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ON THE CAUCHY PROBLEM FOR THE NAVIER–STOKES EQUATIONS WITH NONDECAYING INITIAL DATA

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1. Introduction and main results

We consider the Cauchy problem of the Navier–Stokes equations in $\mathbf{R}^n$ for $n \geq 2$:

$$
\text{(NS)}\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{for} \quad x \in \mathbf{R}^n,\ t > 0, \\ \operatorname{div} u = 0 & \text{for} \quad x \in \mathbf{R}^n,\ t > 0, \\ u|_{t=0} = u_0 & \text{for} \quad x \in \mathbf{R}^n, \end{cases}
$$

where $u = u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and its pressure, while $u_0 = u_0(x) = (u_0^1(x), \cdots, u_0^n(x))$ is the given initial velocity vector field.

There is a large literature on existence of solutions of (NS) in $\mathbf{R}^n$ (e.g. [1],[2],[4],[6], [8],[11],[13],[14],[18],[19]). For a given initial data solutions of (NS) have been constructed in various function spaces. In these works, however, the initial data is required to decay at space infinity. On the other hand, there are many exact solutions which is unbounded at space infinity in some practical flow problems (see [22]).

The goal of this paper is to construct a unique local solution of (NS) without assuming that the initial data u_0 does not decay at $|x| \to \infty$. There might be several freedom to choose function spaces of nondecaying functions. In this paper we do not handle functions growing in the space infinity. We choose an initial data u_0 in $BUC = BUC(\mathbf{R}^n)$, which is the space of all bounded uniformly continuous functions on $\mathbf{R}^n$ or in $L^\infty = L^\infty(\mathbf{R}^n)$, which is the space of all essentially bounded functions on $\mathbf{R}^n$. We do not distinguish the space of vector-valued and scalar functions. This assumption allows us to take nondecaying functions as an initial data such as a constant or a spatially periodic functions. Our goal is to construct a unique local solution of (NS) for given initial data $u_0 \in BUC$ or L^∞. We also study the global regularity of soluitons.

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For this purpose as usual we transform (NS) into the abstract ordinary differential equation for u

$$u_t + Au + \mathbb{P}(u, \nabla)u = 0 \tag{ABS}$$

by eliminating the prssure, where $\mathbb{P}$ is the operator defined by

$$\mathbb{P} = (P_{ij})_{1 \le i,j \le n} \ , \ P_{ij} = \delta_{ij} + R_i R_j \ ;$$

$R_j (j = 1, \cdots, n)$ are the Riesz operator with symbol $\sqrt{-1}\ \xi_j/|\xi|$ (cf. [24]). (The equation (ABS) is formally obtained from (NS) by noting $\mathbb{P}(\nabla p) = 0, \mathbb{P}\Delta u = \Delta u$ since div$u = 0$.) Using the solution operator $e^{t\Delta}$ of the heat equation (i.e., $e^{t\Delta} f = E(t) * f$ where $E(t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$), we transform (ABS) into an integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}((u, \nabla)u)(s) ds \ . \tag{I}$$

Since div$u = \nabla \cdot u = 0$ so that $(u, \nabla)u = \nabla \cdot (u \otimes u)$ and since operators $\mathbb{P}, \nabla, e^{t\Delta}$ are commutative, the above integral equation is expressed in the form

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} \mathbb{P}(u \otimes u)(s) ds \ , \tag{INT}$$

where $u \otimes u$ is the tensor whose ij component equals $u^i u^j$ and $\nabla \cdot F$ denotes the divergence of a tensor $F = (F_{ij})$ defined by $(\nabla \cdot F)_i = \sum_{j=1}^n \partial F_{ij}/\partial x_j$, $(j = 1, \cdots, n)$. The equation (INT) is formally equivalent to (ABS). To fix the idea we mainly consider the initial data in BUC and point out necessary alternations to the case of L^∞. (The space BUC is a closed subspace of the Banach space L^∞ equipped with the maximum norm $||\cdot||_\infty$.) Our goal is to construct a unique local-in-time solution u of (INT) in the space $C([0, T); BUC)$, which is the space of all continuous functions from $[0, T)$ to BUC.

Theorem 1 (Local existence and uniqueness of the solution to (INT)). (i) *For $u_0 \in BUC$ there exists $T_0 > 0$ and a unique solution $u = u(t)$ of (INT) such that*

$$u \in C([0, T_0]; BUC) \ . \tag{1.1}$$

(ii) *The solution $u(t)$ satisfies*

$$t^{1/2}\ \nabla u \in C([0, T_0]; BUC) \ , \tag{1.2}$$

$$\lim_{t \downarrow 0} t^{1/2} ||\nabla u(t)||_\infty = 0 \ , \tag{1.3}$$

$$u \in C^\alpha([\delta, T_0]; BUC) \ , \tag{1.4}$$

$$\nabla u \in C^\alpha([\delta, T_0]; BUC) \ , \tag{1.5}$$

$$(-\Delta)^\beta u \in C([\delta, T_0]; BUC) \tag{1.6}$$

for any α *with* $0 < \alpha < 1/2$, *any* β *with* $0 < \beta < 1$, *and any* δ *with* $0 < \delta < T_0$.

Here $C^\alpha(I; X)$ $(0 < \alpha < 1)$ denotes the space of all Hölder continuous functions of order α defined in an interval with values in a Banach space X. The operator $(-\Delta)^\alpha$, the fractional power of $-\Delta$, is defined as in Remark (ii) right after Proposition A.2.1 in Appendix 2, where it is expressed by A^α. The solution of (INT) constructed in Therorem 1 is often called a mild solution (NS). The difficulty in proving Theorem 1 lies in the fact that the operator $\mathbb{P}$ is not bounded in $BUC(\mathbf{R}^n)$, or $L^\infty(\mathbf{R}^n)$, since the Riesz transforms are unbounded in these spaces although they are bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ (see [25]). For this reason we rather use the form (INT) instead of (I). To overcome this difficulty we use the estimate of the derivative of the heat kernel $E(t) = E(x,t)$ in the Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ which was obtained by Carpio [3]. Such an estimate yields a uniform bound for $t^{1/2}\nabla e^{t\Delta}\mathbb{P}$ as a bounded operator in L^∞ although $\mathbb{P}$ is not bounded in L^∞. Thus we are able to construct a solution using the standard iteration (see e.g. [14]). To prove (1.6) we generalize her estimate to $(-\Delta)^\alpha E$ (Appendix A.2).

Remark. (i) For a lower estimate for $T_0 > 0$, we get

$$T_0 \geq C\ ||u_0||_\infty^{-2}$$

with a numerical constant $C > 0$. This follows from the way of construction in §3. This estimate is consistent with existence result for L^p–data in [6] which says

$$T_0 \geq C||u_0||_p^\sigma \quad \text{with } \sigma = 2/\{(n/p) - 1\}.$$

(ii) For regularity of $u(t)$, we also have

$$\nabla^k u \in C([\delta, T_0]; BUC)$$

for any δ with $0 < \delta < T_0$ and any positive integer k. This follows (1.2) and iterative use of (iv) below.

(iii) If we replace BUC by L^∞ in Theorem 1, we obtain same results as Theorem 1. But we have to replace (1.1),(1.2) by

$$u \in C_w([0, T_0]; L^\infty)\ , \tag{1.1'}$$

$$t^{1/2}\nabla u \in C_w([0, T_0]; L^\infty) \tag{1.2'}$$

for any $0 < \delta < T_0$, and exclude (1.3). Here for an interval $I \subset \mathbf{R}$, $C_w(I; L^\infty)$ denotes the space of all L^∞ valued weakly continuous functions $f(t)$ defined in $t \in I$. The proof paralles that of Theorem 1.

(iv) If we assume $\nabla^k u_0 \in BUC$ for some positive integer k, the solution of (INT) obtained in Theorem 1 satisfies

$$\nabla^k u \in C([0, T_0]; BUC)\ ,$$

$$t^{1/2}\nabla^{k+1} u \in C([0, T_0]; BUC)\ .$$

For the proof we note that if $\nabla^k u_0 \in BUC$ for some k then $\nabla^j u_0 \in BUC$ for all $j \leq k$. Taking ∇^k of both terms of (INT) we have above result using the argument is §3 inductively on k.
(v) Assume $u_0 \in BUC$ is periodic in the direction of x_ℓ-axis, that is,

$$u_0(x + \omega e_\ell) = u_0(x) \quad \text{in} \quad \mathbf{R}^n$$

for some $\omega > 0$ and e_ℓ is the unit normal vector whose ℓ-th component is one. Then the solution $u(t)$ of (INT) in Theorem 1 satisfies the same periodicity condition in space. This follows from the uniqueness of (INT) and translation invariance of (INT).
(vi) From the proof we may choose T_0 (satisfying Remark (i)) so that there is bound for $||u(t)||_\infty$, $t^{1/2}||u(t)||_\infty$:

$$\sup_{0 \leq t \leq T_0} ||u(t)||_\infty \leq 2||u_0||_\infty, \quad \sup_{0 \leq t \leq T_0} t^{1/2}||\nabla u(t)||_\infty \leq C||u_0||_\infty$$

with a constant C depending only on n.

We next check whether our mild solution u really solves (ABS) or (NS) by assuming $\mathrm{div} u_0 = 0$. The main difficulty here is that $\mathbb{P}(u, \nabla)u$ in (ABS) may not belong to BUC at fixed t, so one can not interpret (ABS) as an equation in $C((0,T); BUC)$; u_t may not belong to BUC at fixed t. In fact, taking the derivative with respect to t in (INT) we find a few terms expressed by $\nabla \mathbb{P} v$ for some $v \in BUC$. However, because of the unboundedness $\mathbb{P}$ in BUC we are not sure whether these terms belong to BUC. One should be careful about global uniform regularity of u_t. Nevertheless we are able to prove that our solution u solves (NS) with suitable choice of p.

Theorem 2 (Local existence of the solution to (NS)). *Let $u_0 \in BUC$ satisfy div $u_0 = 0$. Let the solution $u(t) \in C([0, T_0]; BUC)$ of (INT) satisfy $t^{1/2}\nabla u \in C([0, T_0]; BUC)$. Then*

$$u_t + \nabla \cdot q \in C((0, T_0]; BUC),$$

for some tensor $q \in C((0, T_0]; BMO)$. If we set $p(t) = \sum_{i,j=1}^n R_i R_j u^i u^j(t)$ for each $t > 0$, then (u, p) is smooth in $\mathbf{R}^n \times (0, T_0)$ and solves (NS).

Remark. (i) We denote by $BMO = BMO(\mathbf{R}^n)$, the space of all (vector valued) fuctions of bounded mean oscillation (see [5],[12],[25]).
(ii) Taking the divergence of (NS), we formally obtain

$$-\Delta p(t) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u^i(t) u^j(t). \tag{1.7}$$

Recalling the identity

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f = -R_i R_j \Delta f$$

for $f \in L^r(\mathbf{R}^n)$ and $1 < r < \infty$ (see [24;p59]), we observe

$$\tilde{p}(t) = \sum_{i,j=1}^{n} R_i R_j u^i(t) u^j(t) \tag{1.8}$$

$\tilde{p}$ solves (1.7). Since $R_j : L^\infty(\mathbf{R}^n) \to BMO$ and $R_j : BMO \to BMO$ are bounded, $\tilde{p}$ belongs to BMO if $u \in L^\infty(\mathbf{R}^n)$ (see [20],[25],[27]). We note that $\tilde{p}$ is defined only up to an additive constant since the element of BMO is determined up to constant as a locally integrable function in $\mathbf{R}^n$. However, $\nabla\tilde{p}$ in (NS) is uniquely determined as a tempered distribution.

If the solution u does not decay at spatially infinity, the uniqueness as a solution (u, p) of original (NS) is rather subtle. For example we consider a spatially constant function $u = g(t)$. If p is taken as $p(x) = g'(t) \cdot x$ (inner product), then (u, p) evidently solve (NS) no matter how g is taken. This shows nonuniqueness of solution, since u is not determined even if we fix initial data $g(0)$. To get uniqueness we need to impose some condition on p. In [15] Kim and Chae (extending a result of Okamoto [21]) showed that a classical solution (u, p) of (NS) having a bound for $|\nabla u|$ and $|x|^{2/n-1}|p(x)|$ is unique. However, their condition requires some decay of p for $n \geq 3$ which is evidently too strong to include spatially periodic solution. We give the uniqueness by assigning the value of p from u without assuming that p is bounded or decays at space infinity. Our assumption implies $p \in BMO$ which still allows growing p as $|x| \to \infty$ ([25;p.140]).

