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GENERALIZATION OF VARIATIONS AND BAUM-BOTT RESIDUES FOR HOLOMORPHIC FOLIATIONS ON SINGULAR VARIETIES

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GENERALIZATION OF VARIATIONS AND BAUM-BOTT RESIDUES FOR HOLOMORPHIC FOLIATIONS ON SINGULAR VARIETIES

DANIEL LEHMANN AND TATSUO SUWA

1. Introduction

There are already several kinds of residues for holomorphic singular foliations on singular varieties. In this paper, combining the two techniques of using coherent sheaves and the extension of the normal bundle to a local complete intersection, we would like to define:
- a generalization to the case of arbitrary dimension, of the "variation", defined in [KS] for foliations by complex curves on complex surfaces,
- and, closely related to the previous one, a generalization to the case of foliations on singular varieties, of the usual Baum-Bott residues for singular holomorphic foliations on complex manifolds. Here, by a (singular) foliation on a singular subvariety V of a complex manifold M, we mean a foliation $\mathcal{F}_V$ on the regular part of V, that we assumed to be induced by a (singular) foliation $\mathcal{F}$ on M leaving V invariant; in fact, the data is not only $\mathcal{F}_V$, but also the germ of $\mathcal{F}$ along V.

In this framework, the link between the Camacho-Sad index and the variation given in [KS] (see also [Br2]) becomes more or less obvious.

Before all, let us recall the general routine which produces "residue theorems". First, we need a **vanishing theorem** (or **obstruction theory**), of the following type: the existence of some special "object" $\mathcal{D}$ on a space X implies that "some" characteristic classes $\varphi(E) \in H^*(X)$ of "some" bundle $E \to X$ vanish. We are only concerned in this paper with "primary" characteristic classes, which are obstructions to the existence of some object; however the same could be done with higher order characteristic classes which are obstructions to the "compatibility" between several such objects (see for instance [H], [Le5], [GL], [Po]). Also, all homology and cohomology groups in this paper are for real or complex coefficients; corresponding vanishing theorems for integral coefficients may be wrong (see [Bo2]).

When such an object $\mathcal{D}$ is given now only away from some closed subset S of X (called its "singular" set), then it is only the restrictions $\varphi(E)|_{X-S}$ to $H^*(X-S)$ which vanish: therefore, because of the exactness of the sequence

$$\cdots \to H^*(X, X-S) \to H^*(X) \to H^*(X-S) \to \cdots, \tag{1.1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

the classes $\varphi(E)$ will lift to some $\varphi(E,\mathcal{D}) \in H^*(X, X-S)$, and generally in a canonical way once given $\mathcal{D}$ on $X-S$. The classes $\varphi(E,\mathcal{D})$ will be called the **residual characteristic classes**. Recall that, for any open neighborhood U of S, $H^*(X,X-S)$ and $H^*(U,U-S)$ are isomorphic ("excision" property): this implies that $\varphi(E,\mathcal{D})$ depends only on the local behaviour of the given object $\mathcal{D}$ near (but away from) S: it is the reason why, we say that the characteristic classes have been "localized" near S.

If, furthermore, X (hence S) is compact, and if X has a fundamental class $[X]$, we call **residue** the image $\mathrm{Res}_\varphi(\mathcal{D},E;S) = A_S(\varphi(E,\mathcal{D}))$ of $\varphi(E,\mathcal{D})$ by the Alexander homomorphism $u \to u \frown [X]$ from $H^*(X,X-S)$ into $H_{\ell-*}(S)$, ℓ denoting the real dimension of $[X]$. More generally, if we denote by S_α a compact connected component of S having some orientable regular neighborhood U_α not intersecting the other components S_β, we may define $\mathrm{Res}_\varphi(\mathcal{D},E;S_\alpha)$ as the component on $H_{\ell-*}(S_\alpha)$ of $\mathrm{Res}_\varphi(\mathcal{D},E;S) \in H_{\ell-*}(S) = \oplus_\beta H_{\ell-*}(S_\beta)$.

Note that the homomorphism A_S is compatible with the Poincaré homomorphism $P_X : H^*(X) \to H_{\ell-*}(X)$, in such a way that the following diagram is commutative:

$$\begin{array}{ccc} H^*(X,X-S) & \xrightarrow{\pi^*} & H^*(X) \\ {\scriptstyle A_S}\downarrow & & {\scriptstyle P_X}\downarrow \\ H_{\ell-*}(S) = \oplus_\alpha H_{\ell-*}(S_\alpha) & \xrightarrow{\iota_*} & H_{\ell-*}(X) \end{array} \tag{1.2}$$

Therefore, we get the **residue theorem** in $H_*(X)$:

$$\sum_\alpha (\iota_\alpha)_* \,\mathrm{Res}_\varphi(\mathcal{D},E;S_\alpha) = \varphi(E) \frown [X], \tag{1.3}$$

where ι_α denotes the natural inclusion of S_α into X. When P_X is an isomorphism (for instance when X is a compact oriented manifold), we get the corresponding residue theorem in cohomology:

$$\sum_\alpha (\mu_\alpha)_* \,\mathrm{Res}_\varphi(\mathcal{D},E;S_\alpha) = \varphi(E),$$

where $(\mu_\alpha)_* = {P_X}^{-1} \circ (\iota_\alpha)_*$.

However, it may happen that the bundle E depends on the data $\mathcal{D}$. In this case, we first need to extend in some sense E or at least $\varphi(E)$ on all of X, even if $\mathcal{D}$ itself is defined only over $X-S$, for the previous residue theorem to make sense. Moreover, it may happen that there are different ways to do so. Let us recall three examples of this situation:

1) The Bott vanishing theorem ([Bo2]) asserts that the graded algebra $Chern^*(N_\mathcal{F})$ of Chern polynomials of the normal bundle $N_\mathcal{F}$ to a non-singular

holomorphic foliation $\mathcal{F}$ of dimension p on a complex manifold M of dimension n vanishes in dimension $* > 2q = 2(n-p)$. If $\mathcal{F}$ is only defined over $M-S$, Baum and Bott need first to define $\mathcal{N}_{\mathcal{F}}$ as a coherent sheaf (or equivalently as a virtual vector bundle) over all of M and develop a Chern-Weil theory for such coherent sheaves or virtual bundles, before to define and compute their residues [BB1], [BB2].

2) A similar vanishing theorem is proved in [Le2]: if a non-singular submanifold V of M is invariant by $\mathcal{F}$, and if $\mathcal{F}$ has no singularities on V, then the graded algebra $Chern^*(N_V)$ of the normal bundle N_V to V in M vanishes in dimension $* > 2(m-p)$, where m denotes the complex dimension of V, thus $m-p$ is the complex codimension of the induced foliation $\mathcal{F}_V$ on V. The Camacho-Sad index ([CS] for $m=p=q=1$) and its first generalizations ([G], [Li2] for $m=p$ and $q=1$, or [Le2] for arbitrary m,p,q) were concerning the case where $\mathcal{F}$ had singularities on V, V being still non-singular and N_V therefore defined on all of V. When V is now itself singular, but being a local complete intersection, the normal bundle to the regular part of V in M is first extended to all of V and then to a neighborhood of V before studying corresponding residues in [LS1] (generalizing various particular cases in [Li2], [So], [Su2] where the extension of N_V wasnot used).

