leq C(y/x)^\alpha \quad (y \geq x \geq X).$$

Taking $\lambda > 1$ so large that $A := C\lambda^\alpha < 1$, we find

$$U(\lambda x)/U(x) \leq A < 1 \quad (x \geq X).$$

Since $u \in BD$, we may choose finite $b < \beta(u)$, and then by [BGT, Proposition 2.2.1] there exist positive constants c, x_0 such that

$$u(y)/u(x) \geq c(y/x)^b \quad (y \geq x \geq x_0).$$

Thus for large y ,

$$\frac{yu(y)}{c} \int_{1/\lambda}^1 x^b dx \geq \int_{y/\lambda}^y u(x) dx = U(y/\lambda) - U(y) \geq (1/A - 1)U(y),$$

so that $\liminf_{y \rightarrow \infty} yu(y)/U(y) > 0$. A similar calculation on $\int_y^{\lambda y} u(x) dx$ shows that the corresponding lim sup is finite. •

Before going into details, we explain the proof of Theorem 2 in rough outline. We write, for X large enough and $0 < x < \infty$,

$$\tilde{f}(x) := I_{[X, \infty)}(x)f(x), \quad (4.1)$$

$$\tilde{g}(x) := (C * \tilde{f})(x), \quad (4.2)$$

$$\tilde{G}(x) := (D * \tilde{f})(x), \quad (4.3)$$

where

$$C(x) := x^{\nu + \frac{1}{2}} e^{-x}, \quad (4.4)$$

$$D(x) := d_\nu \frac{x^{\nu + \frac{3}{2}}}{(1 + x^2)^{\nu + \frac{3}{2}}}, \quad d_\nu := \frac{2^{\nu+1} \Gamma(\nu + \frac{3}{2})}{\pi^{1/2}}. \quad (4.5)$$

We will obtain as in [BI1]

$$\tilde{G}(x) = (k_\nu * \tilde{g})(x), \quad (4.6)$$

$$\tilde{G}(x)/\tilde{g}(x) \rightarrow c \quad (x \rightarrow \infty), \quad (4.7)$$

$$\int_X^\infty f(t)t^{\nu+\frac{1}{2}}dt = \infty. \quad (4.8)$$

We will also obtain $\tilde{g} \in BI$ and $\rho(\tilde{g}) = \rho$. By (4.8),

$$x^{\nu+\frac{3}{2}}\tilde{G}(x) = d_\nu \int_X^\infty \frac{t^{\nu+\frac{1}{2}}f(t)}{\{1+(t/x)^2\}^{\nu+\frac{3}{2}}}dt \rightarrow \infty \quad (x \rightarrow \infty),$$

whence by (4.7) we have $x^{-(\nu+\frac{3}{2})} = o(\tilde{g}(x))$ as $x \rightarrow \infty$. So (2.7) holds if we replace f by $\tilde{g}$. So this time, by considering $I_{[1,\infty)}\tilde{g}$, we can apply Theorem 1 to obtain $\tilde{g} \in R_\rho$. To deduce $f \in R_\rho$ from this, we will use Karamata's Tauberian theorem. For that, we will need some Tauberian condition on f . Here an extra complication, absent in [BI1, BI2], arises. For there, f was assumed to satisfy the strong Tauberian condition of monotonicity, which is not available now. Fortunately, the desired Tauberian condition can be shown by the same arguments as in Steps 7–10 of the proof of [BGT, Theorem 5.2.1], whence the proof will be complete.

Now we are ready to prove Theorem 2.

Step 1. Since $f \in BI \cup BD$, by [BGT, Proposition 2.2.1] there exists $X > 0$ such that f and $1/f$ are both positive and locally bounded on $[X, \infty)$. Define $\tilde{f}$ by (4.1). Write, for $x \in (0, \infty)$,

$$\tilde{F}(x) := (k_\nu * \tilde{f})(x),$$

$$\bar{f}(x) := I_{(0,X)}(x)f(x),$$

$$\bar{F}(x) := (k_\nu * \bar{f})(x).$$

Pick $\gamma \in (\rho, -1)$. Then there exists $c_1 \in (0, \infty)$ such that

$$0 \leq \tilde{f}(x) \leq c_1 x^\gamma \quad (0 < x < \infty), \quad (4.9)$$

whence

$$|\tilde{F}(x)| \leq c_1 |k_\nu|^\sim(\gamma) x^\gamma \quad (0 < x < \infty).$$