Theorem 3 (Uniqueness of the solution to (NS)). *Let $u_0 \in BUC$ satisfy $\operatorname{div} u_0 = 0$. For each $t > 0$ set $p = \sum_{i,j=1}^{n} R_i R_j u^i u^j$. Then the solution (u, p) of (NS) in the distributional sense which satisfies*

$$u \in C([0,T]; BUC) \tag{1.9}$$

$$u_t,\ \Delta u \in C((0,T]; S') \quad \text{for some } T > 0 \tag{1.10}$$

is unique, where S' denotes the space of tempered distribution.

Remark. The solution u constructed in Theorem 1 and 2 fulfill the last condition by the global uniform regularity of u_t in Theorem 2 and Remark (ii) of Theorem 1. By the way the assumption on p can be weakened. In fact it suffices to assume that

$$p = \sum_{i,j=1}^{n} R_i R_j \pi_{ij}$$

for some $\pi_{ij} \in L^\infty$ satisfying $\nabla\pi_{ij} \in L^\infty$. We shall discuss this problem in our forthcoming paper.

After this work was completed, Cannone informed us of the paper [2] by Cannone and Meyer. In the paper they gave a simple sufficient condition for an abstract Banach space X that guarantees the existence and uniqueness of the solution $u \in C([0,T);X)$ of (INT). This condition is based on the Littlewood–Paley decomposition, and there are several Banach spaces fulfilling the condition. They also proved some spatial regularity results in the Besov-type space on the quadratic term of the mild solution. Since it is easy to check that our space $X = BUC$ satisfies their condition, the existence and uniqueness of a mild solution $u \in C([0,T);BUC)$ of (INT) is obtained by their method although it is not explicitly written in [2]. In the case $X = L^\infty$ is mentioned in [1]. However, they did not obtain regularity in time nor the relation the mild solution to a solution of (NS). We prove some regularity results on the mild solution in time in Theorem 1 (ii) and explain in what sense the constructed mild solution satisfies (NS) by using BMO–functions in Theorem 2.

There is a result of Knightly ([16;Theorem 3]) closely related to our Theorem 1. He proved existence and uniqueness of a mild solution to (NS) for the initial data which is only assumed to be bounded. In his paper a key estimate to show this result is stated, but seems to be not proved at least explicitly. Since our results are also valid for the initial data belonging to $L^\infty(\mathbf{R}^n)$ except the continuity of the solution at $t = 0$ (see Remark(iii) after Theorem 1), our results recover the result by Knightly in a sense although his proof seems to be based on delicate pointwise estimates of growth of a solution while our proof is based on estimates by using norms which is simpler than his proposed method.

This paper is organized as follows. In Section 2 we prepare various estimates for $e^{t\Delta}$ which are used later. In Section 3 we prove Theorem 1 (i) and (1.2). In Section 4 we prove Theorem 1 (ii) except (1.2). Proofs of Theorem 2 and 3 are written respectively in Section 5 and 6. In Appendix 1 we show that Laplacian generates an analytic semigroup in BUC. In Appendix 2 we prove an estimate in the Hardy space for fractional powers of Laplacian of the heat kernel which is stated in Section 2. In Appendix 3 we prove that $f \in BMO$ is smooth if its all derivatives belongs to BMO to prove regularity of (u,p) in Theorem 2.

We are grateful to Professor M. Cannone for informing us of applicability of his theory.

2. Estimates for solutions of the heat equation

We recall various fundamental estimates for the heat semigroup $e^{t\Delta}$. Except estimates involving $\mathbb{P}$, these estimates are well known. Throughout this paper we denote positive numerical constants by C the value of which may differ from one occasion to another. A subscript attached to C indicates the dependence of the parameter denoted as a subscript.

For the Laplace operator we associate a closed linear operator A (densely defined in BUC) of form

$$\begin{aligned} Af =& -\Delta f \quad \text{for } f \in D(A) \text{ (domain of } A\text{)}, \\ D(A) =& \{f \in BUC \ ; \ f \text{ is } C^1 \text{ with } \nabla f \in BUC \\ & \text{and its distributional derivative} \\ & \Delta f \text{ belongs to } BUC\} \end{aligned}$$

As shown in Appendix 1 , $-A$ generates the analytic semigroup in BUC. The theory of analytic semigroups and fractional powers of its generators ([10;p26 Th.1.4.3], [23;p74 Th.6.13]) yields:

Lemma 1. For $0 < \alpha \leq 1$ there is a constant C_α that satisfies

$$||A^\alpha \ e^{-tA} \ f||_\infty \leq C_\alpha \ t^{-\alpha} ||f||_\infty \quad \text{for all } f \in BUC \text{ and } t > 0, \tag{2.1}$$

$$||(e^{-tA} \ - I)f||_\infty \leq C_\alpha \ t^\alpha ||A^\alpha f||_\infty \quad \text{for all } f \in D(A^\alpha) \text{ and } t \geq 0, \tag{2.2}$$

where I denotes the identity operator.

We recall the basic estimate for the heat semigroup $e^{t\Delta}$ acting for $f \in L^\infty$ by

$$\begin{aligned} (e^{t\Delta} f)(x) =& E(t) * f \\ =& \int_{\mathbf{R}^n} E(x-y,t) f(y) dy \end{aligned}$$

where E denotes the heat kernel of form

$$E(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t).$$

Lemma 2.

$$||e^{t\Delta} \ f||_\infty \leq ||f||_\infty \quad \text{for} \quad t \geq 0 \ , f \in L^\infty, \tag{2.3}$$

$$||\nabla \ e^{t\Delta} f||_\infty \leq C \ t^{-1/2} \ ||f||_\infty \quad \text{for} \quad t > 0 \ , f \in L^\infty, \tag{2.4}$$

with some $C > 0$.

Unfortunately $e^{t\Delta}$ is not a continuous (C_0) semigroup in L^∞ as shown in Lemma 5. However, it has several properties similar to those of e^{-tA} defined on BUC. Moreover, by (2.3–4) $e^{t\Delta} f \in BUC$ for $t > 0$. So we use the notation e^{-tA} for $e^{t\Delta}$ even if it is rather abuse of notation.

Remark. The estimate (2.1) is still valid by replacing BUC by L^∞. In fact express $A^\alpha e^{-tA} f = A^\alpha e^{-tA/2}(e^{-tA/2} f)$ and use (2.1) for BUC and (2.3) to get (2.1) for L^∞ since $e^{-tA/2} f \in BUC$ by (2.3–4). Note that (2.1) for L^∞ can be proved directly if we use the explicit representation of $A^\alpha f = \text{constant}\Delta |x|^{2(1-\alpha)-n} * f$,$(0 < \alpha < 1)$ given in Appendix 2. (The case $\alpha = 1$ follows from (2.4).) This integral operator is often written by $(-\Delta)^\alpha$ instead of A^α. A direct method also proves $||(e^{-tA}-I)A^{-\alpha}f||_\infty \le Ct^\alpha ||f||_\infty$ for $f \in L^\infty$.

Lemma 3. Assume that $0 < \alpha < 1$.

(i) There exists a constant C_α that satisfies

$$||A^\alpha E(t)||_{\mathcal{H}^1(\mathbf{R}^n)} \le C_\alpha\, t^{-\alpha} \quad \text{for} \quad t > 0, \tag{2.5}$$

where $\mathcal{H}^1 = \mathcal{H}^1(\mathbf{R}^n)$ is the Hardy space equipped with the norm

$$||f||_{\mathcal{H}^1(\mathbf{R}^n)} = ||f||_{L^1(\mathbf{R}^n)} + \sum_{j=1}^n ||R_j f||_{L^1(\mathbf{R}^n)}. \tag{2.6}$$

(ii) There exists a constant C_α that satisfies

$$||A^\alpha e^{-tA}\mathbb{P} f||_\infty \le C_\alpha\, t^{-\alpha} ||f||_\infty \quad \text{for} \quad t > 0\ , \tag{2.7}$$

where $A^\alpha = (-\Delta)^\alpha$.

Remark. (i)Since $\mathbb{P}$ is not bounded in L^∞, the uniform estimate $||e^{t\Delta}\mathbb{P}f||_\infty \le C||f||_\infty$ fails. Although, $e^{t\Delta}\mathbb{P}f$ has uniform continuity by (2.8), we do not espect the boundedness of $e^{t\Delta}\mathbb{P}f$ for $t > 0$. So we rather use the explicit operator $(-\Delta)^\alpha$ instead of the operator A^α in BUC.

(ii)In [3;Lemma 2.1] Carpio proved that

$$||\nabla E(t)||_{\mathcal{H}^1(\mathbf{R}^n)} \le Ct^{-1/2};$$

(The proof there is misprinted. A full proof is found in [7].) This estimates yields

$$||\nabla\ e^{-tA}\mathbb{P}\ f||_\infty \le C\, t^{-1/2} ||f||_\infty, \quad \text{for} \quad t > 0\ . \tag{2.8}$$

The proof is similar to derive (2.7) from (2.6). We shall only give the proof of (2.7).

Proof of Lemma 3. We shall prove (i) in Appendix 2 so we prove (ii) admitting (2.5). For $f \in L^\infty$, the operator $A^\alpha e^{-tA}\mathbb{P}$ is written in the form

$$\begin{aligned}
i\text{-th component of } A^\alpha e^{-tA}\mathbb{P}f &= A^\alpha e^{-tA} f_i + \sum_{j=1}^n A^\alpha e^{-tA} R_i R_j f_j \\
&= A^\alpha E(t) * f_i + \sum_{j=1}^n A^\alpha R_i R_j E(t) * f_j \\
&= \sum_{j=1}^n (\delta_{ij} + R_i R_j)(A^\alpha E(t)) * f_j \\
&= \sum_{j=1}^n k_{ij}(t) * f_j \quad \text{for} \quad 1 \le i \le n
\end{aligned} \tag{2.9}$$

where the kernel $k_{ij}(t) = (\delta_{ij} + R_i R_j)(A^\alpha E(t))$ and $*$ means a convolution with respect to space variables. Since the Riesz transforms are bounded in $\mathcal{H}^1(\mathbf{R}^n)$ and $||\cdot||_{L^1(\mathbf{R}^n)} \le ||\cdot||_{\mathcal{H}^1(\mathbf{R}^n)}$, we have

$$\begin{aligned} ||k_{ij}(t)||_{L^1(\mathbf{R}^n)} &\le ||k_{ij}(t)||_{\mathcal{H}^1(\mathbf{R}^n)} \\ &\le ||A^\alpha E(t)||_{\mathcal{H}^1(\mathbf{R}^n)}. \end{aligned}$$

Hence (2.5) implies

$$||k_{ij}(t)||_{L^1(\mathbf{R}^n)} \le C_\alpha t^{-\alpha} \quad \text{for} \quad t > 0\,.$$

Thus, applying Young's inequality to (2.9), we obtain (ii). ■

For the proof of Theorem 1, we list all of the estimates which are deduced from Lemmas 1, 2 and 3.

Lemma 4. Assume that $0 < \alpha \le 1$. Then

$$||(e^{-sA} - e^{-tA})\, f||_\infty \le C_\alpha (s-t)^\alpha\, t^{-\alpha}\, ||f||_\infty, \tag{2.10}$$

$$||(\nabla e^{-sA}\mathbb{P} - \nabla e^{-tA}\mathbb{P})\, f||_\infty \le C_\alpha (s-t)^\alpha\, t^{-\alpha-(1/2)}\, ||f||_\infty, \tag{2.11}$$

$$||(\nabla e^{-sA} - \nabla e^{-tA})\, f||_\infty \le C_\alpha (s-t)^\alpha\, t^{-\alpha-(1/2)}\, ||f||_\infty, \tag{2.12}$$

$$||\nabla^2 e^{-tA}\mathbb{P}\,(f\otimes f)||_\infty \le C\, t^{-1/2}\, ||\nabla f||_\infty ||f||_\infty, \tag{2.13}$$

$$||(\nabla^2 e^{-sA}\mathbb{P} - \nabla^2 e^{-tA}\mathbb{P})(f\otimes f)||_\infty \le C_\alpha (s-t)^\alpha\, t^{-\alpha-(1/2)}\, ||\nabla f||_\infty ||f||_\infty \tag{2.14}$$

for $0 < t \le s$, $f \in L^\infty$, where $\nabla^2 = \nabla\nabla$.

Proof. Since $e^{-sA} - e^{-tA} = (e^{-(s-t)A} - I)A^{-\alpha}A^\alpha e^{-tA}$, we get (2.10) immediately from (2.1–2).