3) In the case where $\mathcal{F}$ is globally defined by a holomorphic non singular vector field v on M, the coherent sheaf $\mathcal{N}_{\mathcal{F}}$ is stably equivalent to TM. Therefore the Bott vanishing theorem becomes, for M compact: all Chern numbers of M vanish. Thus, when v is singular, the sum of Baum-Bott residues is here equal to Chern numbers of M ([BB1]; see also [Bo1], [Bo3] when v is non-degenerate). If now v is tangent to a subvariety V of M, we get similarly Chern numbers of V as long as V is non-singular. If V has singularities, but being a local complete intersection, we still can do the same, but replacing now Chern numbers of V (i.e., Chern numbers of the tangent bundle TV) by Chern numbers of the "virtual tangent bundle" $TM|_V - N_V$ (globally defined on V since N_V is defined on all of V): this is done in [LSS], defining residues (which turn out to be equal in the case of isolated singularities to those defined in [GSV] and [SS]). This will be a particular case of the Baum-Bott residues on V that we wish to get; in fact, if the leaves are one dimensional, the foliation $\mathcal{F}$ is defined by a morphism $h : F \to TM$ of vector bundles, where F is some holomorphic line bundle (the previous particular case corresponding to a trivial F): the effective computations done in [LSS] in the case of isolated singularities remain then valid with respect to vector fields defined locally by mean of a local trivialization of F; the only difference is that the sum of these residues will produce now Chern numbers of the virtual normal bundle $TM|_V - N_V - F|_V$ to $\mathcal{F}_V$ in V, instead of the virtual tangent bundle $TM|_V - N_V$ to V.

The first situation we are interested in is that of the restriction $\mathcal{N}_{\mathcal{F}}|_V$ of $\mathcal{N}_{\mathcal{F}}$ to an invariant subvariety V (variations): in this case, the corresponding vanishing theorem asserts that, when $S = \mathrm{Sing}(V) \cup (\mathrm{Sing}(\mathcal{F}) \cap V)$ is empty, $Chern^*(\mathcal{N}_{\mathcal{F}}|_V)$ vanishes not only for $* > 2q = 2(n-p)$ but also for $* > 2(m-p)$. Observe on

the other hand that, for $p = m$, N_V and $\mathcal{N}_{\mathcal{F}}|_V$ are two different extensions of the normal bundle to a leaf in $V - S$, giving rise to two different kind of residues.

The second situation concerns $\mathcal{N}_{\mathcal{F}}|_V - N_V$, which is an extension to V of the normal bundle $N_{\mathcal{F}_V}$ to $\mathcal{F}_V$ over $V - S$. When $S = \emptyset$, the usual Bott vanishing theorem may be used for $\mathcal{F}_V$ on V. Thus, in the general case, corresponding residues are generalizations of the Baum-Bott residues for $\mathcal{F}_V$, to the case where V may be singular.

2. Backgrounds

(A) Mayer-Vietoris complex on a singular variety, and integration

Let M be a complex manifold of dimension n and let V be an analytic subvariety of M of dimension m.

Let S be a closed subset of V containing the singular set $\mathrm{Sing}(V)$ of V, and let $(S_\alpha)_\alpha$ be the connected components of S. For each α, we take a regular neighborhood $\tilde{U}_\alpha$ of S_α in M so that $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$, if $\alpha \neq \beta$. Let $\tilde{U}_0$ be a tubular neighborhood of $V - S$ in M with the projection $\rho : \tilde{U}_0 \longrightarrow U_0 = V - S$. Then we have a covering $\mathcal{U} = \{\tilde{U}_0, (\tilde{U}_\alpha)_\alpha\}$ of $\tilde{U} = \tilde{U}_0 \cup \left(\bigcup_\alpha \tilde{U}_\alpha\right)$. We assume moreover that V is a deformation retract of $\tilde{U}$. Set $\tilde{U}_1 = \bigcup_\alpha \tilde{U}_\alpha$.

Denoting by $A^*(N)$ the differential algebra of de Rham forms on a smooth manifold N, the Mayer-Vietoris complex $A^*(\mathcal{U})$ of V is defined by:

$$A^*(\mathcal{U}) = A^*(V - S) \oplus A^*(\tilde{U}_1) \oplus A^{*-1}(\tilde{U}_{01}),$$

with the differential $D(\alpha_0, \alpha_1, \alpha_{01}) = (d\alpha_0, d\alpha_1, -d\alpha_{01} + \alpha_1 - \rho^*\alpha_0)$, and the cup product

$$(\alpha_0, \alpha_1, \alpha_{01}) \smile (\beta_0, \beta_1, \beta_{01}) = (\alpha_0 \wedge \beta_0, \alpha_1 \wedge \beta_1, \alpha_{01} \wedge \beta_1 + (-1)^{|\alpha|}\rho^*\alpha_0 \wedge \beta_{01}).$$

Recall ([BT]) that the map $\iota : A^*(\tilde{U}) \longrightarrow A^*(\mathcal{U})$ defined by $\iota(\alpha) = (\alpha|_{V-S}, \alpha|_{U_1}, 0)$ induces an algebra isomorphism $H^*(V) \simeq H^*(\tilde{U}) \longrightarrow H(A^*(\mathcal{U}))$ in cohomology. (It is understood that the symbol of restriction of differential forms to the suitable open set is omitted.)

Also let, for each α, $\tilde{R}_\alpha$ be a compact (real) $2n$-dimensional manifold in $\tilde{U}_\alpha$, with C^∞ oriented boundary $\tilde{R}_{0\alpha}$, containing S_α in its interior. We set $\tilde{R}_0 = \tilde{U} - \bigcup_\alpha \mathrm{Int}\,\tilde{R}_\alpha$. We further assume that each $\tilde{R}_{0\alpha} = \partial\tilde{R}_\alpha$ is transversal to V, and we set $R_0 = \tilde{R}_0 \cap V$, $R_\alpha = \tilde{R}_\alpha \cap V$ and $R_{0\alpha} = \tilde{R}_{0\alpha} \cap V$. Recall that every element σ in $A^*(\mathcal{U})$ is expressed as $\sigma = \left(\sigma_0, (\sigma_\alpha)_\alpha, (\sigma_{0\alpha})_\alpha\right)$.

If V is compact, we have the evaluation

$$\int_V : A^{2m}(\mathcal{U}) \longrightarrow \mathbb{C},$$

defined, for σ in $A^{2m}(\mathcal{U})$, by

$$\int_V \sigma = \int_{R_0} \sigma_0 + \sum_\alpha \left(\int_{R_\alpha} \sigma_\alpha + \int_{R_{0\alpha}} \sigma_{0\alpha} \right).$$

Remark 2.1. The Mayer-Vietoris complex imbeds naturally in the total Čech-de Rham complex associated to an open covering $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of a manifold $\tilde{U}$ by two open sets. All what is recalled in this paragraph could be done for a covering $\mathcal{U}$ with arbitrary number of open sets (cf. [BT] for Čech-de Rham complex and [Le3] for integration in the general case). Observe that integration for arbitrary open covering, while not used in this paper, has been intensively used in various papers, such as [Le2], [LS1], [LSS].

