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Reticular Legendrian Singularities

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1 Introduction

In [6] K.Jänich explained the wavefront propagation mechanism on a manifold which is completely described by a positive and positively homogeneous Hamiltonian function on the cotangent bundle and investigated the local gradient models given by the ray length function. Wavefronts generated by an initial wavefront which is a hypersurface without boundary in the manifold is investigated as Legendrian singularities by V.I.Arnold (cf., [1]). In [9], I.G.Scherbak studied the case when the hypersurface has a boundary and she explained the wavefronts generated by the hypersurface with a boundary corresponds to a generalized notion of wavefronts (i.e., the *boundary fronts*).

In this paper we investigate the more general case when the hypersurface has an r-*corner*. In this case each wavefront incident from each edge of the hypersurface gives a *contact regular r-cubic configuration* (cf., Section 5) at a point of the 1-jet bundle which is a generalization of the notion of Legendrian submanifolds. In complex analytic category, the notion of contact regular r-cubic configurations is introduced by Nguyen Huu Duc, Nguyen Tien Dai and F.Pham (cf., [2], [5]). But all contact regular r-cubic configuration in their category is stable.

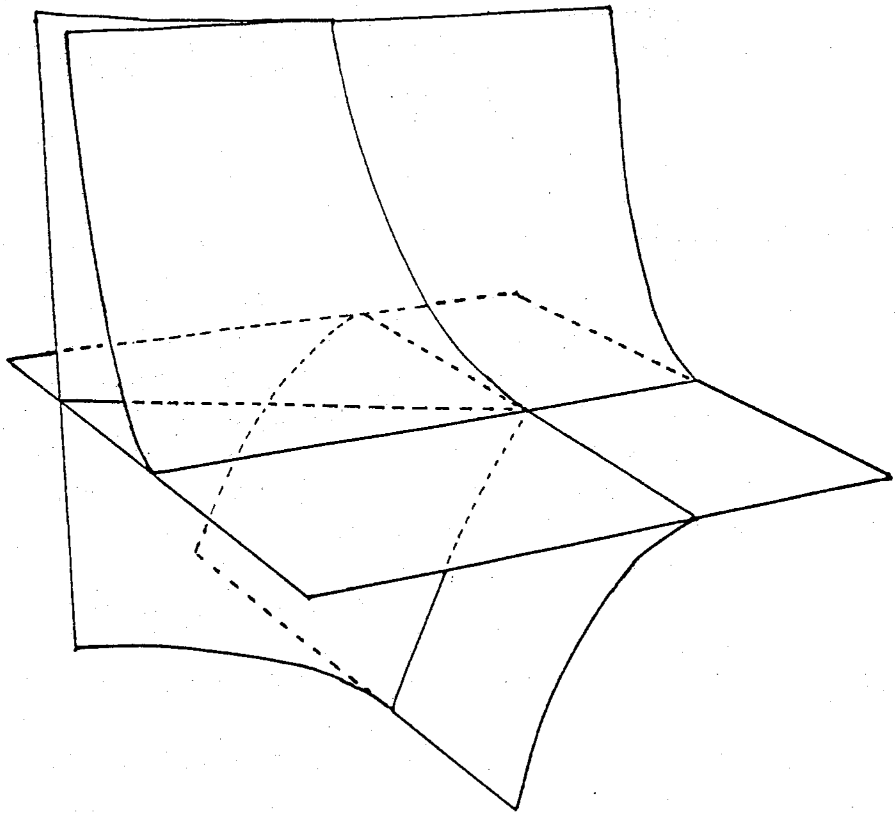
The first topic in this paper is the investigation of the relation between (symplectic) regular r-cubic configurations which has been developed in [10] and contact regular r-cubic configurations.

The second topic is the investigation of the stability of smooth contact regular r-cubic configurations and the classification of *stable wavefonts* given by stable contact regular r-cubic configurations in C^∞-category. In order to realize this purpose we shall define the notion of *reticular Legendrian maps* in Section 7 which is a generalization of the notion of Legendrian maps for our situations. We shall also prove that the equivalence relation among reticular Legendrian maps is equivalent to a certain equivalence relation of corresponding generating families. In this section we shall define the notion of *stability, homotopically stability, infinitesimal stability* of reticular Legendrian maps and prove that these and the stability of corresponding generating families are all equivalent.

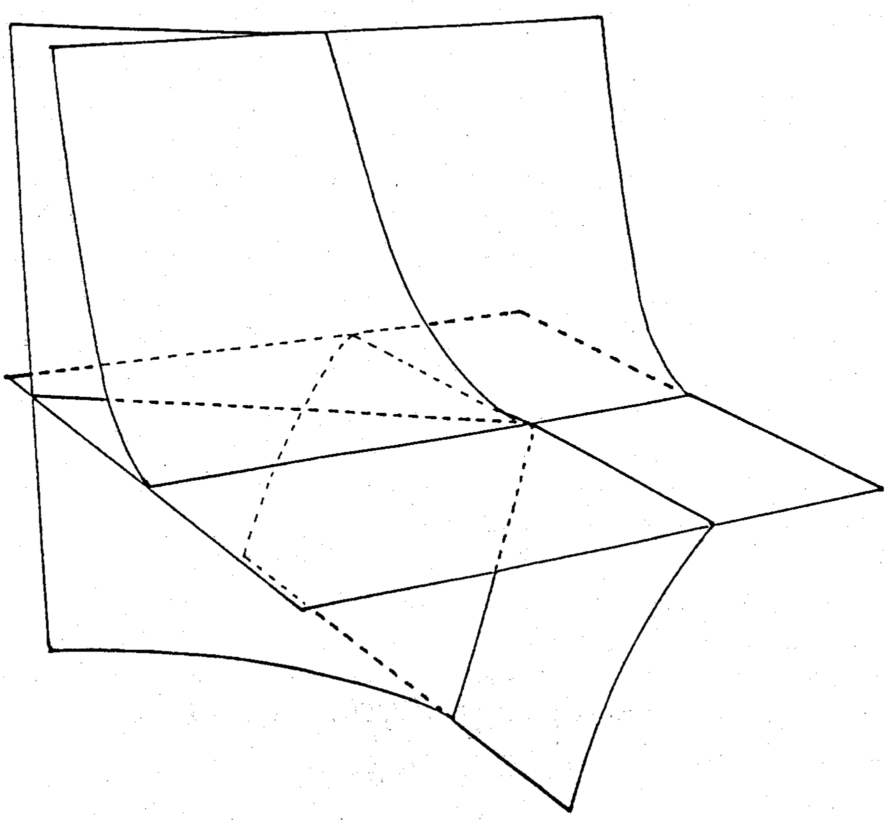
By the above results the classification of stable wavefronts is reduced to the classifications of function germs. In section 8 we classify function germs with respect to *reticular K-equivalence* with *reticular K-codimension* lower than 8 . This gives the classification of stable wavefonts in manifolds of dimension≤ 7.

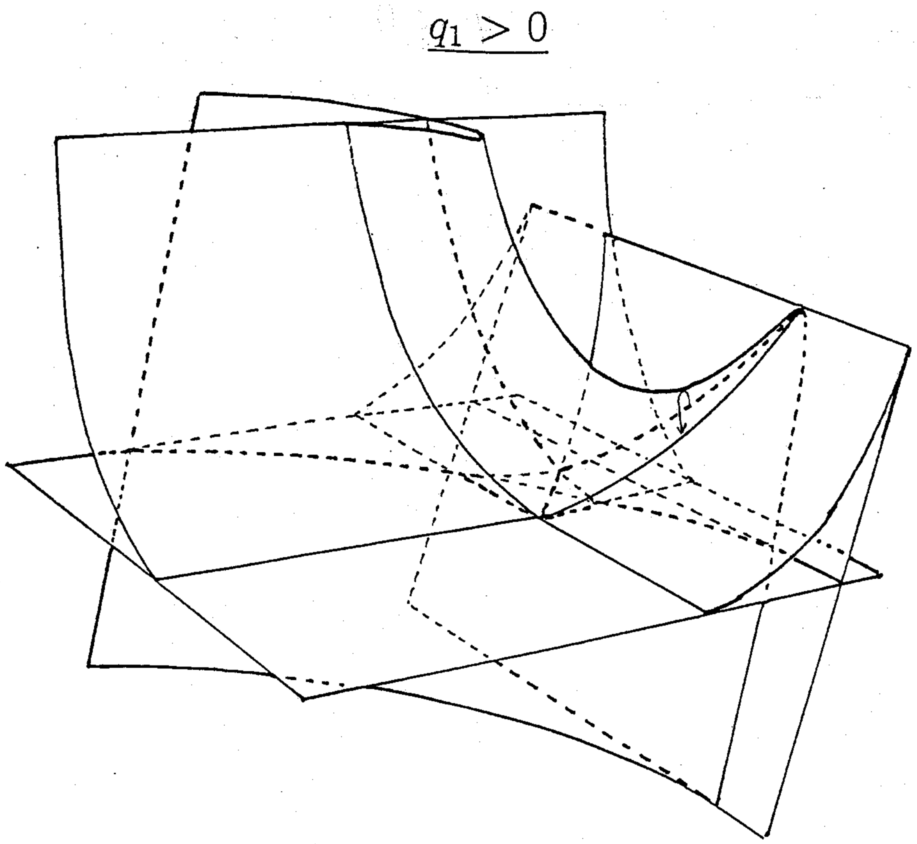
Here, we draw the figure of the wavefront of one of the reticular Legendrian map-germ whose generating family is a reticular versal unfolding of $B_{2,3}^{-,-}$-singularity in the classification list, that is $x_1^2 - x_1x_2 - x_2^3 + q_1x_2^2 + q_2x_2 + q_3x_1 + q_4$. The wavefront given by this generating family is a subset in (q_1, q_2, q_3, q_4)-space around 0. Hence we draw the sections of this wavefront in (q_2, q_3, q_4)-space given by cutting at $q_1 < 0, q_1 = 0, q_1 > 0$ respectively.

$\underline{q_1 < 0}$



$\underline{q_1 = 0}$





2 Preliminaries

Here we shall define several notations and recall basic facts. Let $\mathrm{H}^r = \{(x_1, \cdots, x_r) \in \mathrm{R}^r | x_1 \geq 0, \cdots, x_r \geq 0\}$ be an r-corner. Let $\mathcal{E}(r; l)$ be the ring of smooth function germ at 0 on $\mathrm{H}^r \times \mathrm{R}^l$ for $r, l \in \mathrm{N}$ and $m(r; l) = \{f \in \mathcal{E}(r, l) |\ f(0) = 0\}$ be the maximal ideal of $\mathcal{E}(r; l)$. Let $\mathcal{B}(r; l)$ the set of diffeomorphism germs on $(\mathrm{H}^r \times \mathrm{R}^l, 0)$ preserving $\mathrm{H}^r \cap \{x_\sigma = 0\} \times \mathrm{R}^l$ for all $\sigma \subset I_r = \{1, \cdots, r\}$. We remark that a diffeomorphism germ ϕ on $(\mathrm{H}^r \times \mathrm{R}^l, 0)$ is an element of $\mathcal{B}(r; l)$ if and only if ϕ is written in the following form:

$$\phi(x, y) = (x_1 a_1(x, y), \cdots, x_r a_r(x, y), b_1(x, y), \cdots, b_l(x, y)) \text{ for } (x, y) \in (\mathrm{H}^r \times \mathrm{R}^l, 0),$$

where $a_1, \cdots, a_r, b_1, \cdots, b_r \in \mathcal{E}(r; l)$ and $a_1(0) > 0, \cdots, a_r(0) > 0$.

A function germ $F(x_1, \cdots, x_r, y_1, \cdots, y_k, q_1, \cdots, q_n) \in \mathrm{m}^2(r; k + n)$ is called *S-non-degenerate* if

$$x_1, \cdots, x_r, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k}$$

are independent on $(\mathrm{H}^k \times \mathrm{R}^{k+n}, 0)$, that is

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial q} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial q} \end{pmatrix}_0 = r + k.$$

We remark that $F(x,y,u) \in \mathrm{m}^2(r;k+n)$ is S-non-degenerate only if $r \leq n$.

A function germ $\bar{F}(x_1,\cdots,x_r,y_1,\cdots,y_k,\lambda_1,\cdots,\lambda_{n+1}) \in \mathrm{m}(r;k+n+1)$ is called *C-non-degenerate* if $\frac{\partial \bar{F}}{\partial x}(0)=0$, $\frac{\partial \bar{F}}{\partial y}(0)=0$ and

$$x_1,\cdots,x_r,\bar{F},\frac{\partial \bar{F}}{\partial x_1},\cdots,\frac{\partial \bar{F}}{\partial x_r},\frac{\partial \bar{F}}{\partial y_1},\cdots,\frac{\partial \bar{F}}{\partial y_k}$$

are independent on $(\mathbf{H}^k \times \mathbf{R}^{k+n+1},0)$, that is

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial \bar{F}}{\partial y} & \dfrac{\partial \bar{F}}{\partial \lambda} \\ \dfrac{\partial^2 \bar{F}}{\partial x \partial y} & \dfrac{\partial^2 \bar{F}}{\partial x \partial \lambda} \\ \dfrac{\partial^2 \bar{F}}{\partial y \partial y} & \dfrac{\partial^2 \bar{F}}{\partial y \partial \lambda} \end{pmatrix}_0 = r+k+1.$$

We remark that $\bar{F}(x,y,\lambda) \in \mathrm{m}(r;k+n+1)$ is C-non-degenerate only if $r \leq n$.

Let $\bar{\pi} : PT^*\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be the projective cotangent bundle equipped with the contact structure defined in [1, p.310]. By the trivialization

$$\begin{array}{ccc} PT^*\mathbf{R}^{n+1} & \cong & \mathbf{R}^{n+1} \times P(\mathbf{R}^{n+1}) \\ [\xi_1 d\lambda_1|_\lambda + \cdots + \xi_{n+1} d\lambda_{n+1}|_\lambda] & & ((\lambda_1,\cdots,\lambda_{n+1}),[\xi_1;\cdots;\xi_{n+1}]), \end{array}$$

we call $(\lambda,[\xi])$ *a homogeneous coordinate*, where λ is coordinates of the base space of $\bar{\pi}$.

Let $\tilde{\pi} : J^1(\mathbf{R}^n,\mathbf{R}) \to \mathbf{R}^{n+1}((q,z;p) \mapsto (q,z))$ be the canonical Legendrian bundle equipped with the contact structure defined by the canonical 1-form $\alpha = dz - pdq$, where $(q_1,\cdots,q_n,z;p_1,\cdots,p_n)$ are canonical coordinates of $J^1(\mathbf{R}^n,\mathbf{R})$.

We fix $[\xi^0] \in PT^*\mathbf{R}^{n+1}$. Choose coordinates $(q_1,\cdots,q_n,z)$ of $\mathbf{R}^{n+1}$ (the base space of $\bar{\pi}$ and $\tilde{\pi}$) such that $[\xi^0]=(0,[0;\cdots;0;1])$. Set the affine chart of $PT^*\mathbf{R}^{n+1}$:$U_z = \{((q_1,\cdots,q_n,z),[\xi_1;\cdots;\xi_n;\eta])|\eta \neq 0\}$. Then

$$\psi_z : U_z \xrightarrow{\sim} J^1(\mathbf{R}^n,\mathbf{R})((q,z),[\xi;\eta]) \mapsto (q,z,-\frac{\xi_1}{\eta},\cdots,-\frac{\xi_n}{\eta}))$$

is a Legendrian equivalence. We define

$$p_1 : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n \ (\ (q,z) \mapsto q\ ),$$

$$\tilde{p_1} : J^1(\mathbf{R}^n,\mathbf{R}) \longrightarrow T^*\mathbf{R}^n \ (\ (q,z,p) \mapsto (q,p)\ ).$$

Then the following diagram is commutative:

$$\begin{array}{ccccc} U_z & \xrightarrow{\psi_z} & J^1(\mathbf{R}^n,\mathbf{R}) & \xrightarrow{\tilde{p_1}} & T^*\mathbf{R}^n \\ \bar{\pi}\downarrow & & \tilde{\pi}\downarrow & & \downarrow\pi \\ \mathbf{R}^{n+1} & \xrightarrow{id} & \mathbf{R}^{n+1} & \xrightarrow{p_1} & \mathbf{R}^n. \end{array}$$

We say that function germs $F(x,y,u), G(x,y,u) \in \mathrm{m}(r;k+l)$, where $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$ and $u \in \mathbf{R}^l$, are *reticular K-equivalent* (as l-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r;k+l)$ and a unit $a \in \mathcal{E}(r;k+l)$ satisfying the following:

(1) $\Phi = (\phi,\psi)$, where $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+l},0) \to (\mathbf{H}^r \times \mathbf{R}^k,0)$ and $\psi : (\mathbf{R}^l,0) \to (\mathbf{R}^l,0)$.