It follows from (2.1–2) and (2.8) that

$$\begin{aligned} ||(\nabla e^{-sA}\mathbb{P} - \nabla e^{-tA}\mathbb{P})f||_\infty &= ||(e^{-(s-t)A} - I)\nabla e^{-tA}\mathbb{P}f||_\infty \\ &\le ||(e^{-(s-t)A} - I)A^{-\alpha}||_{\infty\to\infty}\, ||A^\alpha e^{(-t/2)A}||_{\infty\to\infty}\, ||\nabla e^{(-t/2)A}\mathbb{P}f||_\infty \\ &\le C_\alpha (s-t)^\alpha t^{-\alpha-(1/2)} ||f||_\infty, \end{aligned}$$

which yields (2.11). Here $||\cdot||_{\infty\to\infty}$ stands for the operator norm from L^∞ to L^∞ .

It follows from (2.1–2) and (2.4) that

$$\begin{aligned} ||(\nabla e^{-sA} - \nabla e^{-tA})f||_\infty &= ||(e^{-(s-t)A} - I)\nabla e^{-tA}f||_\infty \\ &\le ||(e^{-(s-t)A} - I)A^{-\alpha}||_{\infty\to\infty}\, ||A^\alpha e^{(-t/2)A}||_{\infty\to\infty} ||\nabla e^{(-t/2)A}f||_\infty \\ &\le C_\alpha (s-t)^\alpha t^{-\alpha-(1/2)} ||f||_\infty. \end{aligned}$$

We thus obtain (2.12).

By (2.8) we have

$$
\begin{aligned}
&\|\nabla^2 e^{-tA}\mathbb{P}(f\otimes f)\|_\infty \\
&\quad \le \|\nabla e^{-tA}\mathbb{P}\|_{\infty\to\infty}\ \|\nabla(f\otimes f)\|_\infty \\
&\quad \le Ct^{-1/2}\|\nabla f\|_\infty\|f\|_\infty
\end{aligned}
$$

for (2.13).

By (2.11) we have

$$
\begin{aligned}
&\|(\nabla^2 e^{-sA}\mathbb{P} - \nabla^2 e^{-tA}\mathbb{P})(f\otimes f)\|_\infty \\
&\quad \le \|(\nabla e^{-sA}\mathbb{P} - \nabla e^{-tA}\mathbb{P})\|_{\infty\to\infty}\ \|\nabla(f\otimes f)\|_\infty \\
&\quad \le C_\alpha(s-t)^\alpha t^{-\alpha-(1/2)}\|\nabla f\|_\infty\|f\|_\infty.
\end{aligned}
$$

which yields (2.14). ∎

To obtain the continuity at $t=0$ of the solution $u(t)$ of (INT), we need a continuity of $e^{-tA}f$ at $t=0$ for $f\in BUC$. (cf. Remark (iii) after Theorem 1). The proof is standard but we give it for completeness.

Lemma 5. Assume that $f\in L^\infty$. Then $f\in BUC$ if and only if

$$
e^{-tA}f\to f \quad \text{in} \quad L^\infty \quad \text{as} \quad t\downarrow 0\,. \tag{2.15}
$$

Proof. Assume $f\in BUC$. For arbitrary δ with $0<\delta<1$, we have

$$
e^{-tA}f = \int_{|y|\le\delta} E(y,t)f(x-y)dy + \int_{|y|>\delta} E(y,t)f(x-y)dy.
$$

Taking t tends to 0, the L^∞ norm of the second term tends to 0 by the property of the heat kernel. Since $\int_{\mathbf{R}^n} E(y,t)dy = 1$ and f is bounded continuous , we get

$$
\begin{aligned}
&\Big\|\int_{|y|\le\delta} E(y,t)f(x-y)dy - f(x)\Big\|_\infty \le \\
&\qquad \sup_{x\in\mathbf{R}^n}\Big|\int_{|y|\le\delta} E(y,t)(f(x-y)-f(x))dy\Big| \\
&\qquad\quad + \sup_{x\in\mathbf{R}^n}\Big|\int_{|y|>\delta} E(y,t)f(x)dy\Big| \\
&\qquad \le \sup_{x\in\mathbf{R}^n}\max_{|y|\le\delta}|f(x-y)-f(x)| + \|f\|_\infty\int_{|y|>\delta} E(y,t)dy\ ,
\end{aligned}
$$

which tends to 0 as $t\downarrow 0$ for small $\delta>0$.

Conversely, assume $f \in L^\infty$ and (2.15). For $h > 0$, we have

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^n}\sup_{|y|\le h}|f(x+y)-f(x)| \\
&\quad\le \sup_{x\in\mathbf{R}^n}\sup_{|y|\le h}\{|f(x+y)-(e^{-tA}f)(x+y)| \\
&\quad\quad + |(e^{-tA}f)(x+y)-(e^{-tA}f)(x)| + |(e^{-tA}f)(x)-f(x)|\}.
\end{aligned}$$

The first and third terms of the right–hand side are estimated by

$$2\,||f-e^{-tA}f||_\infty.$$

The second term is estimated by

$$\sup_{x\in\mathbf{R}^n}\sup_{|y|\le h}||\nabla e^{-tA}f||_\infty |y|.$$

Thus we get by (2.4)

$$\sup_{x\in\mathbf{R}^n}\sup_{|y|\le h}|f(x+y)-f(x)| \le 2\,||f-e^{-tA}f||_\infty + hCt^{-1/2}||f||_\infty.$$

We send h to zero first and send t to zero to observe that f is uniformly continuous. ∎

3. Local existence and uniqueness of the solution to (INT)

In this section we prove (i) and (1.2),(1.3) of Theorem 1, using a succesive iteration. The method is standard when BUC is replaced by L^p $1 < p < \infty$(e.g. [6],[14]). Since we handle BUC (not L^∞), we give also a proof of continuity of approximate sequence in time with values in BUC.

We define the approximation $u_j(t)$ $(j = 1, 2, \cdots)$ by

$$\begin{aligned}
u_1(t) &= e^{-tA}u_0 \\
u_{j+1}(t) &= e^{-tA}u_0 + Su_j(t),
\end{aligned} \tag{3.1}$$

where u_0 is the initial data of (NS) and $Sv(t) = \int_0^t \nabla e^{-(t-s)A}(-\mathbb{P})(v\otimes v)(s)ds$. First we set, for $j = 1, 2, \cdots$

$$K_j = K_j(T) = \sup_{0\le t\le T}||u_j(t)||_\infty \text{ and } K_j' = K_j'(T) = \sup_{0\le t\le T} t^{1/2}||\nabla u_j(t)||_\infty;$$

note that $K_0 = ||u_0||_\infty$. From the above iteration scheme we have

$$K_{j+1}(T) \le K_0 + C_1T^{1/2}K_j(T)^2,$$

by (2.3) and (2.8). Similarly (2.4) and (2.13) imply

$$K'_{j+1}(T) \le CK_0 + C_2 T^{1/2} K'_j(T) K_j(T).$$

We take T_1 small so that $\max_j 4C_j T_1^{1/2} K_0 < 1$. Then it is easy to see that

$$\sup_j K_j(T) \le 2K_0 \quad \text{and} \quad \sup_j K'_j(T) \le 2CK_0 \tag{3.2}$$

for any $T \le T_1$.

Second, we show that u_j and $t^{1/2}\nabla u_j$ belong to $C([0,T_1]; BUC)$ for all $j \ge 1$. To prove that $u_j(t_0) \in BUC$ for any t_0 with $0 \le t_0 \le T_1$, it is sufficient to show that $u_j(t_0)$ satisfies the equivalent condition given in Lemma 5. When $t_0 = 0$, $u_j(0) = u_0$ clearly belongs to BUC. For fixed $t_0 \in (0, T_1]$ and arbitrary $t > 0$, it follows from (2.10–11) and (3.2) that

$$\begin{aligned}
||e^{-tA}u_j(t_0) - u_j(t_0)||_\infty &\le ||e^{-(t_0+t)A}u_0 - e^{-t_0 A}u_0||_\infty \\
&+ ||\int_0^{t_0} \{\nabla \cdot e^{-(t_0+t-s)A}(-\mathbb{P}) - \nabla \cdot e^{-(t_0-s)A}(-\mathbb{P})\}(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&\le C_\alpha\, t^\alpha\, t_0{}^{-\alpha} \\
&+ C_\alpha\, t^\alpha \int_0^{t_0} (t_0 - s)^{-\alpha-(1/2)} ds\ ,
\end{aligned}$$

which tends to 0 by letting $t \downarrow 0$ with $0 < \alpha < 1/2$. Thus we see that $u_j(t_0)$ belongs to BUC for $0 \le t_0 \le T_1$.

Similarly, we obtain $t_0^{1/2}\nabla u_j(t_0) \in BUC$ for $0 \le t_0 \le T_1$ as follows. We may assume $0 < t_0 \le T_1$. For $t > 0$, we observe that

$$\begin{aligned}
||e^{-tA}t_0^{1/2}\nabla u_j(t_0) - t_0^{1/2}\nabla u_j(t_0)||_\infty &\le t_0^{1/2}||\nabla e^{-(t_0+t)A}u_0 - \nabla e^{-t_0 A}u_0||_\infty \\
+ t_0^{1/2}||\int_0^{t_0} &\{\nabla^2 \cdot e^{-(t_0+t-s)A}(-\mathbb{P}) - \nabla^2 \cdot e^{-(t_0-s)A}(-\mathbb{P})\}(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&\le C_\alpha\, t_0^{1/2}\, t^\alpha\, t_0^{-\alpha-(1/2)}||u_0||_\infty \\
&+ C_\alpha\, t_0^{1/2}\, t^\alpha \int_0^{t_0} (t_0 - s)^{-\alpha-(1/2)} s^{-1/2} ds\ ,
\end{aligned}$$

which tends to 0 as $t \downarrow 0$. Here we have used (2.12),(2.14) and (3.2).

The continuity in time of $u_j(t)$ and $t^{1/2}\nabla u_j(t)$ is proved as follows. For fixed

$t \in (0,T_1)$ and arbitrary $\delta > 0$, it follows from (2.8),(2.10–11) and (3.2) that

$$\begin{aligned}
&||u_j(t+\delta) - u_j(t)||_\infty \le ||e^{-(t+\delta)A}u_0 - e^{-tA}u_0||_\infty \\
&+ ||\int_0^t \{\nabla \cdot e^{-(t+\delta-s)A}(-\mathbb{P}) - \nabla \cdot e^{-(t-s)A}(-\mathbb{P})\}(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&+ ||\int_t^{t+\delta} \nabla \cdot e^{-(t+\delta-s)A}(-\mathbb{P})(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&\le C_\alpha \delta^\alpha t^{-\alpha} + C_\alpha \delta^\alpha \int_0^t (t-s)^{-\alpha-(1/2)}ds \\
&+ C \int_t^{t+\delta} (t+\delta-s)^{-1/2}ds \ ,
\end{aligned}$$

and for fixed $t \in (0,T_1]$ with $t > \delta$

$$\begin{aligned}
&||u_j(t) - u_j(t-\delta)||_\infty \le ||e^{-tA}u_0 - e^{-(t-\delta)A}u_0||_\infty \\
&+ ||\int_0^{t-\delta} \{\nabla \cdot e^{-(t-s)A}(-\mathbb{P}) - \nabla \cdot e^{-(t-\delta-s)A}(-\mathbb{P})\}(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&+ ||\int_{t-\delta}^{t} \nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\
&\le C_\alpha \delta^\alpha (t-\delta)^{-\alpha} + C_\alpha \delta^\alpha \int_0^{t-\delta} (t-\delta-s)^{-\alpha-(1/2)}ds \\
&+ C \int_{t-\delta}^{t} (t-s)^{-1/2}ds \ .
\end{aligned}$$

Letting $\delta \downarrow 0$, all terms in the rightest hand sides of these inequalities tend to 0 for $0 < \alpha < 1/2$. When $t = 0$, for arbitrary $\delta > 0$ we have

$$\begin{aligned}
||u_j(x,\delta) - u_0||_\infty &\le ||e^{-\delta A}u_0 - u_0||_\infty \\
&+ ||\int_0^\delta \nabla \cdot e^{-(\delta-s)A}(-\mathbb{P})(u_{j-1} \otimes u_{j-1})(s)ds||_\infty .
\end{aligned}$$

Using (2.8) and (3.2), the second term of the right–hand side of the last inequality is estimated by

$$C \int_0^\delta (\delta - s)^{-1/2}ds \ ,$$

which tends to 0 as $\delta \downarrow 0$. The first term also tends to 0 as $\delta \downarrow 0$ by Lemma 5. Thus we conclude that $u_j(t)$ is continuous in $t \in [0,T_1]$ with values in BUC.