Integration above defines the pairing

$$B : A^*(\mathcal{U}) \times A^{2m-*}(\mathcal{U}) \to \mathbb{C}, \tag{2.2}$$

$$B(\sigma, \tau) = \int_V \sigma \smile \tau \quad \text{for } \sigma \in A^*(\mathcal{U}) \text{ and } \tau \in A^{2m-*}(\mathcal{U}).$$

This induces the "Poincaré homomorphism"

$$P_V : H^*(\tilde{U}; \mathbb{C}) \simeq H^*(V; \mathbb{C}) \to H_{2m-*}(\tilde{U}; \mathbb{C}) \simeq H_{2m-*}(V; \mathbb{C}),$$

which coincides with the cap product by the homology class $[V]$.

Denote by $A^*(\mathcal{U}, \tilde{U}_0)$ the subgroup in $A^*(\mathcal{U})$ of elements of the form $\sigma = (0, \sigma_\alpha, \sigma_{0\alpha})$. From the comparison of the long cohomology exact sequence coming from the exact sequence

$$0 \to A^*(\mathcal{U}, \tilde{U}_0) \to A^*(\mathcal{U}) \to A^*(\tilde{U}_0) \to 0$$

with the usual cohomology exact sequence for the pair $(V, V - S)$, we get:

$$H^*(A^*(\mathcal{U}, \tilde{U}_0)) \simeq H^*(V, V - S; \mathbb{C}).$$

Then, (2.2) induces the pairing

$$B : A^*(\mathcal{U}, \tilde{U}_0) \times A^{2m-*}(\tilde{U}_1) \to \mathbb{C},$$

given by

$$B(\sigma, \tau) = \sum_\alpha \left(\int_{R_\alpha} \sigma_a \wedge \tau_\alpha + \int_{R_{0\alpha}} \sigma_{0\alpha} \wedge \tau_\alpha \right) \tag{2.3}$$

for σ in $A^*(\mathcal{U}, \tilde{U}_0)$ and $\tau = (\tau_\alpha)$ in $A^{2m-*}(U_1)$. In cohomology, the pairing defines the "Alexander homomorphism"

$$A_V : H^*(V, V-S) \to H_{2m-*}(\tilde{U}_1; \mathbb{C}) \simeq H_{2m-*}(S; \mathbb{C}) = \oplus_\alpha H_{2m-*}(S_\alpha; \mathbb{C}).$$

We have the commutative diagram:

$$\begin{array}{ccc} H^*(A^*(\mathcal{U}, \tilde{U}_0)) \simeq H^*(V, V-S; \mathbb{C}) & \xrightarrow{j^*} & H^*(A^*(\mathcal{U}) \simeq H^*(V; \mathbb{C}) \\ \downarrow A_V & & \downarrow P_V \\ H_{2m-*}(S; \mathbb{C}) \simeq \oplus_\alpha H_{2m-*}(S_\alpha; \mathbb{C}) & \xrightarrow{i_*} & H_{2m-*}(V; \mathbb{C}) \end{array} \tag{2.4}$$

where j^* denotes the canonical homomorphism. If V is non-singular, the above P_V and A_V are the usual Poincaré and Alexander duality isomorphisms.

(B) Chern-Weil theory for virtual vector bundles and coherent sheaves ([BB2])

Let E be a complex vector bundle over M. Denoting by σ_i the i-th elementary symmetric function, we set $c_i = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i$, which is the i-th Chern polynomial. If ∇ is a connection for E, we have a closed $2i$-form $c_i(\nabla)$ whose class in the (de Rham) cohomology $H^{2i}(M; \mathbb{C})$ is the i-th Chern class $c_i(E)$ of E. The total Chern class is given by $c(E) = 1 + c_1(E) + c_2(E) + \cdots$, which is invertible in the cohomology ring $H^*(M; \mathbb{C})$. More generally, if φ is a symmetric polynomial, we may write $\varphi = P(c_1, c_2, \dots)$ for some polynomial P. We set $\varphi(\nabla) = P(c_1(\nabla), c_2(\nabla), \dots)$, which is a closed form. Furthermore, if we have $p+1$ connections $\nabla_0, \dots, \nabla_p$ for E, there exists a differential form $\varphi(\nabla_0, \dots, \nabla_p)$ alternating in the $p+1$ entries and satisfying

$$\sum_{k=0}^{p} (-1)^k \varphi(\nabla_0, \dots, \widehat{\nabla_k}, \dots, \nabla_p) + (-1)^p d\varphi(\nabla_0, \dots, \nabla_p) = 0 \tag{2.5}$$

([Bo2] p. 65), here we use a sign convention different from that of [Bo2]. In particular, if ∇ and ∇' are two connections for E, then $\varphi(\nabla') - \varphi(\nabla) = d\varphi(\nabla, \nabla')$. Thus the class of $\varphi(\nabla)$ is independent of the choice of the connection ∇ and is the characteristic class $\varphi(E)$ of E with respect to φ. Note that if φ is homogeneous of degree d, then $\varphi(\nabla_0, \dots, \nabla_p)$ above is a $(2d-p)$-form and that $\varphi(E)$ is in $H^{2d}(M; \mathbb{C})$.

Now suppose we are given, for each $i = 0, \dots, q$, a complex vector bundle E_i on M and a connection $\nabla^{(i)}$ for E_i. We denote by $\nabla^\bullet$ the family of connections $(\nabla^{(q)}, \dots, \nabla^{(0)})$ and define the total Chern form $c(\nabla^\bullet)$ of the "virtual bundle" $\xi = \sum_{i=0}^{q} (-1)^i E_i$ with respect to $\nabla^\bullet$ by

$$c(\nabla^\bullet) = \prod_{i=0}^{q} c(\nabla^{(i)})^{\epsilon(i)} \qquad \epsilon(i) = (-1)^i. \tag{2.6}$$

We denote by $c_j(\nabla^\bullet)$ the homogeneous component of degree $2j$ in $c(\nabla^\bullet)$. If φ is a symmetric polynomial (a polynomial in the Chern polynomials of ξ), we may write as a finite sum

$$\varphi = \sum_\ell \varphi_\ell^{(0)} \cdots \varphi_\ell^{(q)},$$

where, for each i and ℓ, $\varphi_\ell^{(i)}$ is a polynomial in the Chern polynomials of E_i. Then we set

$$\varphi(\nabla^\bullet) = \sum_\ell \varphi_\ell^{(0)}(\nabla^{(0)}) \wedge \cdots \wedge \varphi_\ell^{(q)}(\nabla^{(q)}).$$

Now suppose we have $p+1$ families of connections $\nabla_k^\bullet = (\nabla_k^{(q)}, \ldots, \nabla_k^{(0)})$, $k = 0, \ldots, p$. Then we may construct a form $\varphi(\nabla_0^\bullet, \ldots, \nabla_p^\bullet)$ satisfying

$$\sum_{k=0}^{p} (-1)^k \varphi(\nabla_0^\bullet, \ldots, \widehat{\nabla_k^\bullet}, \ldots, \nabla_p^\bullet) + (-1)^p d\varphi(\nabla_0^\bullet, \ldots, \nabla_p^\bullet) = 0. \tag{2.7}$$

In fact, let Δ^p be the standard simplex $\sum_{i=0}^p t_i = 1$, $t_i > 0$ in $\mathbb{R}^{p+1}$, and $\pi : M \times \Delta^p \to M$ be the natural projection. Then, $\varphi(\nabla_0^\bullet, \ldots, \nabla_p^\bullet)$ is equal to the integration of $\varphi(\tilde{\nabla}^\bullet)$ along the fiber Δ^p of the projection π, where $\tilde{\nabla}^\bullet = (\tilde{\nabla}^{(0)}, \tilde{\nabla}^{(1)}, \cdots, \tilde{\nabla}^{(q)})$, each $\tilde{\nabla}^{(r)}$ denoting the connection on $\pi^{-1}(E_k)$ which is the usual affine combination of the connections $\nabla_k^{(r)}$.