(2) $G(x,y,u) = a(x,y,u)\cdot F(\phi(x,y,u),\psi(u))$ for $(x,y,u) \in (\mathbf{H}^r \times \mathbf{R}^{k+l},0)$.

Lemma 2.1 *Let $\bar{F}(x,y,q,z) \in \mathrm{m}(r;k+n+1)$ be a C-non-degenerate function germ. Then $\bar{F}$ is reticular K-equivalent to $-z+F(x,y,q)$, where $F \in \mathrm{m}^2(r;k+n)$ is S-non-degenerate.*

Proof. By taking some coordinate change of (q,z), we may assume that $\frac{\partial \bar{F}}{\partial q}(0)=0$, $\frac{\partial \bar{F}}{\partial z}(0) \neq 0$. By implicit function theorem, there exists $F \in \mathrm{m}(r;k+n)$ such that $\bar{F}(x,y,q,F(x,y,q)) \equiv 0$.It is easy to check that $F \in \mathrm{m}^2(r;k+n)$. Since $\bar{F}|_{\{-z+F=0\}}=0$, there exists $a \in \mathcal{E}(r;k+n+1)$ such that $\bar{F}=a\cdot(-z+F)$. Since $\frac{\partial \bar{F}}{\partial z}(0)=-a(0)$, a is an unit. By Proposition 5.5, F is S-non-degenerate. □

By [1, p.313 Proposition and p.323 Proposition] and [11], we obtain the following Lemma.

Lemma 2.2 *Let $\mathcal{C}^n$ be the set of Legendrian submanifolds of $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ and $\mathcal{S}^n$ be the set of Lagrangian submanifolds of $(T^*\mathbf{R}^n,0)$.Then $\mathcal{C}^n$ and $\mathcal{S}^n$ have the following relations:*
(1) $\tilde{p_1}$ gives a bijection from $\mathcal{C}^n$ to $\mathcal{S}^n$.
(2) Let $\tilde{F}(y,q,z)=-z+F(y,q) \in \mathcal{E}(k+n+1)(F \in \mathrm{m}^2(k+n))$ and $\tilde{L} \in \mathcal{C}^n$. Then $\bar{F}$ is a generating family of $\tilde{L}$ if and only if F is a generating family of $\tilde{p_1}(\tilde{L})$.

Indeed let $\bar{L}$ be a Legendrian submanifold germ of $(PT^*\mathbf{R}^{n+1},[\xi^0])$ and $\bar{F}(y,q,z)=-z+F(y,q_1,\cdots,q_n) \in \mathcal{E}(k+n+1)(F \in \mathrm{m}^2(k+n))$be a generating family of $\bar{L}$. Then

$$\begin{aligned}
\bar{L} &= \{(q_1,\cdots,q_n,z,[\frac{\partial \bar{F}}{\partial q_1};\cdots;\frac{\partial \bar{F}}{\partial q_n};-1])|\frac{\partial \bar{F}}{\partial y}=\bar{F}=0\},\\
\tilde{L}=\psi_z(\bar{L}) &= \{(q_1,\cdots,q_n,F,\frac{\partial F}{\partial q_1},\cdots,\frac{\partial F}{\partial q_n})|\frac{\partial F}{\partial y}=0\},\\
L=\tilde{p_1}(\tilde{L}) &= \{(q_1,\cdots,q_n,\frac{\partial F}{\partial q_1},\cdots,\frac{\partial F}{\partial q_n})|\frac{\partial F}{\partial y}=0\}.
\end{aligned}$$

Under these fact, we identify $(PT^*\mathbf{R}^{n+1},[\xi^0])$ and $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ and identify Legendrian submanifold of $(PT^*\mathbf{R}^{n+1},[\xi^0])$ and that of $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ respectively.

3 Propagation mechanism of wavefronts

The propagation mechanism of wavefronts incident from a hypersurface germ with an r-corner in a smooth manifold is described as follows (cf., [6],[10]): Let M be an $n(=r+k+1)$-dimensional differentiable manifold and $H: T^*M\backslash 0 \to \mathbf{R}$ be a C^∞-function, called a *Hamiltonian function*, which we suppose to be everywhere positive and positively homogeneous of degree one, that is $H(\lambda\xi)=\lambda H(\xi)$ for all $\lambda>0$ and $\xi \in T^*M\backslash 0$. Let X_H denote the corresponding Hamiltonian vector field on $T^*M\backslash 0$, given locally by the Hamiltonian equations:

$$\dot{q_i}=\frac{\partial H}{\partial p_i},\ \dot{p_i}=-\frac{\partial H}{\partial q_i},$$

where (q,p) are local canonical coordinates of T^*M.

We set $E=H^{-1}(1)$ and consider the following canonical projections : $\pi: T^*M \to M$, $\pi_E: \mathbf{R}\times E \to E$, $\pi_{\mathbf{R}}: \mathbf{R}\times E \to \mathbf{R}$. We denote E_q the fiber of the spherical cotangent bundle $\pi|_E$ at $q \in M$.

Let $q_0 \in M$, $t_0 \geq 0$, $\xi_0 \in E_{q_0}$ and η_0 be the image of the phase flow of X_H at (t_0,ξ_0). Since the phase flow of X_H preserves values of H, the local phase flow $\Psi: (\mathbf{R}\times T^*M\backslash 0,(t_0,\xi_0)) \to (T^*M\backslash 0,\eta_0)$ of X_H induces the map $\Phi: (\mathbf{R}\times E,(t_0,\xi_0)) \longrightarrow (\mathbf{R}\times E,(t_0,\eta_0))$ given by

$\Phi(t,\xi)=(t,\Psi(t,\xi))$.

We set $exp=\pi_M\circ\Phi:(\mathbf{R}\times E,(t_0,\xi_0))\to(M,u_0)$, $exp_{q_0}=exp|_{\mathbf{R}\times E_{q_0}}$, $exp^-=\pi_M\circ\Phi^{-1}:(\mathbf{R}\times E,(t_0,\eta_0))\to(M,q_0)$, $exp^-_{u_0}=exp^-|_{\mathbf{R}\times E_{u_0}}$, $\phi_1=(\pi_M,exp):(\mathbf{R}\times E,(t_0,\xi_0))\to(M\times M,(q_0,u_0))$, $\phi_2=(exp^-,\pi_M):(\mathbf{R}\times E,(t_0,\eta_0))\to(M\times M,(q_0,u_0))$, where $u_0=\pi(\eta_0)$.

By [6, 2.2] we have the following Proposition

Proposition 3.1 *If exp_{q_0} is regular then ϕ_1 and ϕ_2 are diffeomorphisms.*

Let exp_{q_0} be regular. We can define the function germ

$$\tau=\pi_{\mathbf{R}}\circ\phi_1^{-1}=\pi_{\mathbf{R}}\circ\phi_2^{-1}:(M\times M,(q_0,u_0))\to(\mathbf{R},t_0).$$

We call τ the *ray length function* associated with the regular point (t_0,ξ_0) of exp_{q_0}. Then the following diagram is commutative:

$$\begin{array}{ccccc}
 & (\mathbf{R}\times E,(t_0,\xi_0)) & \xrightarrow{\Phi} & (\mathbf{R}\times E,(t_0,\eta_0)) & \\
 & \swarrow (\pi_{\mathbf{R}},exp) & \phi_1\searrow \quad \swarrow\phi_2 & (\pi_{\mathbf{R}},exp^-)\searrow & \\
(\mathbf{R}\times M,(t_0.u_0)) & \xleftarrow{(\tau,\pi_2)} & (M\times M,(q_0,u_0)) & \xrightarrow{(\tau,\pi_1)} & (\mathbf{R}\times M,(t_0,q_0))
\end{array}$$

Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0}=0$ with an r-corner defined as the image of an immersion $\iota:(\mathbf{H}^r\times\mathbf{R}^k,0)\to(M,q_0)$. We parameterize V^0 by ι. For each $\sigma\subset I_r=\{1,\cdots,r\}$ we define Λ^0_σ by the set of conormal vectors of $V^0_\sigma:=V^0\cap\{x_\sigma=0\}$ in (E,ξ_0) as the lift of the initial wavefront incident from V^0_σ. Then we regard the set $\tilde{L_\sigma}$ the image of covectors in Λ^0_σ by Φ around time t_0, that is

$$\tilde{L_\sigma}=\{\Phi(t,\xi)\in(\mathbf{R}\times E,(t_0,\eta_0))|(t,\xi)\in(\mathbf{R},t_0)\times\Lambda^0_\sigma\},$$

as the set of the lift of the wavefronts incident from V^0_σ around time t_0. We also regard the union of $\tilde{L_\sigma}$ for all $\sigma\subset I_r$ as the set of the lift of wavefront incident from the hypersurface V^0 around time t_0. We define the wavefront incident from V^0 by

$$\bigcup_{\sigma\subset I_r}(\pi_{\mathbf{R}},\pi_M)(\tilde{L_\sigma}).$$

The family of submanifolds $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ of $(\mathbf{R}\times E,(t_0,\eta_0))$ is 'generated' by the ray length function τ as the following:

Proposition 3.2 *Let V^0 be the hypersurface germ in (M,q_0) satisfying $\xi_0|_{T_{q_0}V^0}=0$ which is the image of an immersion $\iota:(\mathbf{H}^r\times\mathbf{R}^k,0)\to(M,q_0)$. Let $\tilde{L_\sigma}$ be the set of the lift of the wavefronts incident from $V^0_\sigma:=V^0\cap\{x_\sigma=0\}$ around time t_0 for $\sigma\subset I_r$. Define $\bar{F}(x,y,u,t):=-t+\tau\circ(\iota(x,y),u)\in\mathcal{E}(r;k+m+1)$. Then the following hold:*
(1) $\bar{F}$ is C-non-degenerate, that is $\frac{\partial\bar{F}}{\partial x}(0)=0$, $\frac{\partial\bar{F}}{\partial y}(0)=0$ and

$$\operatorname{rank}\begin{pmatrix}\dfrac{\partial\bar{F}}{\partial y} & \dfrac{\partial\bar{F}}{\partial u}\\ \dfrac{\partial^2\bar{F}}{\partial x\partial y} & \dfrac{\partial^2\bar{F}}{\partial x\partial u}\\ \dfrac{\partial^2\bar{F}}{\partial y\partial y} & \dfrac{\partial^2\bar{F}}{\partial y\partial u}\end{pmatrix}_0=r+k+1.$$

(2)

$$\tilde{L_\sigma} = \{(t, d_u\bar{F}(x,y,u)) \in (\mathbf{R}\times T^*M\backslash 0, (t_0,\eta_0))|$$
$$x_\sigma = d_{x_{I_r-\sigma}}\bar{F}(x,y,u) = d_y\bar{F}(x,y,u) = \bar{F} = 0\}$$

for $\sigma \subset I_r$, where we identify (M, u_0) and $(\mathbf{R}^n, 0)$ by coordinates $(u_1, \cdots, u_n)$ of (M, u_0).

Proof. This is immediately followed by [10] Proposition 2.2.

By Theorem 5.6 (2), Proposition 3.2 means that $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ is a *contact regular r-cubic configuration* of $(\mathbf{R}\times T^*M\backslash 0, (t_0,\eta_0))$ with the contact structure defined by the canonical 1-form $dt - pdu$, where p are the fiber coordinates corresponding to u. Hence there exists a contact diffeomorphism $C : (J^1(\mathbf{R}^n, \mathbf{R}), 0) \longrightarrow (\mathbf{R}\times T^*M\backslash 0, (t_0,\eta_0))$ such that

$$\tilde{L_\sigma} = C(\tilde{L_0^\sigma}) \qquad \text{for} \qquad \sigma \subset I_r,$$

where $\tilde{L_\sigma^0} = \{(q,z,p) \in (J^1(\mathbf{R}^n,\mathbf{R}),0) | q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = z = 0, q_{I_r-\sigma} \geq 0\}$ (cf., Section 5).

Small perturbations of the immersion ι implies small perturbations of contact diffeomorphism C. Therefore we investigate the stabilities of contact regular r-cubic configurations with respect to perturbations of corresponding contact diffeomorphisms in a more general situation in Section 7.

4 Results of Reticular Lagrangian singularities

Here we shall recall some results given in [10]. Let (q,p) be canonical coordinates of $(T^*\mathbf{R}^n, 0)$ and $\pi : (T^*\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be the cotangent bundle. Let $H = \{(q_1,\cdots,q_n) \in (\mathbf{R}^n,0) | q_1 \geq 0, \cdots, q_r \geq 0, q_{r+1} = \cdots = q_n = 0\}$ be an r-corner and $H_\sigma = \{(q_1,\cdots,q_n) \in H | q_\sigma = 0\}$ be an edge of H for $\sigma \subset I_r$. We define L_σ^0 the conormal bundle of H_σ, that is

$$L_\sigma^0 = \{(q,p) \in (T^*\mathbf{R}^n, 0) | q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = 0, q_{I_r-\sigma} \geq 0\}.$$

Definition 4.1 *Let $\{L_\sigma\}_{\sigma\subset I_r}$ be a family of 2^r Lagrangian submanifold germs of $(T^*\mathbf{R}^n, 0)$ under canonical symplectic structure of $(T^*\mathbf{R}^n, 0)$. Then $\{L_\sigma\}_{\sigma\subset I_r}$ is called a symplectic regular r-cubic configuration if there exists a symplectomorphism S on $(T^*\mathbf{R}^n, 0)$ such that $L_\sigma = S(L_\sigma^0)$ for all $\sigma \subset I_r$.*

Let $\{L_\sigma\}_{\sigma\subset I_r}$ be a symplectic regular r-cubic configuration and $F(x,y,q) \in \mathrm{m}(r; k+n)^2$ be a function germ which is S-non-degenerate. We call F a *generating family* of $\{L_\sigma\}_{\sigma\subset I_r}$ if $F|_{x_\sigma=0}$ is a generating family of L_σ for $\sigma \subset I_r$, that is

$$L_\sigma = \{(q, \frac{\partial F}{\partial q}(x,y,q)) \in (T^*\mathbf{R}^n, 0) | x_\sigma = \frac{\partial F}{\partial x_{I_r-\sigma}} = \frac{\partial F}{\partial y} = 0\} \text{ for } \sigma \subset I_r.$$

Let $\{L_\sigma^1\}_{\sigma\subset I_r}$ and $\{L_\sigma^2\}_{\sigma\subset I_r}$ be symplectic regular r-cubic configurations. We call $\{L_\sigma^1\}_{\sigma\subset I_r}$ and $\{L_\sigma^2\}_{\sigma\subset I_r}$ are Lagrangian equivalent if there exists a Lagrangian equivalence Θ such that $L_\sigma^2 = \Theta(L_\sigma^1)$ for $\sigma \subset I_r$.

We say that function germs $F(x,y,u), G(x,y,u) \in \mathrm{m}(r; k+n)$, where $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$

and $u \in \mathbf{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r; k+n)$ and $\alpha \in \mathrm{m}(n)$ satisfying the following:
(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+n}, 0) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$ and $\psi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.
(2) $G(x, y, u) = F(\phi(x, y, u), \psi(u)) + \alpha(u)$ for $(x, y, u) \in (\mathbf{H}^r \times \mathbf{R}^{k+n}, 0)$.
We say (Φ, α) a reticular R^+-isomorphism from G to F and if $\alpha = 0$ we say that F and G are reticular R-equivalent.

We say that function germs $F(x, y_1, \cdots, y_{k_1}, u) \in \mathrm{m}(r; k_1 + n)$ and $F(x, y_1, \cdots, y_{k_2}, u) \in \mathrm{m}(r; k_2 + n)$ are *stably reticular R^+-equivalent* if F and G are reticular R^+-equivalent after additions of non-degenerate quadratic forms in the variables y.

Theorem 4.2 (1) *For any symplectic regular r-cubic configuration $\{L_\sigma\}_{\sigma \subset I_r}$, there exists a function germ $F \in \mathrm{m}(r; k+n)^2$ which is a generating family of $\{L_\sigma\}_{\sigma \subset I_r}$.*
(2) *For any S-non-degenerate function germ $F \in \mathrm{m}(r; k+n)^2$, there exists a symplectic regular r-cubic configuration of which F is a generating family.*
(3) *Two symplectic regular r-cubic configuration are Lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

We remark that two S-non-degenerate function germ $F, G \in \mathrm{m}(r; k+n)^2$ are generating families of the same symplectic regular r-cubic configuration, then F and G are reticular R-equivalent.

