Categorical aspects of generating functions (I) : Exponential formulas and Krull-Schmidt categories

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Categorical aspects of generating functions (I): Exponential formulas and Krull-Schmidt categories

TO THE MEMORY OF PROF. MICHIO SUZUKI

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Contents. 1. Introduction. 2. $\mathcal{E}$-structures. 3. Species. 4. Formal exponential functions. 5. Exponential formulas. 6. Examples.

Abstract. In this paper, we study formal power series with exponents in a category. For example, the generating function of a category $\mathcal{E}$ with finite Hom-sets is defined by $\mathcal{E}(t) = \sum t^X/|\mathrm{Aut}(X)|$, where the summation is taken over all isomorphism classes of obejects of $\mathcal{E}$. We can use such power series to enumerate the number of $\mathcal{E}$-structures along a faithful functors(Theorem 4.6). Our theory is closely related to the thory of species (Joyal 1981). A species can be identified with a faithful functor from a groupoid to the category of finite sets (Theorem 3.6). We use mainly the concept of faithful functors with finite fibers instead of that of species, by which we can separate the roles which categories play and which functors play. For example, the exponential formula $\mathcal{E}(t) = \exp(\mathsf{Con}(\mathcal{E})(t))$ means the unique coproduct decomposition property(Theorem 5.8). In the final section, we give some applications of our theory to rather classical enumerations.

1 Introduction

(1.1) Plan. This series of papers aims to develop the theory of generating functions (of polynomial and power series type) from the categorical view point and is planed to study the following objects:

(A) $\mathcal{E}$-structures and species.
(B) Exponential formulas and Krull-Schmidt categories.
(C) Operations on categories and functors.
(D) Polynomials and power series with exponents in a locally finite toposes.
(E) Polynomials and power series with coefficients in a Mackey functor with multiplicative induction.

In the present paper we study (A) and (B); in the next paper we will study (C), and in the third paper we will study (D) and (E). The purpose of (D) and (E), which is also the final purpose of this paper, is to build the "external "theory of polynomials and power series. Here, for example, the external integer ring with respect to a category $\mathsf{Set}_f{}^G$ of finite G-sets and G-maps is nothing but the Burnside ring of the finite group G, that is, the Grothendieck ring of $\mathsf{Set}_f{}^G$ with respect to disjoint union and cartesian product ([DreSi 88], [Yos 87]); on the other hand, the internal integer ring with respect to Set^G is the rational integer ring with trivial G-action which is defined by internal Peano's axiom and internal Grothendieck construction. Of course, the external integer ring with respect to Set_f and the internal integer ring inside Set are both isomorphic to the rational integer ring $\boldsymbol{Z}$. See MacLane-Moerdijk's book [MclMo 92, Section V.5]

(1.2) Species and $\mathcal{E}$-structures. The most famous and successful abstract theory of generating functions is Joyal's theory of species([Joy 81], [BerLL 98]) and its generalizations ([MenNa 93], [Ber 89]), which affect also the present paper. Let Bij_f be the category of finite sets and bijections and Set_f the category of finite sets and maps. Then a species $\boldsymbol{A}$ is now a functor from Bij_f to Set_f; an element of the image $\boldsymbol{A}[E]$ of E under $\boldsymbol{A}$ is called an $\boldsymbol{A}$-structure on label set E; furthermore, its generating function is defined by

$$\boldsymbol{A}(t) := \sum_{n=0}^{\infty} \frac{|\boldsymbol{A}[n]|}{n!} t^n.$$

Here $\boldsymbol{A}[n]$ denotes the value at the n-point set $[n] := \{1, 2, \ldots, n\}$.

On the other hand, for a faithful functor $F : \mathcal{E} \longrightarrow \mathcal{S}$ with finite fibers $|F^{-1}(N)/\cong| < \infty$, an $\mathcal{E}$-structure over N along F is defined to be a pair (X, σ) of an object X of $\mathcal{E}$ and an isomorphism $\sigma : F(X) \longrightarrow N$. The concept of $\mathcal{E}$-structures (or F-structures) is explicitly or implicitly considered in some articles (e.g. Dür [Dur 86], Joyal [Joy 81]). Let $\mathsf{Str}(\mathcal{E}/N)/\cong$ be the set of isomorphism classes of (labeled) $\mathcal{E}$-structures on N. For example, if $\mathcal{E} = \mathsf{Set}_f{}^G$ is the category of finite G-sets, where G is a finitely generated group, and $F : \mathsf{Set}_f{}^G \longrightarrow \mathsf{Set}_f$ is a forgetful functor, then there is a bijective correspondence (Example 2.10):

$$\mathsf{Str}(\mathsf{Set}_f{}^G/[n])/\cong \;\longleftrightarrow \mathrm{Hom}(G, S_n).$$

Furthermore, if $\mathcal{E} = \mathsf{RTree}$ is the category of rooted trees and $F : \mathsf{RTree} \longrightarrow \mathsf{Set}_f$ is the forgetful functor which assigns a rooted tree to the vertex set, then an isomorphism class of RTree-structure on N is bijectively corresponding to a labeled rooted tree on N.

Theorem 3.6. *Let $S : \mathcal{G} \longrightarrow \mathsf{Set}_f$ be a faithful functor with finite fibers from a groupoid $\mathcal{G}$. Then the assignment*

$$A_S : N \longmapsto \mathsf{Str}(\mathcal{E}/N)/\cong, \quad N \in \mathsf{Bij}_f$$

is a species. This construction gives a bijective correspondence between a species and a faithful functor with finite fibers from a groupoid to Set_f.

For example, the functor corresponding to the species of graphs is the functor from the groupoid of graphs and graph isomorphisms to Set_f which assigns its vertex set to a graph; here note that the same species is also obtained from the category of graphs and *all* graph homomorphisms.

The above theorem states that there exists an equivalence

$$\Gamma : (\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}} \longrightarrow [\mathsf{Bij}_f, \mathsf{Set}_f]$$

between the category of species and the category of faithful functors with finite fibers from groupoids to Set_f. This equivalence remind of an equivalence familiar in sheaf theory([MclMo 92, II.6], [Joh 77, 0.2, Chapter 3]):

$$\Gamma : \mathbf{Etale}/X \xrightarrow{\cong} \mathrm{Sheave}(X) \subseteq [\mathcal{O}(X)^{\mathrm{op}}, \mathsf{Set}],$$

where $\mathbf{Etale}/X$ is the category of local homeomorphisms over the space X. See also (2.3.b).

The concept of faithful functors with finite fibers have some advantages in comparison with that of species as follows:

(1) The source category of the functor corresponding to a species via Theorem 3.6 is a groupoid like Bij_f. But this restriction is not essential. Using faithful functors from any category with finite fibers instead of species, we

can generalize the theory of species. Especially we can take a locally finite topos as the source category ([Joh 77], [MclMo 92]).

(2) Considering a faithful functor $S : \mathcal{E} \longrightarrow \mathcal{S}$ instead of a species, we can separate the part which the source category $\mathcal{E}$ plays and the one which the functor S plays. For example, the exponential formula essentially depend on the property that $\mathcal{E}$ is a strict KS-category (see (5.5)); for the functor S we need to assume only its additivity.

(3) When we want to substitute a natural number a for the variable t in the generating function $A(t)$ of polynomial type, the value of t^n at $t = a$ should be the number of mappings from an n-point set $[n]$ to an a-point set $[a]$. However, $|\mathrm{Hom}([n],[a])|$ in Bij_f does not give the expected value a^n in general. In this viewpoint, Set_f is more natural than Bij_f.

(1.3) Generating functions with exponents in a category. The crucial idea used in this paper is that we let the exponents of polynomials or power series range over the objects of a category $\mathcal{E}$. Thus the (exponential) generating function associated to a skeletally small category $\mathcal{E}$ with finite hom-sets may be defined by

$$\mathcal{E}(t) := \sum_{A\in\mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(A)|} t^A,$$

where A runs the isomorphism classes of the objects of $\mathcal{E}$.

Theorem 4.6. *Let $F : \mathcal{E} \longrightarrow \mathcal{S}$ be a faithful functor. Then we have*

$$F(t) := \sum_{A\in\mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(A)|} t^{F(A)} = \sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|} t^N.$$

This formula together with the exponential theorem (Theorem 5.8) has many applications for enumerations of labeled structures.

(1.4) Exponential formulas. In Section 5 we prove the following exponential formula:

Theorem 5.8. *Assume that $\mathcal{E}$ is a strict Krull-Schmidt category with the full subcategory $\mathsf{Con}(\mathcal{E})$ of connected objects. Then the exponential formula holds:*

$$\mathcal{E}(t) = \exp\left(\mathsf{Con}(\mathcal{E})(t)\right).$$

General theories of exponential formulas have been developed in many articles, e.g. Bender-Goldman [BenGo 71], Foata [Foa 74], Beissinger [Bei 85], Dress-Müller [DreMu 97], Dür [Dur 86]. See also [FoaSc 70], [DouRS 72], [Sta 78a], [GarJo 80], [Joy 81], and the books [Aig 79, Section V.2], [Wil 94, Chapter 3], [Sta98].

Among them, Dür's book [Dur 86, Chapter 2] states a general theory of exponential formulas of functors between Krull-Schmidt categories. The definition of Krull-Schmidt categories he used is not the same as the one I adopt in the present paper. For example, the category of finite dimensional vector spaces over a finite field is a Krull-Schmidt category in the sense of Dür, but not in the sense in the present paper. However, our theorem is simple but general enough to apply it, and contains many of the known exponential formulas for non-additive categories, for example, Cayley's formula for the number of labeled rooted trees, Wohlfahrt's formula for the numbers of group homomorphisms $|\mathrm{Hom}(G, S_n)|$, $n = 0, 1, 2, \dots$ (Section 6), and so on.

In the present situation, power series with exponents in a category are inconvenient for composition, substitution and derivation; for example, $(t^A + t^B)^C$ and $d\mathcal{E}(t)/dt$ are both meaningless; in fact, they are not a polynomial or a power series in our sense. To solve this difficulty, we need to extend the concept of power series. In the third paper, we will introduce the "external" ring of power series (and polynomials) associated to a locally small topos ([Bor 94, III], [MclMo 92], [Joh 77]) with coefficients in a Mackey functor with multiplicative transfer, especially a Burnside ring functor ([Tam93], [DreSi 89], [Yos 87]). We will study again the problem on the definition of polynomials and power series in the third paper of this series.

2 $\mathcal{E}$-structures

(2.1) Notation and terminology. Our notation and terminology on categories in this paper are taken from some standard books, e.g. [Mcl 71], [Bor 94].

Here we add some basic but perhaps non-standard notation and terminology. A category is *finite* if it has finitely many objects and morphisms; a category is *skeletally finite* if it is equivalent to a finite category; a category is *locally finite* if every hom-set $\mathrm{Hom}(X, Y)$ is a finite set; a category has *finite automorphism groups* if every automorphism group $\mathrm{Aut}(X)$ is a finite group. A *skeleton* of a category is a full subcategory consisting of complete representatives of the isomorphism classes of objects. A category is *skeletally small* if it is equivalent to a small category; $\mathcal{C}/\cong$ denotes the set of all isomorphism classes of the objects of such a category $\mathcal{C}$ and often identified with a skeleton of $\mathcal{C}$; the isomorphism class containing an object X is denoted by $[X]$ or simply by X if there is no confusion.

A subobject A of an object X is called *complementary* if there is a complement A' in the poset $\mathrm{Sub}(X)$ of subobjects of X, that is,

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d] \\
A' \arrow[r] & X
\end{tikzcd}
\tag{2.1.a}
$$

is a push-out diagram and a pull-back diagram. The poset of complementary subobjects of X is denoted by $\mathrm{Sub}_{\neg\neg}(X)$.

For a category $\mathcal{C}$, we define the groupoid $\mathcal{C}_{iso}$ as the subcategory of $\mathcal{C}$ consisting of all objects of $\mathcal{C}$ and all isomorphisms. The dual category of a category $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$. The notation $X \in \mathcal{C}$ means that X is an object of the category $\mathcal{C}$. The functor category from $\mathcal{C}$ to $\mathcal{D}$ is denoted by $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$. Thus $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ is the category of contravariant functors. The vertical (resp. horizontal) composition of two natural transformations is denoted by using $\circ$ (resp. $*$).

We use the following notation for some important special categories. Set is the category of sets and maps; Set_f is the category of finite sets and maps; $\mathsf{Bij}_f := (\mathsf{Set}_f)_{\mathrm{op}}$ is the category of finite sets and bijections; ${\mathsf{Set}_f}^G$ is the category of finite (left) G-sets and G-maps for a group G; Gpd denotes the category of (infinite)groupoids.

We furthermore use the standard notation of sets. The symbols $\boldsymbol{N}$ (resp. $\boldsymbol{P}$) denotes the set of nonnegative (resp. positive) integers. For a nonnegative integer n, we put

$$[n] := \{1, 2, \dots, n\}, \quad [0] := \emptyset.$$

We write $\mathrm{Sym}(N)$ for the symmetric group on a set N; in particular, $S_n = \mathrm{Sym}([n])$ is the symmetric group of degree n, where we think that S_0 consists of only an identity element.