In particular, if $\nabla_0^\bullet$ and $\nabla_1^\bullet$ are two families of connections, then

$$\varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet) = d\varphi(\nabla_0^\bullet, \nabla_1^\bullet).$$

Hence we see that the class $[\varphi(\nabla^\bullet)]$ of the closed form $\varphi(\nabla^\bullet)$ in $H^*(M; \mathbb{C})$ depends only on ξ and not on the choice of the family of connections $\nabla^\bullet$. We denote this class by $\varphi(\xi)$ and call it the characteristic class of ξ with respect to φ. In the case $\varphi = c_i$ or $\varphi = c = 1 + c_1 + c_2 + \cdots$, this is the i-th Chern class $c_i(\xi)$ or the total Chern class $c(\xi)$ of ξ, respectively. Note that

$$c(\xi) = \prod_{i=0}^{q} c(E_i)^{\epsilon(i)}.$$

Let

$$0 \to E_q \xrightarrow{\psi_q} \cdots \to E_1 \xrightarrow{\psi_1} E_0 \to 0 \tag{2.8}$$

be a sequence of vector bundles on M, and, for each i, let $\nabla^{(i)}$ be a connection for E_i. We say that the family $(\nabla^{(q)}, \ldots, \nabla^{(0)})$ is compatible with the sequence if, for each i, the following diagram is commutative:

$$\begin{array}{ccc} A^0(M, E_i) & \xrightarrow{\nabla^{(i)}} & A^1(M, E_i) \\ {\scriptstyle \psi_i}\downarrow & & \downarrow{\scriptstyle 1 \otimes \psi_i} \\ A^0(M, E_{i+1}) & \xrightarrow{\nabla^{(i+1)}} & A^1(M, E_{i+1}). \end{array}$$

If the above sequence is exact, then the virtual bundle $\xi = \sum_{i=0}^{q}(-1)^i E_i$ is trivial, thus $c(\xi) = 1$. In fact, there is always a family $\nabla^\bullet = (\nabla^{(q)}, \ldots, \nabla^{(0)})$ of connections compatible with the sequence and for such a family we have ([BB2] (4.22) Lemma)

$$c(\nabla^\bullet) = 1.$$

From this we have

Proposition 2.9. *Suppose the sequence (2.8) is exact. Let φ be a symmetric polynomial and $\nabla_k^\bullet = (\nabla_k^{(q)}, \ldots, \nabla_k^{(0)})$, $k = 0, \ldots, p$, families of connections compatible with (2.8). Then*

$$\varphi(\check{\nabla}_0^\bullet, \ldots, \check{\nabla}_p^\bullet) = \varphi(\nabla_0^{(0)}, \ldots, \nabla_p^{(0)}). \quad \textit{In particular} \quad \varphi(\check{\xi}) = \varphi(E_0),$$

where $\check{\nabla}_k^\bullet$ denotes the family of connections $(\nabla_k^{(q)}, \ldots, \nabla_k^{(1)})$ for the virtual bundle $\check{\xi} = \sum_{i=1}^{q}(-1)^{i-1}E_i$. Similarly for the other bundles and "sub"virtual bundles.

Let now $\mathcal{C}$ be a coherent sheaf on a complex manifold M. The singular set $\mathrm{Sing}(\mathcal{C})$ of $\mathcal{C}$ is defined to be the set of points p in M where $\mathcal{C}_p$ fails to be free. If we set $U_0 = M - \mathrm{Sing}(\mathcal{C})$, on U_0, $\mathcal{C}$ is locally free by definition, and there is a (well defined) holomorphic vector bundle $E(\mathcal{C})$ whose sheaf of germs of holomorphic sections is equal to $\mathcal{C}|_{U_0}$. Suppose M is a relatively compact open set in another complex manifold. Take a resolution

$$0 \to \mathcal{A}_M(E_r) \to \cdots \to \mathcal{A}_M(E_0) \to \mathcal{A}_M \otimes_{\mathcal{O}_M} \mathcal{C} \to 0 \tag{2.10}$$

of the coherent sheaf $\mathcal{C}$ by real analytic (complex) vector bundles E_i on M ([AH]), where $\mathcal{A}_M$ denotes the sheaf of germs of real analytic functions on M. Following [BB2], the characteristic classes $\varphi(\mathcal{C})$ are, by definition, the characteristic classes $\varphi(\xi_\mathcal{C})$ of the virtual bundle $\xi_\mathcal{C} = \sum_{i=0}^{r}(-1)^i E_i$, which do not depend on the choice of the above resolution. On U_0 we have the exact sequence of vector bundles

$$0 \to E_r \to \cdots \to E_0 \to E(\mathcal{C}) \to 0. \tag{2.11}$$

Let ∇ be an arbitrary connection for $E(\mathcal{C})$ on U_0. By [BB2] Lemma (4.17), there is a connection $\nabla_0^{(i)}$ on U_0 for each E_i so that the family of connections $(\nabla_0^{(r)}, \ldots, \nabla_0^{(0)}, \nabla)$ is compatible with the sequence (2.11). If we denote by $\nabla_0^\bullet$ the family of connections $(\nabla_0^{(r)}, \ldots, \nabla_0^{(0)})$ on U_0, we have by Proposition 2.9,

$$\varphi(\nabla_0^\bullet) = \varphi(\nabla). \tag{2.12}$$

Let $(S_\alpha)_\alpha$ be the connected components of $\mathrm{Sing}(\mathcal{C})$ and U_α a regular neighborhood of S_α for each α. On U_α, we take an arbitrary family $\nabla_\alpha^\bullet = (\nabla_\alpha^{(r)}, \ldots, \nabla_\alpha^{(0)})$

of connections, each $\nabla_\alpha^{(i)}$ being a connection for E_i on U_α. Then the class $\varphi(\mathcal{C}) = \varphi(\xi_\mathcal{C})$ in $H^{2d}(M;\mathbb{C})$ is represented by the cocycle $\varphi(\nabla_*^\bullet)$ in $A^{2d}(\mathcal{U})$ given by $\varphi(\nabla_*^\bullet) = (\varphi(\nabla), \varphi(\nabla_\alpha^\bullet), \varphi(\nabla_0^\bullet, \nabla_\alpha^\bullet))$. In particular, we get

$$\varphi(\mathcal{C}) = [(0, \varphi(\nabla_\alpha^\bullet), \varphi(\nabla_0^\bullet, \nabla_\alpha^\bullet))] \tag{2.13}$$

if $\varphi(\nabla) = 0$ on U_0 because of some vanishing theorem for $E(\mathcal{C})$ on U_0.

If V denotes a subvariety of M, we shall call characteristic classes of $\mathcal{C}|_V$ the restriction to V of the characteristic classes of $\mathcal{C}$, i.e., characteristic classes of $\xi_\mathcal{C}|_V$.

(C) Singular holomorphic foliations ([BB2], [Su1], [Y])

Let M be a complex manifold of complex dimension n and $\mathcal{O}_M$ (or simply $\mathcal{O}$) the sheaf of germs of holomorphic functions on M. Also, denoting by TM the (holomorphic) tangent bundle of M, let $\Theta_M = \mathcal{O}(TM)$ be the sheaf of holomorphic vector fields on M.

