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RIEMANN-HURWITZ FORMULA FOR MORIOTA-MUMFORD AND SURFACE SYMMETRIES

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RIEMANN-HURWITZ FORMULA FOR MORITA-MUMFORD CLASSES AND SURFACE SYMMETRIES

NARIYA KAWAZUMI AND TAKESHI UEMURA

ABSTRACT. Let a finite group G act on a compact Riemann surface C in a faithful and orientation preserving way. Then we describe the Morita-Mumford classes $e_n(C_G) \in H^{2n}(G;\mathbb{Z})$ of the homotopy quotient (or the Borel construction) C_G of the action in terms of fixed-point data. This fixed-point formula is deduced from a higher analogue of the classical Riemann-Hurwitz formula based on computations of Miller [Mi] and Morita [Mo].

INTRODUCTION.

The last two decades have witnessed a remarkable progress of cohomological study of the mapping class group for a surface. D. Mumford [Mu] and S. Morita [Mo1] independently defined a series of cohomology classes of the mapping class group, whose zeroth term is equal to the Euler number of the surface up to sign. J. Harer has been obtaining various significant results including the homology stability of the mapping class groups [H1]. The Morita-Mumford classes play an important role in the stable cohomology ring of the mapping class groups. It has been revealed by Arbarello, Cornalba, Harer, Miller, Morita and others [AC] [H2-3] [Mi] [Mo1-4] [KM]. The torsion part of the cohomology of the mapping class groups has been highly studied by homotopy theorists including Benson, Charney, Cohen, Lee, Tillmann and Xia [BC] [CC] [CL] [T] [X1,2]. It is notable that Glover and Mislin [GM] have proved there exists a nontrivial torsion class in the $4n$-dimensional stable cohomology group for each $n \geq 1$ by evaluating the even Chern classes of $GL(\mathbb{Z})$ on torsion elements of the mapping class groups.

In the context of surface bundles the Morita-Mumford classes are defined as follows. Let $\pi : X \to B$ be an oriented fiber bundle whose fiber is a 2-dimensional connected closed oriented smooth manifold. We call such a bundle briefly a *surface bundle*. The relative tangent bundle $T_{X/B}$ is the oriented real 2-dimensional vector bundle over the total space X consisting of all the tangent vectors along the fibers. The n-th Morita-Mumford class e_n is, by definition, the Gysin image of the $n+1$-th power of its Euler class $e := e(T_{X/B}) \in H^2(X;\mathbb{Z})$

$$e_n = e_n(X) := \pi_!(e^{n+1}) \in H^{2n}(B;\mathbb{Z}),$$

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which is equal to the pull-back of e_n by the holonomy homomorphism of $\pi_1(B)$ into the mapping class group. Here and throughout this paper we denote the Euler class of an oriented real vector bundle η by $e(\eta)$. If $n = 0$, e_0 is equal to the Euler number of the fibers.

The purpose of the present paper is to study Morita-Mumford classes on finite subgroups of the mapping class groups. We give an explicit formula for the Morita-Mumford classes evaluated on an arbitrary finite subgroup of the mapping class groups in terms of fixed-point data (Theorem B). The authors hope their explicit formula would be widely used to study the torsion part of the cohomology of the mapping class groups.

Our fixed-point formula is deduced from a general formula of Morita-Mumford classes for fiberwise branched coverings of surface bundles. Miller [Mi] and Morita [Mo1], inspired by Atiyah [A], Hirzebruch [Hi] and Kodaira [Ko], computed Morita-Mumford classes of iterated cyclic coverings to prove the stable algebraic independence of e_n's. Following their computations, we prove it under a certain transversality condition (Theorem A) in §1.

Let $\pi_X : X \to B$ and $\pi_Y : Y \to B$ be two surface bundles, and $h : X \to Y$ a continuous map compatible with the projections. Suppose the map h restricted to each fiber is an orientation preserving branched covering. Denote by R the subset of X consisting of all the ramification points of the map h. Now we assume a *transversality condition* that the restriction of π_X to R is a locally trivial fibration, and that there exists a fiber preserving homeomorphism $t : D(T_{X/B}|_R) \hookrightarrow X$ onto an open neighborhood of R such that $t(0_x) = x$ for all $x \in R$. We call it *a fiberwise tubular neighborhood* of R. Here we denote by $D(\eta)$ the open unit disk bundle associated with a vector bundle η with respect to a suitable metric. Let $R_1, R_2, \ldots, R_m$ be the connected components of R, and b_i the ramification degree along R_i. Then