(2.2) Comma categories. For a category $\mathcal{C}$ and its object X, the *comma category* $\mathcal{C}/X$ (or often $\mathcal{C} \downarrow X$) is the category of object over X, that is, an object of this category is a morphism $(\alpha : A \longrightarrow X)$ from an object of $\mathcal{C}$, and a morphism f from $(\alpha : A \longrightarrow X)$ to $(\beta : B \longrightarrow X)$ is a morphism $f : A \longrightarrow B$ in $\mathcal{C}$ such that $\beta \circ f = \alpha$. There is a faithful functor which is often called a localization

$$\Sigma_X : \mathcal{C}/X \longrightarrow \mathcal{C}; (A \longrightarrow X) \longmapsto A \tag{2.2.a}$$

More generally, given functors $F : \mathcal{A} \longrightarrow \mathcal{C}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$, the *comma category* $F \downarrow G$ is defined to be the category of the triples (X, α, Y) where $X \in \mathcal{A}$, $Y \in \mathcal{B}$, and $\alpha : F(X) \longrightarrow G(Y)$; a morphism $(f, g) : (X, \alpha, Y) \longrightarrow (X', \alpha', Y')$ is a pair of $f : X \longrightarrow Y$ and $g : Y \longrightarrow Y'$ such that $G(g) \circ \alpha = \alpha' \circ F(f)$. See [Bor 94, I.1.6].

In particular, for a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and an object A of $\mathcal{C}$, the category $A \downarrow F$ whose object has the form (α, Y), where Y is an object of $\mathcal{F}$ and $\alpha : A \longrightarrow F(A)$, and whose morphism $g : (\alpha, Y) \longrightarrow (\beta, Z)$ is a morphism $g : Y \longrightarrow Z$ of $\mathcal{F}$ such that $F(g) \circ \alpha = \beta$, is also called a comma category.

(2.3) The category of elements. A very important case of a comma category which plays an important role in this series is the *category of elements* of a functor. Let Rel be the category of sets and relations. For a functor $F : \mathcal{C} \longrightarrow \mathsf{Rel}$, an *element* of the functor F is a pair (X, s) of an object X of $\mathcal{C}$ and an element $s \in F(X)$. A morphism $f : (X, s) \longrightarrow (Y, t)$ between elements of F is a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ such that $(s, t) \in F(f) \subseteq F(X) \times F(Y)$. In this paper, we denote the category of the elements of F by $\mathsf{Elts}(F)$. There is a faithful functor called a projection

$$S_F : \mathsf{Elts}(F) \longrightarrow \mathcal{C}; (X, s) \longmapsto X. \tag{2.3.a}$$

Thus we have a functor

$$\Lambda : [\mathcal{C}, \mathsf{Rel}] \longrightarrow \mathsf{Cat/Set};\ F \longmapsto S_F. \tag{2.3.b}$$

In the case where F is a functor $F : \mathcal{C} \longrightarrow \mathsf{Set}$, a morphism $f : (X, s) \longrightarrow (Y, t)$ between elements of F is a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ such that $F(f)(s) = t$. See [Bor 94, I, Definition 1.6.4].

As an example, take the functor

$$F : \mathcal{C} \longrightarrow \mathsf{Rel}; X \longmapsto \mathrm{Aut}(X),$$

where for $f : X \longrightarrow Y$, the relation $F(f)$ is defined by

$$(\sigma, \tau) \in F(f) \iff \tau f = f\sigma \quad (\sigma \in \mathrm{Aut}(X), \tau \in \mathrm{Aut}(Y)).$$

Then an element of F has the form (X, σ), where $X \in \mathcal{C}, \sigma \in \mathrm{Aut}(X)$, and it is called a *cyclic object* in this paper; a cyclic set is often called a permutation set. The category $\mathsf{Elts}(F)$ of cyclic objects is equivalent to the functor category $\mathcal{C}^C$, where C is a infinite cyclic group viewed as a category.

Some books (e.g.[Mcl 71], [MclMo 92]) adopt the symbol $\displaystyle\int_{\mathcal{C}} F$ instead of $\mathsf{Elts}(F)$; however, this notation is not suitable for the present paper because $\mathsf{Elts}(F)$ is the 'derivation'of F !

(2.4) $\mathcal{E}$-structures. Let $F : \mathcal{E} \longrightarrow \mathcal{S}$ be a faithful functor and N an object of $\mathcal{S}$. Then an *$\mathcal{E}$-structure* on N (along F) is a pair (X, σ) of an object X of $\mathcal{E}$ and an isomorphism $\sigma : F(X) \longrightarrow N$ in $\mathcal{S}$; here, N (resp. σ) is called the *label set* (resp. the *labeling*). A *morphism* $f : (X, \sigma) \longrightarrow (Y, \tau)$ between $\mathcal{E}$-structures on N along F is a morphism $f : X \longrightarrow Y$ in $\mathcal{E}$ such that $\tau \circ F(f) = \sigma$. Thus $\mathcal{E}$-structures on N makes a category $\mathsf{Str}(\mathcal{E}/N)$. The set of the isomorphism classes of $\mathcal{E}$-structures on N is denoted by $\mathsf{Str}(\mathcal{E}/N)/\cong$. Thus an element of $\mathsf{Str}(\mathcal{E}/N)/\cong$ has the form $[X, \sigma]$, where X is an object and $\sigma : F(X) \xrightarrow{\cong} N$, and

$$[X, \sigma] = [Y, \tau] \quad \Longleftrightarrow \quad \text{there exists } f : X \longrightarrow Y \text{ such that } \tau \circ F(f) = \sigma.$$

(2.5) Functors with finite fibers. We say that a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has *finite fibers* if for any object Y of $\mathcal{D}$ there are only finitely many isomorphism classes of object X of $\mathcal{C}$ such that $F(X) \cong Y$:

$$|F^{-1}(Y)/\cong| = \sharp\{X \in \mathcal{C} \mid F(X) \cong Y\}/\cong \; < \infty.$$

This notion is essential in this paper as is shown by the following lemma.

(2.6) Lemma. *Let $F : \mathcal{E} \longrightarrow \mathcal{S}$ be a faithful functor into a category $\mathcal{S}$ with each automorphism group finite. Then the following hold:*

(1) *Under one of the following assumptions, F has finite fibers:*

(a) *F has a right adjoint functor and each object X of $\mathcal{E}$ has only finitely many subobjects.*
(b) *F has a left adjoint functor and each object X of $\mathcal{E}$ has only finitely many quotient objects.*

(2) $|\mathsf{Str}(\mathcal{E}/N)/\cong| < \infty$ *for any object N of $\mathcal{S}$ if and only if F has finite fibers.*

PROOF: (1) Under the condition (a), let $R : \mathcal{S} \longrightarrow \mathcal{E}$ be a right adjoint functor to F and $\sigma : F(X) \longrightarrow N$ an isomorphism. Then its adjoint $\hat{\sigma} : X \longrightarrow R(X)$ is a monomorphism; in fact, for any $u, v : A \longrightarrow X$ such that $\hat{\sigma}u = \hat{\sigma}v$, we have that $\sigma F(u) = \sigma F(v)$, and so $u = v$ by the facts that σ is an isomorphism and that F is faithful. Thus X is isomorphic to a subobject of $R(N)$, and so there are only finitely many such X's by the assumption (a). By the duality, (b) also implies the finiteness of each fiber.
(2) Assume first that F has finite fibers. If two $\mathcal{E}$-structures (X, σ) and (Y, τ) are isomorphic, then clearly $X \cong Y$. Thus

$$\begin{aligned} |\mathsf{Str}(\mathcal{E}/N)/\cong| &\le \sum_{X:F(X)\cong N}{}' \; |\mathrm{Aut}(N)| \\ &= |\mathrm{Aut}(N)| \cdot \sharp\{X \in \mathcal{E}/\cong \mid F(X) \cong N\} < \infty, \end{aligned}$$

where the summation is taken over a complete representatives of objects X on $\mathcal{E}$ such that $F(X) \cong N$, as required. The converse follows from the existence of a surjection

$$\mathsf{Str}(\mathcal{E}/N)/\cong \longrightarrow F^{-1}(N)/\cong;\ [X,\sigma] \longmapsto [X].$$

(2.7) Remark. The assumption of the faithfulness of F in the above lemma (and also in the definition of $\mathcal{E}$-structures, Theorem 4.6, Theorem 3.6, Corollary 5.9) can be replaced by the faithfulness on automorphisms. A functor $F : \mathcal{E} \longrightarrow \mathcal{S}$ is called *faithful on automorphisms* if an automorphism σ of an object X of $\mathcal{E}$ such that $F(\sigma) = 1$ is only identity morphism of X.

(2.8) Example—surjections. Let Surj be the category of surjections of finite sets, that is, an object is a surjection $X \overset{p}{\twoheadrightarrow} X'$ between finite sets and a morphism $(f, f') : (X \overset{p}{\twoheadrightarrow} X') \longrightarrow (Y \overset{q}{\twoheadrightarrow} Y')$ is a pairs of maps $f : X \longrightarrow Y$ and $f' : Y \longrightarrow Y'$ such that $f' \circ p = q \circ f$. Define a faithful functor $F : \mathsf{Surj} \longrightarrow \mathsf{Set}_f$ by $(X \twoheadrightarrow X') \longmapsto X$. Then a Surj-structure $(X \overset{p}{\twoheadrightarrow} X', \sigma)$ on N gives a partition $\{\sigma p^{-1}(x')\}_{x' \in X'}$ of N; and conversely a partition $\{B_1, \ldots, B_k\}$ of N gives a Surj-structure $(N \xrightarrow{p} [k], \mathrm{id})$, where $p(x) = i$ if $x \in B_i$. In particular, $|\mathsf{Str}(\mathsf{Surj}/[n])/\cong|$ is equal to the Bell number $b(n)$, that is, the number of equivalence relations on $[n]$.

(2.9) Example—graphs. Let Graph be the category of simple graphs and $V : \mathsf{Graph} \longrightarrow \mathsf{Set}_f$ the functor which assigns a graph to its vertex set. Then an isomorphism class of Graph-structures on a finite set N along V is a graph structure (N, E), where E is a two-point subset of N, on the vertex set N.

(2.10) Example—G-sets. Let G be a finitely generated group and $\mathsf{Set}_f{}^G$ the category of finite G-sets and G-maps. Let H be a finitely generated subgroup of G. Then there is a faithful functor which is defined by the restriction of the action to H:

$$\mathrm{Res}^G_H : \mathsf{Set}_f{}^G \longrightarrow \mathsf{Set}_f{}^H; X \longmapsto X. \tag{2.10.a}$$

Since the restriction functor has a left (and also a right) adjoint functor, it has finite fibers by Lemma 2.6. Let N be a finite H-set with associated permutation representation:

$$\rho_H : H \longrightarrow \mathrm{Sym}(N); h \longmapsto (i \longmapsto hi), \tag{2.10.b}$$

and put

$$\mathrm{Hom}(G, \mathrm{Sym}(N); H, \rho_H) := \{\pi \in \mathrm{Hom}(G, \mathrm{Sym}(N)) \mid \pi_{|H} = \rho_H\}, \tag{2.10.c}$$

Note that since G is finitely generated, $\mathrm{Hom}(G, \mathrm{Sym}(N))$ is a finite set. Then there is a bijection

$$\mathsf{Str}(\mathsf{Set}_f{}^G/N)/\cong \;\longleftrightarrow \mathrm{Hom}(G, \mathrm{Sym}(N); H, \rho_H) \tag{2.10.d}$$

defined by

$$\begin{aligned}&[X,\sigma] \longmapsto (\pi : g \longmapsto (\pi(g) : i \longmapsto \sigma(g\cdot\sigma^{-1}(i)))),\\ &(\pi : G \longrightarrow \mathrm{Sym}(N)) \longmapsto [N, 1],\\ &\quad\text{(the G-action on N is defined by $gi := \pi(g)(i)$).}\end{aligned}$$

In the case where H is trivial and $N = [n] = \{1, 2, \ldots, n\}$, the restriction functor becomes the forgetful functor

$$F : \mathsf{Set}_f{}^G \longrightarrow \mathsf{Set}_f; X \longmapsto X,$$

and we have a bijection

$$\mathsf{Str}(\mathsf{Set}_f{}^G/[n])/\cong \;\longleftrightarrow \mathrm{Hom}(G, S_n). \tag{2.10.e}$$

3 Species.

(3.1) Species. Let Bij_f be the category of finite sets and bijections. Then a *species* is a functor $\boldsymbol{A} : \mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$. A morphism between two species is defined to be a natural transformation. Thus the category of species is the functor category $[\mathsf{Bij}_f, \mathsf{Set}_f]$. The image of a finite set E by the species $\boldsymbol{A}$ is written usually as $\boldsymbol{A}[E]$ instead of $\boldsymbol{A}(E)$ and its element is called an $\boldsymbol{A}$-*structures* with label set E.

Clearly, the cardinality $|\boldsymbol{A}[E]|$ depends only on the cardinality $|E|$ of E. We simply write $\boldsymbol{A}[n]$ for $\boldsymbol{A}[[n]]$, where $[n] = \{1, 2, \ldots, n\}$; then the symmetric group S_n acts on the set $\boldsymbol{A}[n]$. A species $\boldsymbol{A}$ is completely, up to equivalence, determined by the series $\{\boldsymbol{A}[n]\}_{n=0,1,2,\ldots}$, where for each n $\boldsymbol{A}[n]$ is an S_n-set.