Lemma 4.3 *Let U, V be open sets in $\mathbf{R}^n$ such that $0 \in U$ and let $f_0 : U \to V$ be a embedding. Then there exist a neighborhood U_1 of 0 in U and an open ball V around $f_0(0)$ in V and a neighborhood N_1 of f_0 in $C^\infty(U, V)$ with C^∞-topology such that $f|_{U_1}$ is embedding and $V_1 \subset f(U_1)$ for all $f \in N_1$. Moreover*

$$N_1 \longrightarrow C^\infty(V_1, U) \quad (f \mapsto (f|_{U_1})^{-1}|_{V_1})$$

is continuous.

5 Contact regular r-cubic configurations

In this section we shall define *Contact regular r-cubic configurations* and investigate the relations between symplectic and contact regular r-cubic configurations.

Let $(q_1, \cdots, q_n, z, p_1, \cdots, p_n)$ be canonical coordinates of $J^1(\mathbf{R}^n, \mathbf{R})$. Set $\tilde{L_\sigma^0} = \{(q, z, p) \in (J^1(\mathbf{R}^n, \mathbf{R}), 0) | q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = z = 0, q_{I_r - \sigma} \geq 0\}$ for each $\sigma \subset I_r$.

Definition 5.1 Let $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ be a family of 2^r Legendrian submanifold germs of $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$. Then $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ is called *a contact regular r-cubic configuration* if there exists a contact diffeomorphism C on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that $\tilde{L}_\sigma = C(\tilde{L_\sigma^0})$ for all $\sigma \subset I_r$.

Two contact regular r-cubic configurations $\{\tilde{L_\sigma^1}\}_{\sigma \subset I_r}$ and $\{\tilde{L_\sigma^2}\}_{\sigma \subset I_r}$ are said to be *Legendrian equivalent* if there exist Legendrian equivalence Θ of $\tilde{\pi}$(or $\overline{\pi}$) such that $\tilde{L_\sigma^2} = \Theta(\tilde{L_\sigma^1})$ for all $\sigma \subset I_r$..
Remark: The definition of contact regular r-cubic configuration by Nguyen Huu Duc [5, p. 631] is that there exists a contact diffeomorphism C such that $L_\sigma = C(\{q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = z = 0\})$ for all $\sigma \subset I_r$. Then $\{\tilde{L_\sigma}\}_{\sigma \subset I_r}$ is called a contact regular r-cubic configuration.

Definition 5.2 Let $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ be a contact regular r-cubic configuration in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$. Then $\tilde{F}(x,y,q,z)\in \mathrm{m}(r;k+n+1)$ is called *a generating family* of $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ if the following conditions hold:
(1) $\tilde{F}$ is C-non-degenerate.
(2) For each $\sigma\subset I_r$, $\tilde{F}|_{x_\sigma=0}$ is a generating family of $\tilde{L}_\sigma$, that is

$$\tilde{L_\sigma}=\{(q,z,\frac{\partial\tilde{F}}{\partial q}/(-\frac{\partial\tilde{F}}{\partial z}))|\ x_\sigma=\frac{\partial\tilde{F}}{\partial x_{I_r-\sigma}}=\frac{\partial\tilde{F}}{\partial y}=\tilde{F}=0\}.$$

We now consider contact diffeomorphisms and contact diffeomorphism germs on $J^1(\mathbf{R}^n,\mathbf{R})$ and $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ respectively. Let (Q,Z,P) be canonical coordinates on the source and (q,z,p) be canonical coordinates of the target. We define the following notations:
$\imath:(J^1(\mathbf{R}^n,\mathbf{R})\cap\{Z=0\},0)\to(J^1(\mathbf{R}^n,\mathbf{R}),0)$ be the inclusion map on the domain.

$$\begin{aligned}
C(J^1(\mathbf{R}^n,\mathbf{R}),0) &= \{C:(J^1(\mathbf{R}^n,\mathbf{R}),0)\to(J^1(\mathbf{R}^n,\mathbf{R}),0)|C:\ \text{contact diffeomorphism}\}\\
C^\alpha(J^1(\mathbf{R}^n,\mathbf{R}),0) &= \{C\in C(J^1(\mathbf{R}^n,\mathbf{R}),0)|\ C\ \text{preserves the canonical 1-form }\}\\
C_Z(J^1(\mathbf{R}^n,\mathbf{R}),0) &= \{C\circ\imath\ |C\in C(J^1(\mathbf{R}^n,\mathbf{R}),0)\}\\
C_Z^\alpha(J^1(\mathbf{R}^n,\mathbf{R}),0) &= \{C\circ\imath\ |C\in C^\alpha(J^1(\mathbf{R}^n,\mathbf{R}),0)\}
\end{aligned}$$

Let U be an open set in $J^1(\mathbf{R}^n,\mathbf{R})$ and $V=U\cap\{Z=0\}$. Let $\tilde{\imath}:V\to U$ be the inclusion map.

$$\begin{aligned}
C(U,J^1(\mathbf{R}^n,\mathbf{R})) &= \{\tilde{C}:U\to J^1(\mathbf{R}^n,\mathbf{R})|\tilde{C}:\text{contact embedding }\}\\
C^\alpha(U,J^1(\mathbf{R}^n,\mathbf{R})) &= \{\tilde{C}\in C(U,J^1(\mathbf{R}^n,\mathbf{R}))\ |\ C\ \text{preserves the canonical 1-form }\}\\
C_Z(V,J^1(\mathbf{R}^n,\mathbf{R})) &= \{\tilde{C}\circ\tilde{\imath}\ |\tilde{C}\in(U,J^1(\mathbf{R}^n,\mathbf{R}))\}\\
C_Z^\alpha(V,J^1(\mathbf{R}^n,\mathbf{R})) &= \{\tilde{C}\circ\tilde{\imath}\ |\tilde{C}\in C^\alpha(U,J^1(\mathbf{R}^n,\mathbf{R}))\}
\end{aligned}$$

Lemma 5.3 *Let $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ be a contact regular r-cubic configuration in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ defined by $C\in C(J^1(\mathbf{R}^n,\mathbf{R}),0)$. Then there exists $C'\in C^\alpha(J^1(\mathbf{R}^n,\mathbf{R}),0)$ that also defines $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$.*

Proof. Let $C=(q_C,z_C,p_C)$. Define the function a on $C(J^1(\mathbf{R}^n,\mathbf{R}),0)$ by the relation $C^*(dz-pdq)=a(dZ-PdQ)$. Define

$$\phi:(J^1(\mathbf{R}^n,\mathbf{R})\cap\{Z=0\},0)\xrightarrow{\sim}(J^1(\mathbf{R}^n,\mathbf{R})\cap\{Z=0\},0)\ (\ (Q,P)\mapsto(Q,a\circ\imath(Q,P)P)\ ),$$

$$\begin{aligned}
C':(J^1(\mathbf{R}^n,\mathbf{R}),0)&\xrightarrow{\sim}\qquad\qquad\qquad (J^1(\mathbf{R}^n,\mathbf{R}),0)\\
(Q,Z,\bar{P})&\mapsto(q_C\circ\imath\circ\phi^{-1}(Q,\bar{P}),Z+z_C\circ\imath\circ\phi^{-1}(Q,\bar{P}),p_C\circ\imath\circ\phi^{-1}(Q,\bar{P})).
\end{aligned}$$

Then

$$C'^*(dz-pdq)=dZ+((C\circ\imath)\circ\phi^{-1})^*(dz-pdq)=dZ-(\phi^{-1})^*(a\circ\imath(Q,P)PdQ)=dZ-\bar{P}dQ.$$

Therefore $C'\in C^\alpha(J^1(\mathbf{R}^n,\mathbf{R}),0)$. Since $C'(Q,0,a(Q,P)P)=C(Q,0,P)$, C' also defines $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$.

Lemma 5.4 *Let $S(T^*\mathbf{R}^n,0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbf{R}^n,0)$. We define the following maps:*

$$\begin{array}{rcl} C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0) & \to & S(T^*\mathbf{R}^n,0) \\ C=(q_C,z_C,p_C) & \mapsto & (\ S^C:(Q,P)\mapsto(q_C,p_C)(Q,P)\ ) \end{array}$$

$$\begin{array}{rcl} S(T^*\mathbf{R}^n,0) & \to & C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0) \\ S=(q_S,p_S) & \mapsto & (\ C^S:(Q,P)\mapsto(q_S,f^S,p_S)(Q,P)\ ), \end{array}$$

where $f^S(Q,P)$ is uniquely defined by the relation that $S^*(pdq)-PdQ=df^S, f^S(0,0)=0$. Then these maps are well defined and inverse to each other(that is $S^{C^S}=S, C^{S^C}=C$).

Proof. Let $C\in C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0)$ be given. Take $\bar{C}\in C^{\alpha}(J^1(\mathbf{R}^n,\mathbf{R}),0)$ such that $\bar{C}\circ\imath=C$. Since $S^C=(q_C,p_C)$, we have

$$\begin{aligned} (S^C)^*(dp\wedge dq) &= C^*(dp\wedge dq)=C^*(-d(dz-pdq))=-d((\bar{C}\circ\imath)^*(dz-pdq)) \\ &= -d(\imath^*(dZ-PdQ))=-d(-PdQ)=dP\wedge dQ \end{aligned}$$

Hence $S^C\in S(T^*\mathbf{R}^n,0)$.

Conversely let $S=(q_S,p_S)\in S(T^*\mathbf{R}^n,0)$ be given . We define the diffeomorphism $\bar{C}^S$ on $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ by $\bar{C}^S(Q,Z,P)=(q_S(Q,P),Z+f^S(Q,P),p_S(Q,P))$. Then $\bar{C}^S\circ\imath=C^S$ and

$$(\bar{C}^S)^*(dz-pdq)=dZ+df^S-S^*(pdq)=dZ+(S^*(pdq)-PdQ)-S^*(pdq)=dZ-PdQ.$$

Hence $C^S\in C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0)$. On the other hand, by definition, we have

$$S^{C^S}=(q_{C^S},p_{C^S})=(q_S,p_S)\ ,\ C^{S^C}=(q_{S^C},f^{S^C},p_{S^C})=(q_C,f^{S^C},p_C).$$

Since f^{S^C} and z_C satisfy the equation of $z(Q,P)$ that $dz=p_Cdq_C-PdQ$ and $z(0,0)=0$, we have that $f^{S^C}=z_C$. □

Proposition 5.5 *Let $\mathcal{C}^n_r$ be the set of contact regular r-cubic configurations in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ and $\mathcal{S}^n_r$ be the set of symplectic regular r-cubic configurations in $(T^*\mathbf{R}^n,0)$. We define*

$$T_S:\mathcal{C}^n_r\to\mathcal{S}^n_r\ (\ \{C(\tilde{L}^0_\sigma)\}_{\sigma\subset I_r}\mapsto\{S^C(L^0_\sigma)\}_{\sigma\subset I_r}\ ),\text{where } C\in C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0)$$

$$T_C:\mathcal{S}^n_r\to\mathcal{C}^n_r\ (\ \{S(L^0_\sigma)\}_{\sigma\subset I_r}\mapsto\{C^S(\tilde{L}^0_\sigma)\}_{\sigma\subset I_r}\ ),\text{where } S\in S(T^*\mathbf{R}^n,0)$$

Then (1) T_S and T_C are well defined and inverse to each other.
(2) A function germ $F(x,y,q)\in \mathrm{m}^2(r;k+n)$ is S-non-degenerate if and only if $-z+F$ is C-non-degenerate.
(3) A function germ $F(x,y,q)\in \mathrm{m}^2(r;k+n)$ is a generating family of a symplectic regular r-cubic configuration if and only if $-z+F$ is a generating family of the corresponding contact regular r-cubic configuration.

Proof.(1) Let $C=(q_C,z_C,p_C)\in C^{\alpha}_{Z}(J^1(\mathbf{R}^n,\mathbf{R}),0)$ and $S\in S(T^*\mathbf{R}^n,0)$ satisfy that $S=S^C$ (hence $C=C^S$). Since $S=(q_C,p_C)$, we have $S(L^0_\sigma)=\bar{\pi}_1(C(\tilde{L}^0_\sigma))$ for all $\sigma\subset I_r$. Since $S(L^0_\sigma)$ and $C(\tilde{L}^0_\sigma)$ are uniquely determined by each other under $\bar{\pi}_1$ by Lemma 2.2, we have (1).
(2) Let $F(x,y,q)\in \mathrm{m}^2(r;k+n)$. If we define $\bar{F}\in \mathrm{m}(r;k+n+1)$ by $\bar{F}(x,y,q,z)=-z+F(x,y,q)$. Then $\frac{\partial\bar{F}}{\partial x}(0)=\frac{\partial F}{\partial x}(0)=0, \frac{\partial\bar{F}}{\partial y}(0)=\frac{\partial F}{\partial y}(0)=0$ and

$$\begin{pmatrix} \frac{\partial\bar{F}}{\partial y} & \frac{\partial\bar{F}}{\partial q} & \frac{\partial\bar{F}}{\partial z} \\ \frac{\partial^2\bar{F}}{\partial x\partial y} & \frac{\partial^2\bar{F}}{\partial x\partial q} & \frac{\partial^2\bar{F}}{\partial x\partial z} \\ \frac{\partial^2\bar{F}}{\partial y\partial y} & \frac{\partial^2\bar{F}}{\partial y\partial q} & \frac{\partial^2\bar{F}}{\partial y\partial z} \end{pmatrix}_0 = \begin{pmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial q} & -1 \\ \frac{\partial^2 F}{\partial x\partial y} & \frac{\partial^2 F}{\partial x\partial q} & 0 \\ \frac{\partial^2 F}{\partial y\partial y} & \frac{\partial^2 F}{\partial y\partial q} & 0 \end{pmatrix}_0 .$$

This implies (2).
(3) By (2),we have

$$
\begin{aligned}
&\bar{F} = -z + F \text{ is a generating family of } \{C(\tilde{L}_\sigma^0)\}_{\sigma\subset I_r}\\
\Leftrightarrow\quad &\bar{F} \text{ is C-non-degenerate and } \bar{F}|_{x_\sigma=0} \text{ generates } C(\tilde{L}_\sigma^0) \text{ for all } \sigma \subset I_r\\
\Leftrightarrow\quad &F \text{ is S-non-degenerate and } F|_{x_\sigma=0} \text{ generates } S(L_\sigma^0) \text{ for all } \sigma \subset I_r\\
\Leftrightarrow\quad &F \text{ is a generating family of } \{S(L_\sigma^0)\}_{\sigma\subset I_r}.
\end{aligned}
$$

□

The relation between contact and symplectic regular r-cubic configurations is given in the following diagram:

$$
\begin{array}{ccccccc}
(J^1(\mathbf{R}^n,\mathbf{R})\cap\{Z=0\},0) & \overset{C=\xrightarrow{}S}{} & (J^1(\mathbf{R}^n,\mathbf{R}),0) & \qquad & \{\tilde{L}_\sigma^0\}_{\sigma\subset I_r} & \mapsto & \{\tilde{L}_\sigma\}_{\sigma\subset I_r}\\
\tilde{p_1}|_{Z=0}\downarrow & & \downarrow\tilde{p_1} & & \downarrow & & T_S\downarrow\uparrow T_C\\
(T^*\mathbf{R}^n,0) & \overset{S=\xrightarrow{}C}{} & (T^*\mathbf{R}^n,0) & & \{L_\sigma^0\}_{\sigma\subset I_r} & \mapsto & \{L_\sigma\}_{\sigma\subset I_r}
\end{array}
$$

We say that function germs $F(x,y_1,\cdots,y_{k_1},u)\in \mathrm{m}(r;k_1+m)$ and $F(x,y_1,\cdots,y_{k_2},u)\in \mathrm{m}(r;k_2+m)$ are *stably reticular K-equivalent* if F and G are reticular K-equivalent after additions of non-degenerate quadratic forms in the variables y.

The relations between contact regular r-cubic configurations and their generating families are given in the following theorem.