Theorem A. *In the situation stated above we have*

$$e_n(X) = (\deg h)\, e_n(Y) + \sum_{i=1}^{m} \left(1 - {b_i}^{n+1}\right) (\pi_X|_{R_i})_! \left(e(T_{X/B}|_{R_i})^n\right) \in H^{2n}(B; \mathbb{Z}).$$

for any $n \geq 1$, where $\deg h \in \mathbb{Z}$ is the fiberwise mapping degree of h, and $(\pi_X|_{R_i})_! : H^*(R_i; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ is the Gysin map associated with the fibration $\pi_X|_{R_i}$.

It also holds for the case $n = 0$. In fact, since e_0 is the Euler number of the fibers, the formula

$$e_0(X) = (\deg h)\, e_0(Y) + \sum_{i=1}^{m} (1 - b_i) \cdot \sharp(\text{the fiber of } \pi_X|_{R_i}).$$

is exactly the classical Riemann-Hurwitz formula. This is the reason why we regard Theorem A as a higher analogue of the Riemann-Hurwitz formula.

If the surface bundles π_X and π_Y are C^∞-surface bundles, h is a C^∞ map, and the ramification locus R is a C^∞-submanifold of X transverse to the fibers, then the transversality condition stated above is satisfied.

As another typical example satisfying the transversality condition we have Morita-Mumford classes on finite subgroups of the mapping class groups. In view of the affirmative solution of the Nielsen realization problem by Kerckhoff [Ke] any finite subgroup of the mapping class group is realized as a holomorphic automorphism group of a suitable Riemann surface. Therefore we may consider the following situation.

Let G be a finite group and C a closed oriented connected 2-dimensional smooth manifold. Suppose G acts on C in a faithful and orientation preserving way. Then we may regard G as a subgroup of the mapping class group. The universal principal G-bundle $E_G \to B_G$ induces the homotopy quotient (or the Borel construction) of the action $\pi : C_G \to B_G$. The space C_G is, by definition, the quotient of $E_G \times C$ by the diagonal action of G. The map π induced by the first projection provides an oriented fiber bundle with fiber C

$$C \to C_G \xrightarrow{\pi} B_G.$$

Its Morita-Mumford class $e_n(C_G) \in H^*(B_G; \mathbb{Z}) = H^*(G; \mathbb{Z})$ is equal to the restriction of e_n to the subgroup G.

If π_X is the surface bundle $\pi : C_G \to B_G$, π_Y the product bundle $B_G \times (C/G) \to B_G$, and $h : C_G \to B_G \times (C/G)$ the canonical projection, then the transversality condition is satisfied. Therefore Theorem A implies the following fixed-point formula as is shown in §2.

Denote the isotropy group at a point $p \in C$ by G_p. The fixed-point set

$$F := \{p \in C; G_p \neq \{1\}\}$$

is a G-stable finite subset of C, since the action is faithful and orientation preserving. Let ξ_p be the oriented real 2-dimensional vector bundle over B_{G_p} associated with the action of G_p on the tangent space T_pC and $e(\xi_p) \in H^2(B_{G_p}; \mathbb{Z}) = H^2(G_p; \mathbb{Z})$ its Euler class. Since the transfer map $\mathrm{cor}^G_{G_p} : H^*(G_p; \mathbb{Z}) \to H^*(G; \mathbb{Z})$ is invariant under conjugation, the cohomology class $\mathrm{cor}^G_{G_p}(e(\xi_p)^n) \in H^{2n}(G; \mathbb{Z})$ is constant on each G-orbit.

Theorem B. *In the situation stated above we have*

$$e_n(C_G) = \sum_{p \in F/G} \mathrm{cor}^G_{G_p}(e(\xi_p)^n) \in H^{2n}(B_G; \mathbb{Z}) = H^{2n}(G; \mathbb{Z})$$

for any $n \geq 1$.

The right-hand side depends only on the isotropy groups and their actions on the tangent spaces at the fixed-points. Our formula may be regarded as a certain kind of *fixed-point formulas* of characteristic classes. Especially if the action is free, the Morita-Mumford classes $e_n(C_G)$ vanish for all $n \geq 1$.