Similarly, it follows from (2.1–2),(2.8) and (2.12–14) that

$$t^{1/2}\nabla u_j(t) \quad \text{is continuous in } t \in (0,T_1] \text{ with values in } BUC \text{ for all } j \ge 1. \tag{3.3}$$

We shall prove that $t^{1/2}\nabla u_j \to 0$ in BUC as $t \downarrow 0$. For $u_0 \in BUC$ and arbitrary $\eta > 0$, define $u_0^\eta = E(\eta) * u_0$. Then, for any $\varepsilon > 0$ there exists $\eta_0 > 0$ such that

$$||u_0^\eta - u_0||_\infty \le \varepsilon \ ,$$
$$||\nabla u_0^\eta||_\infty \le C\eta^{-1/2}||u_0||_\infty$$

for $0 < \eta < \eta_0$. So, for arbitrary t with $0 < t < \eta_0^2$ we have

$$\begin{aligned} ||t^{1/2}\nabla e^{-tA}u_0||_\infty &\le t^{1/2}||\nabla e^{-tA}(u_0 - u_0^\eta)||_\infty + t^{1/2}||e^{-tA}\nabla u_0^\eta||_\infty \\ &\le Ct^{1/2}t^{-1/2}||u_0 - u_0^\eta||_\infty + t^{1/2}||\nabla u_0^\eta||_\infty \\ &\le C\varepsilon + Ct^{1/2}\eta^{-1/2} \qquad \text{for} \quad 0 < \eta < \eta_0 \ , \end{aligned}$$

where we have used (2.3–4). After setting $\eta = t^{1/2}$, the right-hand side tends to 0 as $t \downarrow 0$. On the other hand we have by (2.13) and (3.2)

$$\begin{aligned} &t^{1/2}|| \int_0^t \nabla^2 \cdot e^{-(t-s)A}\mathbb{P}(u_{j-1} \otimes u_{j-1})(s)ds||_\infty \\ &\qquad \le Ct^{1/2} \int_0^t (t-s)^{-1/2}s^{-1/2}ds \ , \end{aligned}$$

which tends to 0 as $t \downarrow 0$. Hence we have proved that

$$t^{1/2}\nabla u_j(t) \to 0 \quad \text{as} \quad t \downarrow 0 \quad \text{in } BUC \quad \text{for all } j \ge 1. \tag{3.4}$$

Thus by (3.3–4) we conclude that $t^{1/2}\nabla u_j(t)$ is continuous in $t \in [0, T_1]$ for all $j \ge 1$, then we have proved that u_j and $t^{1/2}\nabla u_j$ belong to $C([0, T_1]; BUC)$ for all $j \ge 1$.

Actually, we do not need to check $u_j(t_0)$, $\nabla u_j(t_0) \in BUC$ once we have $u_j \in C([0, T_0]; L^\infty)$, $t^{1/2}\nabla u_j \in C([0, T_0]; L^\infty)$ since BUC is a closed subspace of L^∞.

Finally we set

$$\begin{aligned} L_j(T) &= \sup_{0 \le t \le T} ||u_j(t) - u_{j-1}(t)||_\infty, \\ L_j'(T) &= \sup_{0 \le t \le T} t^{1/2}||\nabla u_j(t) - \nabla u_{j-1}(t)||_\infty. \end{aligned}$$

for any $T \le T_1$ and $j = 1, 2, \cdots$. By (3.1) we have

$$\begin{aligned} u_{j+1}(t) - u_j(t) &= \int_0^t \nabla \cdot e^{-(t-s)A}(-\mathbb{P})\{(u_j \otimes u_j)(s) - (u_{j-1} \otimes u_{j-1})(s)\}ds, \\ \nabla u_{j+1}(t) - \nabla u_j(t) &= \int_0^t \nabla \cdot e^{-(t-s)A}(-\mathbb{P})\{\nabla(u_j \otimes u_j)(s) - \nabla(u_{j-1} \otimes u_{j-1})(s)\}ds. \end{aligned}$$

So, by (2.8) and (3.2) we observe that

$$\begin{aligned} L_{j+1}(T) &\le C_3T^{1/2}K_0L_j(T) \\ L_{j+1}'(T) &\le C_4T^{1/2}K_0(L_j(T) + L_j'(T)) \end{aligned}$$

for $T \le T_1$ with C_3 and C_4 independent of K_0, T. Take $T_0 \le T_1$ small so that $(C_3 + C_4)T_0^{1/2}K_0 < 1/2$. Then $L_{j+1}/L_j(T) < 1/2$, $(L'_{j+1} + L_{j+1})/(L'_j + L_j)(T) < 1/2$ for any $T \le T_0$ and $j \ge 1$. So, $\{L_j(T)\}_{j\ge1}$ and $\{L'_j(T)\}_{j\ge1}$ converge to 0 as $j \to \infty$. Thus, we conclude that the approximation $\{u_j(t)\}_{j\ge1}$ and $\{t^{1/2}\nabla u_j(t)\}_{j\ge1}$ respectively has a unique limit function $u(t), v(t) \in C([0,T_0]; BUC)$. It is easy to see $v = t^{1/2}\nabla u$, and $u(t)$ satisfies (INT) for $0 \le t \le T_0$. Since $t^{1/2}||\nabla u_j||(t) \to 0$ as $t \to 0$ and since $t^{1/2}\nabla u$ is given a uniform limit of a function $t^{1/2}\nabla u_j$ on $[0, T_0]$ (with values in BUC), (1.3) follows. The uniqueness of solution can be proved by estimating the difference $\sup_{0\le t\le T_0} ||u - v||(t)$, of two solutions u and v of (INT). We have thus proved (i) and (1.2),(1.3) of Theorem 1. By the choice of T_0 Remark (i) right after Theorem 1 is also proved.

4. Regularity of the solution to (INT).

In this section we shall prove Theorem 1 (ii) (1.4–6). The proof is standard except that we frequently use (2.11), (2.13–14).

For $t > 0$ we show the Hölder continuity of the solution u. For arbitrary $\delta > 0$ and $\delta \le t < s \le T_0$, it follows from (2.10–11) that

$$\begin{aligned}
||u(s) - u(t)||_\infty \le & ||e^{-sA}u_0 - e^{-tA}u_0||_\infty \\
& + \int_0^t ||\nabla \cdot e^{-(s-\tau)A}\mathbb{P}(u\otimes u)(\tau) - \nabla \cdot e^{-(t-\tau)A}\mathbb{P}(u\otimes u)(\tau)||_\infty d\tau \\
\le & C_\alpha (s-t)^\alpha \delta^{-\alpha} ||u_0||_\infty \\
& + C_\alpha (s-t)^\alpha \int_0^t (t-\tau)^{-\alpha-(1/2)} d\tau \\
\le & C_\alpha (s-t)^\alpha
\end{aligned} \tag{4.1}$$

with $0 < \alpha < 1/2$. This proves (1.4).

We next prove (1.5). For arbitrary $\delta > 0$ and $\delta \le t < s \le T_0$, we estimate

$$\begin{aligned}
||\nabla u(s) - \nabla u(t)||_\infty \le & ||\nabla e^{-sA}u_0 - \nabla e^{-tA}u_0||_\infty \\
& + ||\int_t^s \nabla^2 \cdot e^{-(s-\tau)A}\mathbb{P}(u\otimes u)(\tau) d\tau||_\infty \\
& + ||\int_0^t \{\nabla^2 \cdot e^{-(s-\tau)A}\mathbb{P}(u\otimes u)(\tau) - \nabla^2 \cdot e^{-(t-\tau)A}\mathbb{P}(u\otimes u)(\tau)\} d\tau||_\infty \\
= & f_1(s,t) + f_2(s,t) + f_3(s,t).
\end{aligned} \tag{4.2}$$

By (2.11) we get for any α with $0 < \alpha < 1$

$$\begin{aligned}
f_1(s,t) & \le C_\alpha (s-t)^\alpha \delta^{-\alpha-(1/2)} ||u_0||_\infty \\
& \le C_{\alpha,\delta}(s-t)^\alpha.
\end{aligned} \tag{4.3}$$

On $f_2(s,t)$ it follows from (2.13), (3.2) and mean value theorem that

$$\begin{aligned}f_2(s,t) &\le C\int_t^s (s-\tau)^{-1/2}\tau^{-1/2}d\tau \\ &\le C\delta^{-1/2}\int_t^s (s-\tau)^{-1/2}d\tau \\ &\le C\delta^{-1/2}(s-t)\{s-(t+\theta(s-t))\}^{-1/2} \\ &\le C_\delta(s-t)^{1/2} \end{aligned}\tag{4.4}$$

for some $0<\theta<1$. On $f_3(s,t)$ we obtain by (2.1–2),(2.14) and (3.2)

$$\begin{aligned}f_3(s,t) &\le C_\alpha(s-t)^\alpha\int_0^t (t-\tau)^{-\alpha-(1/2)}\tau^{-1/2}d\tau \\ &\le C_\alpha\delta^{-\alpha}B(\frac{1}{2}-\alpha,\frac{1}{2})(s-t)^\alpha \le C_{\alpha,\delta}(s-t)^\alpha \quad \text{for } 0<\alpha<1/2\ ,\end{aligned}\tag{4.5}$$

here $B(x,y)$ is the Euler beta function for $x,y>0$. Combining (4.1–5) with $0<\alpha<1/2$, we obtain (1.5).

It remains to prove (1.6). For arbitrary $\rho>0$, define

$$F_\rho(t)=-\int_0^{t-\rho}\nabla\cdot e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)ds \qquad \text{for} \quad \rho\le t\le T_0$$

with $F_\rho(t)=0$ for $0\le t\le\rho$, and

$$F(t)=-\int_0^{t}\nabla\cdot e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)ds \qquad \text{for} \quad 0\le t\le T_0\ .$$

Then by (2.8) F_ρ convergences F in BUC uniformly in $[0,T_0]$ as $\rho\downarrow 0$. Moreover F_ρ is continuous in BUC for $0\le t\le T_0$ since we obtain for $0\le t\le t+h\le T_0$ and any α with $0<\alpha<1$ from (2.8) and (2.11) that

$$\begin{aligned}&||F_\rho(t+h)-F_\rho(t)||_\infty \\ &\quad\le\int_0^{t-\rho}||\nabla\cdot e^{-(t+h-\tau)A}\mathbb{P}-\nabla\cdot e^{-(t-\tau)A}\mathbb{P}||_{\infty\to\infty}||(u\otimes u)(\tau)||_\infty d\tau \\ &\qquad+\int_{t-\rho}^{t+h-\rho}||\nabla\cdot e^{-(t+h-\tau)A}\mathbb{P}||_{\infty\to\infty}||(u\otimes u)(\tau)||_\infty d\tau \\ &\quad\le C_\alpha h^\alpha\int_0^{t-\rho}(t-\tau)^{-\alpha-(1/2)}d\tau \\ &\qquad+C\int_{t-\rho}^{t+h-\rho}(t+h-\tau)^{-1/2}d\tau,\end{aligned}$$

and for $0 \leq t-h \leq t \leq T_0$

$$
\begin{aligned}
&||F_\rho(t) - F_\rho(t-h)||_\infty \\
&\qquad \leq \int_0^{t-h-\rho} ||\nabla \cdot e^{-(t-\tau)A}\mathbb{P} - \nabla \cdot e^{-(t-h-\tau)A}\mathbb{P}||_{\infty\to\infty}||(u\otimes u)(\tau)||_\infty d\tau \\
&\qquad\quad + \int_{t-h-\rho}^{t-\rho} ||\nabla \cdot e^{-(t-\tau)A}\mathbb{P}||_{\infty\to\infty}||(u\otimes u)(\tau)||_\infty d\tau \\
&\qquad \leq C_\alpha h^\alpha \int_0^{t-h-\rho} (t-h-\tau)^{-\alpha-(1/2)} d\tau \\
&\qquad\quad + C\int_{t-h-\rho}^{t-\rho} (t-\tau)^{-1/2} d\tau.
\end{aligned}
$$

These terms tend to 0 as $h \downarrow 0$ for $0 < \alpha < 1/2$. Therefore, F is continuous on $[0, T_0]$ with values in BUC. We also have $||F(t)||_\infty \to 0$ as $t \downarrow 0$ by (2.8).