(3.2) Generating functions of a species. There are some kinds of generating functions of a species $\boldsymbol{A}$ ([Joy 81], [BerLL 98]). The (*exponential*) *generating function* of a species $\boldsymbol{A}$ is the formal power series

$$\boldsymbol{A}(x) := \sum_{n=0}^{\infty} \frac{|\boldsymbol{A}[n]|}{n!} x^n. \tag{3.2.a}$$

The *label generating function* is

$$\widetilde{A}(x) := \sum_{n=0}^{\infty} |A[n]/S_n| x^n, \tag{3.2.b}$$

where $A[n]/S_n$ is the set of S_n-orbits.

(3.3) $(\mathsf{Cat}/\mathsf{Set}_f)_{\mathit{fff}}$ and $(\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}}$. A *groupoid* is a (possibly large but skeletally small) category in which all morphisms are isomorphisms; thus it is equivalent to a disjoint union of some groups viewed as categories. A homomorphism of groupoids is a functor between groupoids as categories. All groupoids and homomorphisms form a category Gpd.

Let $(\mathsf{Cat}/\mathsf{Set}_f)_{\mathit{fff}}$ be the category defined by the following data:

(i) An object is a faithful functor $S : \mathcal{C} \longrightarrow \mathsf{Set}_f$ with finite fibers from a (necessarily locally finite and skeletally small) category $\mathcal{C}$.
(ii) A morphism from $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ to $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ is a pair

$$[F, \lambda], \quad \text{where } F : \mathcal{E} \longrightarrow \mathcal{F},\ \lambda : T \circ F \xrightarrow{\cong} S.$$

For two morphisms $[F, \lambda], [F', \lambda'] : (\mathcal{E} \xrightarrow{S} \mathsf{Set}_f) \longrightarrow (\mathcal{F} \xrightarrow{T} \mathsf{Set}_f)$, the equality is defined by

$$[F, \lambda] = [F', \lambda'] \iff \text{there exists } \varphi : F \xrightarrow{\cong} F' \text{ such that } \lambda' \circ (T * \varphi) = \lambda.$$

where $T * \varphi = 1_T * \varphi$ denotes the horizontal composition.
(iii) The composition of

$$(\mathcal{E} \xrightarrow{S} \mathsf{Set}_f) \xrightarrow{[F,\lambda]} (\mathcal{F} \xrightarrow{T} \mathsf{Set}_f) \xrightarrow{[G,\mu]} (\mathcal{G} \xrightarrow{U} \mathsf{Set}_f)$$

is defined by

$$[G, \mu] \circ [F, \lambda] := [G \circ F,\ U \circ G \circ F \xrightarrow{\mu * F} T \circ F \xrightarrow{\lambda} S].$$

Furthermore, $(\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}}$ denotes the full subcategory of $(\mathsf{Cat}/\mathsf{Set}_f)_{\mathit{fff}}$ consisting of faithful functors with finite fibers from groupoids.

(3.4) The groupoid associated to a species. We construct a functor

$$\Lambda : [\mathsf{Bij}_f, \mathsf{Set}_f] \longrightarrow (\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}}. \tag{3.4.a}$$

For a species $\boldsymbol{A} : \mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$, the category of elements

$$\mathsf{Elts}(\boldsymbol{A}) = \{(N, a) \mid N \in \mathsf{Bij}_f,\ a \in \boldsymbol{A}(N)\} \tag{3.4.b}$$

is a groupoid and the projection

$$S_A : \mathsf{Elts}(\boldsymbol{A}) \longrightarrow \mathsf{Set}_f;\ (N, a) \longmapsto N \tag{3.4.c}$$

is clearly faithful and has finite fibers, and so we obtained an object S_A of $(\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}}$. The functor Λ is defined by $\Lambda(\boldsymbol{A}) = S_A$ on objects.

Any morphism $\theta : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ of species induces the functor:

$$\Lambda(\theta) : \mathsf{Elts}(\boldsymbol{A}) \longrightarrow \mathsf{Elts}(\boldsymbol{B})$$

defined by

$$\begin{aligned} &\Lambda(\theta)((N, a)) := (N, \theta(N)(a)), \\ &\Lambda(\theta)((N, a) \xrightarrow{g} (N', a')) := ((N, \theta(N)(a)) \xrightarrow{g} (N', \theta(N')(a'))), \end{aligned}$$

where note that $\theta_{N'}(\boldsymbol{A}(g)(a)) = \boldsymbol{B}(g)(\theta_N(a))$. Since $S_A = S_B \circ \Lambda(\theta)$, we obtain a required functor Λ.

(3.5) The species of $\mathcal{E}$-structures. Let $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ be a faithful functor. For any finite set N and any bijection $g : N \longrightarrow N'$, define a functor $\boldsymbol{A}_S : \mathsf{Bij}_f \longrightarrow \mathsf{Set}$ by

$$\begin{aligned} \boldsymbol{A}_S(N) &:= \mathsf{Str}(\mathcal{E}/N)/\cong \\ \boldsymbol{A}_S(g) &: \mathsf{Str}(\mathcal{E}/N)/\cong\ \longrightarrow \mathsf{Str}(\mathcal{E}/N')/\cong;\ [X, \sigma] \longmapsto [X, g \circ \sigma]. \end{aligned}$$

By Lemma 2.6, if the functor $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ has finite fibers, then $|\boldsymbol{A}_S(N)| = |\mathsf{Str}(\mathcal{E}/N)/\cong| < \infty$, and so the functor $\boldsymbol{A}_S$ is a species.

We can now define a functor

$$\Gamma : (\mathsf{Cat}/\mathsf{Set}_f)_{\mathit{fff}} \longrightarrow [\mathsf{Bij}_f, \mathsf{Set}_f]. \tag{3.5.a}$$

On objects, Γ is defined by

$$\Gamma(S : \mathcal{E} \longrightarrow \mathsf{Set}_f) := \boldsymbol{A}_S. \tag{3.5.b}$$

Furthermore, let $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be another faithful functor with finite fibers and $[F, \lambda] : S \longrightarrow T$ a morphism. Then a morphism of species

$$\Gamma[F, \lambda] : \boldsymbol{A}_S \longrightarrow \boldsymbol{A}_T$$

is defined by

$$\begin{aligned} \Gamma[F, \lambda]_N \ &: \ \mathsf{Str}(\mathcal{E}/N)/\cong \ \longrightarrow \mathsf{Str}(\mathcal{F}/N)/\cong \\ &; \ [X, \sigma] \longmapsto [F(X), \sigma \circ \lambda_X]. \end{aligned}$$

Hence we obtained the required functor Γ.

(3.6) Theorem. $[\mathsf{Bij}_f, \mathsf{Set}_f] \cong (\mathsf{Gpd}/\mathsf{Set}_f)_{\mathit{fff}}$.

PROOF: We prove that the functors Λ and Γ give an equivalence between the above two categories. It will suffice to show that (a) $\Gamma\Lambda \cong \mathrm{Id}$, (b) Γ is faithful, and (c) Λ is dense, that is, any faithful functor with finite fibers from a groupoid to Set_f is isomorphic to the image of a species under Λ

(a) First, let $\boldsymbol{A}$ be a species and put $\boldsymbol{B} := \Gamma\Lambda(\boldsymbol{A})$. Then any element of $\boldsymbol{B}(N)$ has the form $[(N, a), 1]$; in fact, for any $a' \in \boldsymbol{A}(N')$ and $\sigma : N' \xrightarrow{\cong} N$, we have

$$[(N', a'), \sigma] = [(N, \boldsymbol{A}(\sigma)(a')), 1].$$

Furthermore, $[(N, a), 1] = [(N, a'), 1]$ if and only if $a = a'$. Thus there is a bijection between $\boldsymbol{A}(N)$ and $\boldsymbol{B}(N)$ defined by

$$a(\in \boldsymbol{A}(N)) \quad \longleftrightarrow \quad [(N, a), 1](\in \boldsymbol{B}(N)).$$

This correspondence is clearly natural on N and $\boldsymbol{A}$, and so we have the natural isomorphism of functors:

$$\Gamma \circ \Lambda \cong \mathrm{Id}.$$

(b) We next show that Γ is faithful. Let $[F, \lambda]$, $[G, \mu]$ be two morphisms from $S : \mathcal{G} \longrightarrow \mathsf{Set}_f$ to $T : \mathcal{H} \longrightarrow \mathsf{Set}_f$, where $\mathcal{G}$, $\mathcal{H}$ are groupoids and S, T are faithful functors with finite fibers. Assume that

$$\Gamma[F, \lambda] = \Gamma[G, \mu] : \mathsf{Str}(\mathcal{G}/N)/\cong \ \longrightarrow \mathsf{Str}(\mathcal{H}/N)/\cong.$$

Then for any $[X, \sigma] \in \mathsf{Str}(\mathcal{G}/N)/\cong$, we have

$$[F(X), \sigma\, \lambda_X] = [G(X), \sigma\, \mu_X],$$

that is, there is an isomorphism $\tau_X : F(X) \longrightarrow G(X)$ such that $\sigma\lambda_X = \sigma\mu_X T(\tau_X)$. Since σ is isomorphism, we have that $\lambda_X = \mu_X T(\tau_X)$, and so

$$T(\tau_X) = {\mu_X}^{-1}\lambda_X : TF(X) \longrightarrow TG(X).$$

Thus $X \longmapsto (T(\tau_X) : TF(X) \longrightarrow TG(X))_X$ is natural on X because $\lambda : TF \longrightarrow S$ and $\mu : TG \longrightarrow S$ are natural transformations, and so it follows from the faithfulness of T that $X \longmapsto \tau_X$ is also natural. Hence $\tau = (\tau_X) : F \longrightarrow G$ is a natural isomorphism satisfying $\lambda = \mu \circ (T * \tau)$, that is, $[F, \lambda] = [G, \mu]$, as required.

(c) Finally, let $\mathcal{G} \xrightarrow{S} \mathsf{Set}_f$ be a faithful functor with finite fibers from a groupoid. Define the species $\boldsymbol{A}_S$ and a functor T by

$$\begin{aligned} \boldsymbol{A}_S &:= (\Gamma(S) : N \longmapsto \mathsf{Str}(\mathcal{G}/N)/\cong) \\ T &:= \Lambda(\boldsymbol{A}_S) : \mathsf{Elts}(\boldsymbol{A}_S) \longrightarrow \mathsf{Set}_f;\ (N, [X, \sigma]) \longmapsto N. \end{aligned}$$

We want to show that the functor

$$F : \mathcal{G} \longrightarrow \mathsf{Elts}(\boldsymbol{A}_S);\ X \longmapsto (S(X), [X, 1])$$

gives an equivalence of groupoids, which then gives a required isomorphism $[F, 1_S] : S \longrightarrow T$. The functor $[F, 1_S]$ is faithful because so is S. In order to prove that $[F, 1_S]$ is full, let $g : (S(X), [X, 1]) \longrightarrow (S(Y), [Y, 1])$ be a morphism in $\mathsf{Elts}(\boldsymbol{A}_S)$, so that g is a mapping from $S(X)$ to $S(Y)$ with $[X, g] = [Y, 1] \in \mathsf{Str}(\mathcal{G}/S(Y))/\cong$, and so there is an isomorphism $f : X \xrightarrow{\cong} Y$ in $\mathcal{G}$ such that $S(f) = g$. This means that the morphism g comes from f via F. Thus the functor $[F, 1_S]$ is full. Furthermore, let $(N, [X, \sigma])$ be any object of $\mathsf{Elts}(\boldsymbol{A}_S)$. Then there is an isomorphism

$$\sigma : F(X) = (S(X), [X, 1]) \xrightarrow{\cong} (N, [X, \sigma]).$$

Hence F is an equivalence. The theorem was proved.

(3.7) Other kinds of species. Besides Joyal's original species ([Joy 81]), there are many kinds of species; I-species ([MenNa 93]), S-species (or permutation species, [Joy 81], [Ber 89]), symmetric species ([Men 93]), species on digraphs ([Men 96]), polynomial species ([BonRSV 92]), K-species ([Lab 89]), Möbius species ([MenYa 91]), and so on.

Some of them have the corresponding faithful functors with finite fibers. For example, for a finite set I an I-species is a functor $\mathsf{Bij}_f/I \longrightarrow \mathsf{Set}_f$, where Bij_f/I is the comma category over I, and a species on digraphs can be viewed as an $I \times I$-species. An I-species is bijectively corresponding to a faithful functor $\mathcal{G} \longrightarrow \mathsf{Set}_f/I$ with finite fibers from a locally finite groupoid $\mathcal{G}$ to the comma category Set_f/I ($\cong \mathsf{Set}_f{}^I$).

An S-species (or permutation-species) is a functor from $(\mathsf{Set}_f{}^C)_{iso}$, the category of cyclic finite sets (2.3) and isomorphisms, to Set_f ([Ber 89]). Then the faithful functor corresponding to an S-species is a functor from a groupoid to $\mathsf{Set}_f{}^C$.

Similarly, Chen's compositional ([Che 93b]) is also viewed as a functor from Surj_{iso} to Set_f, where Surj_{iso} is the category of surjections between finite sets with bijections as morphisms. The faithful functor corresponding to a compositional $\boldsymbol{M}$ is the one from the category of elements $\mathsf{Elts}(\boldsymbol{M})$, which forms a groupoid, to Set_f.

