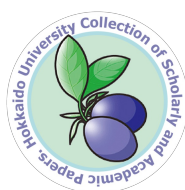




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Fumio Hiroshima

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Ground States of a Model in Quantum Electrodynamics

Fumio Hiroshima *

September 28, 1998

Abstract

The system of one charged non-relativistic particle with a class of external potentials minimally coupled to a massless quantized radiation field without the dipole approximation is considered. An ultraviolet cut-off is imposed on the quantized radiation field, the charged particle has spin $1/2$ and the class of external potentials *contains the Coulomb potential*. It is shown that the ground states of the system exist provided that a coupling constant is in a region, and an expression of the ground state energy of the system is given by a functional integral containing a vector-valued stochastic integral.

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CONTENTS

SECTION 1 INTRODUCTION

SECTION 2 PRELIMINARIES: COMMUTATION RELATIONS AND INEQUALITIES

SECTION 3 DEFINITION OF A HAMILTONIAN

Subsection 3.1 Quantized Fields

Subsection 3.2 Definition of a Hamiltonian (Hypothesis 1,2,3)

SECTION 4 EXISTENCE OF GROUND STATES

Subsection 4.1 Periodic Approximation (Hypothesis 4)

Subsection 4.2 Unitary Transformations

Subsection 4.3 Existence of Ground States with Artificial Mass

Subsection 4.4 Binding (Hypothesis 5)

Subsection 4.5 Overlap and Existence of Ground states (Hypothesis 6)

Subsection 4.6 Convergence of Ground states

SECTION 5 GROUND STATE ENERGY

Subsection 5.1 Functional Integral Representations (Hypothesis 7)

Subsection 5.2 Ground State Energy

SECTION 6 REFERENCES

1 INTRODUCTION

In this paper we prove that the ground states of the system of one charged non-relativistic spin 1/2 particle with a class of external potentials coupled to a massless quantized radiation field exist. The class of external potentials includes the *Coulomb potential*. The quantized radiation field is quantized in the Coulomb gauge and, on which an ultraviolet cut-off is

imposed. With a coupling constant, the particle and the radiation field minimally interact without the dipole approximation. Region of the coupling constant, in which the ground states of the system exist, is given. In the case where the particle has no spin, we give an expression of the infimum of the spectrum of the Hamiltonian of the system (ground state energy) by a functional integral containing a vector valued stochastic integral.

In the past few decades, several articles have been devoted to the study of non-relativistic particles coupled to quantized radiation fields (non-relativistic QED coupled to quantized radiation fields) and has provoked a great deal of controversy [2,3,4,5]. Non-relativistic QED coupled to a quantized radiation field interprets the physical phenomenon "Lamb shift" [8,28]. In [7], instability of excited states and in [9,10], stability of matter of non-relativistic QED coupled to quantized radiation fields are proposed, respectively. We are interested in stability of its ground states (the existence of its ground states). A great deal of effort has been made on showing the existence of ground states of related models [1,6,7,11,16,26,27], little is known, however, about non-relativistic QED coupled to quantized radiation fields.

In the previous paper [16], we have obtained the existence of the ground states of N -non-relativistic particle system coupled to a radiation field and confined by a class of external potentials which have the form "quadratic potential + some potential V_0 ", i.e. $x^2 + V_0$. Considering a particle system with the Coulomb potential is a very natural requirement [10]. Hence this derives us to a question whether the ground states of the system of non-relativistic particles coupled to quantized radiation fields with the Coulomb potential exist or not. One of the goal of this paper is to extend results in [16] to a class of external potential including the Coulomb potential but one particle case. Problems of instability of excited states and stability of ground states can be regarded as a problem of perturbation of "embedded eigenvalues in absolutely continuous spectrum". It is not clear that perturbed eigenvalues embedded in absolutely continuous spectrum, in general, can be expanded with respect to a coupling constant, even in the case where the coupling constant is sufficiently

small. It is much different from perturbation of discrete spectrum [20, Kato-Rellich Theorem]. The other goal of this paper is to give an expression of the ground state energy of the system, where the particle has no spin, from *non-perturbative approach*. By constructing a functional integral representation, we shall show it.