If $0 \leq s < t$, then $\nabla e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)$ belongs to BUC by (1.1) and (2.8), by (2.7) and (1.2) this quantity also belongs to $D(A^\beta)$ for any β with $0 < \beta < 1$, so the Riemann sum for $F_\rho(t)$

$$
\sum_{0\leq s_i\leq t-\rho} \nabla \cdot e^{-(t-s_i)A}\mathbb{P}(u\otimes u)(s_i)\Delta s_i \quad \text{belongs to } D(A^\beta),
$$

and

$$
\begin{aligned}
\lim_{\Delta s\to 0} \sum_{0\leq s_i\leq t-\rho} &\nabla \cdot e^{-(t-s_i)A}\mathbb{P}(u\otimes u)(s_i)\Delta s_i \\
&= \int_0^{t-\rho} \nabla \cdot e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)ds
\end{aligned}
$$

in the topology of BUC, where we denote $\Delta s = \max_{i\geq 0}\Delta s_i$ and $\Delta s_i = s_{i+1} - s_i$ for $0 \leq s_i \leq s_{i+1} \leq t-\rho$. On the other hand for any β with $0 < \beta < 1$ we obtain

$$
\begin{aligned}
\lim_{\Delta s\to 0} A^\beta \sum_{0\leq s_i\leq t-\rho} &\nabla \cdot e^{-(t-s_i)A}\mathbb{P}(u\otimes u)(s_i)\Delta s_i \\
&= \int_0^{t-\rho} A^\beta\nabla \cdot e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)ds.
\end{aligned}
$$

Thus, by the closedness of A^β, $F_\rho(t)$ belongs to $D(A^\beta)$ and

$$
A^\beta F_\rho(t) = -\int_0^{t-\rho} A^\beta\nabla \cdot e^{-(t-s)A}\mathbb{P}(u\otimes u)(s)ds
$$

for $0 < \beta < 1$.

Consider arbitrary $\delta > 0$, then $A^\beta F_\rho(t) \to A^\beta F(t)$ in BUC on $[\delta, T)$ as $\rho \downarrow 0$ if $0 \le \beta < 1$. Indeed by the commutativity of ∇ and $e^{-(t-\tau)A}\mathbb{P}$ we see that

$$\begin{aligned}
&||A^\beta F_\rho(t) - A^\beta F(t)||_\infty \\
&\quad \le || \int_{t-\rho}^t A^\beta \nabla \cdot e^{-(t-\tau)A}(-\mathbb{P})\{(u \otimes u)(\tau) - (u \otimes u)(t)\} d\tau ||_\infty \\
&\qquad + || \int_{t-\rho}^t A^\beta \nabla \cdot e^{-(t-\tau)A}(-\mathbb{P})(u \otimes u)(t) d\tau ||_\infty \\
&\quad \le \int_{t-\rho}^t ||A^\beta e^{-(t-\tau)A}\mathbb{P}||_{\infty\to\infty} \, ||\nabla \cdot \{(u \otimes u)(\tau) - (u \otimes u)(t)\}||_\infty d\tau \\
&\qquad + C \int_{t-\rho}^t ||A^\beta e^{-(t-\tau)A}\mathbb{P}||_{\infty\to\infty} t^{-1/2} t^{1/2} ||\nabla u(t)||_\infty \, ||u(t)||_\infty d\tau \\
&\quad \le C_\beta \delta^{-1/2} \int_{t-\rho}^t (t-\tau)^{\alpha-\beta} d\tau \\
&\qquad + C_\beta \delta^{-1/2} \int_{t-\rho}^t (t-\tau)^{-\beta} d\tau \,.
\end{aligned}$$

Here we have used (2.7) and the Hölder continuity (1.4–5) with $0 < \alpha < 1/2$. Again by closedness of A^β, we conclude that $F(t) \in D(A^\beta)$ for any β with $0 < \beta < 1$ and (1.6) has been proved. The proof of Theorem 1 (ii) (1.4–6) is now complete.

5. Proof of Theorem 2.

Let $u \in C([0, T_0]; BUC)$ be the solution of (INT) given in Theorem 1.

First we differentiate u of (INT) with respect to time in the distributional sense to get

$$u_t(t) = \Delta e^{-tA} u_0 - \nabla \cdot \mathbb{P}(u \otimes u)(t) - \int_0^t \Delta \nabla \cdot e^{-(t-s)A}\mathbb{P}(u \otimes u)(s) ds \quad \text{for } 0 \le t \le T_0 \,, \tag{5.1}$$

that is,

$$\begin{aligned}
&i\text{-th component of } u_t(t) \\
&\quad = \Delta e^{-tA} u_0^i - \sum_{k=1}^n \nabla_{x_k} \sum_{j=1}^n (\delta_{ij} + R_i R_j)\{u^j(t) u^k(t) + \int_0^t \Delta e^{-(t-s)A} u^j(s) u^k(s) ds\} \\
&\quad = \Delta e^{-tA} u_0^i - \sum_{k=1}^n \nabla_{x_k} \sum_{j=1}^n (\delta_{ij} + R_i R_j) F_{jk}(t) \quad \text{for } 1 \le i \le n \,.
\end{aligned}$$

We shall prove $F_{jk}(t) \in C((0, T_0]; BUC)$ in the same manner as §3. To prove that $F_{jk}(t_0) \in BUC$ for any t_0 with $0 < t_0 \le T_0$, it is sufficient to show that

$F_{jk}(t_0)$ satisfies the equivalent condition given in Lemma 5. For fixed $t_0 \in (0, T_0]$ and arbitrary $t > 0$, it follows from (2.12) and a bound of u, $t^{1/2}\nabla u$ that

$$\begin{aligned}
&||e^{-tA}\int_0^{t_0} \Delta e^{-(t_0-s)A}u^j(s)u^k(s)ds - \int_0^{t_0} \Delta e^{-(t_0-s)A}u^j(s)u^k(s)ds|| \\
&\quad \leq \int_0^{t_0} ||\nabla e^{-(t_0+t-s)A} - \nabla e^{-(t_0-s)A}||_{\infty\to\infty}||\nabla(u^j(s)u^k(s))||_\infty ds \\
&\quad \leq C_\alpha t^\alpha \int_0^{t_0} (t_0-s)^{-\alpha-(1/2)}s^{-1/2}ds,
\end{aligned}$$

which tends to 0 as $t \downarrow 0$ choosing $0 < \alpha < 1/2$. The continuity in time of $F_{jk}(t)$ is guaranteed since for $0 < t_0 \leq t_0 + h \leq T_0$ we get from (2.4),(2.12), and a bound of u, $t^{1/2}\nabla u$ that

$$\begin{aligned}
&||\int_0^{t_0+h} \Delta e^{-(t_0+h-s)A}u^j(s)u^i(s)ds - \int_0^{t_0} \Delta e^{-(t_0-s)A}u^j(s)u^i(s)ds|| \\
&\quad \leq C\int_0^{t_0} ||\nabla e^{-(t_0+h-s)A} - \nabla e^{-(t_0-s)A}||_{\infty\to\infty}||\nabla u(s)||_\infty||u(s)||_\infty ds \\
&\quad +C\int_{t_0}^{t_0+h} ||\nabla e^{-(t_0+h-s)A}||_{\infty\to\infty}||\nabla u(s)||_\infty||u(s)||_\infty ds \\
&\quad \leq C_\alpha h^\alpha \int_0^{t_0} (t_0-s)^{-\alpha-(1/2)}s^{-1/2}ds \\
&\quad +C\int_{t_0}^{t_0+h} (t_0+h-s)^{-1/2}ds \to 0 \text{ as } h \downarrow 0,
\end{aligned}$$

and for $0 < t_0 - h \leq t_0 \leq T_0$

$$\begin{aligned}
&||\int_0^{t_0} \Delta e^{-(t_0-s)A}u^j(s)u^i(s)ds - \int_0^{t_0-h} \Delta e^{-(t_0-h-s)A}u^j(s)u^i(s)ds|| \\
&\quad \leq C\int_0^{t_0-h} ||\nabla e^{-(t_0-s)A} - \nabla e^{-(t_0-h-s)A}||_{\infty\to\infty}||\nabla u(s)||_\infty||u(s)||_\infty ds \\
&\quad +C\int_{t_0-h}^{t_0} ||\nabla e^{-(t_0-s)A}||_{\infty\to\infty}||\nabla u(s)||_\infty||u(s)||_\infty ds \\
&\quad \leq C_\alpha h^\alpha \int_0^{t_0-h} (t_0-h-s)^{-\alpha-(1/2)}s^{-1/2}ds \\
&\quad +C\int_{t_0-h}^{t_0} (t_0-s)^{-1/2}ds \ \to 0 \text{ as } h \downarrow 0,
\end{aligned}$$

Here we chose α as $0 < \alpha < 1/2$. Thus, $F_{jk} \in C((0, T_0]; BUC)$, hence we deduce (see [27;Lemma2.1])

$$R_iR_jF_{jk} \in C((0, T_0]; BMO) \quad \text{for } 1 \leq i, j, k \leq n \ .$$

Hence if we define $q(t)$ by

$$(i,k) \text{ component of } q(t) = \sum_{j=1}^{n} R_i R_j F_{jk}(t) \quad \text{for } 1 \le i,k \le n\ ,$$

then we get

$$\begin{aligned} q &\in C((0,T_0];BMO), \\ u_t + \nabla \cdot q &\in C((0,T_0];BUC). \end{aligned}$$

We next calculate Δu of (INT) in the distributional sense to get

$$\Delta u = \Delta e^{-tA} u_0 - \int_0^t \Delta \nabla \cdot e^{-(t-s)A} \mathbb{P}(u \otimes u)(s) ds \quad \text{for } 0 \le t \le T_0\ . \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$u_t(t) - \Delta u(t) + \nabla \cdot \mathbb{P}(u \otimes u)(t) = 0 \quad \text{for } 0 \le t \le T_0$$

in the distributional sense. Since div $e^{-tA}u_0 = 0$ by div $u_0 = 0$ and since div $(\nabla e^{-tA}\mathbb{P}f) = 0$ we see div$u = 0$. Thus, we obtain

$$u_t(t) - \Delta u(t) + (u,\nabla)u + \nabla\Big(\sum_{i,j=1}^{n} R_i R_j u^i(t) u^j(t)\Big) = 0 \quad \text{for } 0 < t < T_0\ .$$

We thus proved that (u,p) solves the Navier-Stokes equation (NS) in the distribution sense. Since $\nabla^m u \in C((0,T_0);BUC)$, $\nabla^m \mathbb{P}(u \otimes u) \in C((0,T_0);BMO)$ for all $m \ge 0$, the smoothness of (u,p) follows from (NS) and Appendix A.3. The proof of Theorem 2 is now complete.

6. Proof of Theorem 3.

Define $p = \sum_{i,j=1}^{n} R_i R_j u^i u^j$, then (NS) is transformed into the form

$$\begin{cases} u_t - \Delta u + \nabla \cdot \mathbb{P}(u \otimes u) = 0 & \text{for } x \in \mathbf{R}^n,\ t > 0, \\ \text{div } u = 0 & \text{for } x \in \mathbf{R}^n,\ t > 0, \\ u|_{t=0} = u_0 & \text{for } x \in \mathbf{R}^n. \end{cases} \tag{6.1}$$

Let u be a solution of (6.1) in the distributional sense satisfying (1.9–1.10) for some

$T > 0$. Then we have for any $f \in S$

$$
\begin{aligned}
&\int_\varepsilon^t \langle\, f, \nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u \otimes u)(s)\ \rangle ds \\
&\qquad = \int_\varepsilon^t \langle\, f, e^{-(t-s)A}(u_t - \Delta u)(s)\ \rangle ds \\
&\qquad = \int_\varepsilon^t \langle\, f, \frac{d}{dt}(e^{-(t-s)A}u) + \Delta e^{-(t-s)A}u\ \rangle ds \\
&\qquad = \langle\, f, u(t) - e^{-(t-\varepsilon)A}u(\varepsilon)\ \rangle \\
&\qquad\quad + \int_\varepsilon^t \langle f, \Delta e^{-(t-s)A}u(s)\rangle ds \\
&\qquad\quad - \int_\varepsilon^t \langle f, \Delta e^{-(t-s)A}u(s)\rangle ds \quad \text{for } \varepsilon \le t \le T.
\end{aligned}
$$

Here we used $\frac{d}{dt}(e^{-tA}u) = \Delta e^{-tA}u + e^{-tA}u_t$ as an element of BUC, of course, as an element of S'. Since $u(t)$, $e^{-(t-\varepsilon)A}u(\varepsilon)$ and $\int_\varepsilon^t \nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u \otimes u)(s)ds$ belong to BUC by (1.9),(2.3) and (2.8), we deduce

$$
u(t) = e^{-(t-\varepsilon)A}u(\varepsilon) + \int_\varepsilon^t \nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u \otimes u)(s)ds \quad \text{for } \varepsilon \le t \le T \tag{6.2}
$$

as an element of BUC. It follows from (1.9–10),(2.1) and (2.8) that

$$
\|\nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u \otimes u)(s)\|_\infty \le C_n(\sup_{\varepsilon \le s \le t} \|u(s)\|_\infty)^2 (t-s)^{-1/2}.
$$

From this estimate it is allowed to send ε to 0 in (6.2) to get

$$
u(t) = e^{-tA}u_0 + \int_0^t \nabla \cdot e^{-(t-s)A}(-\mathbb{P})(u \otimes u)(s)ds \quad \text{for } 0 \le t \le T
$$

in $C([0,T]; BUC)$, which means that u is a solution of (INT). We have proved that a solution of (6.1) with (1.9–10) satisfies (INT). Conversely, Theorem 2 says that the solution u of (INT) given in Theorem 1 satisfies (6.1) with (1.9–10). On the other hand the solution u of (INT) is unique by Theorem 1. Thus the solution of (6.1) is unique. This completes the proof of Theorem 3.