Theorem 5.6 (1) *For any contact regular r-cubic configuration $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$, there exists a function germ $\bar{F}\in \mathrm{m}(r;k+n+1)$ which is a generating family of $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$.*
(2) *For any C-non-degenerate function $\bar{F}\in \mathrm{m}(r;k+n+1)$, there exists a contact regular r-cubic configuration in $(PT^*\mathbf{R}^{n+1},(0,[\frac{\partial\bar{F}}{\partial\lambda}(0)]))$ (or in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$) of which $\bar{F}$ is a generating family.*
(3) *Two contact regular r-cubic configurations are Legendrian equivalent if and only if their generating families are stably reticular K-equivalent.*

Proof. (1) Let a contact regular r-cubic configuration $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ in $(J^1(\mathbf{R}^n,\mathbf{R}),0)$ be given. Set $\{L_\sigma\}_{\sigma\subset I_r}=T_S(\{\tilde{L_\sigma}\}_{\sigma\subset I_r})$ and let $F\in \mathrm{m}^2(r;k+n)$ be a generating family of $\{L_\sigma\}_{\sigma\subset I_r}$. Then $-z+F\in \mathrm{m}(r;k+n+1)$ is a generating family of $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ by Proposition 5.5 (3).
(2) Let a C-non-degenerate function $\bar{F}\in \mathrm{m}(r;k+n+1)$ be given. By Lemma 2.1 and (3)a, we may assume that $\bar{F}$ has the form $\bar{F}(x,y,q,z)=-z+F(x,y,q)(F\in \mathrm{m}^2(r;k+n))$. Then F is a generating family of a symplectic regular r-cubic configuration $\{L_\sigma\}_{\sigma\subset I_r}$ in $(T^*\mathbf{R}^n,0)$ by Proposition 5.5 (2) and Theorem 4.2 (2). Hence $\bar{F}$ is a generating family of $T_C(\{L_\sigma\}_{\sigma\subset I_r})$ by Proposition 5.5 (3).
(3) We suppose the assertions (3)a, (3)b, (3)c, (3)d and prove (3). Let $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ and $\{\bar{L}_\sigma^2\}_{\sigma\subset I_r}$ be contact regular r-cubic configurations in $(PT^*\mathbf{R}^{n+1},(0,[\eta_1]))$ and $(PT^*\mathbf{R}^{n+1},(0,[\eta_2]))$ respectively and $\bar{F}\in \mathrm{m}(r;k_1+n+1)$ and $\bar{G}\in \mathrm{m}(r;k_2+n+1)$ be generating families of $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ and $\{\bar{L}_\sigma^2\}_{\sigma\subset I_r}$ respectively.
($\Leftarrow$) Let $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent. By (3)b, we may assume that $k_1=k_2$. Hence by (3)a, $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ and $\{\bar{L}_\sigma^2\}_{\sigma\subset I_r}$ are Legendrian equivalent.
($\Rightarrow$) Let $\{\bar{L}_\sigma^1\}_{\sigma\subset I_r}$ and $\{\bar{L}_\sigma^2\}_{\sigma\subset I_r}$ be Legendrian equivalent. By (3)c, we may assume that $\bar{F}$ and $\bar{G}$ are generating families of the same contact regular r-cubic configuration. By Lemma 2.1, we may assume that $\bar{F}$ and $\bar{G}$ are written in the form: $\bar{F}=-z+F,\bar{G}=-z+G$, where $F\in \mathrm{m}^2(r;k_1+n),G\in \mathrm{m}^2(r;k_2+n)$. Hence by (3)d, $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent.
a. *Let $\bar{F}\in \mathrm{m}(r;k+n+1)$ be a generating family of a contact regular r-cubic configuration*

$\{\bar{L}^1_\sigma\}_{\sigma\subset I_r}$ *of* $(PT^*\mathbf{R}^{n+1},(0,[\frac{\partial\bar{F}}{\partial\lambda}(0)]))$ *and suppose that* $\bar{F}$ *and* $\bar{G}\in\mathrm{m}(r;k+n+1)$ *are reticular K-equivalent. Then* $\bar{G}$ *is a generating family of a contact regular* r*-cubic configuration* $\{\bar{L}^2_\sigma\}_{\sigma\subset I_r}$ *to which* $\{\bar{L}^1_\sigma\}_{\sigma\subset I_r}$ *is Legendrian equivalent.*

We write the reticular K-equivalence between $\bar{F}$ and $\bar{G}$ by

$$\bar{G}(x,y,\lambda)=\alpha(x,y,\lambda)\bar{F}(x_1a_1(x,y,\lambda),\cdots,x_ra_r(x,y,\lambda),h(x,y,\lambda),g(\lambda)).$$

Since $\frac{\partial\bar{G}}{\partial x}(0)=\alpha(0)(\frac{\partial\bar{F}}{\partial x}+\frac{\partial\bar{F}}{\partial y}\frac{\partial h}{\partial x})(0)=0$, $\frac{\partial\bar{G}}{\partial y}(0)=\alpha(0)(\frac{\partial\bar{F}}{\partial y}\frac{\partial h}{\partial y})(0)=0$ and

$$\begin{pmatrix}\frac{\partial\bar{G}}{\partial y}&\frac{\partial\bar{G}}{\partial\lambda}\\ \frac{\partial^2\bar{G}}{\partial x\partial y}&\frac{\partial^2\bar{G}}{\partial x\partial\lambda}\\ \frac{\partial^2\bar{G}}{\partial y\partial y}&\frac{\partial^2\bar{G}}{\partial y\partial\lambda}\end{pmatrix}_0=\alpha(0)\begin{pmatrix}1&\frac{\partial\alpha}{\partial x}/\alpha&\frac{\partial\alpha}{\partial y}/\alpha\\ &a_1&&\\ 0&\ddots&0\\ &&a_r&\\ 0&\frac{\partial h}{\partial x}&\frac{\partial h}{\partial y}\end{pmatrix}^t_0\begin{pmatrix}\frac{\partial\bar{F}}{\partial y}&\frac{\partial\bar{F}}{\partial\lambda}\\ \frac{\partial^2\bar{F}}{\partial x\partial y}&\frac{\partial^2\bar{F}}{\partial x\partial\lambda}\\ \frac{\partial^2\bar{F}}{\partial y\partial y}&\frac{\partial^2\bar{F}}{\partial y\partial\lambda}\end{pmatrix}_0\begin{pmatrix}\frac{\partial h}{\partial y}&\frac{\partial h}{\partial\lambda}\\ 0&\frac{\partial g}{\partial\lambda}\end{pmatrix}_0,$$

$\bar{G}$ is C-non-degenerate. Set
$(x',y',\lambda')=H(x,y,\lambda)=(x_1a_1(x,y,\lambda),\cdots,x_ra_r(x,y,\lambda),h(x,y,\lambda),\,g(\lambda))$. Then

$$\frac{\partial\bar{G}}{\partial x_l}=\frac{\partial\alpha}{\partial x_l}\bar{F}\circ H+\alpha(\sum_{i=1}^r\frac{\partial\bar{F}}{\partial x_i}\circ H\cdot x_i\frac{\partial a_i}{\partial x_l}+\sum_{j=1}^k\frac{\partial\bar{F}}{\partial y_j}\circ H\cdot\frac{\partial h_j}{\partial x_l}+\frac{\partial\bar{F}}{\partial x_l}\circ H\cdot a_l)$$

$(l=1,\cdots,r)$,

$$\frac{\partial\bar{G}}{\partial y_l}=\frac{\partial\alpha}{\partial y_l}\bar{F}\circ H+\alpha(\sum_{i=1}^r\frac{\partial\bar{F}}{\partial x_i}\circ H\cdot x_i\frac{\partial a_i}{\partial y_l}+\sum_{j=1}^k\frac{\partial\bar{F}}{\partial y_j}\circ H\cdot\frac{\partial h_j}{\partial y_l})$$

$(l=1,\cdots,k)$,

$$\frac{\partial\bar{G}}{\partial\lambda_l}=\frac{\partial\alpha}{\partial\lambda_l}\bar{F}\circ H+\alpha(\sum_{i_1=1}^r\frac{\partial\bar{F}}{\partial x_{i_1}}\circ H\cdot x_{i_1}\frac{\partial a_{i_1}}{\partial\lambda_l}+\sum_{i_2=1}^k\frac{\partial\bar{F}}{\partial y_{i_2}}\circ H\cdot\frac{\partial h_{i_2}}{\partial\lambda_l}+\sum_{i_3=1}^n\frac{\partial\bar{F}}{\partial\lambda_{i_3}}\circ H\cdot\frac{\partial g_{i_3}}{\partial\lambda_l})$$

$(l=1,\cdots,n)$. Therefore for $\sigma\subset I_r$ we have

$$\begin{aligned}&\{\ (\lambda,[\frac{\partial\bar{G}}{\partial\lambda}])\mid x_\sigma=\frac{\partial\bar{G}}{\partial x_\tau}=\frac{\partial\bar{G}}{\partial y}=\bar{G}=0\ \}\\
=&\ \{\ (\lambda,[\alpha\sum_{i=1}^n\frac{\partial\bar{F}}{\partial\lambda_i}(x',y',\lambda')\cdot\frac{\partial g_i}{\partial\lambda}(\lambda)])\mid\\
&\quad x'_\sigma=\frac{\partial\bar{F}}{\partial x_\tau}(x',y',\lambda')=\frac{\partial\bar{F}}{\partial y}(x',y',\lambda')=\bar{F}(x',y',\lambda')=0\ \}\\
=&\ \{\ (g^{-1}(\lambda'),[\sum_{i=1}^n\frac{\partial\bar{F}}{\partial\lambda_i}(x',y',\lambda')\cdot\frac{\partial g_i}{\partial\lambda}(g^{-1}(\lambda'))])\mid\\
&\quad x'_\sigma=\frac{\partial\bar{F}}{\partial x_\tau}(x',y',\lambda')=\frac{\partial\bar{F}}{\partial y}(x',y',\lambda')=\bar{F}(x',y',\lambda')=0\ \}\\
=&\ \bar{g}^*(\bar{L}^1_\sigma)\end{aligned}$$

$(\tau = I_r - \sigma)$. It follows that $\bar{G}$ is a generating family of $\{\bar{g}^*(\bar{L}_\sigma^1)\}_{\sigma \subset I_r}$.
b. *Let $\bar{F}(x,y,\lambda) \in \mathrm{m}(r; k+n+1)$ be a generating family of a contact regular r-cubic configuration $\{\bar{L}_\sigma\}_{\sigma \subset I_r}$ in $(PT^*\mathbf{R}^{n+1}, (0, [\frac{\partial \bar{F}}{\partial \lambda}(0)]))$. If $\bar{F}$ and $\bar{G}$ are stably equivalent, then $\bar{G}$ is also a generating family of $\{\bar{L}_\sigma\}_{\sigma \subset I_r}$.*
c. *Let $\{\bar{L}_\sigma^1\}_{\sigma \subset I_r}$ and $\{\bar{L}_\sigma^2\}_{\sigma \subset I_r}$ be contact regular r-cubic configurations in $(PT^*\mathbf{R}^{n+1}, [\eta_1])$ and $(PT^*\mathbf{R}^{n+1}, [\eta_2])$ respectively and let $\bar{F} \in \mathrm{m}(r; k+n+1)$ be a generating family of $\{\bar{L}_\sigma^1\}_{\sigma \subset I_r}$. If these configurations are Legendrian equivalent, then there exists a generating family $\bar{G}$ of $\{\bar{L}_\sigma^2\}_{\sigma \subset I_r}$ to which $\bar{F}$ is reticular K-equivalent.*

Take a diffeomorphism g on the base of $\bar{\pi}$ such that $\bar{L}_\sigma^2 = \bar{g}^*(\bar{L}_\sigma^1)$ for all $\sigma \subset I_r$. Define $\bar{G}(x,y,\lambda) = \bar{F}(x,y,g(\lambda))$. Then we can prove that $\bar{G}$ is a generating family of $\{\bar{L}_\sigma^2\}_{\sigma \subset I_r}$ by the same method of (3)a.
d. *Let $\bar{F}(x,y^1,q,z) = -z + F(x,y^1,q)$ and $\bar{G}(x,y^2,q,z) = -z + G(x,y^2,q)$ $(F \in \mathrm{m}^2(r; k_1 + n), G \in \mathrm{m}^2(r; k_2+n))$ are two generating families of the same contact regular r-cubic configuration. Then $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent.*

By Proposition 5.5, F and G are generating families of the same symplectic regular r-cubic configuration. Therefore F and G are stably reticular R-equivalent by the remark after Theorem 4.2. Hence $\bar{F}$ and $\bar{G}$ are stably reticular K-equivalent. □

6 Stability of function germs

In order to investigate the stabilities of smooth contact regular r-cubic configurations, we shall prepare the results of the singularity theory of function germs with respect to *reticular K-equivalence.* Basic techniques for the characterization of the stabilities we use in this paper depend heavily on the results in this section, however the all arguments are almost parallel along the ordinary theory of the right-equivalence (cf., [13]), so that we omit the detail.

We denote $J^l(r+k,1)$ the set of l-jets at 0 of germs in $\mathrm{m}(r;k)$ and let $\pi_l : \mathrm{m}(r;k) \to J^l(r+k,1)$ be the natural projection. We denote $j^l f(0)$ the l-jet of $f \in \mathrm{m}(r;k)$.

We say $f, g \in \mathrm{m}(r;l)$ are *reticular K-equivalent* if there exists $\phi \in \mathcal{B}(k;l)$ such that $g = f \circ f$.

Lemma 6.1 *Let $f \in \mathrm{m}(r;k)$ and $O_K^l(j^l f(0))$ be the submanifold of $J^l(r+k,1)$ consist of the image by π_l of the orbit of reticular K-equivalence of f. Put $z = j^l f(0)$. Then*

$$T_z(O_{rK}^l(z)) = \pi_l(\langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle).$$

We say that a function germ $f \in \mathrm{m}(r;k)$ is *reticular K-l-determined* if all function germ which has same l-jet of f is reticular K-equivalent to f.

Lemma 6.2 *Let $f \in \mathrm{m}(r;k)$ and let*

$$\mathrm{m}(r;k)^{l+1} \subset \mathrm{m}(r;k)(\langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle) + \mathrm{m}(r;k)^{l+2},$$

then f is reticular K-l-determined. Conversely let $f \in \mathrm{m}(r;k)$ be reticular K-l-determined, then

$$\mathrm{m}(r;k)^{l+1} \subset \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}\rangle_{\mathcal{E}(r;k)} + \mathrm{m}(r;k)\langle\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle.$$

Let $F \in \mathrm{m}(r; k+n_1)$, $G \in \mathrm{m}(r; k+n_2)$ be unfoldings of $f \in \mathrm{m}(r;k)$. We say that F *is reticular K-f-induced from G* if there exist smooth map germs $\phi : (\mathbf{H}^r \times \mathbf{R}^{k+n_2}, 0) \to (\mathbf{H}^r \times \mathbf{R}^k, 0)$, $\psi : (\mathbf{R}^{n_2}, 0) \to (\mathbf{R}^{n_1}, 0)$ and $\alpha \in \mathcal{E}(0; n_2)$ satisfying the following conditions:
(1) $\phi((\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^{k+n_2}) \subset (\mathbf{H}^r \cap \{x_\sigma = 0\}) \times \mathbf{R}^k$ for $\sigma \subset I_r$.
(2) $G(x,y,v) = \alpha(v) \cdot F(\phi(x,y,v), \psi(v))$ for $x \in \mathbf{H}^r$, $y \in \mathbf{R}^k$ and $v \in \mathbf{R}^{n_2}$.

Definition 6.3 Here we give several definitions of the stabilities of unfoldings. Let $f \in \mathrm{m}(r;k)$ and $F \in \mathrm{m}(r; k+n)$ be an unfolding of f.

We define a smooth map germ

$$j_1^l F : (\mathbf{R}^{r+k+n}, 0) \longrightarrow (J^l(r+k, 1), j^l f(0))$$

as follow: Let $\tilde{F} : U \to \mathbf{R}$ be a representative of F. For each $(x,y,u) \in U$, We define $F_{(x,y,u)} \in \mathrm{m}(r;k)$ by $F_{(x,y,u)}(x',y') = F(x+x', y+y', u) - F(x,y,u)$. Now define $j_1^l F(x,y,u)$ =the l-jet of $F_{(x,y,u)}$. $j_1^l F$ depends only on the germ at 0 of F. We say that F is *reticular K-l-transversal* if $j_1^l F|_{x=0}$ is transversal to $O_K^l(j^l f(0))$. It is easy to check that F is reticular K-l-transversal if and only if

$$\mathcal{E}(r;k) = \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)} + W_F + \mathrm{m}(r;k)^{l+1},$$

where $W_F = \langle \frac{\partial F}{\partial u_1}|_{u=0}, \cdots, \frac{\partial F}{\partial u_n}|_{u=0}\rangle_{\mathbf{R}}$.