Definition 2.14. We call singular foliation $\mathcal{F}$ on M the data of a coherent subsheaf $\Theta_\mathcal{F}$ of Θ_M, which is integrable (or involutive), i.e., such that

$$[\Theta_{\mathcal{F},p},\ \Theta_{\mathcal{F},p}] \subset \Theta_{\mathcal{F},p} \qquad \text{for } p \in M. \tag{*}$$

Since Θ_M is locally free, in this case the coherence simply means that the sheaf $\Theta_\mathcal{F}$ is locally finitely generated. We set $\mathcal{N}_\mathcal{F} = \Theta_M/\Theta_\mathcal{F}$ and define $S(\mathcal{F})$ by

$$S(\mathcal{F}) = \mathrm{Sing}(\mathcal{N}_\mathcal{F}).$$

Note that $S(\mathcal{F})$ is an analytic set, containing $\mathrm{Sing}(\Theta_\mathcal{F})$. We may describe $S(\mathcal{F})$ explicitly as follows. Let U be a sufficiently small coordinate neighborhood with coordinates $(z_1, \ldots, z_n)$ and let $v_1, \ldots, v_r$ be generators of $\mathcal{F}$ on U. We write $v_\lambda = \sum_{j=1}^n f_{\lambda j}(z) \frac{\partial}{\partial z_j}$. Then

$$S(\mathcal{F}) \cap U = \{\, z \in U \mid \mathrm{rank}\,(f_{\lambda j}(z)) < p \,\},$$

where p is the rank of (the locally free part of) $\Theta_\mathcal{F}$.

Remarks 2.15. If p is the rank of $\Theta_\mathcal{F}$, then $\mathcal{F}$ defines a non-singular foliation of dimension p on $M - S(\mathcal{F})$. We call $\mathcal{F}$ a singular foliation of dimension p. We call $\Theta_\mathcal{F}$ the tangent sheaf to the foliation, and $\mathcal{N}_\mathcal{F} = \Theta_M/\Theta_\mathcal{F}$ the normal sheaf.

Let M be a complex manifold of dimension n and $\mathcal{F}$ a foliation on M of dimension p. We recall the exact sequence

$$0 \to \Theta_\mathcal{F} \to \Theta_M \xrightarrow{\eta} \mathcal{N}_\mathcal{F} \to 0. \tag{2.16}$$

Note that from the integrability of $\Theta_{\mathcal{F}}$, the map

$$\Theta_{\mathcal{F}} \times \mathcal{N}_{\mathcal{F}} \longrightarrow \mathcal{N}_{\mathcal{F}} \tag{2.17}$$

assigning $\eta([v,w])$ to $(v,\eta(w))$ $(v \in \Theta_{\mathcal{F}},\ w \in \Theta_M)$ is well-defined. On the complement $M - \mathrm{Sing}(\Theta_{\mathcal{F}})$, the sheaf $\Theta_{\mathcal{F}}$ is locally free and there is an involutive subbundle F of TM such that $\Theta_{\mathcal{F}}|_{M-\mathrm{Sing}(\mathcal{F})} = \mathcal{O}(F)$. Also, if we set $M' = M - S(\mathcal{F})$ $(\subset M - \mathrm{Sing}(\Theta_{\mathcal{F}}))$, there is a vector bundle N_F such that $\mathcal{N}_{\mathcal{F}}|_{M'} = \mathcal{O}(N_F)$. We have $N_F = TM/F$ on M'. The map (2.17) induces an action of F on N_F:

$$\alpha : A^0(M', F) \times A^0(M', N_F) \to A^0(M', N_F)$$

and N_F becomes an F-bundle.

Recall that a connection ∇ for N_F is called an F-connection (or Bott connection) if
(i) $\nabla(\eta(w))(v) = \eta([v,w])$ for $w \in A^0(M', TM)$ and $v \in A^0(M', F)$ and
(ii) ∇ is of type $(1,0)$.

(D) Local complete intersections

Let V be a subvariety of dimension m in a complex manifold M of dimension n. We say that V is a "set-theoretic local complete intersection defined by a holomorphic section" if there exist a holomorphic vector bundle $E \to M$ of rank $k = n - m$ over M and a holomorphic section s of E, generically transverse to the zero section, so that $V = s^{-1}(0)$. Thus V is a set-theoretic local complete intersection and the restriction $E|_{V_0}$ coincides with the normal bundle of the regular part V_0 of V. The following proposition is known (see for instance [Pa] when $k = 1$):

Proposition 2.18. *For any connection ∇ on E, the morphism $\nabla s : TM \to E$ has a restriction to V_0 whose kernel is equal to TV_0, and has a restriction to V which does not depend on the choice of ∇. Thus, the restriction of ∇s to V_0 is surjective, and naturally identified with the canonical projection of $TM|_{V_0}$ onto the normal bundle of V_0 in M.*

We denote by N_V the restriction $E|_V$ of E to V: it is therefore an extension to V of the bundle normal to V_0 in M.

If we assume furthermore that the components of the section s with respect to some local trivialization of E generate locally the ideal $\mathcal{I}_V$ of holomorphic functions vanishing on V, then V is not only a set-theoretic local complete intersection, but also a local complete intersection. In this case, the vector bundle N_V is well defined, i.e., does not depend on the choice of E (e.g., [LS1]).

3. Characteristic classes of $\mathcal{N}_F|_V$; variations

Let M be a complex manifold and let V be an analytic subvariety of M. We denote by $\mathrm{Sing}(V)$ the singular set of V and by $\mathcal{I}_V$ the ideal sheaf of $\mathcal{O}_M$ defining

V. We consider the sheaf $\Theta_M(\log V)$ of so called "logarithmic vector fields" of V ([Sa], [BR]) defined by:

$$\Theta_M(\log V) = \{ v \in \Theta_M \mid v(\mathcal{I}_V) \subset \mathcal{I}_V \}.$$

Note that a germ v in Θ_M is in $\Theta_M(\log V)$ if and only if it is tangential to the regular part $V - \mathrm{Sing}(V)$.

We shall call (singular) foliation on V the data $\mathcal{F}_V = (\mathcal{F}, V)$ of a singular foliation $\mathcal{F}$ on M and a subvariety V of M invariant by $\mathcal{F}$, i.e., such that

(1) $\Theta_{\mathcal{F}} \subset \Theta_M(\log V)$ and
(2) $V \not\subset S(\mathcal{F})$.

We assume once for all, for simplification, V to be compact. We denote by n, m and p the complex dimensions of M, V and $\mathcal{F}$, respectively. When using the material of section 2-A, then we shall take $S = \mathrm{Sing}(V) \cup (S(\mathcal{F}) \cap V)$.

We shall be interested here in computing the restriction to V of the characteristic classes of the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{F}$. The interesting fact is that we have more characteristic classes producing residues than on M. In fact, recall that there is an action of F on N_F. If V is non-singular and if $S(\mathcal{F}) \cap V = \emptyset$, this action induces an action of $F_V = F|_V$ on $N_F|_V$. Hence we have the following vanishing theorem:

Proposition 3.1. *If V is non-singular and if $S(\mathcal{F}) \cap V = \emptyset$, then*

$$Chern^*(\mathcal{N}_{\mathcal{F}}|_V) = 0, \quad \text{for } * > 2(m-p).$$

In fact, if ∇ is an F_V-connection for $N_F|_V$, then $\varphi(\nabla) = 0$ for any symmetric polynomial φ of degree $> m - p$.