4 Formal Exponential functions

(4.1) The space of formal power series. Let R be a commutative $\boldsymbol{Q}$-algebra and $\mathcal{E}$ a skeletally small category. Then $R[[\mathcal{E}^{\mathrm{op}}/\cong]]$ denotes the R-module of formal power series with exponents in $\mathcal{E}$:

$$f(t) = {\sum_{N\in\mathcal{E}}}' a_N\, t^N, \quad a_N \in R. \tag{4.1.a}$$

Here $\sum'$ means the summation on all isomorphism classes of objects of $\mathcal{E}$ and the variable t^N is regarded as the isomorphism class containing N in the dual category $\mathcal{E}^{\mathrm{op}}$, and so $t^M = t^N$ if $M \cong N$. When the above summation is finite, $f(t)$ is called a *polynomial*; the R-module of such polynomials is denoted by $R[\mathcal{E}^{\mathrm{op}}/\cong]$. We do not consider multiplication of formal power series yet (5.1).

For the category of finite sets, since an isomorphism class of finite sets is determined by its cardinality, we usually denote t^N by t^n, where $n := |N|$. Thus we identify the power series with exponents in Set_f with the usual power series in one variable:

$${\sum_{N\in\mathcal{E}}}' a_N\, t^N = \sum_{n=0}^{\infty} a_n t^n, \tag{4.1.b}$$

where $a_n = a_{[n]}$, $[n] = \{1, \dots, n\}$ and $t^n = t^{[n]}$.

(4.2) The exponential generating functions. The (*exponential*) *generating function* of a skeletally small and locally finite category $\mathcal{E}$ is the formal (infinite) sum

$$\mathcal{E}(t) := {\sum_{X\in\mathcal{E}}}' \frac{t^X}{|\mathrm{Aut}(X)|} \in R[[\mathcal{E}^{\mathrm{op}}/\cong]]. \tag{4.2.a}$$

(4.3) Other generating functions. The *ordinary generating function* of a skeletally small and locally finite category $\mathcal{E}$ is the formal (infinite) sum

$$\widetilde{\mathcal{E}}(t) := \sum_{X\in\mathcal{E}}{}' t^X \in R[[\mathcal{E}^{\mathrm{op}}/\cong]]. \tag{4.3.a}$$

We furthermore define the *Dirichlet generating function* of $\mathcal{E}$ as the formal sum

$$\mathcal{E}(t)_{\mathrm{Dir}} := \widetilde{\mathcal{E}^{\mathrm{op}}}(-t) = \sum_{X\in\mathcal{E}}{}' X^{-t}. \tag{4.3.b}$$

From the view point of the theory of species, the ordinary generating function is corresponding to the label generating series ([BerLL 98]). See Corollary 4.8.

(4.4) Remark. Let $\mathcal{E}_{iso}$ be the subcategory of $\mathcal{E}$ consisting of all objects and all isomorphisms of $\mathcal{E}$. Then clearly we have

$$\mathcal{E}(t) = \mathcal{E}_{iso}(t),\ \widetilde{\mathcal{E}}(t) = \widetilde{\mathcal{E}_{iso}}(t). \tag{4.4.a}$$

Thus it seems apparently that the non-isomorphic morphisms in $\mathcal{E}$ have no meaning for generating functions. However, it is essential to consider a more general category $\mathcal{E}$ not a groupoid if we apply a functor on a generating function or substitute some element for the variable t^X, For example, the "value" of $\mathcal{E}(t)$ at $T\in\mathcal{E}$

$$\mathcal{E}(t)|_{t=T} := \sum_{X\in\mathcal{E}}{}' \frac{|\mathrm{Hom}(X,T)|}{|\mathrm{Aut}(X)|} \tag{4.4.b}$$

is not the same as

$$\mathcal{E}_{iso}(t)|_{t=T} = 1.$$

See (5.3).

(4.5) A linear map induced from a functor. Let $F : \mathcal{E} \longrightarrow \mathcal{S}$ be a faithful functor with finite fibers between skeletally small categories. Then the functor F induces a linear map

$$F_* : R[[\mathcal{E}^{\mathrm{op}}/\cong]] \longrightarrow R[[\mathcal{S}^{\mathrm{op}}/\cong]] \tag{4.5.a}$$

defined by

$$F_*\left(\sum_{X\in\mathcal{E}}{}' a_X\, t^X\right) := \sum_{X\in\mathcal{E}}{}' a_X\, t^{F(X)} = \sum_{N\in\mathcal{S}}{}' \left(\sum_{X\in\mathcal{E}:F(X)\cong N}{}' a_X\right) t^N$$

Note that the inner sum is finite by the condition that F has finite fibers. The series

$$F(t) := F_*(\mathcal{E}(t)) = \sum_{X\in\mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(X)|} t^{F(X)} \tag{4.5.b}$$

is called the *exponential generating function* of $\mathcal{E}$ along F, which corresponds to the generating function of species. Similarly, the series

$$\begin{aligned}\widetilde{F}(t) &:= F_*(\widetilde{\mathcal{E}}(t)) = \sum_{X\in\mathcal{E}}{}' t^{F(X)} \\ &= \sum_{N\in\mathcal{S}}{}' \sharp\{X \in \mathcal{E}/\cong \mid F(X) \cong N\} \cdot t^N \end{aligned} \tag{4.5.c}$$

is called the *ordinary generating function* of $\mathcal{E}$ along F, which corresponds to the label generating function of species.

If $F : \mathcal{G} \longrightarrow \mathsf{Set}_f$ is a faithful functor with finite fibers from a groupoid $\mathcal{G}$ associated to a species $\boldsymbol{A}$, then F and $\boldsymbol{A}$ have the same generating functions: $F(t) = \boldsymbol{A}(t)$.

(4.6) Theorem. *Let $F : \mathcal{E} \longrightarrow \mathcal{S}$ be a faithful functor with finite fibers between skeletally small categories. Then*

$$\sum_{X\in\mathcal{E}}{}' \frac{c_{F(X)}}{|\mathrm{Aut}(X)|} t^{F(X)} = \sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|} c_N t^N. \tag{4.6.a}$$

In particular,

$$F(t) = \sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|} t^N. \tag{4.6.b}$$

Proof. For two $\mathcal{E}$-structures, note that $[X,\sigma] = [X',\sigma']$ implies $X \cong X'$ and that $[X,\sigma] = [X,\sigma']$ if and only if there is an automorphism $f \in \mathrm{Aut}(X)$ such that $\sigma' \circ F(f) = \sigma$, that is, σ and σ' are contained in a common $F(\mathrm{Aut}(X))$-orbit. Furthermore, since F is faithful, $F(\mathrm{Aut}(X)) \cong \mathrm{Aut}(X)$. Thus we have

$$\begin{aligned}|\mathsf{Str}(\mathcal{E}/N)/\cong| &= \sharp\{[X,\sigma] \mid X \in \mathcal{E}/\cong,\ \sigma : F(X) \xrightarrow{\cong} N\} \\ &= \sum_{X\in\mathcal{E}}{}' \sharp\{[X,\sigma] \mid \sigma : F(X) \xrightarrow{\cong} N\} \\ &= \sum_{X\in\mathcal{E}}{}' \frac{|\mathrm{Iso}(F(X), N)|}{|F(\mathrm{Aut}(X))|} \\ &= \sum_{X\in\mathcal{E}:F(X)\cong N}{}' \frac{|\mathrm{Aut}(N)|}{|\mathrm{Aut}(X)|}. \end{aligned} \tag{4.6.c}$$

Hence

$$\sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|} c_N t^N = \sum_{N\in\mathcal{S}}{}' \sum_{X\in\mathcal{E}:F(X)\cong N}{}' \frac{1}{|\mathrm{Aut}(X)|} c_N t^N \quad (4.6.d)$$

$$= \sum_{X\in\mathcal{E}}{}' \frac{c_{F(X)}}{|\mathrm{Aut}(X)|} t^{F(X)} \quad (4.6.e)$$

as required. In particular, when $c_N = 1$ for all $N \in \mathcal{S}$, the last summation is equal to $F(t)$.

(4.7) Corollary. *Let $\mathcal{E}$ be a skeletally small category with finite automorphism groups. Let $\mathcal{E}^C$ be the category of cyclic objects* (2.3) *with projection functor*:

$$F : \mathcal{E}^C \longrightarrow \mathcal{E}; (X,\alpha) \longrightarrow X.$$

Then we have

$$F_*(\mathcal{E}(t)) = \widetilde{\mathcal{E}}(t). \quad (4.7.a)$$

PROOF. This formula directly follows from the faithfulness of F and the existence of the bijection

$$\mathsf{Str}(\mathcal{E}^C/N)/\cong \;\longrightarrow \mathrm{Aut}(N).$$

defined by

$$\begin{aligned} [(X,\alpha),\sigma] &\longmapsto \sigma\alpha\sigma^{-1}, \quad (\alpha \in \mathrm{Aut}(X), \sigma \in \mathrm{Iso}(X,N)) \\ \beta &\longmapsto [(N,\beta),1_N] \quad (\beta \in \mathrm{Aut}(N)). \end{aligned}$$

(4.8) Corollary. *Let $\boldsymbol{A} : \mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$ be a species accompanied with a faithful functor $S : \mathcal{G} \longrightarrow \mathsf{Set}_f$ with finite fibers from a groupoid $\mathcal{G}$. Then the exponential (resp. ordinary) generating function of $\boldsymbol{A}$ and the exponential (resp. label) generating function of S coincide each other*:

$$\boldsymbol{A}(t) = S(t), \; \widetilde{\boldsymbol{A}}(t) = \widetilde{S}(t). \quad (4.8.a)$$

(4.9) Example—some species. Here there are some faithful functors from groupoids associated to special species.

(a) The faithful functor with finite fibers corresponding to the zero species $\mathbf{0}$: $E \longmapsto \emptyset$ is the functor $\emptyset \longrightarrow \mathsf{Set}_f$ from the empty category. Its exponential generating function is 0.

(b) Let $\mathbf{1}$ be a species such that $\mathbf{1}[E] = \{\emptyset\}$ if $E = \emptyset$ and $\mathbf{1}[E] = \emptyset$ otherwise. Then the functor corresponding to I is the functor $\mathbf{1} \longrightarrow \mathsf{Set}_f : * \longmapsto \emptyset$, where $\mathbf{1}$ is the category with only one object $*$ and only one morphism. Its exponential function is $1 = t^{\emptyset}$.
(c) A singleton species X is defined by $X[E] = \{E\}$ if $|E| = 1$ and $X[E] = \emptyset$ otherwise. The corresponding functor is the functor $\mathbf{1} \longrightarrow \mathsf{Set}_f : * \longmapsto \{*\}$, where $\mathbf{1}$ is the category with only one object $*$ and only one morphism. Its exponential generating function is t.
(d) A uniform species U is defined by $U[E] = \{E\}$ for all finite set E. Then the corresponding functor is the embedding $\mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$ and its exponential function is $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$.

In some species $\boldsymbol{A} : \mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$ can be extended to a functor from Set_f. For example, the identity functor $\mathsf{Set}_f \longrightarrow \mathsf{Set}_f$ is an extension of the uniform species U and they have the same exponential function e^t. Contrarily the singleton species X, for which $X[E] = \{E\}$ if $|E| = 1$ and $= \emptyset$ otherwise, can not be extended to Set_f. Since the category Set_f of finite sets and mappings has richer structures than the category Bij_f of finite sets and bijections, it is significant to study faithful functors with finite fibers instead of species.

(4.10) Example—finite groups. Let Gp be the category of finite groups and homomorphisms; and let $F : \mathsf{Gp} \longrightarrow \mathsf{Set}_f$ be the forgetful functor. Then an isomorphism class of Gp-structures on a finite set N bijectively corresponds to a group structure μ on N:

$$\mu : N \times N \longrightarrow N \tag{4.10.a}$$

where, of course, the multiplication μ must satisfy the axiom of groups. Thus the exponential function of the functor F has the form

$$(\mathsf{Gp} \xrightarrow{F} \mathsf{Set}_f)(t) = \sum_{G \in \mathsf{Gp}} \frac{1}{|\mathrm{Aut}(G)|} t^{|G|} \tag{4.10.b}$$

and the number of group structure on an n-point set $[n]$ is equal to

$$|\mathsf{Str}(\mathsf{Gp}/[n])/\cong| = n! \sum_{|G|=n} \frac{1}{|\mathrm{Aut}(G)|}. \tag{4.10.c}$$

Such summations on finite groups (or finite abelian groups) appear in several area, e.g. [Hal 38], [CohLe 84]. In particular,

$$n^{-n^2} n! \sum_{|G|=n} \frac{1}{|\mathrm{Aut}(G)|} \tag{4.10.d}$$

presents the probability where a binary operation $\mu : N \times N \longrightarrow N$ satisfies the group axiom. Furthermore, when $n = p^m$ is a power of prime, we have

$$\sum_{|A|=p^m} \frac{1}{|\mathrm{Aut}(A)|} = \sum_{\mathrm{rk}(A)=m} \frac{1}{|A|} = p^{-m} \prod_{i=1}^{m} \frac{1}{1-p^{-i}}, \tag{4.10.e}$$

where the first (resp. second) summation is taken over all abelian p-groups of order p^m (resp. rank m). See [Hal 38], [Yos 92].