We say that F is *reticular K-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbf{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbf{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbf{H}^r \times \mathbf{R}^{k+n}}$ at $(0, y_0, u_0')$ is reticular K-equivalent to F for some $(0, y_0, u_0') \in U$.

We say that F is *reticular K-versal* if F is reticular K-f-induced from all unfolding of f.

We say that F is *reticular K-infinitesimal versal* if

$$\mathcal{E}(r;k) = \langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)} + W_F.$$

We say that F is *reticular K-infinitesimal stable* if

$$\begin{aligned}&\mathcal{E}(r;k+n)\\ =\ &\langle F, x_1\frac{\partial F}{\partial x_1}, \cdots, x_r\frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k}\rangle_{\mathcal{E}(r;k+n)} + \langle\frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_n}\rangle_{\mathcal{E}(n)}.\end{aligned}$$

We say that F is *reticular K-homotopically stable* if for any smooth path-germ $(\mathbf{R}, 0) \to \mathcal{E}(r; k+n), t \mapsto F_t$ with $F_0 = F$, there exists a smooth path-germ $(\mathbf{R}, 0) \to \mathcal{B}(r; k+n) \times \mathcal{E}(n), t \mapsto (\Phi_t, \alpha_t)$ with $(\Phi_0, \alpha_0) = (id, 1)$ such that each (Φ_t, α_t) is a reticular K-isomorphism and $F_0 = \alpha_t \cdot F_t \circ \Phi_t$.

Theorem 6.4 (Transversality lemma) *Let U be a neighborhood of 0 in $0 \in \mathbf{R}^{r+k+n}$ with the coordinates $(x_1, \cdots, x_r, y_1, \cdots, y_k, u_1, \cdots, u_n)$ and A be a submanifold of $J^l(r+k, 1)$. Then the set*

$$T_A = \{F \in C^\infty(U, \mathbf{R}) \mid j_1^l F|_{x=0} \textit{ is transversal to } A\}$$

is dense in $C^\infty(U, \mathbf{R})$ with respect to C^∞-topology, where $j_1^l F(x,y,u)$ is the l-jet of the map $(x', y') \mapsto F(x+x', y+y', u)$ at 0.

The transversality we used is a slightly different for the ordinary one [13], however we can also prove this theorem by the method along the ordinary method.

Theorem 6.5 *Let $F \in \mathrm{m}(r; k+n)$ be an unfolding of $f \in \mathrm{m}(r; k)$. Then the following are equivalent.*
(1) *F is reticular K-stable.*
(2) *F is reticular K-versal.*
(3) *F is reticular K-infinitesimal versal.*
(4) *F is reticular K-infinitesimal stable.*
(5) *F is reticular K-homotopically stable.*

For $f \in \mathrm{m}(r; k)$ we define the *reticular K-codimension of f* by the $\mathbf{R}$-dimension of the vector space

$$\mathcal{E}(r; k)/\langle f, x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r}, \frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_k}\rangle_{\mathcal{E}(r;k)}.$$

By the above theorem if $a_1, \cdots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space, then $f + a_1 v_1 + \cdots a_n v_n \in \mathrm{m}(r; k+n)$ is a reticular K-stable unfolding of f.

7 Reticular Legendrian maps

Our purpose in this section is to investigate the stabilities of smooth contact regular r-cubic configurations. At first, we define the reticular Legendrian maps and their equivalence relation.

Let $\tilde{\mathbf{L}^0} = \{(q, z, p) \in J^1(\mathbf{R}^n, \mathbf{R}) | q_1 p_1 = \cdots = q_r p_r = q_{r+1} = \cdots = q_n = z = 0, q_{I_r} \geq 0\}$ be a representative of the union of $\tilde{L_\sigma^0}$ for all $\sigma \subset I_r$. We call the map germ

$$(\tilde{\mathbf{L}^0}, 0) \xrightarrow{i} (J^1(\mathbf{R}^n, \mathbf{R}), 0) \xrightarrow{\tilde{\pi}} (\mathbf{R}^n \times \mathbf{R}, 0)$$

a *reticular Legendrian map* if there exists a contact diffeomorphism C on $(J^1(\mathbf{R}^n, \mathbf{R}),\ 0)$ such that $i = C|_{\tilde{\mathbf{L}^0}}$. C is called an *extension* of i. We call $\{i(\tilde{L_\sigma^0})\}_{\sigma \subset I_r}$ the contact regular r-cubic configuration associated with $\tilde{\pi} \circ i$. We call F a *generating family* of $\tilde{\pi} \circ i$ if F is a generating family of $\{i(\tilde{L_\sigma^0})\}_{\sigma \subset I_r}$. A homeomorphism germ $\phi : (\tilde{\mathbf{L}^0}, 0) \longrightarrow (\tilde{\mathbf{L}^0}, 0)$ is called a *reticular diffeomorphism* if there exists a diffeomorphism Φ on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that $\phi = \Phi|_{\tilde{\mathbf{L}^0}}$ and $\phi(\tilde{L_\sigma^0}) = \tilde{L_\sigma^0}$ for all $\sigma \subset I_r$. Two reticular Legendrian maps $\tilde{\pi} \circ i_1, \tilde{\pi} \circ i_2 : (\tilde{\mathbf{L}^0}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ are called *Legendrian equivalent* if there exists a reticular diffeomorphism ϕ and a Legendrian equivalence Θ on $\tilde{\pi}$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
(\tilde{\mathbf{L}^0}, 0) & \xrightarrow{i_1} & (J^1(\mathbf{R}^n, \mathbf{R}), 0) & \xrightarrow{\tilde{\pi}} & (\mathbf{R}^n \times \mathbf{R}, 0) \\
\phi \downarrow & & \Theta \downarrow & & g \downarrow \\
(\tilde{\mathbf{L}^0}, 0) & \xrightarrow{i_2} & (J^1(\mathbf{R}^n, \mathbf{R}), 0) & \xrightarrow{\tilde{\pi}} & (\mathbf{R}^n \times \mathbf{R}, 0)
\end{array}$$

where g is the diffeomorphism of the base of $\tilde{\pi}$ induced from Θ.

Under this equivalence relation, we have the following theorem as a corollary of Theorem 5.6.

Theorem 7.1 (1) *For any reticular Legendrian map* $\tilde{\pi}\circ i$, *there exists a function germ* $F \in \mathrm{m}(r; k+n+1)$ *which is a generating family of* $\tilde{\pi}\circ i$.
(2) *For any C-non-degenerate function germ* $F \in \mathrm{m}(r; k+n+1)$, *there exists a reticular Legendrian map of which* F *is a generating family.*
(3) *Two reticular Legendrian maps are Legendrian equivalent if and only if their generating families are stably reticular K-equivalent.*

Here we give several definitions of the stabilities of reticular Legendrian maps.

Stability: Let $\tilde{\pi}\circ i : (\tilde{\mathrm{L}}^0, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ be a reticular Legendrian map. $\tilde{\pi}\circ i$ is called *stable* if the following condition holds: For any extension $C_0 \in C(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ of i and any representative $\tilde{C}_0 \in C(U, J^1(\mathbf{R}^n, \mathbf{R}))$, there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ in C^∞-topology such that for all $\tilde{C} \in N_{\tilde{C}_0}$ $\tilde{\pi}\circ \tilde{C}|_{\mathrm{L}^0 \ at\ x_0}$ and $\tilde{\pi}\circ i$ are Legendrian equivalent for some $x_0 = (0; 0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in U$.

Let $\tilde{\pi}\circ i$ is a reticular Legendrian map. By Lemma 5.3, we may assume that there exists an extension $C \in C^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ of i_0. Therefore we may consider the following other definitions of stabilities of reticular Legendrian maps: (1) The definition given by replacing $C(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and $C(U, J^1(\mathbf{R}^n, \mathbf{R}))$ to $C^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and $C^\alpha(U, J^1(\mathbf{R}^n, \mathbf{R}))$ of the original definition respectively. (2) The definition given by replacing to $C_Z(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and $C_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ respectively. (3) The definition given by replacing to $C^\alpha_Z(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and $C^\alpha_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ respectively.

Lemma 7.2 *The original definition and these definitions of stabilities of reticular Legendrian maps are all equivalent.*

Proof. (original)$\Rightarrow$(1). Let $C_0 \in C^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 \in C^\alpha(U, J^1(\mathbf{R}^n, \mathbf{R}))$ be a representative of C_0. Take a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ in $C(U, J^1(\mathbf{R}^n, \mathbf{R}))$ for which the hypothesis of the original definition holds. Set $N'_{\tilde{C}_0} = N_{\tilde{C}_0} \cap C^\alpha(U, J^1(\mathbf{R}^n, \mathbf{R}))$. Then the hypothesis of the definition of (1) holds for $N'_{\tilde{C}_0}$.
(1)$\Rightarrow$(3). Let $C_0 \in C^\alpha_Z(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 \in C^\alpha_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ be a representative of C_0. We construct the continuous map $C^\alpha_Z(V, J^1(\mathbf{R}^n, \mathbf{R})) \to C^\alpha(V \times \mathbf{R}, J^1(\mathbf{R}^n, \mathbf{R}))$ $(\tilde{C} \mapsto \tilde{C}')$ by the following: Let $\tilde{C} = (z_{\tilde{C}}, q_{\tilde{C}}, p_{\tilde{C}}) \in C^\alpha_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$. Then $\tilde{C}'$ is defined by $\tilde{C}'(Q, Z, P) = (q_{\tilde{C}}(Q, P), Z + z_{\tilde{C}}(Q, P), p_{\tilde{C}}(Q, P))$. Then $\tilde{C}'^*(dz - pdq) = dZ + \tilde{C}^*(dz - pdq) = dZ - PdQ$. Hence this map is well defined. Take a neighborhood $N_{\tilde{C}_0'}$ of $\tilde{C}_0{}'$ in $C(V \times \mathbf{R}, J^1(\mathbf{R}^n, \mathbf{R}))$ for which the hypothesis of the definition of (1) holds. Let $N'_{\tilde{C}_0}$ be the inverse image of $N_{\tilde{C}_0'}$ by the preceding map. Then the hypothesis of the definition of (3) holds for $N'_{\tilde{C}_0}$.
(3)$\Rightarrow$(2). Let $C_0 \in C_Z(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 \in C_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ be a representative of C_0. Define

$$C_Z(V, J^1(\mathbf{R}^n, \mathbf{R})) \to C^\infty(V, J^1(\mathbf{R}^n, \mathbf{R}) \cap \{Z = 0\})\ (\tilde{C} \mapsto \phi_{\tilde{C}} : (Q, P) \mapsto (Q, f_{\tilde{C}}P)\ ),$$

where $f^{\tilde{C}} \in C^\infty(V, \mathbf{R})$ is defined by $\tilde{C}^*(dz - pdq) = -f^{\tilde{C}}PdQ$. Then this map is continuous because $f^{\tilde{C}}P_i = (f^{\tilde{C}}(PdQ))(\frac{\partial}{\partial Q_i}) = -(dz - pdq)(\tilde{C}_* \frac{\partial}{\partial Q_i}) = -\frac{\partial z_{\tilde{C}}}{\partial Q_i} + p_{\tilde{C}} \frac{\partial q_{\tilde{C}}}{\partial Q_i}$ $(i = 1, \cdots, n)$. We may assume $\phi_{\tilde{C}_0}$ is embedding by shrinking V if necessary. By Lemma 4.3 there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ and a neighborhood V_1 of 0 in V and a neighborhood W of 0 in $J^1(\mathbf{R}^n, \mathbf{R}) \cap \{Z = 0\}$ such that

$$N_{\tilde{C}_0} \to \mathrm{Emb}(W, V)\ (\tilde{C} \mapsto (\phi_{\tilde{C}}|_{V_1})^{-1}|_W\ )$$

is well defined and continuous. Therefore we may define the following continuous map:

$$N_{\tilde{C}_0} \to C^{\alpha}_{Z}(W, J^1(\mathbf{R}^n, \mathbf{R}))\ (\tilde{C} \mapsto \tilde{C} \circ (\phi_{\tilde{C}}|_{V_1})^{-1}|_W\ ).$$

Take a neighborhood N of $\tilde{C}_0 \circ (\phi_{\tilde{C}_0}|_{V_1})^{-1}|_W$ for which the hypothesis of the definition of (3) holds. Let $N'_{\tilde{C}_0}$ be the inverse image of N by the preceding map. Then the hypothesis of the definition of (2) holds for $N'_{\tilde{C}_0}$.
(2)$\Rightarrow$(original). Let $C_0 \in C(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 \in C(U, J^1(\mathbf{R}^n, \mathbf{R}))$ be a representative of C_0. Let $V = U \cap \{Z = 0\}$ and $\tilde{C}_0{}' = \tilde{C}_0|_{Z=0}$. Take a neighborhood $N_{\tilde{C}_0{}'}$ of $\tilde{C}_0{}'$ in $C_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ for which the hypothesis of the definition of (2) holds. Because $C(U, J^1(\mathbf{R}^n, \mathbf{R})) \to C_Z(V, J^1(\mathbf{R}^n, \mathbf{R}))$ $(\tilde{C} \mapsto \tilde{C}|_{Z=0})$ is continuous, if we set $N'_{\tilde{C}_0}$ the inverse image of $N_{\tilde{C}_0{}'}$ by the preceding map then the hypothesis of the original definition holds for $N'_{\tilde{C}_0}$. □

Homotopical Stability: Let $\bar{\pi} \circ i : (\tilde{\mathbf{L}}^0, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ be a reticular Legendrian map. A map germ $\bar{i} : (\tilde{\mathbf{L}}^0 \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, P, t) \mapsto \bar{i}_t(Q, P))(\bar{i}_0 = i)$ is called a *reticular Legendrian deformation* of i if there exists a one-parameter family of contact diffeomorphisms $\bar{C} : (J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, Z, P, t) \mapsto \bar{C}_t(Q, Z, P))$ such that $\bar{i}_t = \bar{C}_t|_{\tilde{\mathbf{L}}^0}$ for t near 0. We call $\bar{C}$ an *extension* of $\bar{i}$. Let $\phi : (\tilde{\mathbf{L}}^0, 0) \to (\tilde{\mathbf{L}}^0, 0)$ be a reticular diffeomorphism. A map germ $\bar{\phi} : (\tilde{\mathbf{L}}^0 \times \mathbf{R}, 0) \to (\tilde{\mathbf{L}}^0, 0)((Q, P, t) \mapsto \bar{\phi}_t(Q, P))(\bar{\phi}_0 = \phi)$ is called a *one-parameter deformation of reticular diffeomorphisms* of ϕ if there exists a one-parameter family of diffeomorphisms $\bar{\Phi} : (J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, Z, P, t) \mapsto \bar{\Phi}_t(Q, Z, P))$ such that $\bar{\phi}_t = \bar{\Phi}_t|_{\tilde{\mathbf{L}}^0}$ for t near 0 and each $\bar{\phi}_t$ is a reticular diffeomorphism. We call $\bar{\Phi}$ an *extension* of $\bar{\phi}$. A reticular Legendrian map $\bar{\pi} \circ i : (\tilde{\mathbf{L}}^0, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ is called *homotopically stable* if for any reticular Legendrian deformation $\bar{i} = \{\bar{i}_t\}$ of i there exist a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ of $id_{(\tilde{\mathbf{L}}^0, 0)}$ and a one-parameter family of Legendrian equivalences $\bar{\Theta} = \{\bar{\Theta}_t\}$ with $\bar{\Theta}_0 = id_{(J^1(\mathbf{R}^n, \mathbf{R}), 0)}$ such that $\bar{i}_t = \bar{\Theta}_t \circ i \circ \bar{\phi}_t$ for t near 0.

Infinitesimal Stability: A vector field v on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ is called *tangent* to $(\tilde{\mathbf{L}}^0, 0)$ if $v|_{\tilde{L}^0_\sigma}$ is tangent to $\tilde{L}^0_\sigma$ for all $\sigma \subset I_r$. A function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ is called fiber preserving if there exists function germs $h_0, \cdots, h_n$ on the base of $\bar{\pi}$ such that $H(q, z, p) = \sum_{i=1}^n h_i(q, z)p_i + h_0(q, z)$. A reticular Legendrian map $\bar{\pi} \circ i : (\tilde{\mathbf{L}}^0, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ is called *infinitesimal stable* if for any function germ f on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ there exists a fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and a vector field v on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that v is tangent to $(\tilde{\mathbf{L}}^0, 0)$ and $X_f \circ i = X_H \circ i + i_* v$, where X_f and X_H are the contact hamiltonian vector field of f and H respectively and $i_* v$ is defined by $i_* v = (C_* v) \circ i$ for an extension $C \in C(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ of i.

