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ρ -Contraction And 2×2 Matrix

by

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Abstract. In this paper, the following is proved. When $|a| \leq 1$ and $|b| \leq 1$,
 $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ is a ρ -contraction if and only if

$$\begin{aligned} & |c|^2 + |a - b|^2 \\ & \leq \inf_{\zeta \in D} \left| \frac{\{\rho + (1 - \rho)\bar{a}\bar{\zeta}\}\{\rho + (1 - \rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2 \end{aligned}$$

where D is the open unit disc.

§1. Introduction

For $\rho > 0$, a bounded linear operator A on a Hilbert space $\mathcal{H}$ is a ρ -contraction if and only if the powers of A admit a representation

$$A^n h = \rho P U^n h \quad (h \in \mathcal{H} ; n = 1, 2, \dots),$$

where U is a unitary operator (called a unitary ρ -dilation) on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and where P is the projection from $\mathcal{K}$ to $\mathcal{H}$. A is a contraction, that is, the norm $\|A\| \leq 1$ if and only if A is a 1-contraction. This is a theorem of Sz.-Nagy [4]. The numerical radius $w(A)$ of A is not bigger than 1 if and only if A is a 2-contraction. This is a theorem of Berger [1]. Sz.-Nagy and Foias [5] gave intrinsic characterizations of operators of ρ -contractions. Unfortunately, even if $\mathcal{H}$ is of two dimension, it is difficult to determine when A is a ρ -contraction except $\rho = 1$. In this paper, we assume that $\mathcal{H}$ is a two dimensional Hilbert space. Assuming A is triangular, we determine when A is a ρ -contraction. In fact, for $\rho = 2$, the numerical radius is given.

In this paper, D is the open unit disc and $\bar{D}$ is its closure. $\mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ denotes a set of all holomorphic functions from $\bar{D}$ into $\bar{D}$ which vanish at $\alpha \in D$ and $\mathcal{H}ol(\bar{D})$ denotes a set of all holomorphic contractions on $\bar{D}$.

In the proof of Theorem, we use three elementary well known results.

$$(1) \sup\{|f'(\alpha)| ; f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})\} = \frac{1}{1 - |\alpha|^2}.$$

$$(2) \sup\{|f(\beta)| ; f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})\} = \left| \frac{\alpha - \beta}{1 - \bar{\alpha}\beta} \right| \quad \text{when } \beta \in D.$$

(3) $f(A)$ is a contraction for any $f \in \mathcal{H}ol(\bar{D})$ if and only if $f(A)$ is a contraction for any $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$.

Proof. (1) If $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ then $|(f \circ \phi_{-\alpha})'(0)| \leq 1$ by the Schwarz's lemma where $\phi_{-\alpha}(z) = (z + \alpha)/(1 + \bar{\alpha}z)$. Hence the supremum is less than equal to $1/(1 - |\alpha|^2)$. The supremum is attained by $\phi_{\alpha}(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. (2) If $f \in \mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ then $f(z) = \phi_{\alpha}(z)g(z)$ and $|g(z)| \leq 1$. Hence the supremum is attained by $\phi_{\alpha}(z)$. (3) The 'only if' part is clear. If $f \in \mathcal{H}ol(\bar{D})$ and $f(\alpha) = \beta$, $g = \phi_{\beta} \circ f$ belongs to $\mathcal{H}ol(\bar{D}, \alpha, \bar{D})$ and it follows that $\|g(A)\| \leq 1$ or, equivalently $\|f(A)\| = \|\phi_{-\beta}(g(A))\| \leq 1$ (see [3]).

§2. Theorem

In order to give Theorem, we use a theorem of Misra [3, Theorem 1.1] and a result of Ando and the second author [6]. Corollary 1 is known. Corollary 2 is new and with a theorem of Berger [1], it gives a necessary and sufficient condition for $w(A) \leq 1$.

Theorem. Suppose a and b are complex numbers in the closed unit disc $\bar{D}$. Then, $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ is a ρ -contraction if and only if

$$|c|^2 + |a - b|^2 \leq \inf_{\zeta \in D} \left| \frac{\{\rho + (1 - \rho)\bar{a}\zeta\}\{\rho + (1 - \rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2$$

Proof. By [6], A is a ρ -contraction if and only if for any $\zeta \in D$, $g_\zeta(A)$ is a contraction where $g_\zeta(z) = z\zeta/\{\rho + (1 - \rho)z\zeta\}$. If $a = b$, by [3, Theorem 1.1], $g_\zeta(A)$ is a contraction if and only if

$$|c| \cdot |g'_\zeta(a)| \leq (\sup\{|f'(g_\zeta(a))|; f \in \mathcal{H}ol(\bar{D}, g_\zeta(a), \bar{D})\})^{-1}.$$

Hence by (1)

$$\begin{aligned} |c| &\leq \frac{1 - |g_\zeta(a)|^2}{|g'_\zeta(a)|} \\ &= \frac{|\rho + (1 - \rho)a\zeta|^2 - |a\zeta|^2}{|\zeta\rho|}. \end{aligned}$$

Thus A is a ρ -contraction if and only if

$$|c| \leq \inf_{\zeta \in D} \left| \frac{|\rho + (1 - \rho)a\zeta|^2 - |a\zeta|^2}{\zeta\rho} \right|.$$

This implies the theorem when $a = b$.