(4.11) Example—G-sets. Let G be a finitely generated group and $\mathsf{Set}_f{}^G$ the category of finite G-sets and G-maps with the forgetful functor

$$F : \mathsf{Set}_f{}^G \longrightarrow \mathsf{Set}_f; X \longmapsto X.$$

Let $[n, \rho]$ denote the G-set correspond to the homomorphism $\rho : G \longrightarrow S_n$; note that $[n, \rho] \cong [n, \rho']$ as G-sets if and only if there is an inner automorphism ι of S_n such that $\rho' = \iota \circ \rho$. Thus we have

$$\begin{aligned} \mathsf{Set}_f{}^G(t) &= \sum_X{}' \frac{1}{|\mathrm{Aut}(X)|} t^X = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\rho \in \mathrm{Hom}(G,S_n)} t^{[n,\rho]}, && (4.11.a) \\ F(t) &= \sum_X{}' \frac{1}{|\mathrm{Aut}(X)|} t^{F(X)}, \\ &= \sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} t^n. && (4.11.b) \end{aligned}$$

Similarly, for the restriction functor Res_H^G to a subgroup H, we have

$$\begin{aligned} (\mathrm{Res}_H^G)_*(\mathsf{Set}_f{}^G(t)) &= \sum_X{}' \frac{1}{|\mathrm{Aut}(X)|} t^{X_H} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\rho \in \mathrm{Hom}(H,S_n)} |\mathrm{Hom}(G, S_n; H, \rho)| \, t^{[n,\rho]}. \end{aligned}$$

5 Exponential formulas

(5.1) The polynomial ring. Let $\mathcal{E}$ be a skeletally small category with finite coproducts, especially an initial object $\emptyset$. Then the set $\mathcal{E}^{\mathrm{op}}/\cong$ of isomorphism classes of objects of the dual category forms a multiplicative monoid with a unit element 1 by the product operation

$$t^M \cdot t^N := t^{M+N}, \quad t^{\emptyset} = 1, \tag{5.1.a}$$

where $\emptyset$ is an initial object of $\mathcal{E}$, and so the R-module $R[\mathcal{E}^{\mathrm{op}}/\cong]$ of polynomials (4.1) of the form:

$$f(t) = \sum_{N\in\mathcal{E}}{}' a_N t^N, \quad a_N \in R \tag{5.1.b}$$

where the summation is finite, becomes a semi-group algebra. We call $R[\mathcal{E}^{\mathrm{op}}/\cong]$ the *polynomial ring* with exponents in $\mathcal{E}$. When $\mathcal{E} = \mathrm{Set}_f$, this ring is canonically isomorphic to the usual polynomial ring $R[t]$ in one variable t by the identification $t^N = t^{|N|}$.

(5.2) The ring of formal power series. Let $\mathcal{E}$ be a skeletally finite category with finite coproducts. In order to extend the multiplication of polynomials to that of power series, we need the following additional assumption:

Assumption. *For any object N of $\mathcal{E}$, there exist only finite number of isomorphism classes of pairs (A, B) of $\mathcal{E}$ such that $N \cong A + B$.*

Under this assumption, the R-module $R[[\mathcal{E}^{\mathrm{op}}/\cong]]$ of infinite R-linear combinations of t^N ($N \in \mathcal{E}/\cong$) becomes an R-algebra (the complete semigroup algebra) equipped with the product defined by

$$\left(\sum_{N\in\mathcal{E}}{}' a_N t^N\right)\cdot\left(\sum_{N\in\mathcal{E}}{}' b_N t^N\right) := \sum_{N\in\mathcal{E}}{}' c_N t^N,$$
$$c_N := \sum_{A+B\cong N}{}' a_A \cdot b_B,$$

where (A, B) runs over all pairs of the isomorphism classes of the objects in $\mathcal{E}$ such that $A + B \cong N$. By the above assumption, the summation (5.2.a) is finite and the product is well-defined.

(5.3) Substitution. We next consider substituting an element of a ring for the variable "t "of a power series

$$f(t) = \sum_{N\in\mathcal{E}}{}' a_N t^N, \quad (a_N \in R).$$

Let $\widetilde{R}$ be a commutative ring containing R as a subring. A *multiplicative invariant* of $\mathcal{E}$ is a map $\tau : \mathcal{E} \longrightarrow \widetilde{R}$ satisfying the following conditions:

(i) $M \cong N \implies \tau(M) = \tau(N)$,
(ii) $\tau(\emptyset) = 1, \tau(M + N) = \tau(M)\cdot\tau(N)$,

Then we can substitute τ for t in $f(t)$:

$$f(\tau) := \sum_{N\in\mathcal{E}}{}' a_N \tau(N). \tag{5.3.a}$$

Here this series can not converge to an element of $\widetilde{R}$ in general. However, except for the problem of convergence, the substitution $f(t) \longmapsto f(\tau)$ is a ring homomorphism. This series (5.3.a) converges in $\widetilde{R}$ with discrete topology if and only if

(iii) $\sharp\{N \in \mathcal{E}/\cong \mid a_N \neq 0, \tau(N) = c\} < \infty$ for any $c \in \widetilde{R}$.

(5.4) Examples of invariants. (1) A typical example of multiplicative invariant of a skeletally small and locally finite category with finite coproducts is the *Yoneda invariant* associated to $X \in \mathcal{E}$:

$$\eta_X : N \longmapsto |\mathrm{Hom}(N, X)|.$$

(2) If $F : \mathcal{E} \longrightarrow \mathrm{Set}_f$ is a functor preserving coproducts, then

$$\tau_F : N \longmapsto x^{|N|} \in R[x],$$

where x is an indeterminate, is a multiplicative invariant.

(3) Assume that $\mathcal{E}$ is a (locally finite and skeletally small) full subcategory of a topos closed under subobjects and coproducts. Let denote by $\mathrm{Sub}(X)$ the set of subobjects of X. Then

$$\mathrm{sub} : N \longmapsto |\mathrm{Sub}(N)|$$

is a multiplicative invariant. When $\mathcal{E}$ is a subcategory of a topos closed under subobjects, this invariant is equal to the Yoneda invariant associated to the subobject classifier Ω.

(4) Assume that $\mathcal{E}$ is a strict KS-category (see (5.5) below). Let $\mathrm{Sub}_{\neg\neg}(X)$ denote the set of complementary subobjects of X (see (2.1)). Then

$$\mathrm{sub}_{\neg\neg} : N \longmapsto |\mathrm{Sub}_{\neg\neg}(N)|$$

is a multiplicative invariant. When $\mathcal{E}$ is a subcategory of a topos closed under cimplementary subobjects, ths invariant equals the Yoneda invariant associated to $\mathbf{1} + \mathbf{1}$.

(5) The product of two multiplicative invariants is also a multiplicative invariant. Furthermore, if $\tau : \mathcal{E} \longrightarrow \widetilde{R}$ is a multplicative invariant and z is a new indeterminate, then

$$\tau^z : N \longmapsto \tau(N)^z$$

is a multiplicative invariant into the ring of "Dirichlet series". The universal multiplicative invariant of $\mathcal{E}$ is given by $N \longmapsto t^N$ into the polynomial ring $R[\mathcal{E}^{\mathrm{op}}/\cong]$.

(5.5) Strict KS-categories. Let $\mathcal{E}$ be a category with any finite coproducts. An *connected* object J is a non-initial object satisfying the condition that $J = A+B$ implies that A or B is an initial object. The full subcategory of connected objects of $\mathcal{E}$ is denoted by $\mathsf{Con}(\mathcal{E})$. The category $\mathcal{E}$ is a (*strict*) *KS-category* if the following strict Krull-Schmidt type property:

(KS) *Any object X is isomorphic to a coproduct of some finite number of connected objects, and a coproduct decomposition of X is unique in the following sense: if $X = I_1 + \cdots + I_m = J_1 + \cdots + J_n$ are two coproduct decompositions into connected objects with canonical injections $i_\alpha : I_\alpha \longrightarrow X$ and $j_\beta : J_\beta \longrightarrow X$, then $m = n$ and there exist a permutation $\pi \in S_n$ and isomorphisms $f_\alpha : I_\alpha \xrightarrow{\cong} J_{\pi(\alpha)}$ such that $j_{\pi(\alpha)}\, f_\alpha = i_\alpha$ for all $\alpha = 1, \ldots, n$.*

Let $\mathcal{E}$ be a strict KS-category. Then any object X can be uniquely represented as a finite coproduct

$$X \cong \coprod_J{}' \, n_J(X)\, J, \tag{5.5.a}$$

where J's are representatives of isomorphism classes of connected objects; furthermore, $n_J(X)\, J$ is the coproduct of $n_J(X)$ copies of J and $n_J(X) \neq 0$ only for finitely many J. The above coproduct decomposition is called a *KS-decomposition* of X with *KS-type* $(n_J(X))$.

(5.6) Remark. The uniqueness of the usual Krull-Schmidt decomposition for categories with finite biproducts states that if $M = I_1 \oplus \cdots \oplus I_m = J_1 \oplus \cdots \oplus J_n$ is a bi-product (or simply coproduct) decomposition into *indecomposable* objects with canonical injections $i_\alpha : I_\alpha \longrightarrow M$ and $j_\beta : J_\beta \longrightarrow M$, then $m = n$ and there exist a permutation $\pi \in S_n$, isomorphisms $f_\alpha : I_\alpha \longrightarrow J_{\pi(\alpha)}$ and an automorphism $f \in \mathrm{Aut}(M)$ such that $f \circ i_\alpha = j_{\pi(\alpha)} \circ f_\alpha$ for all α. We can build an analogue theory for semi-simple additive KS-categories. In his book [Dur 86], Dür defined KS-categories and proved some exponential formulas for such categories; his definition of KS-categories contains the injectivity of injection into coproducts and the disjointness of coproducts (see (2.1.a)); these conditions hold for many important categories, for example, the category of finite sets, the category of finite dimensional vector spaces over a finite field (see [Dur 86, 2.15, 2.28]).

However, in this paper *a KS-category means always a strict one.*

(5.7) Lemma. *Let $X = \coprod_\alpha n_\alpha\, J_\alpha$ be a KS-decomposition of X. Then*

$$\mathrm{Aut}(X) \cong \prod_\alpha (\mathrm{Aut}(J_\alpha))\; wr\, S_{n_\alpha}, \tag{5.7.a}$$

where wr stands for a wreath product. In particular,

$$|\mathrm{Aut}(X)| = \prod_{\alpha} |\mathrm{Aut}(J_\alpha)|^{n_\alpha} n_\alpha!. \tag{5.7.b}$$

PROOF: Take the canonical injections

$$j_{\alpha\beta} : J_\alpha \longrightarrow X = \coprod_{\alpha} n_\alpha J_\alpha \quad (\beta = 1, \dots, n_\alpha).$$

Then the required group isomorphism is defined by the correspondence

$$f \longleftrightarrow ((f_{\alpha,\beta}), \pi),$$

where

$$\pi = (\pi_\alpha) \in \prod_{\alpha} S_{n_\alpha} \subseteq S_{\Sigma n_\alpha}$$

as follows. Given f, the family of morphisms $(f_{\alpha,\beta})$ and π are uniqully determined by the following commutative diagram:

$$\begin{tikzcd} J_\alpha \arrow[r, "f_{\alpha,\beta}"] \arrow[d, "j_{\alpha,\beta}"'] & J_\alpha \arrow[d, "i_{\alpha,\pi(\beta)}"] \\ X \arrow[r, "f"] & X. \end{tikzcd}$$

Conversely, given $((f_{\alpha,\beta}), \pi)$, the above commutative diagram uniquly determines the endomorphism f.

(5.8) Theorem (exponential formula). *Let $\mathcal{E}$ be a skeletally small KS-category and let $\mathcal{J} := \mathsf{Con}(\mathcal{E})$ be the full subcategory of connected objects of $\mathcal{E}$. Then*

$$\mathcal{E}(t) = \exp(\mathcal{J}(t)). \tag{5.8.a}$$

Here the power series $\mathcal{J}(t)$ is viewed as an element of $\boldsymbol{Q}[[\mathcal{E}^{\mathrm{op}}]]$ through the canonical embedding $\mathcal{J} \subseteq \mathcal{E}$.

PROOF: By the definition of KS-decomposition and Lemma 5.7, we have

$$\begin{aligned}
\exp\left(\mathcal{J}(t)\right) &= \exp\left({\sum_{J\in\mathcal{J}}}' \frac{t^J}{|\mathrm{Aut}(J)|}\right) \\
&= {\prod_{J\in\mathcal{J}}}' \exp\left(\frac{t^J}{|\mathrm{Aut}(J)|}\right) \\
&= {\prod_{J\in\mathcal{J}}}' \left(\sum_{n=0}^{\infty} \frac{t^{nJ}}{n!\,|\mathrm{Aut}(J)|^n}\right) \\
&= \sum_{(n_J)} \prod_{J\in\mathcal{J}} \frac{t^{n_J J}}{n_J!\,|\mathrm{Aut}(J)|^{n_J}} \\
&= \sum_{(n_J)} \frac{1}{\prod_J n_J!\,|\mathrm{Aut}(J)|^{n_J}}\, t^{\coprod n_J J} \\
&= \sum_{(n_J)} \frac{1}{|\mathrm{Aut}(\coprod_J n_J J)|}\, t^{\coprod n_J J} \\
&= {\sum_{X\in\mathcal{E}}}' \frac{t^X}{|\mathrm{Aut}(X)|} = \mathcal{E}(t),
\end{aligned}$$

where (n_J) in the above summations runs over all $(\mathcal{J}/\cong)$-indexed non-negative integers with ${\sum_J}' n_J < \infty$, proving the theorem.