Lemma 7.3 *For any one-parameter family of Legendrian equivalences $\bar{\Theta} : (J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)$ with $\bar{\Theta}_0 = id$, there exists a fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that $X_H = \frac{d\bar{\Theta}}{dt}|_{t=0}$. Conversely for any fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$, the flow $\bar{\Theta}$ of X_H with the initial condition $\bar{\Theta}_0 = id$ is a one-parameter family of Legendrian equivalences.*

Theorem 7.4 *Let $\bar{\pi} \circ i : (\tilde{\mathbf{L}}^0, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0) \to (\mathbf{R}^n \times \mathbf{R}, 0)$ be a reticular Legendrian map with the generating family $F(x, y, q, z) \in \mathfrak{m}(r; k + n + 1)$. Let $f = F|_{\{q=z=0\}}$. Then the*

following are equivalent.
(1) *F is a reticular K-stable unfolding of f.*
(2) *$\tilde{\pi} \circ i$ is homotopically stable.*
(3) *$\tilde{\pi} \circ i$ is infinitesimal stable.*
(4) *For any function germ f on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$, there exists a fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that $f \circ i = H \circ i$.*
(5) *$\tilde{\pi} \circ i$ is stable.*

Proof. We shall prove (1)$\Leftrightarrow$(5), (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1).
(1)$\Rightarrow$(5). Let $C_0 \in C_Z^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be an extension of i_0 and $\tilde{C}_0 = (q_{\tilde{C}_0}, z_{\tilde{C}_0}, p_{\tilde{C}_0}) \in C_Z^\alpha(U, J^1(\mathbf{R}^n, \mathbf{R}))$ be a representative of C_0. We shall construct the map (1) which maps a contact embedding near $\tilde{C}_0$ to a function near a representative of F.
For each $\tilde{C} \in C_Z^\alpha(U, J^1(\mathbf{R}^n, \mathbf{R}))$ we define

$$P_{\tilde{C}} = \{(Q, P; q, z, p) \in U \times J^1(\mathbf{R}^n, \mathbf{R}) \,|(z, q, p) = \tilde{C}(Q, P)\}$$

$$\pi_{\tilde{C}} : U \to \mathbf{R}^{2n} \ (\ (Q, P) \mapsto (Q, p_{\tilde{C}}(Q, P)) \ ).$$

By taking some Legendrian equivalence of $\tilde{\pi} \circ i$ and shrinking U if necessary, we may assume $\pi_{\tilde{C}_0}$ is embedding. By Lemma 4.3 there exists a neighborhood $N_{\tilde{C}_0}$ of $\tilde{C}_0$ and a neighborhood U_1 of 0 in U and a convex neighborhood V of 0 in $\mathbf{R}^{2n}$ such that

$$N_{\tilde{C}_0} \to \mathrm{Emb(V, U)} \ (\ \tilde{\mathrm{C}} \mapsto (\pi_{\tilde{\mathrm{C}}}|_{\mathrm{U}_1})^{-1}|_{\mathrm{V}} \ )$$

is well defined and continuous. Let $\tilde{C} \in N_{\tilde{C}_0}$. Since $dz - pdq = -PdQ$ on $P_{\tilde{C}}$, there exists $H_{\tilde{C}} \in C^\infty(V, \mathbf{R})$ $(H_{\tilde{C}}(0,0) = 0)$ such that

$$P_{\tilde{C}}|_{(Q,p)\in V} = \{(Q, -\frac{\partial H_{\tilde{C}}}{\partial Q}(Q, p); -\frac{\partial H_{\tilde{C}}}{\partial p}(Q, p), H_{\tilde{C}} - \langle\frac{\partial H_{\tilde{C}}}{\partial p}(Q, p), p\rangle, p)\}.$$

Since $H_{\tilde{C}}(Q, p) = z - \langle q, p\rangle$ on $P_{\tilde{C}}$, the map

$$N_{\tilde{C}_0} \to C^\infty(V, \mathbf{R}) \ (\tilde{C} \mapsto H_{\tilde{C}})$$

is continuous. Let $V' = V \cap \{Q_{r+1} = \cdots = Q_n = 0\}$. Now we consider the continuous map

$$\phi : N_{\tilde{C}_0} \to C^\infty(V' \times \mathbf{R}^{n+1}, \mathbf{R}) \ (\tilde{C} \mapsto \tilde{F}_{\tilde{C}}(x, y, q, z) = -z + H_{\tilde{C}}(x, 0; y) + \langle y, q\rangle), \quad (1)$$

where $x = (x_1, \cdots, x_r), y = (y_1, \cdots, y_n), q = (q_1, \cdots, q_n)$. This map maps a contact embedding near $\tilde{C}_0$ to a function near $\tilde{F}_0 = \tilde{F}_{\tilde{C}_0}$. We may assume that $\tilde{F}_0$ is a representative of F. Since F is stable by hypothesis, there exists a neighborhood $N_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{G} \in N_{\tilde{F}_0}$ $\tilde{G}|_{\mathbf{H}^r \times \mathbf{R}^{2n+1}}$ at $(0, y^0, q^0, z^0)$ and F are reticular K-equivalent for some $(0, y^0, q^0, z^0) \in V' \times \mathbf{R}^{n+1}$. Set $N'_{\tilde{C}_0} = \phi^{-1}(N_{\tilde{F}_0})$. Let $\tilde{C} \in N'_{\tilde{C}}$ be given. Take $(0, y^0, q^0, z^0) \in V' \times \mathbf{R}^{n+1}$ such that the preceding condition holds for $\tilde{F}_{\tilde{C}}$. If we denote $\{\tilde{L}_\sigma^{\tilde{C}}\}_{\sigma \subset I_r}$ the contact regular r-cubic configuration defined by $F_{\tilde{C}} = \tilde{F}_{\tilde{C}}|_{\mathbf{H}^r \times \mathbf{R}^{2n+1}}$ at $(0, y^0, q^0, z^0)$, then for each $\sigma \subset I_r$

$$\tilde{L}_\sigma^{\tilde{C}} = \{(q_0 + q, z_0 + z, \frac{\partial F_{\tilde{C}}}{\partial q}(x, y^0 + y, q^0 + q, z^0 + z))|$$

$$x_\sigma = \frac{\partial F_{\tilde{C}}}{\partial x_{I_r-\sigma}} = \frac{\partial F_{\tilde{C}}}{\partial y} = F_{\tilde{C}} = 0\}$$

$$= \{(q_0+q, H_{\tilde{C}}(x,0;y_0+y) + \langle y_0+y, q_0+q\rangle, y_0+y)|$$
$$x_\sigma = \frac{\partial H_{\tilde{C}}}{\partial x_{I_r-\sigma}} = \frac{\partial H_{\tilde{C}}}{\partial y} + q_0 + q = 0\}$$
$$= \{(-\frac{\partial H_{\tilde{C}}}{\partial y}, H_{\tilde{C}} - \langle y_0+y, \frac{\partial H_{\tilde{C}}}{\partial y}\rangle, y_0+y)|x_\sigma = \frac{\partial H_{\tilde{C}}}{\partial x_{I_r-\sigma}}(x,0;y_0+y) = 0\}$$
$$= \{(-\frac{\partial H_{\tilde{C}}}{\partial p}, H_{\tilde{C}} - \langle y_0+p, \frac{\partial H_{\tilde{C}}}{\partial p}\rangle, y_0+p)|$$
$$Q_\sigma = \frac{\partial H_{\tilde{C}}}{\partial Q_{I_r-\sigma}}(Q;y_0+p) = Q_{r+1} = \cdots = Q_n = 0, Q_{I_r-\sigma} \geq 0\}$$
$$= \tilde{C}(\tilde{\tilde{L}}_\sigma \ at \ (0;0,P^0)),$$

where $(0;0,P^0) = \tilde{C}^{-1}(q_0,z_0,y_0)$ and $\tilde{\tilde{L}}_\sigma = \{(Q,Z,P) \in J^1(\mathbf{R}^n,\mathbf{R})|Q_\sigma = P_{I_r-\sigma} = Q_{r+1} = \cdots = Q_n = Z = 0, Q_{I_r-\sigma} \geq 0\}$. Since $F_{\tilde{C}}$ and F_0 are reticular K-equivalent, this implies that $\bar{\pi} \circ \tilde{C}|_{\tilde{\mathbf{L}}^0 \ at \ (0;0,P^0)}$ and $\bar{\pi} \circ i$ are Legendrian equivalent.
(5)$\Rightarrow$(1). Let a reticular Legendrian map $\bar{\pi} \circ i$ be given and C_0 be an extension of i. Then we may assume that $C_0 \in C^\Omega_Z(J^1(\mathbf{R}^n,\mathbf{R}),0)$ by Lemma 7.2. By considering some Legendrian equivalence of $\bar{\pi} \circ i$, we may assume that there exists a function germ $T_0 \in \mathrm{m}^2(2n)$ such that

$$\{(Q,P;C_0(Q,P))\} = \{(Q, -\frac{\partial T_0}{\partial Q}(Q,p); -\frac{\partial T_0}{\partial p}(Q,p), T_0(Q,p) - \langle\frac{\partial T_0}{\partial p}(Q,p),p\rangle, p)\}$$

Define a generating family $F_0(x,y,q,z) \in \mathrm{m}(r;n+n+1)$ of $\bar{\pi} \circ i$ by $F_0(x,y,q,z) = -z + T(x,0;y) + \langle y,q\rangle$. It is enough to prove that F_0 is a reticular K-stable unfolding of $F_0|_{\{q=z=0\}}$. Let $\tilde{F}_0 \in C^\infty(U,\mathbf{R})$ be a representative of F_0. By shrinking U if necessary, we may assume that there exist neighborhood U_1,U_2,U_3 of 0 in $(\mathbf{R}^n;Q),(\mathbf{R}^n;y),(\mathbf{R}^{n+1};(q,z))$ respectively and $\tilde{T}_0(Q,y) \in C^\infty(U_1 \times U_2,\mathbf{R})$ such that the following conditions hold:
(a) $\tilde{T}_0$ is a representative of T_0,
(b) $U = (U_1 \cap \{Q_{r+1} = \cdots = Q_n = 0\}) \times U_2 \times U_3$ and $\tilde{F}_0(x,y,q,z) = -z + \tilde{T}_0(x,0;y) + \langle y,q\rangle$,
(c) The map $U_1 \times U_2 \times U_3 \to U_1 \times U_2 \times \mathbf{R}^{n+1}$ $((Q,y,q,z) \mapsto (Q,y,\frac{\partial \tilde{T}_0}{\partial y}(Q,y)+q, -z + \tilde{T}(Q,y) + \langle y,q\rangle))$ is an embedding,
(d) The map $U_1 \times U_2 \to U_1 \times \mathbf{R}^n$ $((Q,y) \mapsto (Q, -\frac{\partial \tilde{T}_0}{\partial Q}(Q,y))$ is an embedding.

Define $\bar{F}_0 \in C^\infty(U_1 \times U_2 \times U_3,\mathbf{R})$ by $\bar{F}_0(Q,y,q,z) = -z + \tilde{T}(Q,y) + \langle y,q\rangle$. Since the map

$$C^\infty(U,\mathbf{R}) \to C^\infty(U_1 \times U_2 \times U_3,\mathbf{R}) \ (\ \tilde{F} \mapsto \bar{F}(Q,y,q,z) = \bar{F}_0(Q,y,q,z) + (\tilde{F} - \tilde{F}_0)(Q',y,q,z) \ )$$

$(Q' = (Q_1,\cdots,Q_{r+1}))$ is continuous, the map

$$C^\infty(U,\mathbf{R}) \to C^\infty(U_1 \times U_2 \times U_3, U_1 \times U_2 \times \mathbf{R}^{n+1}) \ (\ \tilde{F} \mapsto \phi_{\tilde{F}}(Q,y,q,z) = (Q,y,\frac{\partial \bar{F}}{\partial y},\bar{F}) \ )$$

is also continuous. Since $\phi_{\tilde{F}_0}$ is embedding by (c), there exist a neighborhood $N^1_{\tilde{F}_0}$ of $\tilde{F}_0$ and a neighborhood U' of 0 in $U_1 \times U_2 \times U_3$ and an open ball V around 0 in $U_1 \times U_2 \times \mathbf{R}^{n+1}$ such that

$$N^1_{\tilde{F}_0} \to \mathrm{Emb}(V, U_1 \times U_2 \times U_3) \ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_V \ )$$

is well defined and continuous. Let $V_1 = V \cap (U_1 \times U_2 \times \{0\})$. Then the map

$$N^1_{\tilde{F}_0} \to \mathrm{Emb}(V_1, U_1 \times U_2 \times U_3)\ (\ \tilde{F} \mapsto (\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}\ )$$

is also continuous. We denote $(\phi_{\tilde{F}}|_{U'})^{-1}|_{V_1}(Q, y)$ by $(Q, y, q_{\tilde{F}}(Q, y), z_{\tilde{F}}(Q, y))$. Then the map

$$N^1_{\tilde{F}_0} \to C^\infty(V_1, U_1 \times \mathbf{R}^n)\ (\ \tilde{F} \mapsto \psi_{\tilde{F}}(Q, y) = (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}(Q, y), z_{\tilde{F}}(Q, y)))\ )$$

is also continuous. Since $\psi_{\tilde{F}_0}$ is embedding by (d), there exist a neighborhood $N^2_{\tilde{F}_0}$ of $\tilde{F}_0$ and a neighborhood V_2 of 0 in V_1 and an open ball W around 0 in $U_1 \times \mathbf{R}^n$ such that

$$N^2_{\tilde{F}_0} \to \mathrm{Emb}(W, V_1)\ (\ \tilde{F} \mapsto (\psi_{\tilde{F}}|_{V_2})^{-1}|_W\ )$$

is well defined and continuous. We denote $(\psi_{\tilde{F}}|_{V_2})^{-1}|_W(Q, P)$ by $(Q, y_{\tilde{F}}(Q, P))$. Then the map

$$N^2_{\tilde{F}_0} \to \mathrm{Emb}(W, U_1 \times U_2 \times U_3)\ (\ \tilde{F} \mapsto (\ (Q, P) \mapsto (Q, y_{\tilde{F}}(Q, P), q_{\tilde{F}}(Q, y_{\tilde{F}}), z_{\tilde{F}}(Q, y_{\tilde{F}}))\ )$$

is also continuous. Hence the map

$$N^2_{\tilde{F}_0} \to C^\alpha_Z(W, J^1(\mathbf{R}^n, \mathbf{R}))\ (\tilde{F} \mapsto \tilde{C}_{\tilde{F}}(Q, P) = (q_{\tilde{F}}, z_{\tilde{F}}, \frac{\partial \bar{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}}, z_{\tilde{F}}))\ ) \tag{2}$$

is well defined and continuous. This map maps a function near $\tilde{F}_0$ to a contact embedding near a representative $\tilde{C}_{\tilde{F}_0}$ of C_0. Hence by hypothesis, there exists a neighborhood $N^3_{\tilde{F}_0}$ of $\tilde{F}_0$ such that for any $\tilde{F} \in N^3_{\tilde{F}_0}$ $\bar{\pi} \circ (\tilde{C}_{\tilde{F}}|_{\tilde{\mathbf{L}}^0 \text{ at } (Q^0, P^0)})$ and $\bar{\pi} \circ i$ are Legendrian equivalent for some $(Q^0, P^0) = (0; 0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in W$. Therefore if we set $(Q^0, y^0, q^0, z^0) = (Q^0, y_{\tilde{F}}(Q^0, P^0), q_{\tilde{F}}(Q^0, y_{\tilde{F}}), z_{\tilde{F}}(Q^0, y_{\tilde{F}}))$ then $\tilde{F}$ at $(0, y^0, q^0, z^0)$ and F_0 are reticular K-equivalent. $\square$