We shall show the strategy of this paper. A good place to start is to survey [16]. Let the Hamiltonian of the system with an ultraviolet cut-off $\hat{\rho}$ be $H(\hat{\rho})$, which is a self-adjoint operator acting in a Hilbert space $\mathcal{H}$. One first introduces artificial mass $m > 0$ of photon and define $H^m(\hat{\rho})$. In addition to this, $H^m(\hat{\rho})$ is periodically approximated [6,7,12,16,24] with an index $V > 0$: $H_V^m(\hat{\rho})$. It is established that the spectrum of $H_V^m(\hat{\rho})$ is purely discrete in the neighborhood of its ground state energy. The ground states of $H(\hat{\rho})$ has been constructed through (1) a norm resolvent convergence from $H_V^m(\hat{\rho})$ to $H^m(\hat{\rho})$ as $V \rightarrow \infty$ (thus one knows that the ground states of $H^m(\hat{\rho})$ exist); (2) a non zero weak limit of a ground state of $H^m(\hat{\rho})$ as $m \rightarrow 0$ (the limit is just a ground state of $H(\hat{\rho})$). In [16] it plays an important role to show (1) that particles are confined an external potential with the form $x^2 + V_0$. Then in our case it seems, however, to be difficult to show (1). In order to avoid such sufferings, we shall find a family of unitary transformations U_V with the index V . That goes to the very heart of solving our problem. We shall show that, instead of $H_V^m(\hat{\rho})$, the operator $U_V H_V^m(\hat{\rho}) U_V^{-1}$ converges in a norm resolvent sense to an operator which is unitarily equivalent to $H^m(\hat{\rho})$. Then we conclude that $H^m(\hat{\rho})$ has discrete spectrum in the neighborhood of its ground state energy. The question we must consider next is to specify a region of the coupling constant, in which the ground states of $H(\hat{\rho})$ exist. The particle decoupled to quantized radiation fields is governed by

$$H_{el} = -\frac{1}{2}\Delta + V,$$

where Δ is the Laplacian in $L^2(\mathbf{R}^3)$ and V an external potential. We take an interest in two standard examples: (1) the essential spectrum of H_{el} is $[E, \infty)$ and its ground state energy

is isolated; (2) the spectrum of H_{el} is purely discrete. More conditions on the coupling constant for (1) than that for (2) is technically needed. Unfortunately we can not remove such additional conditions from (1) [27]. In order to give an expression of the ground state energy of Hamiltonian with no spin, $H_0(\hat{\rho})$, we shall show that $(\Psi, \Omega(0))_{\mathcal{H}} \neq 0$, where $(\cdot, \cdot)_{\mathcal{H}}$ denotes the scalar product in $\mathcal{H}$, Ψ is a ground state of a decoupled Hamiltonian and $\Omega(0)$ a ground state of $H_0(\hat{\rho})$. Hence we get

$$\inf \sigma(H_0(\hat{\rho})) = \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} \log(\Psi, e^{-\beta H_0(\hat{\rho})} \Psi)_{\mathcal{H}}. \quad (1.1)$$

We shall present the right hand side of (1.1) by a functional integral containing a vector valued stochastic integral. This paper is organized as follows: In Section 2, we show fundamental inequalities for later use. In Section 3, we define non-relativistic QED coupled to a quantized radiation field and we give Hypothesis 1,2 and 3. Subsection 4.1 is devoted to define periodically approximate Hamiltonians. We give Hypothesis 4 and consider the spectrum in the neighborhood of their ground state energy. In Subsection 4.2 we introduce a family of unitary transformations. Subsection 4.3 offers the key to proving the existence of ground states. We shall show convergence of operators in a norm resolvent sense. In Subsection 4.4 we give Hypothesis 5 and some binding. In Subsection 4.5 we give Hypothesis 6 and state main theorem (Theorem 4.16). In Subsection 4.6 we show a weak convergence of ground states. In subsection 5.1 we construct a functional integral representation of $e^{-tH_0(\hat{\rho})}$. In Subsection 5.2, we give an expression of the ground state energy of the system by a functional integral. Section 6 is devoted to references.