On the other hand, there exists c_2 such that

$$|k_\nu(x)| \leq c_2 x^{-(\nu+\frac{3}{2})} \quad (0 < x < \infty),$$

whence

$$|\bar{F}(x)| \leq c_2 x^{-(\nu+\frac{3}{2})} \int_0^X |f(t)| t^{\nu+\frac{1}{2}} dt.$$

As a consequence, for some $c_3 \in (0, \infty)$,

$$|F(x)| \leq c_3 \{x^{-(\nu+\frac{3}{2})} + x^\gamma\} \quad (0 < x < \infty).$$

Step 2. Let C and D as in (4.4) and (4.5). Then as in the proof of [BI1, Lemma 2], we have

$$(C * \bar{F})(x) = (D * \bar{f})(x) \quad (0 < x < \infty),$$

the integral converging absolutely. By (4.9), $\tilde{f} \in L^1[0, \infty)$, while $C(x \cdot) \in L^1[0, \infty)$ for any $x > 0$. So Parseval's formula (cf. [MO, Theorem III]) yields

$$(C * \tilde{F})(x) = (D * \tilde{f})(x) \quad (0 < x < \infty).$$

Combining, we obtain

$$(D * f)(x) = (C * F)(x) \quad (0 < x < \infty), \quad (4.10)$$

the integral converging absolutely.

Step 3. Define $\tilde{g}$ and $\tilde{G}$ by (4.2) and (4.3). We write $G(x) := (D * f)(x)$ for $0 < x < \infty$. By (4.10),

$$G(x) = (C * F)(x) \quad (0 < x < \infty). \quad (4.11)$$

As in [BI1, §3], we have (4.8), whence

$$\tilde{G}(x)/G(x) \rightarrow 1 \quad (x \rightarrow \infty).$$

On the other hand, since $1/f$ is locally bounded on $[X, \infty)$, as in [BI1, §3], it follows from (2.6) and (4.11) that

$$G(x)/\tilde{g}(x) \rightarrow c \quad (x \rightarrow \infty),$$

whence (4.7) follows.

Step 4. Since $|k_\nu|^\sim(\gamma) < \infty$ and $|C|^\sim(\gamma) < \infty$, it follows from (4.9) that

$$|k_\nu| * (|C| * \tilde{f})(x) \leq c_1 |k_\nu|^\sim(\gamma) |C|^\sim(\gamma) x^\gamma < \infty.$$

Therefore, by Fubini's theorem,

$$\tilde{G}(x) = (D * \tilde{f})(x) = ((k_\nu * C) * \tilde{f})(x)$$

$$= (k_\nu * (C * \tilde{f}))(x) = (k_\nu * \tilde{g})(x),$$

whence (4.6).

By the definition (4.2), $x^{-(\nu+\frac{1}{2})}\tilde{g}(x)$ is decreasing, whence $\tilde{g} \in BI$. As in [BI1, Lemma 5], $\rho(\tilde{g}) = \rho$.

As we saw above, (2.7) holds if we replace f by $\tilde{g}$:

$$\int_0^1 \tilde{g}(t)k_\nu(x/t)dt/t = o(\tilde{g}(x)) \quad (x \rightarrow \infty).$$

Therefore from (4.7) we have for $\bar{g} := I_{[1,\infty)}\tilde{g}$,

$$(k_\nu * \bar{g})(x)/\bar{g}(x) \rightarrow c \quad (x \rightarrow \infty).$$

By [BI1, Proposition 3] and [BI2, §5], the kernel k_ν satisfies the conditions (2.1) and (2.2). Therefore, by Theorem 1, $c = \check{k}(\rho)$ and $\bar{g} \in R_\rho$, whence $\tilde{g} \in R_\rho$.

Step 5. Write

$$U(x) := \int_x^\infty \tilde{f}(t)t^{-(\nu+\frac{3}{2})}dt \quad (0 < x < \infty).$$

Then by Karamata's Tauberian theorem (cf. [BGT, Theorem 1.7.1']) $\tilde{g} \in R_\rho$ yields $U \in R_{\rho-\nu-\frac{1}{2}}$. In particular, $\beta(U) = \alpha(U) = \rho - \nu - \frac{1}{2} < 0$. So if we write

$$q(x) := x^{\nu+\frac{1}{2}}U(x) \quad (0 < x < \infty),$$

then Proposition 4.1 yields, for some $M > 1$ and $Y > 0$,

$$M^{-1} \leq f(x)/q(x) \leq M \quad (x \geq Y).$$

As a consequence, $f \in OR$ (see [BGT, §2.0.2] for the definition of OR).