The construction of (2) is summarized in the following diagram:

$$\begin{array}{ccccc}
U_1 \times U_2 \times U_3 & = U_1 \times U_2 \times U_3 & U_1 \times \mathbf{R}^n & \supset & W \\
(Q, y, q, z) & (Q, y, q_{\tilde{F}}, z_{\tilde{F}}) & \to (Q, -\frac{\partial \bar{F}}{\partial Q}(Q, y, q_{\tilde{F}}, z_{\tilde{F}})) & = & (Q, P) \\
\phi_{\tilde{F}} \downarrow & \uparrow & \nearrow & & \downarrow \tilde{C}_{\tilde{F}} \\
(Q, y, \frac{\partial \bar{F}}{\partial y}, \bar{F}) & (Q, y, 0, 0) & \psi_{\tilde{F}} & & (q_{\tilde{F}}, z_{\tilde{F}}, \frac{\partial \bar{F}}{\partial q}(Q, y_{\tilde{F}}, q_{\tilde{F}}, z_{\tilde{F}})) \\
U_1 \times U_2 \times \mathbf{R}^{n+1} & \supset\ V_1 \times \{0\} & & & J^1(\mathbf{R}^n, \mathbf{R})
\end{array}$$

(1)$\Rightarrow$(2). Let $\bar{i} : (\tilde{\mathbf{L}}^0 \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, P, t) \mapsto \bar{i}_t(Q, P))$ be a reticular Legendrian deformation of i. Take a one-parameter family of contact diffeomorphisms $\bar{C} : (J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, Z, P, t) \mapsto \bar{C}_t(Q, Z, P))$ such that $\bar{i}_t = \bar{C}_t|_{\tilde{\mathbf{L}}^0}$ for t near 0. By using analogous methods of the proof of Lemma 5.3, we may assume that $\bar{C}_t \in C^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ for t near 0. Moreover we may assume that there exists a smooth function germ $\bar{T} : (\mathbf{R}^{2n} \times \mathbf{R}, 0) \to (\mathbf{R}, 0)((Q, p, t) \mapsto \bar{T}_t(Q, p))$ such that

$$\{(Q, P; \bar{C}_t(Q, 0, P); t)\} = \{(Q, -\frac{\partial \bar{T}_t}{\partial Q}(Q, p); -\frac{\partial \bar{T}_t}{\partial p}(Q, p), \bar{T}_t(Q, p) - \langle \frac{\partial \bar{T}_t}{\partial p}(Q, p), p \rangle, p; t)\}.$$

Define $F(x, y, q, z, t) \in \mathcal{E}(r; n + n + 1 + 1)$ by $F(x, y, q, z, t) = F_t(x, y, q, z) = -z + \bar{T}_t(x, 0; y) + \langle y, q \rangle$. Then F_t is a generating family of $\bar{i}_t$ for t near 0. By hypothesis there

exists a one-parameter family of reticular K-equivalences $\{(a_t, \Psi_t)\}$ such that $F_t = a_t \cdot F_0 \circ \Psi_t$ for t near 0. If Ψ_t is written in the form $\Psi_t(x, y, q, z) = (*, *, g_t^1(q, z), \cdots, g_t^{n+1}(q, z))$ then we can prove that $\bar{i}_t(\tilde{L_\sigma^0}) = \bar{g}_t^* \circ \bar{i}_0(\tilde{L_\sigma^0})$ for all $\sigma \subset I_r$ by the same method of the proof of theorem 5.6-(3)a. Define the one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ by $\bar{\phi}_t = C_0^{-1} \circ \bar{g}_t^{*-1} \circ C_t|_{\tilde{\mathbf{L}^0}}$. Then we have $\bar{i}_t = \bar{g}_t^* \circ \bar{i}_0 \circ \bar{\phi}_t$ for t near 0.
(2)$\Rightarrow$(3). Let a function germ f on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be given. Take an extension $C \in C(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ of i. Let X_f be the contact hamiltonian vector field of f on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and $\bar{C} = \{\bar{C}_t\}$ be the flow of X_f with the initial condition $\bar{C}_0 = C$. Since $\bar{i} = \{\bar{i}_t = \bar{C}_t|_{\tilde{\mathbf{L}^0}}\}$ is a reticular Legendrian deformation of i, there exists a one-parameter family of Legendrian equivalences $\bar{\Theta} = \{\bar{\Theta}_t\}$ with $\bar{\Theta}_0 = id_{(J^1(\mathbf{R}^n, \mathbf{R}), 0)}$ and a one-parameter deformation of reticular diffeomorphisms $\bar{\phi} = \{\bar{\phi}_t\}$ of $id_{(\tilde{\mathbf{L}^0}, 0)}$ such that $\bar{i}_t = \bar{\Theta}_t \circ i \circ \bar{\phi}_t$ for t near 0. Let $\bar{\Phi} : (J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}, 0) \to (J^1(\mathbf{R}^n, \mathbf{R}), 0)((Q, Z, P, t) \mapsto \bar{\Phi}_t(Q, Z, P))$ be an extension of $\bar{\phi}$ and set $v = \frac{d\bar{\Phi}_t}{dt}|_{t=0}$. Then v is tangent to $(\tilde{\mathbf{L}^0}, 0)$ and by Lemma 7.3 there exists a fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that $\frac{d\bar{\Theta}_t}{dt}|_{t=0} = X_H$. Then

$$X_f \circ i = \frac{d\bar{C}_t}{dt}|_{t=0} \circ i = \frac{d\bar{\Theta}_t}{dt}|_{t=0} \circ i + (C_* \frac{d\bar{\Phi}_t}{dt}|_{t=0}) \circ i = X_H \circ i + i_* v.$$

This implies that $\tilde{\pi} \circ i$ is infinitesimal stable.
(3)$\Rightarrow$(4). Let a function germ f on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ be given. By hypothesis, there exists a fiber preserving function germ H on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ and a vector field v on $(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ such that v is tangent to $\tilde{\mathbf{L}^0}$ and $X_f \circ i = X_H \circ i + i_* v$. Set $i_\sigma = i|_{\tilde{L_\sigma^0}}, v_\sigma = v|_{\tilde{L_\sigma^0}}$ for each $\sigma \subset I_r$. Then it is easy to prove that $(f - H) \circ i_\sigma = 0$ because $X_f \circ i_\sigma = X_H \circ i_\sigma + (i_\sigma)_* v_\sigma$. Therefore $f \circ i = H \circ i$.
(4)$\Rightarrow$(1)Take an extension $C = (q, z, p) \in C_Z^\alpha(J^1(\mathbf{R}^n, \mathbf{R}), 0)$ of i. We may assume that there exists a function germ $T(Q, p)$ on $(\mathbf{R}^{2n}, 0)$ such that

$$\{(Q, P; C(Q, P))\} = \{(Q, -\frac{\partial T}{\partial Q}(Q, p); -\frac{\partial T}{\partial p}(Q, p), T(Q, p) - \langle \frac{\partial T}{\partial p}(Q, p), p \rangle, p)\}$$

Define a generating family $F(x, y, q, z) \in \mathcal{E}(r; n + n + 1)$ of $\tilde{\pi} \circ i$ by $F(x, y, q, z) = -z + T(x, 0; y) + \langle y, q \rangle$. Since $(Q, p) \mapsto (q(Q, p), p)$ is invertible, there exists $I \subset \{1, \cdots, n\}(|I| = r)$ such that $\phi : (x, y) \mapsto (q_I(x, 0; y), y)$ is also invertible. On the other hand since $(Q, p) \mapsto (Q, P(Q, p))$ is invertible, $\psi : (x, y) \mapsto (x, P(x, 0; y))$ is also invertible. Set $C' = \phi \circ \psi^{-1}$. Let $f := F|_{q=z=0}$. Set $g(q, z, y) = g'(q, y) := f \circ \phi^{-1}(q_I, y)$. Since $C(x, 0; P)|_{x_1 P_1 = \cdots = x_r P_r = 0} = i(x, 0; P)$, there exists a fiber preserving function germ $H(q, z, p) = \sum_{i=1}^n h_i(q, z) p_i + h_0(q, z)$ such that

$$g \circ C(x, 0; P)|_{x_1 P_1 = \cdots = x_r P_r = 0} = H \circ C(x, 0; P).$$

Therefore there exist $a_1, \cdots, a_r \in \mathcal{E}(r; n)$ such that

$$g \circ C(x, 0; P) = H \circ C(x, 0; P) + \sum_{j=1}^r x_j P_j a_j(x, P).$$

Hence

$$\begin{aligned} & f(x, y) \\ = \ & (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) \circ \psi = g' \circ C' \circ \psi = g \circ C(x, 0; P(x, 0; y)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} h_i(q(x,0;y), z(x,0;y))y_i + h_0(q(x,0;y), z(x,0;y)) + \sum_{j=1}^{r} x_j P_j(x,0;y)a'_j(x,y) \\
&= \sum_{i=1}^{n} h_i(-\frac{\partial T}{\partial y}, T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y)y_i + h_0(-\frac{\partial T}{\partial y}, T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y) \\
&\qquad + \sum_{j=1}^{r} x_j(-\frac{\partial T}{\partial x_j}(x,0;y))a'_j(x,y) \\
&\equiv \sum_{i=1}^{n} h_i(0)y_i + h_0(0) \bmod \langle \frac{\partial T}{\partial y}(x,0;y), (T - \langle \frac{\partial T}{\partial y}, y\rangle)(x,0;y), x\frac{\partial T}{\partial x}(x,0;y)\rangle_{\mathcal{E}(r;n)}, \\
&\equiv \sum_{i=1}^{n} h_i(0)y_i + h_0(0) \bmod \langle T(x,0;y), x\frac{\partial T}{\partial x}(x,0;y), \frac{\partial T}{\partial y}(x,0;y)\rangle_{\mathcal{E}(r;n)},
\end{aligned}
$$

where $a'_j(x,0;y) = a_j(x,0;-\frac{\partial T}{\partial Q}(x,0;y))$ $(j = 1,\cdots,r)$. This implies that F is a reticular K-infinitesimal versal unfolding of f. Hence F is a reticular K-stable unfolding of f. □

8 Classification of function germs

We now start the classification of function germs with reticular K-codimension lower than 8 with respect to reticular K-equivalence. By Lemma 7.1 and Lemma 7.2 in [10], we have only to classify residual singularities, that is function germs in $\mathrm{m}(r;k)^2$ whose restriction to $x = 0$ is an element of $\mathrm{m}(0;k)^3$. $j_{y^\alpha,x^\beta}f(0) \approx g$ denotes quasihomogeneous equivalence of jets and $f \approx g$ means f is reticular K-equivalent to g and $\Rightarrow$ means 'see' or 'implies'.

Let $f \in \mathrm{m}(r;k)^2$ be a residual singularity with the reticular K-codimension lower than 8. We set $\phi(y) = f(0,y) \in \mathrm{m}(0;k)^3$.

The case $r = 1, k = 0$. $f \approx x^n$ $(n = 2,\cdots,7)$.

The case $r = 1, k = 1$. One of the five:

$$
\begin{array}{lcl}
j^2f(0) \approx xy + x^2 \text{ or } xy & \Rightarrow & f \approx xy + \varepsilon y^n \ (\varepsilon^{n+1} = 1, n = 3,\cdots,7), \\
j_{y^3,x^2}f(0) \approx y^3 + x^2 & \Rightarrow & f \approx y^3 + x^2, \\
j_{y^3,x^2}f(0) \approx x^2 & \Rightarrow & (1), \\
j_{y^3,x^2}f(0) \approx y^3 & \Rightarrow & (3), \\
j_{y^3,x^2}f(0) \approx 0 & \Rightarrow & (5).
\end{array}
$$

(1) $j_{y^3,x^2}f(0) = x^2 \Rightarrow$ one of the five:

$$
\begin{array}{lcl}
j_{y^4,x^2}f(0) \approx y^4 + axy^2 \pm x^2(a^2 \neq 4) & \Rightarrow & f \approx y^4 + axy^2 \pm x^2(a^2 \neq 4), \\
j_{y^4,x^2}f(0) \approx (y^2 \pm x)^2 & \Rightarrow & f \approx y^5 + (y^2 \pm x)^2 \text{ or } y^6 \pm (y^2 \pm x)^2, \\
j_{xy^2,x^2}f(0) \approx xy^2 \pm x^2 & \Rightarrow & f \approx y^5 + xy^2 \pm x^2, \\
j_{y^5,x^2}f(0) \approx y^5 + x^2 & \Rightarrow & f \approx y^5 \pm xy^3 + x^2, \\
j_{y^5,x^2}f(0) \approx x^2 \text{ or } 0 & \Rightarrow & (2).
\end{array}
$$

(2) $j_{y^6,x^2}f(0)$ is adjacent to $y^6 + axy^3 \pm x^2(a^2 \neq \pm 4)$ and hence the codimension of $f \geq \dim\mathcal{E}(1;1)/(\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, y\frac{\partial f}{\partial y}\rangle_{\mathbf{R}} + \langle x^3, x^2y, xy^4, y^7\rangle_{\mathcal{E}(1;1)}) \geq 12 - 3 = 9$.

(3) $j_{y^3,x^2}f(0) = y^3 \Rightarrow$ one of the five:

$$
\begin{array}{lll}
j^3f(0) \approx y^3 + ax^2y + 2x^3 (a \neq -3) & \Rightarrow & f \approx y^3 + ax^2y + 2x^3 (a \neq -3), \\
j^3f(0) \approx y^3 + xy^2 & \Rightarrow & f \approx y^3 + xy^2 \pm x^4 \text{ or } y^3 + xy^2 \pm x^5, \\
j^3f(0) \approx y^3 + x^2y & \Rightarrow & f \approx y^3 + x^2y, \\
j_{y^3,x^4}f(0) \approx y^3 + x^4 & \Rightarrow & f \approx y^3 \pm x^3y + x^4, \\
j_{y^3,x^4}f(0) \approx y^3 & \Rightarrow & (4).
\end{array}
$$

(4) f is adjacent to $y^3 \pm x^4y + x^5$ and this has codimension 9.
(5) $j_{y^3,x^2}f(0) = 0 \Rightarrow$ one of the four:

$$
\begin{array}{lll}
j^3f(0) \approx xy^2 \pm x^3 & \Rightarrow & f \approx y^4 \pm xy^2 \pm x^3 \text{ or } y^5 \pm xy^2 \pm x^3, \\
j^3f(0) \approx xy^2 & \Rightarrow & f \approx y^4 \pm xy^2 \pm x^4, \\
j^3f(0) \approx x^2y & \Rightarrow & f \approx y^4 + xy^3 \pm x^2y, \\
j^3f(0) \approx x^3 \text{ or } 0 & \Rightarrow & (6).
\end{array}
$$

(6) $j^3f(0) = x^3$ or $0 \Rightarrow f$ is adjacent to $y^4 + xy^3 \pm x^3$ and this has codimension 8.
The case $r = 1, k = 2$ One of the two:

$$
\begin{array}{lll}
j^3\phi \neq 0 & \Rightarrow & (7), \\
j^3\phi = 0 & \Rightarrow & (22).
\end{array}
$$

(7). $j^3\phi \neq 0 \Rightarrow$ on of the four:

$$
\begin{array}{lll}
\phi \in D_4 & \Rightarrow & (8), \\
\phi \in D_5 & \Rightarrow & (12), \\
\phi \in D_6 & \Rightarrow & (16), \\
\phi \in E_6 & \Rightarrow & (19).
\end{array}
$$

(8). $\phi = y_1^2y_2 \pm y_2^3 \Rightarrow$ one of the four:

$$
\begin{array}{lllll}
j_{y_1^2y_2,y_2^3,xy_2}f(0) & \approx & y_1^2y_2 \pm y_2^3 + xy_1 + axy_2, & & \\
 & & \qquad a^2 \pm 1 \neq 0 & \Rightarrow & f \approx y_1^2y_2 \pm y_2^3 + xy_1 + axy_2, \\
j_{y_1^2y_2,y_2^3,xy_2}f(0) & \approx & y_1^2y_2 \pm y_2^3 \pm xy_2 & \Rightarrow & (9), \\
j_{y_1^2y_2,y_2^3,x^2}f(0) & \approx & y_1^2y_2 \pm y_2^3 + x^2 & \Rightarrow & (10), \\
j_{y_1^2y_2,y_2^3,x^2}f(0) & \approx & y_1^2y_2 \pm y_2^3 & \Rightarrow & (11).
\end{array}
$$