Appendix 1

We show that the Laplace operator generates an analytic semigroup in $BUC(\mathbf{R}^n)$ with $n \ge 1$ for the convenience of readers although this fact is well known at least when $n = 1$ (see [9; p 23 Ex. 2.18.7],[26]).

For $f(x) \in BUC$, we define the heat operator:

$$
T(t)f = \int_{\mathbf{R}^n} E(x-y,t)f(y)dy, \quad \text{for} \quad t > 0
$$

with $T(0)f = f$ at $t = 0$, where

$$E(x,t) = (4\pi t)^{-n/2} \exp(-\frac{|x|^2}{4t}).$$

We define an operator in BUC by

$$Af = -\Delta f, \quad f \in BUC$$

with the domain

$$D(A) = \{f \in BUC \mid \nabla f, \Delta f \in BUC\},$$

where Δf is defined as a distribution. Even if f is vector valued the operator A and $T(t)$ operates componentwise we may assume that BUC is the space of bounded uniformly continuous scalar functions in $\mathbf{R}^n$ in the whole argument in Appendices.

Proposition A.1.1. *$\{T(t)\}_{t\geq 0}$ is an analytic contraction semigroup in BUC.*

Proof. It is easy to see that $T(t)f$ is a holomorphic function for Re $t > 0$ with values in BUC and that $T(t)T(s) = T(t+s)$ for $t, s > 0$. The continuity of $T(t)f$ at $t = 0$, that is, $T(t)f \to f$ as $t \downarrow 0$ in BUC has been proved in Lemma 5. By (2.4) it is clear that $\{T(t)\}_{t\geq 0}$ is a contraction semigroup. ■

We next show that the operator $-A$ generates the semigroup $T(t)$. For this purpose we consider for each $f \in BUC$

$$(I - \Delta)u = f \quad \text{in } \mathbf{R}^n. \tag{E}$$

Lemma A.1.2. Let $f \in BUC$. Then there exists a unique solution $u \in D(A)$ of (E) in the distributional sense.

Proof. We may assume that $n \geq 2$ since the case $n = 1$ is trivial. To solve

$$(I - \Delta)u = f \quad \text{in } \mathbf{R}^n$$

we set

$$u(x) = \int_{\mathbf{R}^n} G(|x-y|)f(y)dy, \tag{A.1.1}$$

where

$$G(\rho) = (2\pi)^{-\nu-1}\rho^{-\nu}K_\nu(\rho), \ \rho \geq 0, \ \nu = (n-2)/2,$$

and $K_\nu(z)$ is the modified Bessel function with the integral representation

$$K_\nu(z) = \frac{1}{2}(\frac{z}{2})^\nu \int_0^\infty \exp(-t - \frac{z^2}{4t})t^{-\nu-1}dt \tag{A.1.2}$$

for Re $\nu > -1/2$, $|\arg z| < \pi/4$ (see [17; p119,(5.10.25)]). Then u is a continuously differentiable function and $u, \nabla u \in BUC$ for $f \in BUC$ since the estimate

$$G(\rho) = \begin{cases} O(\rho^{2-n}) & \text{if} \quad n = 3, 4, \cdots \\ O(\log \frac{1}{\rho}) & \text{if} \quad n = 2 \end{cases} \qquad \text{as } \rho \to 0 \tag{A.1.3}$$

is obtained by (A.1.2) (see [17; p 136, (5.16.4)]). We prove that u satsifies (E) in the distributional sense. This is easy to see at least formally since $G(|x|)$ is the fundamental solution of $I - \Delta$. We give its proof for completeness. For a test function $\varphi \in C_0^\infty$ it follows from Fubini's theorem that

$$\begin{aligned}
\langle \nabla_x u, \ \nabla_x \varphi \rangle &= \int_{\mathbf{R}^n} \nabla_x \int_{\mathbf{R}^n} G(|x-y|) f(y) dy \cdot \nabla_x \varphi(x) \ dx \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \nabla_x G(|x-y|) f(y) dy \cdot \nabla_x \varphi(x) \ dx \\
&= \int_{\mathbf{R}^n} f(y) dy \int_{\mathbf{R}^n} \nabla_x G(|x-y|) \nabla_x \varphi(x) dx \ .
\end{aligned} \tag{A.1.4}$$

Here $\langle f, g \rangle = \int_{\mathbf{R}^n} f g dx$ and C_0^∞ denotes the space of compactly supported smooth functions in $\mathbf{R}^n$. Integrating by parts yields

$$\begin{aligned}
\int_{\Omega_\varepsilon} \nabla_x G(|x-y|) \nabla_x \varphi(x) dx &= - \int_{\Omega_\varepsilon} \Delta_x G(|x-y|) \varphi(x) dx \\
&\quad + \int_{\partial \Omega_\varepsilon} \varphi(x) \nabla_x G(|x-y|) \cdot n(x) dS \\
&= a_{1,\varepsilon}(y) + a_{2,\varepsilon}(y)
\end{aligned} \tag{A.1.5}$$

Here $\Omega_\varepsilon = \{x \in \mathbf{R}^n ; |x-y| \geq \varepsilon\}$ for $\varepsilon > 0$, and $\partial \Omega_\varepsilon$ is the boundary of Ω_ε; $n(x)$ is the normal outer vector of $\partial \Omega_\varepsilon$ and dS is the surface element of $\partial \Omega_\varepsilon$. Since $\Delta G(x) = G(x)$ by using Bessel's differential equation (see [17; p110, (5.7.7)]), we have

$$a_{1,\varepsilon}(y) = - \int_{\Omega_\varepsilon} G(|x-y|) \varphi(x) dx. \tag{A.1.6}$$

For $a_{2,\varepsilon}(y)$ we later prove

$$\lim_{\varepsilon \downarrow 0} a_{2,\varepsilon}(y) = \varphi(y). \tag{A.1.7}$$

Letting $\varepsilon \downarrow 0$ in (A.1.5) combined with (A.1.6–7), there holds

$$\int_{\mathbf{R}^n} \nabla_x G(|x-y|) \nabla_x \varphi(x) dx = - \int_{\mathbf{R}^n} G(|x-y|) \varphi(x) dx + \varphi(y).$$

Hence by (A.1.4) it follows from the Fubini's theorem that

$$\begin{aligned}
\langle \nabla_x u, \ \nabla_x \varphi \rangle &= - \int_{\mathbf{R}^n} f(y) \int_{\mathbf{R}^n} G(|x-y|) \varphi(x) dx dy + \int_{\mathbf{R}^n} f(y) \varphi(y) dy \\
&= - \int_{\mathbf{R}^n} \varphi(x) \int_{\mathbf{R}^n} G(|x-y|) f(y) dy dx + \int_{\mathbf{R}^n} f(y) \varphi(y) dy \\
&= - \langle \ u, \varphi \ \rangle + \langle \ f, \varphi \ \rangle \ ,
\end{aligned}$$

which says that u is a solution of (E) in the distributional sense. It remains to prove (A.1.7). We estimate

$$\begin{aligned} |a_{2,\varepsilon}(y)-\varphi(y)| &\le \Big|\, \varphi(y)\int_{\partial\Omega_\varepsilon} \nabla G(|x-y|)\cdot n(x)dS-\varphi(y)\,\Big| \\ &\quad + \int_{\partial\Omega_\varepsilon} |\varphi(x)-\varphi(y)|\;|\nabla G(|x-y|)\cdot n(x)|\;dS \\ &= b_{1,\varepsilon}(y)+b_{2,\varepsilon}(y)\ . \end{aligned}$$

Setting $x-y=\varepsilon\omega$ with $\omega\in\mathbf{R}^n$ and $|\omega|=1$, we have

$$\begin{aligned} b_{1,\varepsilon}(y) &= \Big|\, \varphi(y)\int_{S^{n-1}} \nabla G(|\varepsilon\omega|)\cdot n(y+\varepsilon\omega)\varepsilon^{n-1}dS-\varphi(y)\,\Big| \\ &\le ||\varphi||_\infty \,\Big|\int_{S^{n-1}} G'(\varepsilon)\cdot n(y+\varepsilon\omega)\varepsilon^{n-1}dS-1\Big| \\ &= ||\varphi||_\infty |\,|S^{n-1}|\;G'(\varepsilon)\varepsilon^{n-1}-1|, \end{aligned}$$

where $S^{n-1}=\{x\in\mathbf{R}^n;|x|=1\}$ and $|S^{n-1}|$ is the surface area of S^{n-1}. By the recurrence formula of $K_\nu(\rho)$ (see [17; p110,(5.7.9)]) we obtain

$$\begin{aligned} G'(\varepsilon)\varepsilon^{n-1} &= (2\pi)^{-\nu-1}\frac{d}{d\varepsilon}(\varepsilon^{-\nu}K_\nu(\varepsilon))\varepsilon^{n-1} \\ &= (2\pi)^{-\nu-1}(\varepsilon^{-2\nu-1}\varepsilon^{\nu+1}K_{\nu+1}(\varepsilon))\varepsilon^{n-1} \\ &= (2\pi)^{-\nu-1}\varepsilon^{-2\nu+n-2}\varepsilon^{\nu+1}K_{\nu+1}(\varepsilon). \end{aligned}$$

Hence it follows from (A.1.2) that

$$\begin{aligned} |S^{n-1}|\;G'(\varepsilon)\;\varepsilon^{n-1} &= ((2\pi)^{-\nu-1}/\Gamma(\nu+1))(2\pi)^{-\nu-1}\;\varepsilon^{-2\nu+n-2} \\ &\quad\times 2^\nu\int_0^\infty s^\nu\exp(-s-\frac{\rho^2}{4s})ds \\ &\to 1 \quad \text{as} \quad \varepsilon\downarrow 0\ . \end{aligned}$$

Similarly by (A.1.3) we estimate

$$\begin{aligned} b_{2,\varepsilon}(y) &\le ||\varphi(x)-\varphi(y)||_\infty\;|G'(\varepsilon)|\;\varepsilon^{n-1}\;|S^{n-1}| \\ &\le ||\varphi(y+\varepsilon\omega)-\varphi(y)||_\infty\;C\;|S^{n-1}| \\ &\to 0 \quad \text{as} \quad \varepsilon\downarrow 0\ , \end{aligned}$$

with some constant C independent of ε. This proves (A.1.7).

Note that the solution u of (E) given by (A.1.1) is in $D(A)$ since the equation (E) implies $\Delta u\in BUC$. It remains to prove the uniqueness of solution of (E) in $D(A)$. Suppose that $u=\Delta u$ for $u\in D(A)$. We set $u=v\eta$ with $\eta(x)=|x|^2+2n+1$ to get

$$(\eta-\Delta\eta)v=\eta\Delta v+2\sum_{i=1}^n \frac{\partial v}{\partial x_i}\frac{\partial \eta}{\partial x_i} \quad \text{in } \mathbf{R}^n$$

from $u=\Delta u$. If $u\not\equiv 0$, then v has either local positive maximum or negative minimum since $u\in BUC(\mathbf{R}^n)$. At these points we get $\eta\le\Delta\eta$ which contradicts the definition of η. Thus we get $u\equiv 0$. ■

Proposition A.1.3. *$-A$ is a generator of the analytic semigroup $\{T(t)\}_{t\geq 0}$.*

Proof. Let S be the generator of $\{T(t)\}_{t\geq 0}$, that is,

$$Sf = \lim_{t\downarrow 0} \frac{1}{t}\{T(t)f - f\}$$

for $f \in D(S)$ with

$$D(S) = \{f \in BUC \mid \lim_{t\downarrow 0} \frac{1}{t}\{T(t)f - f\} \quad \text{exists}\}.$$

Here the limit is taken in the strong topology of BUC. We have to show $S = -A$.