(5.9) Corollary. *Let $\mathcal{E}$ be a skeletally small KS-category, $\mathcal{J} = \mathsf{Con}(\mathcal{E})$ the full subcategory of connected objects of $\mathcal{E}$ and $F : \mathcal{E} \longrightarrow \mathcal{S}$ a faithful functor with finite fibers preserving finite coproducts. Then the following holds:*

$$\sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E})/\cong|}{|\mathrm{Aut}(N)|}\, t^N = \exp\left(\sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{J})/\cong|}{|\mathrm{Aut}(N)|}\, t^N\right). \tag{5.9.a}$$

PROOF: Since the functor F has finite fibers, it follows from Theorem 4.6 that

$$\begin{aligned}
F_*(\mathcal{E}(t)) &= (\mathcal{E} \xrightarrow{F} \mathcal{S})(t) = \sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|}\, t^N, \\
F_*(\mathcal{J}(t)) &= (\mathcal{J} \xrightarrow{F} \mathcal{S})(t) = \sum_{N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{J}/N)/\cong|}{|\mathrm{Aut}(N)|}\, t^N.
\end{aligned}$$

Furthermore, F preserves coproducts, and so the linear map

$$F_* : R[[\mathcal{E}^{\mathrm{op}}/\cong]] \longrightarrow R[[\mathcal{S}^{\mathrm{op}}/\cong]];\ t^X \longmapsto t^{F(X)} \tag{5.9.b}$$

gives an R-algebra homomorphism. Thus

$$F_*(\exp(\mathcal{J}(t)) = \exp(F_*(\mathcal{J}(t)). \tag{5.9.c}$$

Now the statement follows from taking the image of the exponential formula $\mathcal{E}(t) = \exp(\mathcal{J}(t))$ (Theorem 5.8) by F_*.

(5.10) Theorem (binomial theorem). *Let $\mathcal{E}$ be a skeletally small KS-category and let $\mathcal{J} := \mathsf{Con}(\mathcal{E})$ be the full subcategory of connected objects of $\mathcal{E}$. Then*

$$\widetilde{\mathcal{E}}(t) = \prod_{I\in\mathcal{J}}{}' \left(\frac{1}{1-t^I}\right) \tag{5.10.a}$$

$$= \exp\left(\sum_{n\geq 1}\frac{1}{n}\sum_{I\in\mathcal{J}}{}' t^{nI}\right). \tag{5.10.b}$$

PROOF. Using KS-decomposition, we have

$$\begin{aligned}\widetilde{\mathcal{E}}(t) &= \sum_{(n_I)} t^{\coprod n_I I} \\ &= \sum_{(n_I)}\prod_{I\in\mathcal{J}}{}'(t^I)^{n_I} \\ &= \prod_{I\in\mathcal{J}}{}'\sum_{n\geq 0}(t^I)^n \\ &= \prod_{I\in\mathcal{J}}{}'\frac{1}{1-t^I},\end{aligned}$$

where (n_J) runs over nonnegative integers indexed by $\mathcal{J}/\cong$ such that $\sum n_J < \infty$ as before. (5.10.b) follows from (5.10.a) by an easy calculation.

6 Examples

(6.1) Lemma. *Assume*

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}t^n = \exp\left(\sum_{n=1}^{\infty}\frac{b_n}{n!}t^n\right).$$

Then the following hold:

(1) $a_n = n! \sum_{\mu \vdash n} \prod_i \frac{1}{\mu_i!} \left(\frac{b_i}{i!}\right)^{\mu_i} \quad (n \geq 1), \quad a_0 = 1,$

where the summation is taken over $(\mu_1, \mu_2, \dots)$ *of non-negative integers such that* $\sum_i i\mu_i = n$.

(2) $b_n = -n! \sum_{\mu \vdash n} (-1 + \sum_i \mu_i)! \prod_i \frac{1}{\mu_i!} \left(\frac{-a_i}{i!}\right)^{\mu_i} \quad (n \geq 1).$

(3) $a_{n+1} = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i+1} \quad (n \geq 0).$

PROOF. Trivial.

(6.2) Bell polynomials. Here is a typical but rather old example ([Rio 58 , 2.8]). Let Surj be the category of surjections of finite sets (Example 2.8). A connected object in this category has the form of

$$J(n) := ([n] \longrightarrow [1]), \quad [n] := \{1, 2, \dots, n\}, \quad (n \geq 1). \tag{6.2.a}$$

Then the isomorphism classes of objects of $(X \overset{p}{\twoheadrightarrow} X')$ such that $|X| = n$ are bijectively correspondent to the partitions of n; in fact, the surjection p gives a partition $\mu = 1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n}$, where μ_i is the number of elements $x' \in X'$ such that $p^{-1}(x')$ has i elements, and so (μ_i) is the KS-type of $(X \overset{p}{\twoheadrightarrow} X')$; and conversely a partition $\mu = 1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n}$ gives an object

$$\coprod_{i=1}^{n} \mu_i \, J(i) \tag{6.2.b}$$

of Surj. Since $\mathrm{Aut}(J(i)) \cong S_i$, we have

$$\mathrm{Aut}(\coprod_i \mu_i \, J(i)) \cong \prod_i S_i \,\mathrm{wr}\, S_{\mu_i}. \tag{6.2.c}$$

Thus

$$\begin{aligned} \mathsf{Surj}(t) &= \sum_{[X \twoheadrightarrow X']}{}' \frac{1}{|\mathrm{Aut}(X \twoheadrightarrow X')|} t^{[X \twoheadrightarrow X']} \\ &= \sum_{n=0}^{\infty} \sum_{\mu \vdash n} \frac{1}{\prod_i \mu_i! \, i!^{\mu_i}} t^{\coprod \mu_i \, J(i)} \\ &= \sum_{n=0}^{\infty} \sum_{\mu \vdash n} \frac{1}{\mu_1! \cdots \mu_n!} \left(\frac{t^{J(1)}}{1!}\right)^{\mu_1} \cdots \left(\frac{t^{J(n)}}{n!}\right)^{\mu_n}, \end{aligned} \tag{6.2.d}$$

where $\mu = 1^{\mu_1} \cdots n^{\mu_n}$ runs over the partitions of n. This summation is conventionally written as if it is a power series in one variable (for example [Ber 89]):

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{1}{\lambda!} t^{\lambda}. \tag{6.2.e}$$

On the other hand, the exponential generating function associated to $\mathcal{J} := \mathsf{Con}(\mathsf{Surj})$, the category of connected objects in Surj, is as follows:

$$\mathcal{J}(t) = \sum_{i=1}^{\infty} \frac{t^{J(i)}}{|\mathrm{Aut}(J(i))|} = \sum_{i=1}^{\infty} \frac{t^{J(i)}}{i!}. \tag{6.2.f}$$

Thus the exponential formula gives the following identity (Lemma 6.1 (1)):

$$\sum_{n=0}^{\infty} \sum_{\mu \vdash n} \frac{1}{\mu_1! \cdots \mu_n!} \left(\frac{t^{J(1)}}{1!} \right)^{\mu_1} \cdots \left(\frac{t^{J(n)}}{n!} \right)^{\mu_n} = \exp \left(\sum_{i=1}^{\infty} \frac{t^{J(i)}}{i!} \right) \tag{6.2.g}$$

This identity is usually written by using so-called Bell polynomials. Substituting $f\, g_i\, t^i$ for each indeterminate $t^{J(i)}$, we have

$$\mathsf{Surj}(t) = \sum_{n=0}^{\infty} \frac{Y_n(f g_1, \ldots, f g_n)}{n!} t^n, \tag{6.2.h}$$

where the polynomial

$$Y_n(f g_1, \ldots, f g_n) := \sum_{\mu \vdash n} \frac{n!}{\mu_1! \cdots \mu_n!} \left(\frac{f\, g_1}{1!} \right)^{\mu_1} \cdots \left(\frac{f\, g_n}{n!} \right)^{\mu_n}. \tag{6.2.i}$$

in variables $f, g_1, g_2, \ldots, g_n$ is just the Bell polynomial. Thus the exponential formula gives the following well-known formula:

$$\sum_{n=0}^{\infty} \frac{Y_n(f\, g_1, \ldots, f g_n)}{n!} t^n = \exp \left(\sum_{i=1}^{\infty} \frac{f g_i\, t^i}{i!} \right). \tag{6.2.j}$$

Furthermore, the binomial theorem (Theorem 5.10) simply means the follow trivial identity:

$$\widetilde{\mathsf{Surj}}(t) = \sum_{[X \twoheadrightarrow X']}{}' t^{[X \twoheadrightarrow X']} = \prod_{i \geq 1} \frac{1}{1 - t^{J(i)}}. \tag{6.2.k}$$

Substituting the multiplicative invariant

$$t^{J(i)} \longmapsto \begin{cases} t & (i \leq m) \\ 0 & (i > m), \end{cases}$$

we have the usual binomial identity:

$$\begin{aligned}(1-t)^{-m} &= \sum_{n_1,\dots,n_m \geq 0} t^{\sum n_i} \\ &= \sum_{k=0}^{\infty} \sharp\{(n_1,\dots,n_m) \mid \sum n_i = k\}\, t^k \\ &= \sum_{k=0}^{\infty} \binom{m+k-1}{k} t^k.\end{aligned}$$

Furthermore, substituting $t^{|X|}$ for $t^{[X \twoheadrightarrow X']}$ in (6.2.k), we have

$$\sum_{[X \twoheadrightarrow X']}{}' \; t^{|X|} = 1 + \sum_{n=1}^{\infty} p(n)t^n = \prod_{i \geq 1} \frac{1}{1-t^i}, \tag{6.2.l}$$

where $p(n)$ is the partition number; here remember that the isomorphism class of the surjection p is determined by the partition corresponding to p.

(6.3) Bell numbers. A Bell number $b(n)$ is the number of equivalence relations on $[n]$. Then as is well known, the exponential generating function of the Bell numbers satisfies the following:

$$\sum_{n=0}^{\infty} \frac{b(n)}{n!} t^n = \exp(e^x - 1). \tag{6.3.a}$$

This formula directly follows from our exponential formula. Let Surj be the category of surjections $(X \xrightarrow{p}\!\!\!\!\rightarrow X')$ and $F : \mathsf{Surj} \longrightarrow \mathsf{Set}_f$ a faithful functor defined by $F(X \stackrel{p}{\twoheadrightarrow} X') := X$. Any Surj-structure $((X \stackrel{p}{\twoheadrightarrow} X'), (\sigma : X \xrightarrow{\cong} N))$ on N is isomorphic to $((N \stackrel{p\sigma^{-1}}{\twoheadrightarrow} X'), 1)$; furthermore, two Surj-structures $((N \stackrel{p}{\twoheadrightarrow} X'), 1)$ and $((N \stackrel{q}{\twoheadrightarrow} X''), 1)$ are isomorphic if and only if there is a bijection $g : X' \longrightarrow X''$ such that $g \circ p = q$. Thus an isomorphism class of Surj-structure on N is bijectively corresponding to an isomorphism class of surjection from N, that is, an equivalence relation on N, and so we have

$$(\mathsf{Surj} \xrightarrow{F} \mathsf{Set}_f)(t) = \sum_{n=0}^{\infty} \frac{|\mathsf{Str}(\mathsf{Surj}/[n])/\cong|}{n!} t^n = \sum_{n=0}^{\infty} \frac{b(n)}{n!} t^n, \tag{6.3.b}$$

where $[n] := \{1,\dots,n\}$. On the other hand, an object $(X \twoheadrightarrow X')$ of Surj is connected if and only if X' is a one-point set, and so $\mathcal{J} = \mathsf{Con}(\mathsf{Surj})$ is equivalent to the category of nonempty finite sets. Thus we have

$$(\mathcal{J} \xrightarrow{F} \mathsf{Set}_f)(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1. \tag{6.3.c}$$

Now, the required formula (6.3.a) on the Bell number generating function follows from the exponential formula.