(9) $j_{y_1^2y_2,y_2^3,xy_2}f(0) = y_1^2y_2 \pm y_2^3 \pm xy_2 \Rightarrow f \approx y_1^2y_2 \pm y_2^3 \pm xy_2 + xy_1^2$ or $y_1^2y_2 \pm y_2^3 \pm xy_2 + xy_1^3$.
(10). $j_{y_1^2y_2,y_2^3,x^2}f(0) = y_1^2y_2 \pm y_2^3 + x^2 \Rightarrow f \approx y_1^2y_2 \pm y_2^3 + x^2 \pm xy_2^2$.
(11) $f \in \mathrm{m}^3(1;2)$. Therefore the codimension of $f \geq \mathcal{E}(1;2)/(\langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}\rangle_{\mathbf{R}} + \mathrm{m}^3(1;2)) \geq 10 - 2 = 8$.
(12). $\phi = y_1^2y_2 + y_2^4 \Rightarrow$ one of the three:

$$
\begin{array}{lllll}
j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 + y_2^4 \pm xy_2 & \Rightarrow & (13), \\
j_{y_1^2y_2,y_2^4,xy_1}f(0) & \approx & y_1^2y_2 + y_2^4 + xy_1 & \Rightarrow & (14), \\
j_{y_1^2y_2,y_2^4,xy_1}f(0) & \approx & y_1^2y_2 + y_2^4 & \Rightarrow & (15).
\end{array}
$$

(13). $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2 + y_2^4 \pm xy_2 \Rightarrow f \approx y_1^2y_2 + y_2^4 \pm xy_2 + xy_1$ or $y_1^2y_2 + y_2^4 \pm xy_2 + xy_1^2$.
(14) $j_{y_1^2y_2,y_2^4,xy_1}f(0) = y_1^2y_2 + y_2^4 + xy_1 \Rightarrow f \approx y_1^2y_2 + y_2^4 + xy_1 \pm xy_2^2$.
(15) $j_{y_1^2y_2,y_2^4,xy_1}f(0) = y_1^2y_2 + y_2^4$. Then f is adjacent to $y_1^2y_2 + y_2^4 + \varepsilon x^2 + xy_2^2(a + \delta y_2)$ $(a^2 \neq 4\varepsilon)$ and this has codimension 9.
(16) $\phi = y_1^2y_2 \pm y_2^5 \Rightarrow$ one of the two:

$$\begin{array}{lclcl} j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 \pm xy_2 & \Rightarrow & (17), \\ j_{y_1^2y_2,y_2^4,xy_2}f(0) & \approx & y_1^2y_2 & \Rightarrow & (18). \end{array}$$

(17). $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2 \pm xy_2 \Rightarrow f \approx y_1^2y_2 \pm y_2^5 \pm xy_2 + xy_1$.
(18) $j_{y_1^2y_2,y_2^4,xy_2}f(0) = y_1^2y_2$. Then f is adjacent to $y_1^2y_2 + \varepsilon y_2^5 + xy_1 + xy_2^2(a+\delta y_2)$ $(a^2 \neq -\varepsilon)$ and this has codimension 10
(19). $\phi = y_1^3 + y_2^4$. $\Rightarrow$ one of the two:

$$\begin{array}{lclcl} j_{y_1^3,y_2^4,xy_2}f(0) & \approx & = y_1^3 + y_2^4 \pm xy_2 & \Rightarrow & (20), \\ j_{y_1^3,y_2^4,xy_2}f(0) & \approx & = y_1^3 + y_2^4 & \Rightarrow & (21). \end{array}$$

(20). $j_{y_1^3,y_2^4,xy_2}f(0) = y_1^3 + y_2^4 \pm xy_2 \Rightarrow f \approx y_1^3 + y_2^4 \pm xy_1 + xy_2$
(21). $j_{y_1^3,y_2^4,xy_2}f(0) = y_1^3 + y_2^4$. Then f is adjacent to $y_1^3 + y_2^4 \pm xy_1 \pm xy_2^2$ and this has codimension 8.
(22) Since $\phi \in \mathrm{m}^4(0;2)$, we have the codimension of $\phi \geq \mathcal{E}(0;2)/(\langle \frac{\partial \phi}{\partial y_1}, \frac{\partial \phi}{\partial y_2}\rangle_{\mathbf{R}} + \mathrm{m}^4(0;2)) \geq 10 - 2 = 8$. Therefore f has codimension ≥ 9.
<u>The case $r = 1, k \geq 3$.</u> We need only to prove that the codimension of $f \geq 8$ in the case $r = 1, k = 3$. Since the codimension of $\phi \geq \mathcal{E}(0;3)/(\langle \frac{\partial \phi}{\partial y_1}, \frac{\partial \phi}{\partial y_2}, \frac{\partial \phi}{\partial y_3}\rangle_{\mathbf{R}} + \mathrm{m}^3(0;3)) \geq 10 - 3 = 7$. Therefore the codimension of $f \geq 7 + 1 = 8$.
<u>The case $r = 2, k = 0$.</u> One of the five:

$$\begin{array}{lcl} j^2f(0) \approx x_1^2 + ax_1x_2 \pm x_2^2 (a^2 \neq \pm 4) & \Rightarrow & f \approx x_1^2 + ax_1x_2 \pm x_2^2 (a^2 \neq \pm 4), \\ j^2f(0) \approx (x_1 \pm x_2)^2 & \Rightarrow & f \approx (x_1 \pm x_2)^2 \pm x_2^n (n = 3, \cdots, 6), \\ j^2f(0) \approx x_1^2 \pm x_1x_2 \text{ or } \pm x_1x_2 + x_2^2 \text{ or } x_1x_2 & \Rightarrow & f \approx x_1^n \pm x_1x_2 \pm x_2^m \\ & & (n, m \geq 2,\ 5 \leq n+m \leq 8), \\ j^2f(0) \approx x_1^2 \text{ or } x_2^2 & \Rightarrow & (23), \\ j^2f(0) \approx 0 & \Rightarrow & (26). \end{array}$$

(23) We investigate only the case $j^2f(0) = x_1^2$. But the case $j^2f(0) = x_2^2$ is calculated analogously.
One of the two:

$$\begin{array}{lcl} j_{x_1^2,x_2^3}f(0) \approx x_1^2 \pm x_2^3 & \Rightarrow & f \approx x_1^2 \pm x_1x_2^2 \pm x_2^3 \text{ or } x_1^2 \pm x_2^3, \\ j_{x_1^2,x_2^3}f(0) \approx x_1^2 & \Rightarrow & (24). \end{array}$$

(24) One of the three:

$$\begin{array}{lcl} j_{x_1^2,x_2^4}f(0) \approx x_1^2 + ax_1^2x_2 \pm x_2^4 & \Rightarrow & f \approx x_1^2 + ax_1^2x_2 \pm x_2^4 \pm x_1x_2^3, \\ j_{x_1^2,x_2^4}f(0) \approx x_1^2 \pm x_1^2x_2 & \Rightarrow & f \approx x_1^2 \pm x_1^2x_2 \pm x_2^5, \\ j_{x_1^2,x_2^4}f(0) \approx x_1^2 & \Rightarrow & (25). \end{array}$$

(25) $j_{x_1^2,x_2^4}f(0) = x_1^2 \Rightarrow f$ is adjacent to $x_1^2 \pm x_1x_2^3 \pm x_2^5$ and this has codimension 8.
(26) $j^2f(0) = 0 \Rightarrow$ Since $f \in \mathrm{m}^3(2,0)$, the codimension of $f \geq \mathcal{E}(2,0)/(\langle x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}\rangle_{\mathbf{R}} + \mathrm{m}^4(2;0)) \geq 10 - 2 = 8$.
<u>The case $r = 2, k = 1$.</u> One of the five:

$$\begin{array}{lcl} j^2f(0) \approx x_1y \pm x_2y \pm x^2 \text{ or } x_1y \pm x_2y & \Rightarrow & f \approx y^n \pm x_1y \pm x_2y + x_2^2 \\ & & (n \geq 3, m \geq 2, m+n \leq 8), \\ j^2f(0) \approx x_1y + x^2 \text{ or } x_2y + x_1^2 & \Rightarrow & (27), \\ j^2f(0) \approx x_1y \text{ or } x_2y & \Rightarrow & (29), \\ j_{x_1^2,x_2^2,y^3}f(0) \approx x_1^2 + ax_1x_2 \pm x_2^2 (a^2 \neq \pm 4) & \Rightarrow & f \approx y^3 + \varepsilon x_2^2y + x_1^2 + ax_1x_2 + \delta x_2^2 \\ & & (a^2 \neq 4\delta), \\ \text{others} & \Rightarrow & (30). \end{array}$$

(27) We investigate only the case $j^2f(0) = x_1y + x^2$. But the case $j^2f(0) = x_2y + x_1^2$ is calculated analogously.

One of the four:

$$
\begin{array}{lcl}
j_{y^3,x_1y,x_2^2}f(0)f \approx y^3 \pm x_1y + x_2^2 & \Rightarrow & f \approx y^3 \pm x_1y \pm x_2y^2 + x_2^2, \\
j_{y^4,x_1y,x_2^2}f(0) \approx y^4 + ax_2y^2 + x_1y \pm x_2^2 & \Rightarrow & f \approx y^4 + ax_2y^2 \pm x_2^2y + x_1y \pm x_2^2, \\
j_{y^4,x_1y,x_2^2}f(0) \approx x_2y^2 \pm x_1y \pm x_2^2 & \Rightarrow & y^5 \pm x_2y^2 \pm x_1y + x_2^2, \\
j_{y^4,x_1y,x_2^2}f(0) \approx \pm x_1y + x_2^2 & \Rightarrow & (28).
\end{array}
$$

(28) $j_{y^4,x_1y,x_2^2}f(0) = \pm x_1y + x_2^2 \Rightarrow f$ is adjacent to $y^5 \pm x_2y^3 \pm x_1y + x_2^2$ and this has codimension 8.

(29) $j^2f(0) = x_1y \Rightarrow f$ is adjacent to $y^3 + ax_2^2y + 2x_2^3 \pm x_2^2y^2 \pm x_1y (a \neq -3)$ and this has codimension 8.

$j^2f(0) = x_2y \Rightarrow f$ is adjacent to $y^3 + ax_1^2y + 2x_1^3 \pm x_1^2y^2 \pm x_2y (a \neq -3)$ and this has codimension 8.

(30) $j^2f(0)$ is adjacent to $f_0 = (x_1 \pm x_2)^2$ or $x_1^2 \pm x_1x_2$ or $\pm x_1x_2 + x_2^2 \Rightarrow f$ is adjacent to $f_0 + y^3 + ax_2^2y \pm 2x_2^3 (a \neq -3)$ and this has codimension 8.

The classification list of singularities with reticular K-codimension lower than 8

$r = 1$

k	Normal form	codim	Conditions	Notation
0	x^n	n	$n = 2, \cdots, 7$	B_n
1	$xy + \varepsilon y^n$	n	$\varepsilon^{n+1} = 1, n = 3, \cdots, 7$	C_n^ε
	$y^3 + x^2$	4		F_4
	$y^4 + axy^2 \pm x^2$	6	$a^2 \neq \pm 4$	$K_{4,2}^{\pm,a}$
	$y^5 + (y^2 \pm x)^2$	6		$K_{1,1}^{\#,\pm}$
	$y^6 + \varepsilon(y^2 + \delta x)^2$	7		$K_{1,2}^{\#,\varepsilon,\delta}$
	$y^5 \pm xy^3 + x^2$	7		$K_{5,3}^{1,\pm}$
	$y^3 + ax^2y + 2x^3$	6	$a \neq -3$	$F_{1,0}^a$
	$y^3 + xy^2 \pm x^4$	6		$F_6^\pm$
	$y^3 + xy^2 \pm x^5$	7		$F_7^\pm$
	$y^3 \pm x^2y$	6		$F_{1,0}^{',\pm}$
	$y^3 \pm x^3y + x^4$	7		$F_7^{',\pm}$
	$y^4 + \varepsilon xy^2 + \delta x^3$	6		$K_{4,3}^{\varepsilon,\delta}$
	$y^5 + xy^2 \pm x^2$	6		$K_{5,2}^\pm$
	$y^5 + xy^2 \pm x^3$	7		$K_{5,3}^\pm$
	$y^4 + \varepsilon xy^2 + \delta x^4$	7		$K_{4,4}^{\varepsilon,\delta}$
	$y^4 + xy^3 \pm x^2y$	7		$K_{4,2}^{2,\pm}$
2	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	6	$a^2 \pm 1 \neq 0$	$D_{4,1}^{\pm,a}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2 + xy_1^2$	6		$D_{4,2}^{\varepsilon,\delta}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2 + xy_1^3$	7		$D_{4,3}^{\varepsilon,\delta}$
	$y_1^2y_2 + \varepsilon y_2^3 + \delta xy_2^2 + x^2$	7		$D_4^{2,\varepsilon,\delta}$
	$y_1^2y_2 + y_2^4 + xy_1 \pm xy_2$	6		$D_{5,1}^\pm$
	$y_1^2y_2 + y_2^4 + \varepsilon xy_1^2 + \delta xy_2$	7		$D_{5,2}^{\varepsilon,\delta}$
	$y_1^2y_2 + y_2^4 + xy_1 \pm xy_2^2$	7		$D_5^{1,\pm}$
	$y_1^2y_2 + \varepsilon y_2^5 + xy_1 + \delta xy_2$	7		$D_{6,1}^{\varepsilon,\delta}$
	$y_1^3 + y_2^4 \pm xy_1 + xy_2$	7		$E_{6,0}^\pm$

where $\varepsilon = \pm 1, \delta = \pm 1$.

$r=2$

k	Normal form	codim	Conditions	Notation
0	$x_1^2+ax_1x_2\pm x_2^2$	4	$a^2\neq\pm4$	$B_{2,2}^{\pm,a}$
	$(x_1+\varepsilon x_2)^2+\delta x_2^n$	$n+1$	$n=3,\cdots,6$	$B_{2,2,n}^{\varepsilon,\delta}$
	$x_1^n+\varepsilon x_1x_2+\delta x_2^m$	$n+m-1$	$n,m\geq 2,\ 5\leq n+m\leq 8$	$B_{n,m}^{\varepsilon,\delta}$
	$x_1^2+\varepsilon x_1x_2^2+\delta x_2^3$	5		$B_{2,3'}^{\varepsilon,\delta}$
	$x_2^2+\varepsilon x_1^2x_2+\delta x_1^3$	5		$B_{3,2'}^{\varepsilon,\delta}$
	$x_1^2\pm x_2^3$	6		$B_{2,3,0}^{\pm}$
	$x_2^2\pm x_1^3$	6		$B_{3,2,0}^{\pm}$
	$x_1^2+ax_1^2x_2+\varepsilon x_2^4+\delta x_1x_2^3$	7		$B_{2,4'}^{\varepsilon,\delta,a}$
	$x_2^2+ax_1x_2^2+\varepsilon x_1^4+\delta x_1^3x_2$	7		$B_{4,2'}^{\varepsilon,\delta,a}$
	$x_1^2+\varepsilon x_1^2x_2+\delta x_2^5$	7		$B_{2,5'}^{\varepsilon,\delta}$
	$x_2^2+\varepsilon x_1x_2^2+\delta x_1^5$	7		$B_{5,2'}^{\varepsilon,\delta}$
1	$y_1^n+\varepsilon x_1y+\delta x_2y+x_2^m$	$n+m-1$	$n\geq 3, m\geq 2,\ m+n\leq 8$	$C_{n,m}^{\varepsilon,\delta}$
	$y^3+\varepsilon x_1y+\delta x_2y^2+x_2^2$	5		$C_{3,2,1}^{\varepsilon,\delta}$
	$y^3+\varepsilon x_2y+\delta x_1y^2+x_1^2$	5		$C_{3,2,2}^{\varepsilon,\delta}$
	$y^4+ax_2y^2+\varepsilon x_2^2y+x_1y+\delta x_2^2$	7		$C_{4,2,1}^{\varepsilon,\delta,a}$
	$y^4+ax_1y^2+\varepsilon x_1^2y+x_2y+\delta x_1^2$	7		$C_{4,2,2}^{\varepsilon,\delta,a}$
	$y^5+\varepsilon x_2y^2+\delta x_1y+x_2^2$	7		$C_{5,2,1}^{\varepsilon,\delta}$
	$y^5+\varepsilon x_1y^2+\delta x_2y+x_1^2$	7		$C_{5,2,2}^{\varepsilon,\delta}$
	$y^3+\varepsilon x_2^2y+x_1^2+ax_1x_2+\delta x_2^2$	7	$a^2\neq 4\delta$	$C_{3,2'}^{\varepsilon,\delta,a}$

where $\varepsilon=\pm1, \delta=\pm1$.

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