First we prove easier inclusion $D(A) \subset D(S)$. Suppose that $f \in D(A)$. For $0 < \varepsilon < t$ by Fubini's theorem and the integration by parts we see that

$$\begin{aligned} T(t)f - T(\varepsilon)f &= \int_{\mathbf{R}^n} E(x-y,t) - E(x-y,\varepsilon)f(y)dy \\ &= \int_{\mathbf{R}^n} f(y)dy \int_\varepsilon^t \frac{\partial}{\partial s}E(x-y,s)ds \\ &= \int_\varepsilon^t ds \int_{\mathbf{R}^n} \Delta E(x-y,s)f(y)dy \\ &= \int_\varepsilon^t ds \int_{\mathbf{R}^n} E(x-y,s)\Delta f(y)dy \ . \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we obtain from Lemma 5

$$T(t)f - f = \int_0^t T(s)\Delta f ds \ .$$

Applying Lemma 5 again we now get

$$\lim_{t\downarrow 0} \frac{1}{t}\{T(t)f - f\} = \Delta f$$

since $\Delta f \in BUC$. This means $f \in D(S)$ and $-Af = Sf$. We next prove the other inclusion $D(S) \subset D(A)$. Suppose that $f \in D(S)$. Since $(I-S)f \in BUC$, there is a unique solution $w \in D(A)$ of the following equation

$$(I + A)w = (I - S)f \tag{A.1.8}$$

by Lemma A.1.2. Because $w \in D(A) \subset D(S)$ and $-Aw = Sw$ we have

$$(I - S)w = (I - S)f \ .$$

Now $\{\lambda \in \mathbb{C} | \ \mathrm{Re}\ \lambda > 0\}$ is in the resolvent set of the generator S by the theory on contraction semigroups (see [23; p11 Corollary 3.6], [26;p282 Th. 12–4]). Hence 1 is in the resolvent set of S, then w must agree with f. This implies that $f \in D(A)$ since $w \in D(A)$. By (A.1.8) we also observe $-Af = Sf$. This completes the proof. ■

Appendix 2

We prove Lemma 3 (i), that is,

Proposition A.2.1. *For $n \geq 2$ let A be the operator defined in Appendix 1. Then for any α with $0 < \alpha < 1$, there exists positive constant C_α such that*

$$||A^\alpha E(t)||_{\mathcal{H}^1(\mathbf{R}^n)} \leq C_\alpha t^{-\alpha} \quad \textit{for} \quad t > 0 \tag{A.2.1}$$

where $\mathcal{H}^1 = \mathcal{H}^1(\mathbf{R}^n)$ is the Hardy space.

Remark. (i) In this section we use the norm of $\mathcal{H}^1(\mathbf{R}^n)$ of form

$$|| \sup_{s>0} |E(x,s) * f(x)| \, ||_{L^1(\mathbf{R}^n)},$$

which is equivalent to (2.6) (see [5]).
(ii) For $0 < \alpha < 1$, the integral kernel of the operator $A^{\alpha-1}$ is $K(n,\alpha)|x|^{-n+2(1-\alpha)}$ (see [24;p117]), then we have

$$\begin{aligned} A^\alpha E(t) &= AA^{\alpha-1}E(t) \\ &= -\Delta_x \int_{\mathbf{R}^n} \frac{K(n,\alpha)}{|x-y|^{n-2(1-\alpha)}} E(y,t)dy \end{aligned}$$

where $1/K(n,\alpha) = \pi^{n/2}2^{2(1-\alpha)}\Gamma(1-\alpha)/\Gamma(\frac{n}{2}+\alpha-1)$. This form is valid even for $n = 2$ and for $n = 1$ with $1/2 < \alpha < 1$.

For the proof of Proposition A.2.1 we need pointwise estimates on the heat kernel.

Lemma A.2.2 Let A be the operator defined in Apendix 1. Then for any α with $0 < \alpha < 1$ and any ℓ with $0 < \ell < n + 2\alpha$, there exists positive constants $C_{n,\alpha}$ and $C_{\ell,n,\alpha}$ such that

$$|A^\alpha E(x,t)| \leq \begin{cases} C_{n,\alpha} t^{-(n/2)-\alpha} & \text{for} \quad |x| \leq 1 \\ C_{\ell,n,\alpha}|x|^{-\ell} t^{(\ell-n-2\alpha)/2} & \text{for} \quad |x| > 1 \, . \end{cases} \tag{A.2.2}$$

Proof. Here we argue in the same way as the proof of Lemma 2.3 in [7]. Setting $x = t^{1/2}z$, it is easy to see that

$$A^\alpha E(x,t) = t^{-(n/2)-\alpha} A^\alpha E(z,1). \tag{A.2.3}$$

So, to prove (A.2.2) it is sufficient to show for $t = 1$, that is,

$$|A^\alpha E(x,1)| \leq \begin{cases} C_{n,\alpha} & \text{for} \quad |x| \leq 1 \\ C_{\ell,n,\alpha}|x|^{-\ell} & \text{for} \quad |x| > 1 \end{cases} \tag{A.2.4}$$

with some $C_{n,\alpha} > 0$ and $C_{\ell,n,\alpha} > 0$ which are independent of t. In fact if (A.2.4) is valid, we have

$$\begin{aligned} |A^\alpha E(x,t)| &\leq C_{\ell,n,\alpha} \, t^{-(n/2)-\alpha} \, |z|^{-\ell} \\ &= C_{\ell,n,\alpha} \, t^{-(n/2)-\alpha} \, t^{\ell/2} \, |x|^{-\ell} \end{aligned}$$

for $|x| > 1$ and

$$|A^\alpha E(x,t)| \le C_{n,\alpha} t^{-(n/2)-\alpha}$$

for $|x| \le 1$.

Let ψ_1 be a smooth function in $\mathbf{R}^n$ such that supp $\psi_1 \subset \{x \in \mathbf{R}^n; |x| \le 1\}$, $0 \le \psi_1 \le 1$ and $\psi_1(x) = 1$ for $|x| < 1/2$, and let $\psi_2 \equiv 1 - \psi_1$. Then we have

$$\begin{aligned} A^\alpha E(x,1) =& - K(n,\alpha)\Delta_x \int_{\mathbf{R}^n} \frac{\psi_1(x-y)}{|x-y|^{n-2(1-\alpha)}} E(y,1)dy \\ & - K(n,\alpha)\Delta_x \int_{\mathbf{R}^n} \frac{\psi_2(x-y)}{|x-y|^{n-2(1-\alpha)}} E(y,1)dy \\ =& K(n,\alpha)(I_1(x) + I_2(x)) \ . \end{aligned}$$

For the term I_1, we have

$$\begin{aligned} I_1(x) =& - \Delta_x \int_{|y|\le 1} \frac{\psi_1(y)}{|y|^{n-2(1-\alpha)}} E(x-y,1)dy \\ =& - \frac{1}{(4\pi)^{n/2}} \int_{|y|\le 1} \frac{\psi_1(y)}{|y|^{n-2(1-\alpha)}} (\frac{|x-y|^2}{4} - \frac{n}{2})\exp(-\frac{|x-y|^2}{4})dy \end{aligned}$$

since supp $\psi_1(x) \subset \{x \in \mathbf{R}^n; |x| \le 1\}$. We note that $|x-y| \le |x| + 1$ and $|x-y|^2 \ge (1/2)|x|^2 - 1$ for $|y| \le 1$. Then we obtain

$$\begin{aligned} |I_1(x)| \le& C_n(\frac{(|x|+1)^2}{4} + \frac{n}{2})\ e^{-|x|^2/8} \int_{|y|\le 1} \frac{1}{|y|^{n-2(1-\alpha)}} dy \\ \le& C_{n,\alpha}|x|^{-\ell} \end{aligned} \tag{A.2.5}$$

for all $x \in \mathbf{R}^n$ and any $\ell \ge 0$, and any α with $0 < \alpha < 1$.

For the estimate of I_2, differentiating the kernel $\psi_2(x)|x|^{-n+2(1-\alpha)}$ we get

$$\begin{aligned} I_2(x) =& - \int_{\mathbf{R}^n} \frac{(\Delta\psi_2)(x-y)}{|x-y|^{n-2(1-\alpha)}} E(y,1)dy \\ & - (2n - 4 + 4\alpha) \int_{\mathbf{R}^n} \sum_{i=1}^n \frac{(\partial_{x_i}\psi_2)(x-y)}{|x-y|^{n+2\alpha}} (x_i - y_i)E(y,1)dy \\ & - (4\alpha^2 + 2\alpha n - 4\alpha) \int_{\mathbf{R}^n} \frac{\psi_2(x-y)}{|x-y|^{n+2\alpha}} E(y,1)dy \\ =& - J_1(x) - (2n - 4 + 4\alpha)J_2(x) - (4\alpha^2 + 2\alpha n - 4\alpha)J_3(x). \end{aligned}$$

For the terms J_1 and J_2, since the supports of $\partial_{x_i}\psi_2$ and $\Delta\psi_2$ are included in $1/2 \le |x| \le 1$ we can estimate similarly on the term I_1 as follows.

$$\begin{aligned} |J_1(x)| \le& \int_{1/2\le|y|\le 1} \frac{|(\Delta\psi_2)(y)|}{|y|^{n-2(1-\alpha)}} |E(x-y,1)|dy \\ \le& C_n||\Delta\psi_2||_\infty e^{-|x|^2/8} \int_{1/2\le|y|\le 1} \frac{1}{|y|^{n-2(1-\alpha)}} dy \\ \le& C_{n,\alpha}|x|^{-\ell}, \end{aligned} \tag{A.2.6}$$

$$
\begin{aligned}
|J_2(x)| \leq & \int_{1/2\leq|y|\leq 1} \frac{|(\nabla\psi_2)(y)|}{|y|^{n+2\alpha}}|y|\ |E(x-y,1)|dy \\
\leq & C_n||\nabla\psi_2||_\infty\ e^{-|x|^2/8}\int_{1/2\leq|y|\leq 1}\frac{|y|}{|y|^{n+2\alpha}}dy \\
\leq & C_{n,\alpha}|x|^{-\ell}
\end{aligned} \tag{A.2.7}
$$

for all $x \in \mathbf{R}^n$ and any $\ell \geq 0$ and any α with $0 < \alpha < 1$.

For the term J_3, we have

$$
J_3(x) = \int_{|x-y|\geq 1/2} \frac{\psi_2(x-y)}{|x-y|^{n+2\alpha}}E(y,1)dy \tag{A.2.8}
$$

since the support of $\psi_2(x)$ is included in $\{x \in \mathbf{R}^n; |x| \geq 1/2\}$. If $|x| > 1$, we get

$$
\begin{aligned}
|J_3(x)| = & \frac{|x|^\ell}{|x|^\ell}\ |\int_{|x-y|\geq 1/2}\frac{\psi_2(x-y)}{|x-y|^{n+2\alpha}}E(y,1)dy\ | \\
\leq & \frac{C_\ell}{|x|^\ell}\int_{|x-y|\geq 1/2}\frac{(|x-y|^\ell+|y|^\ell)}{|x-y|^{n+2\alpha}}E(y,1)\ dy\ ,
\end{aligned}
$$

here we used $|\psi_2| \leq 1$ and $|x|^\ell \leq C_\ell(|x-y|^\ell + |y|^\ell)$ with some constant $C_\ell > 0$.