Suppose $a \neq b$. Note that $g_\zeta(a) \neq g_\zeta(b)$ for $\zeta \neq 0$. Hence A is a ρ -contraction if and only if

$$\begin{bmatrix} f \circ g_\zeta(a) & c \frac{f \circ g_\zeta(a) - f \circ g_\zeta(b)}{a - b} \\ 0 & f \circ g_\zeta(b) \end{bmatrix}$$

is a contraction for any $f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$ by the von Neumann's inequality and (3). As in [3, Theorem 1.1], this is equivalent to

$$|c|^2/|a - b|^2 \leq \frac{1}{|f \circ g_\zeta(a)|^2} - 1$$

for any $f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$. It is easy to see that by (2)

$$\begin{aligned} &\sup\{|f \circ g_\zeta(a)|; f \in \mathcal{H}ol(\bar{D}, g_\zeta(b), \bar{D})\} \\ &= \left| \frac{g_\zeta(a) - g_\zeta(b)}{1 - \overline{g_\zeta(a)}g_\zeta(b)} \right|^{-1} \\ &= \left| \frac{\{\rho + (1 - \rho)\bar{a}\zeta\}\{\rho + (1 - \rho)b\zeta\} - \bar{a}b|\zeta|^2}{(a - b)\rho\zeta} \right|. \end{aligned}$$

This implies the theorem when $a \neq b$.

Corollary 1. *In Theorem, A is a 1-contraction if and only if*

$$\begin{aligned} |c|^2 &\leq \inf_{\zeta \in D} \left| \frac{1 - \bar{a}b|\zeta|^2}{\zeta} \right|^2 - |a - b|^2 \\ &= (1 - |a|^2)(1 - |b|^2). \end{aligned}$$

Corollary 2. *In Theorem, A is a 2-contraction if and only if*

$$\begin{aligned} |c|^2 &\leq \inf_{|\zeta|=1} |2 - (\bar{a}\zeta + b\zeta)|^2 - |a - b|^2 \\ &= 4 \inf_{|\zeta|=1} \{(1 - \operatorname{Re}(a\zeta))(1 - \operatorname{Re}(b\zeta))\}. \end{aligned}$$

§3. Remarks

1. An operator A on H is called a quadratic operator if A satisfies quadratic polynomial, that is, $A^2 + rA + sI = 0$ for some complex numbers r and s . In the finite dimensional case, if $t^2 + rt + s = (t - a)(t - b)$ then A is unitarily similar to a matrix of the form

$$\begin{bmatrix} aI_l & C \\ 0 & bI_m \end{bmatrix} \quad (*)$$

where C is an $l \times m$ matrix. Since the ρ -contraction is invariant under the unitary similarity, we may assume that a quadratic operator A has a form of (*). Then we can show A is a ρ -contraction if and only of

$$\begin{aligned} &\|C\|^2 + |a - b|^2 \\ &\leq \inf_{\zeta \in D} \left| \frac{\{\rho + (1 - \rho)\bar{a}\zeta\}\{\rho + (1 - \rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2. \end{aligned}$$

In fact, by the singular value decomposition of C (cf. [2]), there is an $l \times m$ matrix $\Sigma = [\sigma_{ij}]$ where $\sigma_{ij} = 0$ for all $i \neq j$, and $\sigma_{11} \geq \sigma_{22} \geq \dots \geq \sigma_{qq} \geq 0$ ($q = \min\{l, m\}$) and there are unitary matrices $U \in M_l$, $V \in M_m$ such that $UCV^* = \Sigma$. Here $\sigma_{11}, \sigma_{22}, \dots, \sigma_{qq}$ are the decrasing ordered singlar values of A , specially, $\sigma_{11} = \|C\|$. Then A is unitarily similar

to a matrix $\begin{bmatrix} aI_l & \Sigma \\ 0 & bI_m \end{bmatrix}$, and this matrix is unitarily similar to the direct sum of 2×2 matrices of form $C_i = \begin{bmatrix} a & \sigma_{ii} \\ 0 & b \end{bmatrix}$ and possibly with aI_k or bI_k with $k = |l - m|$. It follows that A is a ρ -contraction if and only if C_1 is. From Theorem in this paper our assertion is led.

2. In the rest of this paper, let $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$. For some special a and b , we compute the infimums in Theorem and Corollary 2 as that in Corollary 1.

- (1) When $|a| = 1$ or $|b| = 1$, A is a ρ -contraction if and only if $c = 0$.
- (2) When $a = b$, A is a ρ -contraction if and only if $|c| \leq (\rho - 2)|a|^2 + 2(1 - \rho)|a| + \rho$.
- (3) When $b = 0$, A is a ρ -contraction if and only if $|c|^2 \leq \rho(\rho - 2)|a|^2 + 2\rho(1 - \rho)|a| + \rho^2$.
- (4) When $a \geq 0$ and $b \geq 0$, A is a ρ -contraction if and only if $|c|^2 \leq \{\rho + (1 - \rho)(a + b) + (\rho - 2)ab\}^2 - (a - b)^2$.
- (5) When $a = -b$ is real, A is a ρ -contraction if and only if $|c|^2 + 4a^2 \leq \{\rho - (2 - \rho)a^2\}^2$.

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