Remark 3.2. The usual Bott vanishing theorem asserts $Chern^*(\mathcal{N}_{\mathcal{F}}) = 0$ only for $* > 2(n-p)$.

Let S, $(S_\alpha)_\alpha$, $\tilde{U}_\alpha$, $\tilde{U}_0$, $\tilde{R}_\alpha$ and $\tilde{R}_0$ etc. be as above. For a homogeneous symmetric polynomial φ of degree d, the characteristic class $\varphi(\mathcal{N}_{\mathcal{F}})$ on $\tilde{U}$ is represented by the cocycle $\varphi(\nabla^\bullet_*)$ in $A^{2d}(\mathcal{U})$ given by

$$\varphi(\nabla^\bullet_*) = (\varphi(\nabla^\bullet_0), \varphi(\nabla^\bullet_\alpha), \varphi(\nabla^\bullet_0, \nabla^\bullet_\alpha))$$

(section 2, (B)). The bundle $N_F|_{\tilde{U}_0}$ is isomorphic to $\rho^*(N_F|_{U_0})$. Thus, we take as ∇ the connection equivalent to the pull-back by ρ of an F_V-connection for $N_F|_{U_0}$. Then by Proposition 3.1,

$$\varphi(\nabla^\bullet_0) = \varphi(\nabla) = 0.$$

Hence $\varphi(\nabla^\bullet_*)$ is in $A^{2d}(\mathcal{U}, \tilde{U}_0)$. We denote its class $[\varphi(\nabla^\bullet_*)]$ in $H^{2d}(A^\bullet(\mathcal{U}, \tilde{U}_0)) = H^{2d}(V, V - S; \mathbb{C})$ by $\varphi_S(\mathcal{N}_{\mathcal{F}})$, and call it the "localization" of $\varphi(\mathcal{N}_{\mathcal{F}})$. It is sent to $\varphi(\mathcal{N}_{\mathcal{F}})$ by the canonical homomorphism $j^* : H^{2d}(V, V - S; \mathbb{C}) \to H^{2d}(V; \mathbb{C})$.

We have, for each α, the residue $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha)$ in $H_{2m-2d}(S_\alpha; \mathbb{C})$ as the α-component of the image of $\varphi_S(\mathcal{N}_\mathcal{F})$ by the Alexander homomorphism A_V. Thus $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha)$ is represented by a $(2m-2d)$-cycle C in S_α such that

$$\int_C \tau_\alpha = \int_{R_\alpha} \varphi(\nabla^\bullet_\alpha) \wedge \tau_\alpha + \int_{R_{0\alpha}} \varphi(\nabla^\bullet_0, \nabla^\bullet_\alpha) \wedge \tau_\alpha \tag{3.3}$$

for any closed $(2m-2d)$-form τ_α on $\tilde{U}_\alpha$. In particular, if $d = m$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha) = \int_{R_\alpha} \varphi(\nabla^\bullet_\alpha) + \int_{R_{0\alpha}} \varphi(\nabla^\bullet_0, \nabla^\bullet_\alpha). \tag{3.4}$$

Thus we have proved the following theorem.

Theorem 3.5. *Let V be a subvariety of dimension m in M and $\mathcal{F}$ a singular foliation of dimension p on M which leaves V invariant. Also, let φ be a homogeneous symmetric polynomial of degree $d > m - p$.*
(1) For each compact connected component S_α of the singular set $S(\mathcal{F}_V)$, we have the residue

$$\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha) \in H_{2m-2d}(S_\alpha; \mathbb{C}),$$

which is determined by the local behavior of $\mathcal{F}$ near S_α and is given by (4.3).
(2) If V is compact,

$$\sum_\alpha (i_\alpha)_* \mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha) = \varphi(\mathcal{N}_\mathcal{F}) \frown [V] \qquad \text{in} \qquad H_{2m-2d}(V; \mathbb{C}),$$

where i_α denotes the embedding $S_\alpha \hookrightarrow V$ and the sum is taken over all the components S_α of S.

We call $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_\mathcal{F}|_V; S_\alpha)$ the **variation** of $\mathcal{F}$ with respect to V and φ at S_α.

Particular case $p = 1$

Suppose there is a line bundle F on M such that $\Theta_\mathcal{F} = \mathcal{O}(F)$. We may consider the sequence (2.16) as a locally free resolution of $\mathcal{N}_\mathcal{F}$ and in (2.11), we may set $r = 1$, $E_0 = TM$ and $E_1 = F$. Note that in this case, we have residues only for homogeneous symmetric polynomial of degree m and they are complex numbers.

Since the characteristic classes of $\mathcal{N}_\mathcal{F}$ are exactly those of the virtual bundle $TM - F$, we have the following

Theorem 3.6. *Let $\mathcal{F}$ be a one dimensional foliation on a complex manifold M leaving invariant a subvariety V of dimension m and φ a homogeneous symmetric polynomial of degree m.*

(1) *For each compact connected component S_α of the singular set $S = S(\mathcal{F}_V)$, we have the residue $\mathrm{Res}_\varphi(\mathcal{F}, (TM - F)|_V; S_\alpha)$, which is a complex number determined by the local behavior of $\mathcal{F}$ near S_α and is given by (3.4).*
(2) *If V is compact,*

$$\sum_\alpha \mathrm{Res}_\varphi(\mathcal{F}, (TM - F)|_V; S_\alpha) = \varphi(TM - F) \frown [V],$$

where the sum is taken over all the components S_α of S.

Remark 3.7. Suppose $n = 2$ and $m = 1$. Then V is a (complex) curve C in a non-singular surface M and each S_α consists of a point a. Moreover, for φ, we have essentially only c_1. The residue $\mathrm{Res}_{c_1}(\mathcal{F}, (TM - F)|_C; a)$ coincides with the "variation" $\mathrm{Var}(\mathcal{F}, C; a)$ defined in [KS] and Theorem 3.6 above reduces to Theorem 2.2 in [KS].

Suppose there is a generator v of $\mathcal{F}$ in a neighborhood $\tilde{U}$ of S_α in M. Then,

$$\mathrm{Res}_\varphi(\mathcal{F}, (TM - F)|_V; S_\alpha) = \mathrm{Res}_\varphi(v, TM|_V; S_\alpha),$$

where the right hand side is defined as in [LSS] (see also [LS2]). This also shows that, if h is a non-vanishing holomorphic function on U, then

$$\mathrm{Res}_\varphi(hv, TM|_V; S_\alpha) = \mathrm{Res}_\varphi(v, TM|_V; S_\alpha).$$

If $\mathcal{F}$ is generated by a global vector field v, then F is trivial and the results reduce to those of v for $TM|_V$ ([LS2]). See below for the formula for $\mathrm{Res}_\varphi(v, TM|_V; a)$ when S_α consists of a point a.

4. Baum-Bott residues for singular varieties

Let again $\mathcal{F}_V = (\mathcal{F}, V)$ be as above. In this section we assume V to be a set-theoretic local complete intersection defined by a holomorphic section of a vector bundle $E \to M$ of rank $k = n - m$ (see section 2, (D)). Let $N_V = E|_V$.