2 PRELIMINARIES: COMMUTATION RELATIONS AND INEQUALITIES

Let us start with mathematical definitions of notations which we shall most use. For a complex Hilbert space $\mathcal{K}$, the associated norm and the scalar product is denoted by $\|f\|_{\mathcal{K}}$

and $(f, g)_K, f, g \in K$, respectively, where the scalar product is linear in g and antilinear in f . The domain of an operator T in K is denoted by $D(T)$. The spectrum (resp. the discrete spectrum) of an operator T is denoted by $\sigma(T)$ (resp. $\sigma_d(T)$) and the ground state energy of T by $G[T] = \inf \sigma(T)$.

We say that nonnegative self-adjoint operators A and B in a Hilbert space K are “strongly commutative” if and only if

$$e^{-tA}e^{-sB} = e^{-sB}e^{-tA}, s, t \geq 0. \quad (2.1)$$

By the definition (2.1) and the Stone theorem [19], we see that if $\psi \in D(A)$ then $e^{-sB}\psi \in D(A), s \geq 0$ with

$$Ae^{-sB}\psi = e^{-sB}A\psi. \quad (2.2)$$

In addition to this, by (2.2) and again the Stone theorem, we see that if $\psi \in D(A) \cap D(B)$ and $B\psi \in D(A)$, then $A\psi \in D(B)$ with

$$AB\psi = BA\psi. \quad (2.3)$$

Lemma 2.1 *Let A and B be strongly commutative nonnegative self-adjoint operators in K . Then $(A + \epsilon)^{-\alpha}$ and $(B + \eta)^{-\beta}$ commute, where $0 \leq \alpha, \beta$ and $0 < \epsilon, \eta$.*

Proof: By the spectral theorem of self-adjoint operators, we see that

$$\begin{aligned} (A + \epsilon)^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA} e^{-\epsilon t} dt, \\ (B + \eta)^{-\beta} &= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-tB} e^{-\eta t} dt, \end{aligned}$$

where Γ denotes the Gamma function: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ and the integral is defined in the sense of Bochner integral. Then we see that

$$\begin{aligned} (\phi, (A + \epsilon)^{-\alpha} (B + \eta)^{-\beta} \psi)_K &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s\epsilon} e^{-t\eta} (e^{-sA} \phi, e^{-tB} \psi)_K ds dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s\epsilon} e^{-t\eta} (e^{-tB} \phi, e^{-sA} \psi)_K ds dt \\ &= (\phi, (B + \eta)^{-\beta} (A + \epsilon)^{-\alpha} \psi)_K. \end{aligned}$$

Then the proof is complete. $\square$

Lemma 2.2 *Let A and B be strongly commutative nonnegative self-adjoint operators in $\mathcal{K}$, and $\psi \in D(A)$. Then $(B + \eta)^{-\beta}\psi \in D(A)$ with*

$$A(B + \eta)^{-\beta}\psi = (B + \eta)^{-\beta}A\psi. \quad (2.4)$$

Proof: Noting (2.2), we see that, for $\phi \in D(A)$,

$$\begin{aligned} (A\phi, (B + \eta)^{-\beta}\psi)_{\mathcal{K}} &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\eta} (A\phi, e^{-tB}\psi)_{\mathcal{K}} ds dt \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s\eta} (\phi, e^{-tB}A\psi)_{\mathcal{K}} ds dt \\ &= (\phi, (B + \eta)^{-\beta}A\psi)_{\mathcal{K}}, \end{aligned}$$

which implies that $(B + \eta)^{-\beta}\psi \in D(A)$ and (2.4) hold. $\square$

Proposition 2.3 *Let A and B be strongly commutative nonnegative self-adjoint operators in $\mathcal{K}$ and $\|A^{\frac{1}{2}}\psi\|_{\mathcal{K}} \leq \|B^{\frac{1}{2}}\psi\|_{\mathcal{K}}$ for $\psi \in D(B^{\frac{1}{2}})$. Moreover we suppose that $\psi \in D(A) \cap D(B)$, $A\psi \in D(B)$. Then*