Step 6. For any fixed $B > 1$,

$$\int_0^B f(u)k_\nu(x/u)du/u = o(q(x)) \quad (x \rightarrow \infty).$$

For, as $x \rightarrow \infty$,

$$|\int_0^B f(u)k_\nu(x/u)du/u| \leq c_2 x^{-(\nu+\frac{3}{2})} \int_0^B |f(t)|t^{\nu+\frac{1}{2}}dt = o(q(x)).$$

Then, as in Steps 7–9 of the proof of [BGT, Theorem 5.2.1], we obtain

$$\lim_{\lambda \downarrow 1} \liminf_{x \rightarrow \infty} \inf_{\sigma \in [1, \lambda)} f(\sigma x)/f(x) = 1,$$

which implies that $\log f$ is slowly decreasing. Thus, by the Monotone Density Theorem, $U \in R_{\rho-\nu-\frac{1}{2}}$ implies $f \in R_\rho$, as desired. This completes the proof. •

§5. Remarks

In this section, we show that the kernel k_ν defined by (2.4) fails to satisfy the conditions of Jordan [J, Theorems 1 and 1a] if ν is large enough. Recall the Mellin transform $\check{k}_\nu$ of our kernel k_ν from (W).

Write

$$\Psi(x) = \Gamma'(x)/\Gamma(x) \quad (0 < x < \infty)$$

for the logarithmic derivative of the gamma function (digamma function). Then Ψ is increasing on $(0, \infty)$. Now

$$\log 2 + \frac{1}{2}\Psi(1/2) + \frac{1}{2}\Psi(1) = -\gamma < 0,$$

$$\log 2 + \frac{1}{2}\Psi(3/4) + \frac{1}{2}\Psi(5/4) = -\gamma - 2\log 2 + 2 > 0,$$

where γ is Euler's constant. So we may define $\nu_1 \in (\frac{1}{2}, 1)$ by

$$\log 2 + \frac{1}{2}\Psi\left(\frac{1}{4} + \frac{\nu_1}{2}\right) + \frac{1}{2}\Psi\left(\frac{3}{4} + \frac{\nu_1}{2}\right) = 0$$

($\nu_1 = 0.9616 \dots$ by calculation using *Mathematica*).

For $x \in (-\nu - \frac{3}{2}, \nu + 1)$,

$$\log \check{k}_\nu(x) = (x + \frac{1}{2})\log 2 + \log \Gamma\left(\frac{3}{4} + \frac{\nu}{2} + \frac{x}{2}\right) - \log \Gamma\left(\frac{1}{4} + \frac{\nu}{2} - \frac{x}{2}\right),$$

whence

$$\check{k}'_\nu(x)/\check{k}_\nu(x) = \log 2 + \frac{1}{2}\Psi\left(\frac{3}{4} + \frac{\nu}{2} + \frac{x}{2}\right) + \frac{1}{2}\Psi\left(\frac{1}{4} + \frac{\nu}{2} - \frac{x}{2}\right).$$

So

$$\check{k}'_\nu(-1) = \check{k}_\nu(-1) \left\{ \log 2 + \frac{1}{2}\Psi\left(\frac{1}{4} + \frac{\nu}{2}\right) + \frac{1}{2}\Psi\left(\frac{3}{4} + \frac{\nu}{2}\right) \right\}.$$

Since k oscillates infinitely often near zero (and also $\check{k}_\nu(-1)$ is finite), Jordan's Theorem 1 cannot be applied to k_ν at all. If $\nu \leq \nu_1$, then $\check{k}'_\nu(-1) \leq 0$, whence by [BI1, §5], $\check{k}_\nu$ is decreasing on $(-\nu - \frac{3}{2}, -1)$. Further, $|\check{k}_\nu((-\nu - \frac{3}{2})+)| = \infty$. So Jordan's Theorem 1a applies to k_ν for all $\rho \in (-\nu - \frac{3}{2}, -1)$ if $-\frac{1}{2} < \nu \leq \nu_1$.

On the other hand, if $\nu > \nu_1$, then $\check{k}'_\nu(-1) < 0$, whence by [BI2, §5] there exists $b_1 \in (-\nu - \frac{3}{2}, -1)$ such that $\check{k}_\nu$ is decreasing on $(-\nu - \frac{3}{2}, b_1]$ and increasing on $[b_1, -1)$. So Jordan's Theorem 1a still applies to k_ν if $\nu > \nu_1$ and $\rho \in (-\nu - \frac{3}{2}, b_1]$. However, if $\nu > \nu_1$ and $\rho \in (b_1, -1)$, then neither of Jordan's theorems apply to k_ν . Our Theorems 1 and 2 cover this last case.

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