We wish now to compute the characteristic classes of $\mathcal{N}_\mathcal{F}|_V - N_V$, i.e., the characteristic classes of the virtual vector bundle $\mathcal{N}_{\mathcal{F}_V} = \xi_{\mathcal{N}_\mathcal{F}}|_V - N_V$ which is stably equivalent to $N_{F_V} = TU_0/F_V$ on $U_0 = V - S$.

The vanishing theorem we are using is simply the usual Bott vanishing theorem on V for $\mathcal{F}_V$:

Proposition 4.1. *If S is empty, that is if V is non-singular and if $S(\mathcal{F}) \cap V = \emptyset$, then $Chern^*(\mathcal{N}_{\mathcal{F}_V}) = 0$ for $* > 2(m - p)$.*

On U_0 we have the exact sequence

$$0 \to N_{F_V} \to N_F|_{U_0} \to N_{U_0} \to 0.$$

Each bundle in the above sequence admits a natural action of F_V so that the family of actions is compatible with the sequence. Proposition 5.1 is proved taking an F_V-connection for N_{F_V}. We may also prove it taking F_V-connections for $N_F|_{U_0}$ and N_{U_0}.

Let $(S_\alpha)_\alpha$, $\tilde{U}_\alpha$, $\tilde{U}_0$, $\tilde{R}_\alpha$ and $\tilde{R}_0$ etc. be as before. Also, let $\nabla_0^\bullet$, $\nabla_\alpha^\bullet$ and ∇ be as in section 3. Let ∇^N be an F_V-connection for N_{U_0} and ∇_0^E a connection for E on $\tilde{U}_0$ extending ∇^N. On each $\tilde{U}_\alpha$ we choose an arbitrary connection ∇_α^E for E. We set $\tilde{\nabla}_0^\bullet = (\nabla_0^\bullet, \nabla_0^E)$ and $\tilde{\nabla}_\alpha^\bullet = (\nabla_\alpha^\bullet, \nabla_\alpha^E)$. Then the class $\varphi(\mathcal{N}_{\mathcal{F}_V})$ is represented by the cocycle $\varphi(\tilde{\nabla}_*^\bullet)$ in $A^{2d}(\mathcal{U})$ given by

$$\varphi(\tilde{\nabla}_*^\bullet) = (\varphi(\tilde{\nabla}_0^\bullet), \varphi(\tilde{\nabla}_\alpha^\bullet), \varphi(\tilde{\nabla}_0^\bullet, \tilde{\nabla}_\alpha^\bullet)).$$

If $d > m - p$, then by Proposition 2.9,

$$\varphi(\tilde{\nabla}_0^\bullet) = 0.$$

Hence $\varphi(\tilde{\nabla}_*^\bullet)$ is in $A^{2d}(\mathcal{U}, \tilde{U}_0)$. We denote its class $[\varphi(\tilde{\nabla}_*^\bullet)]$ in $H^{2d}(A^\bullet(\mathcal{U}, \tilde{U}_0)) = H^{2d}(V, V - S; \mathbb{C})$ by $\varphi_S(\mathcal{N}_{\mathcal{F}_V})$ and call it the localization of $\varphi(\mathcal{N}_{\mathcal{F}_V})$. It is sent to $\varphi(\mathcal{N}_{\mathcal{F}_V})$ by the canonical homomorphism $j^* : H^{2d}(V, V - S; \mathbb{C}) \to H^{2d}(V; \mathbb{C})$. We have, for each α, the residue $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha)$ in $H_{2m-2d}(S_\alpha; \mathbb{C})$ as the α-component of the image of $\varphi_S(\mathcal{N}_{\mathcal{F}_V})$ by the Alexander homomorphism A_V. Thus $\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha)$ is represented by a $(2m - 2d)$-cycle C in S_α such that

$$\int_C \tau_\alpha = \int_{R_\alpha} \varphi(\tilde{\nabla}_\alpha^\bullet) \wedge \tau_\alpha + \int_{R_{0\alpha}} \varphi(\tilde{\nabla}_0^\bullet, \tilde{\nabla}_\alpha^\bullet) \wedge \tau_\alpha \tag{4.2}$$

for any closed $(2m - 2d)$-form τ_α on $\tilde{U}_\alpha$. In particular, if $d = m$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha) = \int_{R_\alpha} \varphi(\tilde{\nabla}_\alpha^\bullet) + \int_{R_{0\alpha}} \varphi(\tilde{\nabla}_0^\bullet, \tilde{\nabla}_\alpha^\bullet). \tag{4.3}$$

Suppose V is compact. If we compute the class $\varphi(\mathcal{N}_{\mathcal{F}_V})$ in $H^{2d}(\tilde{U}; \mathbb{C})$, which corresponds to $\varphi(\mathcal{N}_{\mathcal{F}_V}) \frown [V]$ in $H_{2m-2d}(V; \mathbb{C})$ by P_V, using F_V-connections, we see that it has a natural reduction in $H^{2d}(A^*(\mathcal{U}, \tilde{U}_0))$ whose image by A_V is the sum $\sum_\alpha \mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha)$ of the residues. Thus we have proved the following theorem.

Theorem 4.4. *Let V be a set-theoretic local complete intersection of dimension m in M and $\mathcal{F}$ a singular foliation of dimension p on M which leaves V invariant. Also, let φ be a homogeneous symmetric polynomial of degree $d > m - p$.*
(1) For each compact connected component S_α of the singular set $S(\mathcal{F}_V)$, we have the residue

$$\mathrm{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha) \in H_{2m-2d}(S_\alpha; \mathbb{C}),$$

which is determined by the local behavior of $\mathcal{F}$ near S_α and is given by (5.2).
(2) *If V is compact,*

$$\sum_\alpha (i_\alpha)_* \operatorname{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha) = \varphi(\mathcal{N}_\mathcal{F} - E) \frown [V] \qquad \text{in} \qquad H_{2m-2d}(V; \mathbb{C}),$$

where i_α denotes the embedding $S_\alpha \hookrightarrow V$ and the sum is taken over all the components S_α of S.

We call $\operatorname{Res}_\varphi(\mathcal{F}, \mathcal{N}_{\mathcal{F}_V}; S_\alpha)$ the **singular Baum-Bott residue** of $\mathcal{F}_V$ at S_α with respect to φ. If V is non-singular, it coincides with the usual Baum-Bott residue of $\mathcal{F}_V$.

Particular case $p = 1$

Since the characteristic classes of $\mathcal{N}_{\mathcal{F}_V}$ are exactly those of the virtual bundle $(TM - F - E)_V$, we have the following

Theorem 4.5. *Let $\mathcal{F}$ be a one dimensional foliation on a complex manifold M leaving invariant a set-theoretic local complete intersection V of dimension m in M, and let φ be a homogeneous symmetric polynomial of degree m.*
(1) *For each compact connected component S_α of the singular set $S = S(\mathcal{F}_V)$, we have the residue $\operatorname{Res}_\varphi(\mathcal{F}, (TM - F - E)|_V; S_\alpha)$, which is a complex number determined by the local behavior of $\mathcal{F}$ near S_α and is given by (5.3).*
(2) *If V is compact,*

$$\sum_\alpha \operatorname{Res}_\varphi(\mathcal{F}, (TM - F - E)|_V; S_\alpha) = \varphi(TM - F - E) \frown [V],$$

where the sum is taken over all the components S_α of S.