1. Riemann-Hurwitz Formula for Morita-Mumford Classes.

In this section we shall prove Theorem A following Miller [Mi] and Morita [Mo]. Let $\pi_X : X \to B$, $\pi_Y : Y \to B$, $h : X \to Y$, R and R_i be as in Introduction. We abbreviate as $e_X := e(T_{X/B})$, $e_Y := e(T_{Y/B})$ and $\eta_i := T_{X/B}|_{R_i}$. From the transversality condition we have a fiberwise tubular neighborhood $t_i : D(\eta_i) \to X$. Since $\{X - R, t_i(D(\eta_i)); 1 \leq i \leq m\}$ is an open covering of X, we obtain an excision isomorphism

$$t^* : H^*(X, X - R) \overset{\cong}{\to} \bigoplus_{i=1}^{m} H^*(E(\eta_i), E_0(\eta_i)).$$

Here $E(\eta_i)$ is the total space of the bundle η_i and $E_0(\eta_i) = E(\eta_i) - (\text{zero section})$.

Let $\phi_i : H^{*-2}(R_i) \to H^*(E(\eta_i), E_0(\eta_i))$ be the Thom isomorphism associated with η_i, and $U_i \in H^2(X, X - R)$ such as $t^*(U_i) = \phi_i(1)$. $H^2(X, X - R)$ is $\mathbb{Z}$-free with free basis $\{U_1, U_2, \ldots, U_m\}$. Clearly we have

$$t^*(U_iU_j) = \begin{cases} \phi_i(e(\eta_i)), & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.1}$$

Since $T_{X/B}$ is isomorphic to $h^*T_{Y/B}$ on $X - R$, the difference $e_X - h^*e_Y$ is in the image of the inclusion homomorphims $j^* : H^2(X, X - R) \to H^2(X)$, i.e., we have $e_X - h^*e_Y = j^*(\sum_{i=1}^m a_iU_i) \in H^2(X)$ for some $a_i \in \mathbb{Z}$. Restricting it to each R_i, we obtain

$$e(\eta_i) - b_ie(\eta_i) = a_ie(\eta_i) \in H^2(R_i), \tag{1.2}$$

where b_i is the ramification degree along R_i. From (1.1) and (1.2)

$$\begin{aligned} {e_X}^{n+1} &= h^*{e_Y}^{n+1} + j^*\left(\sum_{i=1}^{m}\sum_{k=1}^{n+1}\binom{n+1}{k}(h^*e_Y|_{R_i})^{n+1-k}\,{a_i}^k{U_i}^k\right) \\ &= h^*{e_Y}^{n+1} + j^*\left(\sum_{i=1}^{m}\sum_{k=1}^{n+1}\binom{n+1}{k}{b_i}^{n+1-k}{a_i}^k e(\eta_i)^n U_i\right) \\ &= h^*{e_Y}^{n+1} + j^*\left(\sum_{i=1}^{m}\left(1 - {b_i}^{n+1}\right)e(\eta_i)^n U_i\right) \end{aligned}$$

for any $n \geq 1$. Hence we obtain

$${e_X}^{n+1} = h^*{e_Y}^{n+1} + j^*(t^*)^{-1}\left(\sum_{i=1}^{m}\left(1 - {b_i}^{n+1}\right)\phi_i\left(e(\eta_i)^n\right)\right) \tag{1.3}$$

for $n \geq 1$.

We denote the fibers of the bundles $\pi_X : X \to B$, $\pi_X|_{(X-R)} : X - R \to B$ and $\pi_X|_{R_i} : R_i \to B$ over a point $s_0 \in B$ by C, C^0 and $R_i|s_0$ respectively. The Serre spectral sequence of the pair of fiber bundles $(X, X - R)$ induces an isomorphism

$$\pi_{X*} : H^*(X, X - R) \overset{\cong}{\to} H^{*-2}(B; H^2(C, C^0)).$$

As a $\pi_1(B)$-module, $H^2(C, C^0)$ decomposes itself into $\bigoplus_{i=1}^m H^0(R_i|s_0)$. The cohomology group $H^{*-2}(B; H^0(R_i|s_0))$ is naturally isomorphic to $H^{*-2}(R_i)$, and the integration map $H^0(R_i|s_0) \to \mathbb{Z}$ induces the Gysin map $(\pi_X|_{R_i})_! : H^{*-2}(R_i) \to H^{*-2}(B)$. Therefore we obtain a commutative diagram

$$\begin{array}{ccccc} H^*(X, X-R) & \xrightarrow{\pi_{X*}} & H^{*-2}(B; H^2(C, C^0)) & = & \bigoplus_{i=1}^m H^{*-2}(R_i) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \oplus(\pi_X|_{R_i})_!} \\ H^*(X) & \longrightarrow & H^{*-2}(B; H^2(C)) & \xrightarrow{<\,,[C]>} & H^{*-2}(B). \end{array} \tag{1.4}$$