(6.4) Wohlfahrt formula. Let G be a finitely generated group and $\mathsf{Set}_f{}^G$ the category of finite G-sets and G-maps with the forgetful functor

$$F : \mathsf{Set}_f{}^G \longrightarrow \mathsf{Set}_f; X \longmapsto X$$

Any connected finite G-set is isomorphic to the homogeneous G-set G/H for a subgroup H of G of finite index, and $G/H \cong G/K$ if and only if H and K are conjugate in G. Since $\mathrm{Aut}(G/H) \cong N_G(H)/H$ and the number of subgroups conjugate to a subgroup H is equal to $(G : N_G(H))$, we have

$$(\mathsf{Con}(\mathsf{Set}_f{}^G))(t) = \sum_{G/H}{}' \frac{t^{G/H}}{|\mathrm{Aut}(G/H)|} = \sum_{H \leq_f G} \frac{t^{G/H}}{(G : H)}, \tag{6.4.a}$$

where H runs over subgroups of G of finite index. Thus we have the following exponential formula by Theorem 5.8:

$$\mathsf{Set}_f{}^G(t) = \sum_{X}{}' \frac{t^X}{|\mathrm{Aut}(X)|} = \exp\left(\sum_{H \leq_f G} \frac{t^{G/H}}{(G : H)}\right). \tag{6.4.b}$$

A pplying Corollary 5.9 to the functor F, we have the Wohlfahrt formula ([Woh 77]):

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} t^n = \exp\left(\sum_{H \leq_f G} \frac{t^{(G:H)}}{(G : H)}\right). \tag{6.4.c}$$

In general, $|\mathrm{Hom}(G, S_n)|$ expresses the number of solutions to some simultaneous equation in S_n. For example, let G be a free abelian group of rank 2, so that $|\mathrm{Hom}(G, S_n)|$ equals the number of pairs (x, y) in S_n such that xy = yx; in this case, G has $\sigma(m)$ subgroups of index m, where $\sigma(m)$ is the summation of all divisors of n, Thus we have

$$\sum_{n=0}^{\infty} \frac{\sharp\{(x, y) \in S_n \times S_n \mid xy = yx\}}{n!} t^n = \exp\left(\sum_{m=1}^{\infty} \frac{\sigma(m)}{m} t^m\right).$$

Note that, as is well-known, for any finite group G

$$\frac{1}{|G|}\sharp\{(x,y)\in G\times G \mid xy=yx\}$$

equals the number of conjugacy classes of G. Thus we have

$$\sum_{n=0}^{\infty} p(n)t^n = \exp\left(\sum_{m=1}^{\infty}\frac{\sigma(m)}{m}t^m\right), \tag{6.4.d}$$

where $p(n)$ denotes the number of partitions of n. This formula is another expression of the following fundamental identity([Rio 58 , Chapter 6 Section 2], [Aig 79, Proposition 5.16]):

$$\sum_{n=1}^{\infty} p(n)t^n = \prod_{n=1}^{\infty}\frac{1}{1-t^n}.$$

On generating functions related to group theory, refer to Yoshida [Yos 96].

(6.5) Even permutation representations. Let G be a finitely generated group. An *even G-set* is a finite G-set corresponding to a group homomorphism $G \longrightarrow A_n$ into an alternating group A_n of some degree. A transitive G-set G/H associated to a subgroup H is even if and only if

$$|\langle x\rangle\backslash G/H| \equiv (G:H) \pmod 2 \tag{6.5.a}$$

for any 2-element x of G. We define the *signature* of a finite G-set by

$$sgn(X) := \begin{cases} +1 & \text{if } X \text{ is an even } G\text{-set,} \\ -1 & \text{else.} \end{cases}$$

Clearly, it satisfies $sgn(X+Y) = sgn(X)\cdot sgn(Y)$. The category of even G-sets is not a KS-category, and so we can not apply our exponential formula directly.

However, the generating function for even G-sets can be easily obtained from the exponential formula:

$$\sum_X{}' \frac{t^X}{|\mathrm{Aut}(X)|} = \exp\left(\sum_{H\le_f G}\frac{t^{G/H}}{(G:H)}\right),$$

In fact, substituting $sgn(X)\, t^X$ for t^X, we have

$$\sum_X{}' \frac{sgn(X)\, t^X}{|\mathrm{Aut}(X)|} = \exp\left(\sum_{H\le_f G}\frac{sgn(G/H)\, t^{G/H}}{(G:H)}\right), \tag{6.5.b}$$

and so

$$\sum_{X\,:\,\text{even}}{}' \frac{t^X}{|\mathrm{Aut}(X)|} = \frac{1}{2}\exp\left(\sum_{H\leq_f G}\frac{t^{G/H}}{(G:H)}\right) + \frac{1}{2}\exp\left(\sum_{H\leq_f G}\frac{sgn(G/H)t^{G/H}}{(G:H)}\right), \quad (6.5.c)$$

where X runs over a complete set of representatives of the isomorphism classes of even G-sets and H runs over all subgroups of G of finite index.

By the same way as the proof of the Wohlfahrt formula, we obtain the following required formula for the number of homomorphisms into alternating groups:

$$\sum_{n=0}^{\infty}\frac{|\mathrm{Hom}(G,A_n)|}{n!}t^n = \frac{1}{2}\exp\left(\sum_{H\leq_f G}\frac{t^{(G:H)}}{(G:H)}\right) + \frac{1}{2}\exp\left(\sum_{H\leq_f G}\frac{sgn(G/H)t^{(G:H)}}{(G:H)}\right). \quad (6.5.d)$$

In particular, when G is a cyclic group of order m, we have

$$\sum_{n=0}^{\infty}\frac{\sharp\{\pi\in A_n \mid \pi^m=1\}}{n!}t^n = \frac{1}{2}\exp\left(\sum_{k|m}\frac{t^k}{k}\right) + \frac{1}{2}\exp\left(\sum_{k|m}\frac{(-1)^{k-1}t^k}{k}\right). \quad (6.5.e)$$

See Riordan [Rio 58 , p.89 Ex.22]. A essentially same formula as (6.5.d) for an abelian group G is also proved by Takegahara [Tak98].

(6.6) Cyclic sets. For an infinite cyclic group C, a finite C-set X is viewed as a *cyclic set* (X,π), that is, a finite set X equipped with a permutation π on X (see (2.3)). A morphism $f:(X,\pi)\longrightarrow(Y,\tau)$ is a map $f:X\longrightarrow Y$ such that $\tau\circ f=f\circ\pi$. The category of cyclic sets is denoted by $\mathsf{Set}_f{}^C$. The *type* of the cyclic set (X,π) is the type of the permutation π. The isomorphism classes of cyclic sets are completely characterized by their types. If $1^{\mu_1}2^{\mu_2}\cdots$ is the type of (X,π), then (X,π) has the KS-decomposition

$$(X,\pi)\cong\mu_1\,[1]+\cdots+\mu_i\,[i]+\cdots, \quad (6.6.a)$$

where $[i] := \{1, \dots, i\}$ is the unique connected cyclic set with cyclic permutation $(1, 2, \dots, i)$. Since $\mathrm{Aut}([i])$ is cyclic group of order i, KS-decomposition gives

$$|\mathrm{Aut}(\coprod_i \mu_i[i])| = \prod_i i^{\mu_i}\mu_i!. \tag{6.6.b}$$

Thus

$$\mathsf{Set}_f{}^C(t) = \sum_{(\mu_i)} \prod_i \frac{1}{\mu_i!}\left(\frac{t_i}{i}\right)^{\mu_i}, \quad (t_i := t^{[i]}), \tag{6.6.c}$$

$$\mathsf{Con}(\mathsf{Set}_f{}^C)(t) = \sum_{i=1}^{\infty} \frac{1}{i} t_i. \tag{6.6.d}$$

By the exponential formula we have

$$\sum_{(\mu_i)} \prod_i \frac{1}{\mu_i!}\left(\frac{t_i}{i}\right)^{\mu_i} = \exp\left(\sum_{i=1}^{\infty} \frac{1}{i} t_i\right). \tag{6.6.e}$$

Now for any non-negative integer m, define a faithful endo-functor as follows:

$$F_m : \mathsf{Set}_f{}^C \longrightarrow \mathsf{Set}_f{}^C;\ (X, \pi) \longrightarrow (X, \pi^m)$$

The end-functor F_m of $\mathsf{Set}_f{}^C$ is called a Frobenius functor; it has a left adjoint functor V_m called a Verschiebung functor, and so F_m has finite fibers by Lemma 2.6. Refer to [Men 93], [Che 93a], [Che 93b, 2.8], [Nav 87, 1.4], [Haz 78] for Frobenius and Verschiebung functors; and refer to [DreSi 89] for the category of cyclic sets, especially its Burnside ring and its relations with Witt vectors, Necklace rings and lambda rings.

Let (N, ρ_μ) be a cyclic set of type μ with $|N| = n$.. Then there is a bijection

$$\mathsf{Str}(\mathsf{Set}_f{}^C/(N, \rho_\mu))/\cong \ \longleftrightarrow \{\pi \in \mathrm{Sym}(N) \mid \pi^m = \rho_\mu\}$$

defined by

$$\begin{aligned} &[(X, \pi), \sigma] \longmapsto \sigma\pi\sigma^{-1}, \\ &\pi \longmapsto [(N, \pi), 1]. \end{aligned}$$

Thus we have

$$\begin{aligned} (\mathsf{Set}_f{}^C \xrightarrow{F_m} \mathsf{Set}_f{}^C)(t) &= \sum_{(N,\rho_\mu)}{}' \frac{\sharp\{\pi \in \mathrm{Sym}(N) \mid \pi^m = \rho_\mu\}}{|\mathrm{Aut}(N, \rho_\mu)|} t^{(N,\rho_\mu)} \\ &= \sum_{n=0}^{\infty} \sum_{\mu \vdash n} \frac{\sharp\{\pi \in S_n \mid \pi^m = \rho_\mu\}}{\mu_1! \cdots \mu_n!} \prod_i \left(\frac{t^{[i]}}{i!}\right)^{\mu_i}. \end{aligned}$$

Next, let $\mathcal{J} := \mathsf{Con}(\mathsf{Set}_f{}^C)$ be the category of connected cyclic sets $[n]$ $(n = 1, 2, \dots)$, so that $F_m([n]) \cong d\,[n/d]$, where $d := (m, n)$, the greatest common divisor. Thus

$$
\begin{aligned}
(\mathcal{J} \xrightarrow{F} \mathsf{Set}_f{}^C)(t) &= \sum_{n=1}^{\infty} \frac{1}{|\mathrm{Aut}([n])|} t^{F([n])} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} t^{(m,n)\,[n/(m,n)]} \\
&= \sum_{d|m}^{\infty} \sum_{(k,m)=1} \frac{1}{dk} t^{d\,[k]},
\end{aligned}
$$

where k runs over nonnegative integers relatively prime to m. Substituting t_n for $t^{J(n)}$, we have the following exponential formula:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\mu \vdash n} \frac{\sharp\{\pi \in S_n \mid \pi^m = \rho_\mu\}}{\mu_1! \cdots \mu_n!} \prod_i \left(\frac{t_i}{i!}\right)^{\mu_i} \\
&= \sum_{n=0}^{\infty} \sum_{\mu \vdash n} \sharp\{\pi \in S_n \mid \pi^m \text{ is of type } \mu\} \prod_i \left(\frac{t_i}{(i-1)!}\right)^{\mu_i} \\
&= \exp\left(\sum_{d|m} \sum_{(k,m)=1} \frac{1}{dk} t_k{}^d \right). \qquad (6.6.\mathrm{f})
\end{aligned}
$$

For $m = 1$, this formula gives the well-known exponential formula for the cycle indicators of symmetric groups ([Rio 58 , 4.2], [Aig 79, Prop.5.13]).

(6.7) Labeled trees. One of the most famous classical application of exponential formula may be the enumeration of labeled tree. A rooted tree is a finite tree with a distinguished vertex called the root; and a rooted forest is a disjoint union of finite number of rooted trees. Let RForest be the category of rooted forests (resp. rooted trees) in which an object of this category is an (unlabeled) rooted forest and a morphism between two rooted forests is a mapping between vertices preserving edges and roots; similarly let RTree be the category of rooted trees. Then RForest is a KS-category with Con(RForest) = RTree, and so

$$
\mathsf{RForest}(t) = \exp\left(\mathsf{RTree}(t)\right). \qquad (6.7.\mathrm{a})
$$

The category RForest has some good properties and it has a strong resembrance to Set_f and $\mathsf{Set}_f{}^G$. In fact, RForest is almost a topos and its object is viewed as a contravariant functor from the poset $\{0 < 1 < 2 < \cdots\}$ to

Set_f; unfortunately, exponentials and a subobject classifier are infinite rooted forests.

Let $V : \mathsf{RForest} \longrightarrow \mathsf{Set}_f$ be a faithful functor which assigns the vertex set to a rooted forest. Then a $\mathsf{RForest}$-structure on $[n] = \{1, \ldots, n\}$ is a labeled rooted forest with n vertices. Let v_n (resp. u_n) be the number of labeled rooted forests (resp. labeled rooted trees) on n vertices. Then the exponential formula (Corollary 5.9) gives

$$\sum_{n=0}^{\infty} \frac{v_n}{n!} t^n = \exp\left(\sum_{n=1}^{\infty} \frac{u_n}{n!}\right) \tag{6.7.b}$$

Furthermore, since there is an equivalence of categories

$$\Gamma : \mathsf{RForest} \xrightarrow{\cong} \mathsf{RTree};\ F \longmapsto F \cup \{*\}, \tag{6.7.c}$$

where $*$ is a new root, we have $v_n = u_{n+1}$ and Cayley's famous formula:

$$\begin{aligned} u(t) &= t\exp(u(t)), & \text{(6.7.d)}\\ u(t) &:= \sum_{n=1}^{\infty} \frac{u_n}{n!} t^n = t + t^2 + \frac{3}{2}t^3 + \cdots . & \text{(6.7.e)} \end{aligned}$$

See [Aig 79, Prop. 5.11], [Sta98, Chapter 5]; furthermore, for the exponential formula for labeled graphs, see [Gil56],

Next, by the binomial theorem (Theorem 5.10) we have

$$\begin{aligned} \widetilde{\mathsf{RForest}}(t) &:= \sum_F{}' t^F \\ &= \prod_T{}' \frac{1}{1-t^T} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_T{}' t^{nT}\right)\right), \end{aligned} \tag{6.7.f}$$

where F (resp. T) runs over all non-isomorphic rooted forest (resp.tree). Let $f(t)$ (resp. $c(x)$) be the ordinary generating function of unlabeled forests (trees), that is,

$$\begin{aligned} f(t) &:= \sum_{n=0}^{\infty} f_n t^n, \\ c(t) &:= \sum_{n=0}^{\infty} c_n t^n, \end{aligned}$$

where f_n (resp. c_n) is the number of non-isomorphic rooted forests (trees) with n vertices. Note that $t\,f(t) = c(t)$. Since the vertex functor $F \longmapsto V(F)$ preserves finite coproducts, the specialization $t^F \longmapsto t^{|V(F)|}$ gives the following well-known formula:

$$f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} c(t^n)\right), \tag{6.7.g}$$

The proof found in Harary-Palmer's book [HarPa 73, Section 3.1] needs Polya's enumeration theorem.