Suppose there is a generator v of $\mathcal{F}$ in a neighborhood $\tilde{U}$ of S_α in M. Then,

$$\operatorname{Res}_\varphi(\mathcal{F}, (TM - F - E)|_V; S_\alpha) = \operatorname{Res}_\varphi(v, TM|_V - N_V; S_\alpha),$$

where the right hand side is defined as in [LSS] (see also [LS2]). This also shows that, if h is a non-vanishing holomorphic function on U, then

$$\operatorname{Res}_\varphi(hv, TM|_V - N_V; S_\alpha) = \operatorname{Res}_\varphi(v, TM|_V - N_V; S_\alpha).$$

If $\mathcal{F}$ is generated by a global vector field v, then F is trivial and the results reduce to those of v for $TM|_V - N_V$ ([LSS]). See below for the formula for the residue $\operatorname{Res}_\varphi(v, TM|_V - N_V; a)$ when S_α consists of a point a.

Remarks 4.6. 1°. When $\varphi = c_m$, $\operatorname{Res}_{c_m}(v, TM|_V - N_V; S_\alpha)$ coincides with the virtual index $\operatorname{Vir}(v|_V, S_\alpha)$ of the restriction v at S_α ([LSS]), which is also equal to the "GSV index" $\operatorname{GSV}(v|_V, a)$ of v at a when S_α consists of a point a. Note that these are invariants of the foliation $\mathcal{F}$ generated by v.

2°. Suppose $n = 2$ and $m = 1$. Then V is a (complex) curve C in a non-singular surface M and each S_α consists of a point a. Moreover, for φ, we have essentially only c_1 and $\mathrm{Res}_{c_1}(\mathcal{F}, (TM - F - E)|_C; a) = \mathrm{Vir}(\mathcal{F}_C, a) = \mathrm{GSV}(\mathcal{F}_C, a)$. The formula in Theorem 5.5, (2) above reduces to the one in [Br1] Lemma 3, which is equivalent to the one in [KS] Theorem 2.6. From the above argument, we also have the following formula in [KS] Lemma 2.3 (cf. also [Br2] Proposition 5):

$$\mathrm{Vir}(\mathcal{F}_C, a) = \mathrm{Var}(\mathcal{F}, C; a) - \mathrm{CS}(\mathcal{F}, C; a),$$

where $\mathrm{CS}(\mathcal{F}, C; a) = \mathrm{Res}_{c_1}(\mathcal{F}, N_C; a)$.

5. Formulas for the residues

The residues for isolated singularities are already computed in [LSS], [LS2]. Here we quote the results.

Suppose S_α consists of an isolated point a in V and let v be a holomorphic vector field on a neighborhood of a in M. We also assume that V is a complete intersection near a. Then we have the following lemma ([LS1] Theorem 2).

Lemma 5.1. *There exists a local coordinate system $(z_1, ..., z_n)$ near a in M such that, if we express v as*

$$v = \sum_{i=1}^{n} a_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}$$

and if we denote by $(f_1, \ldots, f_k)$, $k = n - m$, a system of defining functions of V near a, then the set of common zeros of the holomorphic functions $a_1, ..., a_m, f_1, ..., f_k$ consists only of a.

We set

$$A = \frac{\partial(a_1, \ldots, a_n)}{\partial(z_1, \ldots, z_n)},$$

the Jacobian matrix of $(a_1, \ldots, a_n)$ with respect to $(z_1, \ldots, z_n)$. If we let $f = (f_1, \ldots, f_k)$ be a system of local reduced defining functions of V near p, the condition that V is invariant by v is expresed as

$$v(f_i) = \sum_{i=1}^{k} c_{ij} f_j,$$

for some holomorphic functions c_{ij} ([St], [BR]). Set $C = (c_{ij})$. We set $\mathbb{C}[c_1, c_2, \ldots] = Chern^*$ and $\mathbb{C}[c_1, c_2, \ldots]_d$ the homogeneous component of degree d.

Theorem 5.2. *In the above situation, if $(z_1, \ldots, z_n)$ denotes a coordinate system as in Lemma 6.1, we have:*
(I) *For a polynomial φ in $\mathbb{C}[c_1, \ldots, c_n]_m$,*

$$\mathrm{Res}_\varphi(v, TM|_V; a) = \int_\Gamma \frac{\varphi(-A) dz_1 \wedge \cdots \wedge dz_m}{a_1 \cdots a_m}.$$

(II) *Let φ be a polynomial in $\mathbb{C}[c_1,\dots,c_m]_m$ and write $\varphi=\sum_i \varphi_i\cdot\varphi_i'$ with φ_i in $\mathbb{C}[c_1,\dots,c_n]$ and φ_i' in $\mathbb{C}[c_1,\dots,c_k]$ so that $\varphi(TM-E)=\sum_i \varphi_i(TM)\cdot\varphi_i'(E)$. Then*

$$\mathrm{Res}_\varphi(v,TM|_V-N_V;a)=\sum_i\int_\Gamma \frac{\varphi_i(-A)\varphi_i'(-C)dz_1\wedge\cdots\wedge dz_m}{a_1\cdots a_m},$$

where Γ denotes the m-cycle in V given by

$$\Gamma=\{\,z\mid |a_1(z)|=\cdots=|a_m(z)|=\varepsilon,\ f_1(z)=\cdots=f_k(z)=0\,\},$$

for a small positive number ε, which is oriented so that $d\theta_1\wedge\cdots\wedge d\theta_m$ is positive, $\theta_i=\arg a_i$.

In particular, when $\varphi=c_m$, denoting by $[c(-A)\cdot c(-C)^{-1}]_d$ the holomorphic function given as the coefficient of t^d in the formal power series expansion of $\det\left(I-t\frac{\sqrt{-1}}{2\pi}A\right)\cdot\det\left(I-t\frac{\sqrt{-1}}{2\pi}C\right)^{-1}$ in t, we have

$$\mathrm{Vir}(v,a)=\int_\Gamma\frac{[c(-A)\cdot c(-C)^{-1}]_m dz_1\wedge\cdots\wedge dz_m}{a_1\cdots a_m}. \tag{5.3}$$

If a is a regular point of V, we may take a coordinate system $(z_1,\dots,z_n)$ about p so that $f_1=z_{m+1},\dots,f_k=z_n$ define V near a. Then the invariance condition for V by v is that $a_{m+1},\dots,a_n$ vanish on V. Thus $(z_1,\dots,z_n)$ is a coordinate system as in Lemma 5.1. On the other hand, c_{ij} and $\frac{\partial a_{m+i}}{\partial z_{m+j}}$ are equal on V. Hence denoting by A' the Jacobian matrix $\frac{\partial(a_{m+1},\dots,a_n)}{\partial(z_{m+1},\dots,z_n)}$, we have, on V, $C=A'$ and $\frac{\partial a_{m+i}}{\partial z_j}=0$ for $i=1,\dots,k$ and $j=1,\dots,m$. Hence denoting by A'' the Jacobian matrix $\frac{\partial(a_1,\dots,a_m)}{\partial(z_1,\dots,z_m)}$, the formula in Theorem 5.2 (II) reduces to

$$\mathrm{Res}_\varphi(v,TM|_V-N_V;a)=\int_\Gamma\frac{\varphi(-A'')dz_1\wedge\cdots\wedge dz_m}{a_1\cdots a_m},$$

which coincides with usual Baum-Bott residue for V (see also [Le2] for more general Grothendieck residues when V is non-singular).

See [LS2] for more explicit computations.

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