The composite of the lower arrows is equal to the Gysin map $\pi_{X!}$ in the definition of the Morita-Mumford classes, and that of the upper ones equal to $\left(\bigoplus \phi_i{}^{-1}\right) \circ t^*$. Consequently, from (1.3), we obtain

$$\begin{aligned} e_n(X) =& \pi_{X!}\left(e_X{}^{n+1}\right) = \pi_{X!}\left(h^* e_Y{}^{n+1} + j^*(t^*)^{-1}\left(\sum_{i=1}^m \left(1 - b_i{}^{n+1}\right)\phi_i\left(e(\eta_i)^n\right)\right)\right) \\ =& (\deg h) e_n(Y) + \left(\oplus\left(\pi_X|_{R_i}\right)_!\right)\pi_{X*}(t^*)^{-1}\left(\sum_{i=1}^m \left(1 - b_i{}^{n+1}\right)\phi_i\left(e(\eta_i)^n\right)\right) \\ =& (\deg h) e_n(Y) + \sum_{i=1}^m \left(1 - b_i{}^{n+1}\right)\left(\pi_X|_{R_i}\right)_!\left(e(\eta_i)^n\right). \end{aligned}$$

This completes the proof of Theorem A.

2. Fixed-point Formula for Surface Symmetries.

In order to prove Theorem B, fix a complete system of representatives $\{p_1, p_2, \ldots, p_m\} \subset F$ with respect to the action of G. We abbreviate as $G_i = G_{p_i}$ and $\xi_i = \xi_{p_i}$. The quotient E_G/G_i can serve as the classifying space B_{G_i}.

Consider the canonical projection

$$h : C_G = (E_G \times C)/G \to B_G \times (C/G), \quad (x, z) \bmod G \mapsto (x \bmod G, z \bmod G).$$

The homotopy quotient $F_G := (E_G \times F)/G$ may be regarded as a closed subset of C_G, and coincides with the ramification locus R of the branched covering h. Clearly $\pi|_{F_G} : F_G \to B_G$ is a locally trivial fibration.

Each representative p_i corresponds to a connected component of R. We introduce a map $f_i : B_{G_i} \to C_G$ ("multi-valued section of π") by $f_i(x \bmod G_i) := (x, p_i) \bmod G$. From a G-stable decomposition $F = \coprod_{i=1}^m G \cdot p_i$ we find $F_G = \coprod_{i=1}^m f_i(B_{G_i})$. Choose a sufficiently small G_i-stable open disk $D_i \subset C$ centered at each p_i. The map defined by

$$t_i : (E_G \times D_i)/G_i \to C_G, \quad (x, z) \bmod G_i \mapsto (x, z) \bmod G$$

can be regarded as a tubular neighborhood of the connected component $f_i(B_{G_i})$. Thus the branched covering h satisfies the transversality condition in Theorem A.

The relative tangent bundle T_{C_G/B_G} is equal to the homotopy quotient of the tangent bundle TC, $T_{C_G/B_G} = (E_G \times TC)/G$. Hence the disk bundle $(E_G \times D_i)/G_i$ is isomorphic to the unit disk bundle of $\xi_i = (E_G \times T_{p_i}C)/G_i$. Clearly $e(T_{B_G \times (C/G)/B_G})^{n+1} = 0$ for $n \geq 1$. Since $b_i = \sharp G_i$, we have $b_i e(\xi_i) = 0 \in H^2(B_{G_i})$. Therefore Theorem A implies

$$e_n(C_G) = \sum_{i=1}^{m} \left(\pi|_{f_i(B_{G_i})}\right)_! (e(\xi_i)^n)$$

for $n \geq 1$.

Conceptually a transfer map results from a Gysin map. In this section, however, we replace the Gysin map $\left(\pi|_{f_i(B_{G_i})}\right)_!$ with the transfer map $\mathrm{cor}_{G_i}^G$ in an explicit manner. For we believe it will be good for future actual computations.