(6.8) Further Cayley type formulas. We use the notation as in the preceding paragraph. By using the correspondence (6.7.c), we rewrite the exponential formula (6.7.a) as follows:

$$\sum_T{}' \frac{t^{T-\{*\}}}{|\mathrm{Aut}(T)|} = \exp\left(\sum_T{}' \frac{t^T}{|\mathrm{Aut}(T)|}\right), \tag{6.8.a}$$

where T runs over all non-isomorphic trees and $T - \{*\}$ is the forest obtained by removing its root $*$ from the tree T.

Let $\mathrm{Sub}(F)$ denotes the set of the sub-rooted forests of a forest F; note that an empty forest $\emptyset$ is a sub-rooted forest of any rooted forests (and rooted trees) and that each rooted tree has at least two sub-rooted forests. Let $ht(F)$ denote the height of F. Then

$$\tau : F \longmapsto \begin{cases} |\mathrm{Sub}(F)|^z\, t^{|V(F)|} & (ht(F) \geq h) \\ 0 & (\text{otherwise}) \end{cases}$$

is a multiplicative invariant. Thus substituting τ for (6.8.a), we have

$$\sum_{ht(T)\leq h+1}{}' \frac{(|\mathrm{Sub}(T)|-1)^z}{|\mathrm{Aut}(T)|} t^{|V(T)|} = t \exp\left(\sum_{ht(T)\leq h}{}' \frac{|\mathrm{Sub}(T)|^z}{|\mathrm{Aut}(T)|} t^{|V(T)|}\right). \tag{6.8.b}$$

Here we used the following formula for the forest $T - \{*\}$:

$$ht(T - \{*\}) = ht(T) - 1$$
$$|\mathrm{Sub}(T - \{*\})| = |\mathrm{Sub}(T)| - 1.$$

As a special case, put

$$u_h(t) := \sum_{ht(T)\leq h}{}' \frac{t^{|V(T)|}}{|\mathrm{Aut}(T)|}.$$

Then $u_0(t) = t$ and

$$u_{h+1}(t) = t\exp(u_h(t)). \tag{6.8.c}$$

We denote by $\mathrm{Sub}^*(T)$ the set of sub-rooted trees of a tree T, so that

$$|\mathrm{Sub}^*(T)| = |\mathrm{Sub}(T)| - 1.$$

Letting $z = 1$ and $h \to \infty$ in (6.8.b), we have

$$u^*(t) = t\exp(u^*(t) + u(t)), \tag{6.8.d}$$

where

$$\begin{aligned} u^*(t) \quad &:= \quad {\sum_T}' \frac{|\mathrm{Sub}^*(T)|}{|\mathrm{Aut}(T)|} t^{|V(T)|} \\ &= \quad t + 4\frac{t^2}{2!} + 30\frac{t^3}{3!} + 332\frac{t^4}{4!} + \cdots \end{aligned}$$

and the series $u(t)$ is defined in (6.7.e). Note that if we let u_n^* be the number of the pairs (T, F) of a rooted tree T on $[n]$ and a sub-rooted tree F, then

$$u^*(t) = \sum_{n=1}^{\infty} \frac{u_n^*}{n!} t^n.$$

By Cayley's formula (6.7.d), we see that the equation (6.8.d) means that $u^*(t)$ as a function of $u(t)$ satisfies Cayley's equation:

$$u^*(t) = u(t)\exp(u^*(t)),$$

and so we have the following formula:

$$u^*(t) = u(u(t)). \tag{6.8.e}$$

(6.9) A mass formula for self dual codes. For simplicity, we study only binary codes. See MacWilliams–Sloane's book [McwSl 77] as a general reference for error-correcting code theory. Let $\boldsymbol{F}_2 = \{0, 1\}$ be a 2-element field. For a finite set N, let ${\boldsymbol{F}_2}^N$ be the vector space of N-indexed elements $(v_i)_{i\in N}$, $v_i \in \boldsymbol{F}_2$. The *weight* of a vector $v(v_i)_i$ of $\boldsymbol{F}^N$ is defined by

$$wt(v) := \sharp\{i \in N \mid v_i \neq 0\}. \tag{6.9.a}$$

A map $f : M \longrightarrow N$ between finite sets induces a linear map

$$\hat{f} : {\boldsymbol{F}_2}^M \longrightarrow {\boldsymbol{F}_2}^N;\ (u_i)_{i\in M} \longrightarrow (\sum_{i\in f^{-1}(j)} u_i)_{j\in N}. \tag{6.9.b}$$

Thus we have a functor from the category of finite sets to the category of $\boldsymbol{F}_2$-vector spaces:

$$\mathsf{Set}_f \longrightarrow \mathsf{Vect}_{\boldsymbol{F}_2};\ N \longmapsto \boldsymbol{F}_2{}^N,\ f \longmapsto \hat{f}. \tag{6.9.c}$$

Now, a (binary linear) *code* (N, C), or simply C, is a pair of a coordinates N and a subspace C of $\boldsymbol{F}_2{}^N$. When $|N| = n$, a code $C \subseteq \boldsymbol{F}_2{}^N$ is called a code of *length* n. The dual code $C^\perp$ consists of all vectors v with $u \cdot v = \sum_i u_i v_i = 0$ for all $u \in C$; clearly $\dim C^\perp = |N| - \dim C$. A code C is self-dual if $C = C^\perp$; for a self-dual code C, we have $\dim C = |N|/2$. A morphism $f : (M, B) \longrightarrow (N, C)$ of codes is defined as a map $f : M \longrightarrow N$ such that $\hat{f}(B) \subseteq C$. Codes and morphisms form a category Code. We denote by sdCode the full subcategory of self-dual codes. Both categories are KS-categories; the coproduct of codes (M, B) and (N, C) is given by

$$(M, B) + (N, C) = (M + N, C \oplus D). \tag{6.9.d}$$

A connected object in Code is usually called *indecomposable*; we denote by $\mathsf{Con}(\mathsf{sdCode})$ the category of indecomposable self-dual codes. We are interested in the enumeration of self-dual codes. The exponential formula for self-dual codes has the following form:

$$\mathsf{sdCode}(t) = \exp(\mathsf{Con}(\mathsf{sdCode})), \tag{6.9.e}$$

Let $\mathcal{C}(n)$ be the set of non-isomorphic self-dual codes of length n, and let $\mathcal{I}(n)$ be its subset of indecomposable codes. Let a_n be the number of self-dual codes of length n and b_n the number of indecomposable self-dual codes on length n, so that

$$a_n = \sum_{C \in \mathcal{C}(n)} \frac{n!}{|\mathrm{Aut}(C)|}, \quad b_n = \sum_{C \in \mathcal{I}(n)} \frac{n!}{|\mathrm{Aut}(C)|}, \tag{6.9.f}$$

where $\mathrm{Aut}(C)$ denotes the automorphism group of a code C. Note that the number of codes of length n isomorphic to a self-dual code C is equal to $n!/|\mathrm{Aut}(C)|$. The following mass formula ([McwST 72], [PleSl 75]) is useful for the classification problem of self-dual codes of small length ([PleSl 75]):

$$a_0 = a_2 = 1,\ a_{2k} = \prod_{i=1}^{k-1} (2^i + 1),\ a_{2k+1} = 0, \tag{6.9.g}$$

Unfortunately, there is no enumeration formula for $\mathcal{C}(n)$ itself.

There is a faithful functor

$$F : \mathsf{sdCode} \longrightarrow \mathsf{Set}_f;\ (N, C) \longmapsto N.$$

A sdCode-structure on a finite set N is exactly a self-dual code in ${\boldsymbol{F}_2}^N$, and so the exponential generating function of F is

$$f(t) := (\mathsf{sdCode} \xrightarrow{F} \mathsf{Set}_f)(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n. \tag{6.9.h}$$

Similarly, we have

$$g(t) := (\mathsf{Con}(\mathsf{sdCode}) \xrightarrow{F} \mathsf{Set}_f)(t) = \sum_{n=1}^{\infty} \frac{b_n}{n!} t^n. \tag{6.9.i}$$

Then by the exponential formula, we have

$$f(t) = \exp(g(t)), \quad \text{or} \quad g(t) = \log(f(t)). \tag{6.9.j}$$

Thus we can calculate the total number b_n of indecomposable self-dual codes of length n by using Lemma 6.1 and the fact that $a_{2k+1} = b_{2k+1} = 0$ as follows:

$$\begin{aligned} a_{2k} &= (2k)! \sum_{\mu \vdash k} \prod_i \frac{1}{\mu_i!} \left(\frac{b_{2i}}{(2i)!} \right)^{\mu_i}, \\ b_{2k} &= -(2k)! \sum_{\mu \vdash k} (-1 + \sum \mu_i)! \prod_{i \geq 1} \left(\frac{(-1)^{\mu_i}}{\mu_i!(2i)!^{\mu_i}} (2^i + 1)^{\sum_{j>i} \mu_j} \right) \end{aligned}$$

(6.10) Remark. There is anther definition of the category of codes. In his paper [Ass 98], Assmus defined a code homomorphism $\varphi : (M, C) \longrightarrow (N, D)$ by a linear map $\varphi : C \longrightarrow D$ satisfying

$$wt\,(\varphi(u)) \leq wt\,(u), \quad (u \in C).$$

For example, for a subset $M \subseteq N$, the punctuation $C' \subseteq \boldsymbol{F}_2^M$ of a code $C \subseteq \boldsymbol{F}_2^N$ consists of codewords of the form $(u_i)_{i \in M}$, where $(u_i)_{i \in N}$ is an element of C. Then the canonical projection $C \longrightarrow C'$ is a code homomorphism in the sense of Assmus; however, there is no morphism in Code from (N, C) to (M, C') unless $M = N$. The category Code$'$ (resp. sdCode$'$) of codes (resp. self-dual codes) and Assumus' code homomorphisms also makes a strict Krull-Schmidt category, which contains Code (resp. sdCode) as a subcategory. We have the same exponential formula by using sdCode$'$ instead of sdCode.

(6.11) A mass formula for weight enumerators. The weight enumerator $w_C(x,y)$ of a code $C \subseteq \boldsymbol{F}_2^N$ is a polynomial in two-variables x, y defined as follows:

$$w_C(x,y) := \sum_{u \in C} x^{|N|-wt(u)} y^{wt(u)}. \tag{6.11.a}$$

For a self-dual code C, two identity hold:

$$w_C(x,y) = w_C\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right), \; w_C(x,y) = w_C(x,-y)$$

([McwSl 77, 5.2]). We introduce the following two power series:

$$\begin{aligned} F_{x,y}(t) &:= \sum_{n=0}^{\infty} \sum_{C \in \mathcal{C}(n)}{}' \frac{w_C(x,y)}{|\mathrm{Aut}(C)|} t^n, \\ G_{x,y}(t) &:= \sum_{n=0}^{\infty} \sum_{D \in \mathcal{I}(n)}{}' \frac{w_D(x,y)}{|\mathrm{Aut}(D)|} t^n, \end{aligned}$$

where $\mathcal{C}(n)$ is the set of non-isomorphic self-dual codes of length n and $\mathcal{I}$ is its subset of indecomposable codes. Since $w_{C \oplus D}(x,y) = w_C(x,y)\, w_D(x,y)$ and $n(C \oplus D) = n(C) + n(D)$, where $n(C)$ denotes the length of C, we have a multiplicative invariant of sdCode:

$$\omega : \mathsf{sdCode} \longrightarrow \boldsymbol{Q}[[x,y,t]]; \; C \longmapsto w_C(x,y) t^{n(C)}. \tag{6.11.b}$$

Subtituting ω for the exponential formula (6.9.e), we have

$$F_{x,y}(t) = \exp(G_{x,y}(t)). \tag{6.11.c}$$

On the other hand, the mass formula ([PleSl 75, Section 4], [McwSl 77, 19.6]) for $w_C(x,y)$ on all self-dual codes of even length n states

$$\begin{aligned} W_n(x,y) &:= n! \sum_{C \in \mathcal{C}(n)} \frac{w_C(x,y)}{|\mathrm{Aut}(C)|} \\ &= 2^{(n/2)-1} a_{n-2} \left[x^n + y^n + \left(\frac{x+y}{\sqrt{2}}\right)^n + \left(\frac{x-y}{\sqrt{2}}\right)^n \right] \quad (n \geq 4), \end{aligned}$$

where a_n is the number of self-dual codes of length n given in (6.9.g); here, $W_n(x,y) = 0$ for an odd n and

$$W_0(x,y) = 1, \; W_2(x,y) = x^2 + y^2.$$

Thus using (6.11.c) and Lemma 6.1, we can theoretically calculate the summation of all the weight enumerators of indecomposable self-dual codes of length n:

$$W_n^I(x,y) := n! \sum_{D \in \mathcal{I}(n)} \frac{w_D(x,y)}{|\mathrm{Aut}(D)|}.$$

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