Restricting $\ell \geq 0$ to $0 \leq \ell < n + 2\alpha$,

$$
|J_3(x)| \leq \frac{C_\ell}{|x|^\ell}\int_{|x-y|\geq 1/2}(2^{n+2\alpha-\ell} + 2^{n+2\alpha}|y|^\ell)E(y,1)dy
$$

since $|x-y| \geq 1/2$. Hence we deduce

$$
|J_3(x)| \leq C_{\ell,n,\alpha}|x|^{-\ell} \tag{A.2.9}
$$

for $0 \leq \ell < n + 2\alpha$, $0 \leq \alpha < 1$ and $|x| \leq 1$. When $|x| \leq 1$, we easily have by (A.2.8)

$$
\begin{aligned}
|J_3(x)| \leq & 2^{n+2\alpha}\int_{|x-y|\geq 1/2} E(y,1)dy \\
\leq & C_{n,\alpha}\ .
\end{aligned} \tag{A.2.10}
$$

Combining estimates (A.2.5–7) and (A.2.9–10), we obtain (A.2.4) and Lemma A.2.2 has been proved. ■

Proof of Proposition A.2.1. Since $E(s) * E(1) = E(1+s)$ for $s > 0$, we have

$$
\begin{aligned}
&||\ \sup_{s>0}|E(x,s) * A^\alpha E(x,1)|\ ||_{L^1(\mathbf{R}^n)} \\
&\qquad =||\ \sup_{s>0}|A^\alpha E(x,1+s)|\ ||_{L^1(\mathbf{R}^n)}.
\end{aligned} \tag{A.2.11}
$$

Since $0 < \ell < n + 2\alpha$ and $0 < \alpha < 1$, Lemma A.2.2 implies

$$\begin{aligned} |A^\alpha E(x, 1+s)| \leq & C_{\ell,n,\alpha}|x|^{-\ell}(1+s)^{(\ell-n-2\alpha)/2} \\ \leq & C_{\ell,n,\alpha}|x|^{-\ell} \end{aligned} \tag{A.2.12}$$

for $|x| > 1$ and $s > 0$ where $C_{\ell,n,\alpha} > 0$ is independent of s and x. For $|x| \leq 1$ and $s > 0$, similarly we get

$$\begin{aligned} |A^\alpha E(x, 1+s)| \leq & C_{n,\alpha}(1+s)^{(-n-2\alpha)/2} \\ \leq & C_{n,\alpha} \end{aligned} \tag{A.2.13}$$

where $C_{n,\alpha} > 0$ is independent of s and x. Hence we deduce from (A.2.11–13) that

$$|| \sup_{s>0} |E(x,s) * A^\alpha E(x,1)| \, ||_{L^1(\mathbf{R}^n)} \leq C_{\ell,n,\alpha} \, .$$

Thus we conclude

$$||A^\alpha E(x,1)||_{\mathcal{H}^1(\mathbf{R}^n)} \leq C_{\ell,n,\alpha} \, .$$

So, setting $x = t^{1/2}z$, again by (A.2.3) we get

$$\begin{aligned} ||A^\alpha E(x,t)||_{\mathcal{H}^1(\mathbf{R}^n)} &= t^{-(n/2)-\alpha}||A^\alpha E(z,1)||_{\mathcal{H}^1(\mathbf{R}^n)} \\ &= t^{-\alpha}||A^\alpha E(x,1)||_{\mathcal{H}^1(\mathbf{R}^n)} \\ &\leq C_{\ell,n,\alpha}t^{-\alpha}. \end{aligned}$$

■

Appendix 3

In this Appendix we prove that

$$\text{If } \nabla^m f \in BMO \text{ for all } m = 0, 1, 2, \cdots \quad \text{then } \nabla f \in C^\infty$$

Theorem A.3.1. *(i)Assume that $|x|f \in L^1(\mathbf{R}^n)$, $|x|^{n+1}f \in L^\infty(\mathbf{R}^n)$ and $f \in L^\infty(\mathbf{R}^n)$. Then $f \in \mathcal{H}^1$ if $\int_{\mathbf{R}^n} f dx = 0$.*
(ii)There exist a constant $C > 0$ such that

$$||f||_{\mathcal{H}^1} \leq C(|| \, |x|f||_\infty + ||f||_\infty)$$

for all $f \in L^\infty$ satisfying $|x|f \in L^1$, $|x|^{n+1}f \in L^\infty$ and $\int_{\mathbf{R}^n} f dx = 0$.

Corollary A.3.2 (i)If $f \in L^\infty$ satisfies $\mathrm{spt} f \subset B_R = \{x \in \mathbf{R}^n; |x| < R\}$, then $f \in \mathcal{H}^1$ if $\int f dx = 0$.

(ii)There exist a constant $C > 0$ such that

$$||f||_{\mathcal{H}^1} \le C(R^{n+1} + 1)||f||_\infty$$

for all $f \in L^\infty(\mathbf{R}^n)$ satisfying spt$f \subset B_R$ and $\int f dx = 0$.

Remark. Corollary A.3.2 seems to be well-known. For example, (i) is found in [25, III, §5.5]. We give here a complete proof of Theorem A.3.1 (and Corollary A.3.2) for reader's convenience.

Corollary A.3.3 (i)If $f \in W^{1,p}(\mathbf{R}^n)$ such that $\int f dx = 0$, spt$f \subset B_R$ with $p > n$, then $f \in \mathcal{H}^1$.

(ii)There exist a constant $C = C(p, n, R) > 0$ such that

$$||f||_{\mathcal{H}^1} \le C||f||_{W_p^1(\mathbf{R}^n)}$$

for $\int f dx = 0$, spt$f \subset B_R$, $f \in W^{1,p}(\mathbf{R}^n)$.

Proof of Theorem A.3.1. We first show that:
(A) There exist a constant $C_1 > 0$ (independent of t) such that

$$||\ |x|^{n+1} e^{t\Delta} f||_\infty \le C_1(||\ |x|f||_1 + ||\ |x|^{n+1} f||_\infty)$$

for all $t > 0$, $f \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ satisfying $\int f dx = 0$, where $||\cdot||_1$ denotes the L^1 norm.

In fact, since $\int f dx = 0$ we see that

$$\begin{aligned} |x|^{n+1} \int G_t(x-y) f(y) dy &= \int |x|^{n+1}(G_t(x-y) - G_t(x)) f(y) dy \\ &= \int (|x-y|^{n+1} G_t(x-y) - |x|^{n+1} G_t(x)) f(y) dy \\ &\quad + \int (|x|^{n+1} - |x-y|^{n+1}) G_t(x-y) f(y) dy. \end{aligned}$$

Thus we get

$$\begin{aligned} ||\ |x|^{n+1} e^{t\Delta} f||_\infty &\le \sup_x \int |\ |x-y|^{n+1} G_t(x-y) - |x|^{n+1} G_t(x)\ ||f| dy \\ &+ \sup_x \int |y|^{n+1} |f(y)| G_t(x-y) dy \\ &= I + II. \end{aligned}$$

Evidently,

$$II \le ||\ |x|^{n+1} f||_\infty$$

To estimate I we use

$$H_t(x-y) - H_t(x) = -\int_0^1 (\nabla H_t)(x-\theta y) \cdot y d\theta$$

$$\text{with } H_t(x) = |x|^{n+1} G_t(x)$$

to get

$$\begin{aligned}\int | \, |x-y|^{n+1} G_t(x-y) - |x|^{n+1} G_t(x) \, || f(y)| dy \\ \leq \int_0^1 \int_{B_R} |\nabla H_t(x-\theta y)||yf(y)| dy d\theta.\end{aligned}$$

Note that both $|| \, |x|^n G_t||_\infty$ and $|| \, |x|^{n+1} \nabla G_t(x)||_\infty$ is bound as a function of $t > 0$, so,

$$\begin{aligned}||\nabla H_t(x)||_\infty &\leq (n+1) || \, |x|^n G_t(x)||_\infty + || \, |x|^{n+1} \nabla G_t(x)||_\infty \\ &\leq C_0\end{aligned}$$

with $C_0 > 0$ independent of $t > 0$. We thus obtain

$$\begin{aligned}\int_{B_R} | \, |x-y|^{n+1} G_t(x-y) - |x|^{n+1} \, | \, |yf(y)| dy \\ \leq \int_0^1 C_0 \int_{B_R} |yf(y)| dy d\theta \\ \leq C_0 || \, |x| f||_1.\end{aligned}$$

This yields $I \leq || \, |x| f||_1$ and completes the proof of (A).

We next estimate $\mathcal{H}^1$ norm of f as follows

$$\begin{aligned}||f||_{\mathcal{H}^1} &= || \sup_{t>0} |e^{t\Delta} f(x)| \, ||_1 \\ &\leq \int_{|x|\geq 1} \frac{1}{|x|^{n+1}} |x|^{n+1} \sup_{t>0} |e^{t\Delta} f(x)| dx + \int_{|x|\leq 1} \sup_{t>0} |e^{t\Delta} f(x)| dx \\ &\leq C' \sup_{|x|\geq 1} |x|^{n+1} \sup_{t>0} |e^{t\Delta} f(x)| + C'' \sup_{t>0} ||e^{t\Delta} f||_\infty \\ &\leq C' \sup_{t>0} || \, |x|^{n+1} (e^{t\Delta} f)(x) \, ||_\infty + C'' \sup_{t>0} ||e^{t\Delta} f||_\infty \\ &\leq C_2 \{ || \, |x| f||_1 + || \, |x|^{n+1} f||_\infty + ||f||_\infty \}\end{aligned}$$

by (A) and $||e^{t\Delta} f||_\infty \leq ||f||_\infty$. The proof is now complete. ■

Corollary A.3.2 immediately follows from Theorem A.3.1 since

$$|| \, |x|^{n+1} f||_\infty \leq R^{n+1} ||f||_\infty \quad \text{and } || \, |x| \, f||_1 \leq R \, ||f||_1.$$

Corollary A.3.2 implies Corollary A.3.3 by L^p-Sobolev inequality.

Remark. If $f \in BMO$ and $\nabla f \in BMO$, then ∇f is well-defined as a distribution. (For $\varphi \in C_0^\infty$ we shall define $\langle \varphi, \nabla f \rangle$.)

Let B_R be taken so that $\mathrm{spt}\varphi \subset B_R$, and let $\pi \in C_0^\infty$ satisfy $\pi \geq 0$, $\int \pi dx = 1$, and $\mathrm{spt}\pi \subset B_R$. We define

$$\langle \varphi, \nabla f \rangle = \langle \varphi - \alpha\pi, \nabla f \rangle + \alpha \langle -\nabla \pi, f \rangle \tag{A.3.1}$$
$$\text{with } \alpha = \int \varphi dx$$

where brakets in the right hand side is the duality pairing of $\mathcal{H}^1$ and BMO [5]. Note that $\int(\varphi - \alpha\pi)dx = 0$ so that $\varphi - \alpha\pi \in \mathcal{H}^1$ by Theorem A.3.1, so the first term is well-defined since $\nabla f \in BMO$. Note also that $\nabla \pi \in \mathcal{H}^1$ so the second term is also well-defined since $f \in BMO$.

Theorem A.3.4. *If $f \in BMO$ and $\nabla f \in BMO$, then ∇f is a Radon measure.*

Corollary A.3.5 If $\nabla^m f \in BMO$ for all $m \geq 0$, then $\nabla f \in C^\infty$.

Proof of Theorem A.3.4. It suffices to show

$$\sup_{\varphi \in C_0^\infty(B_R), ||\varphi||_\infty \leq 1} |\langle \varphi, \nabla f \rangle| < \infty \tag{A.3.2}$$

for each B_R. We estimate the right-hand of (A.3.1). The second term is estimated by

$$|\int \varphi dx \langle -\nabla\pi, f \rangle| \leq C||\nabla\pi||_{\mathcal{H}^1} ||f||_{BMO} ||\varphi||_\infty R^n \leq C||\nabla\pi||_{\mathcal{H}^1} ||f||_{BMO} R^n$$

for $||\varphi||_\infty \leq 1$. Thus we may assume $\int \varphi dx = 0$ in the proof of (A.3.2). By Corollary A.3.2 we know

$$||\varphi||_{\mathcal{H}^1} \leq C||\varphi||_{L^\infty(B_R)}$$

for $\varphi \in L^\infty(B_R), \int \varphi dx = 0$, so that

$$\sup_{\varphi \in C_0^\infty(B_R), ||\varphi||_\infty \leq 1, \int \varphi dx = 0} |\langle \varphi, \nabla f \rangle| \leq \sup_{||\varphi||_{\mathcal{H}^1} \leq 1/2} |\langle \varphi, \nabla f \rangle| \leq C||\nabla f||_{BMO}.$$

The proof is now complete.

Proof of Corollary A.3.5. By Theorem A.3.4 $\nabla^m f$ for $m \geq 1$ is a Radon measure. Applying Sobolev inequalities for mollified function $\nabla f * \rho_\varepsilon$, we get local uniform convergence of all derivatives of $\nabla f * \rho_\varepsilon$ as $\varepsilon \to 0$. Since $\nabla f * \rho_\varepsilon$ is smooth for $\varepsilon > 0$, so is ∇f.

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