Consider the relative cohomology $H^2(C, C^0)$, where $C^0 = C - F$. Let $\nu_i \in H_2(C, C^0)$ be the image of the positive generator of $H_2(D_i, D_i - \{p_i\})$. The evaluation at ν_i induces a G-isomorphism

$$\nu_* : H^2(C, C^0) \overset{\cong}{\to} \bigoplus_{i=1}^{m} \mathrm{Hom}_{\mathbb{Z}[G_i]}(\mathbb{Z}[G], \mathbb{Z})$$

by the universal mapping property of coinduced modules, and an isomorphism

$$\nu_* : H^{*-2}(B_G; H^2(C, C^0)) \overset{\cong}{\to} \bigoplus_{i=1}^{m} H^{*-2}(B_{G_i}; \mathbb{Z})$$

by Shapiro's Lemma (cf. e.g., Brown [B].) Therefore we have a commutative diagram

$$\begin{array}{ccc} H^{*-2}(B_G; H^2(C, C^0)) & \xrightarrow{\nu_*} & \bigoplus_{i=1}^{m} H^{*-2}(B_{G_i}) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \oplus \mathrm{cor}_{G_i}^G} \\ H^{*-2}(B_G; H^2(C)) & \xrightarrow{<\ ,[C]>} & H^{*-2}(B_G) \end{array}$$

from the definition of the transfer map $\mathrm{cor}_{G_i}^G$. Comparing it with the commutative diagram (1.4), we obtain

$$e_n(C_G) = \sum_{i=1}^{m} \left(\pi|_{f_i(B_{G_i})}\right)_! (e(\xi_i)^n) = \sum_{i=1}^{m} \mathrm{cor}_{G_i}^G (e(\xi_i)^n)$$

for $n \geq 1$. This completes the proof of Theorem B.

3. Applications.

Let G be a perfect finite subgroup of the mapping class group of genus g, and C a compact Riemann surface on which G acts as a holomorphic automorphism group. Then, for any element $\gamma \in G$, we have

$$e_1(C_{\langle\gamma\rangle}) = 0 \in H^2(\langle\gamma\rangle;\mathbb{Z}),$$

since $H^2(G;\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(H_1(G;\mathbb{Z}),\mathbb{Z}) = 0$. Here we denote by $\langle\gamma\rangle$ the subgroup of the mapping class group generated by γ. Therefore we obtain

Lemma. *If $e_1(C_{\langle\gamma\rangle}) \neq 0 \in H^2(\langle\gamma\rangle;\mathbb{Z})$, there is no perfect finite subgroup containing γ in the mapping class group.*

As an example, consider two complex plane curves

$$w^2 = 1 - z^{2g+1}, \quad {w_1}^2 = z_1\left({z_1}^{2g+1} - 1\right)$$

for $g \geq 1$. Glueing them each other by the map $z_1 = z^{-1}$, $w_1 = z^{-g-1}w$, we obtain a hyperelliptic curve C of genus g. $\zeta := \exp\left(\frac{2\pi\sqrt{-1}}{2g+1}\right)$ defines an automorphism of the curve by $z \mapsto \zeta z$ and $w \mapsto w$. It induces an element γ of order $2g+1$ of the mapping class group of genus g. Let $u_0 \in H^2(\langle\gamma\rangle;\mathbb{Z})$ be the Euler class associated with the complex 1-dimensional G-module given by multiplication by ζ. ${u_0}^n$ generates the group $H^{2n}(\langle\gamma\rangle;\mathbb{Z}) \cong \mathbb{Z}/(2g+1)$ for each n. Then Theorem B implies

$$e_n(C_{\langle\gamma\rangle}) = {u_0}^n + {u_0}^n + ((-g-1)u_0)^n = (2+g^n){u_0}^n \in H^{2n}(\langle\gamma\rangle;\mathbb{Z}).$$

for any $n \geq 1$. Especially $e_1(C_{\langle\gamma\rangle}) \neq 0$ if $g \geq 2$. Hence the element γ is not contained in any perfect finite subgroup of the mapping class group of genus $g \geq 2$ from the lemma stated above. Moreover we obtain ${e_1}^n \neq 0$ for any $n \geq 1$ as a torsion element of the cohomology group of the mapping class group of genus g, provided that $2g+1$ is not a power of 3.

The second author has found some finite cyclic subgroups satisfying $e_{\mathrm{odd}} = 0$ and $e_2 \neq 0$. The details will be appear elsewhere. It would be interesting that there would exist a finite subgroup satisfying $e_1 = e_2 = 0$ and $e_3 \neq 0$.

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