$$\|A\psi\|_{\mathcal{K}} \leq \|B\psi\|_{\mathcal{K}}. \quad (2.5)$$

Proof: Note that $B\psi \in D(A)$ and $AB\psi = BA\psi$. Since $(A + \epsilon)\psi \in D(B)$, Lemma 2.2 yields that $(A + \epsilon)^{-\frac{1}{2}}(A + \epsilon)\psi \in D(B)$. Then $(B + \epsilon)^{-\frac{1}{2}}(B + \epsilon)(A + \epsilon)^{-\frac{1}{2}}(A + \epsilon)\psi = (B + \epsilon)^{\frac{1}{2}}(A + \epsilon)^{\frac{1}{2}}\psi$ is well defined. On the other hand,

$$\begin{aligned} &(B + \epsilon)^{-\frac{1}{2}}(B + \epsilon)(A + \epsilon)^{-\frac{1}{2}}(A + \epsilon)\psi \\ &= (B + \epsilon)^{-\frac{1}{2}}(A + \epsilon)^{-\frac{1}{2}}(B + \epsilon)(A + \epsilon)\psi \text{ (by (2.4))} \\ &= (A + \epsilon)^{-\frac{1}{2}}(B + \epsilon)^{-\frac{1}{2}}(B + \epsilon)(A + \epsilon)\psi \text{ (by Lemma 2.1)} \\ &= (A + \epsilon)^{-\frac{1}{2}}(B + \epsilon)^{-\frac{1}{2}}(A + \epsilon)(B + \epsilon)\psi \text{ (by (2.3))} \\ &= (A + \epsilon)^{-\frac{1}{2}}(A + \epsilon)(B + \epsilon)^{-\frac{1}{2}}(B + \epsilon)\psi \text{ (by (2.4))} \\ &= (A + \epsilon)^{\frac{1}{2}}(B + \epsilon)^{\frac{1}{2}}\psi. \end{aligned}$$

Hence we have

$$(A + \epsilon)^{\frac{1}{2}}(B + \epsilon)^{\frac{1}{2}}\psi = (B + \epsilon)^{\frac{1}{2}}(A + \epsilon)^{\frac{1}{2}}\psi. \quad (2.6)$$

Then

$$\begin{aligned} \|(A + \epsilon)\psi\|_{\mathcal{K}} &= \|(A + \epsilon)^{\frac{1}{2}}(A + \epsilon)^{\frac{1}{2}}\psi\|_{\mathcal{K}} \\ &\leq \|(B + \epsilon)^{\frac{1}{2}}(A + \epsilon)^{\frac{1}{2}}\psi\|_{\mathcal{K}} = \|(A + \epsilon)^{\frac{1}{2}}(B + \epsilon)^{\frac{1}{2}}\psi\|_{\mathcal{K}} \\ &\leq \|(B + \epsilon)\psi\|_{\mathcal{K}}. \end{aligned} \quad (2.7)$$

Taking $\epsilon \rightarrow 0$ on the both sides of (2.7), we get (2.5). $\square$

Generally we get the following proposition:

Proposition 2.4 *Let operators A and B satisfy the same assumption as that of Proposition 2.3. Let $\mathbf{D}_{\frac{1}{2}}$ be such that $\mathbf{D}_{\frac{1}{2}} \subset D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ and, both $(A + \epsilon)^{\frac{1}{2}}$ and $(B + \epsilon)^{\frac{1}{2}}$, where ϵ runs from 0 to some $0 < \epsilon_0$, leave $\mathbf{D}_{\frac{1}{2}}$ invariant and let $\mathbf{D}_1$ be such that $\mathbf{D}_1 \subset D(A) \cap D(B)$ and, both A and B leave $\mathbf{D}_1$ invariant. Then we have*

$$(1) \quad \|A^{\frac{m}{2}}\psi\|_{\mathcal{K}} \leq \|B^{\frac{m}{2}}\psi\|_{\mathcal{K}}, \psi \in \mathbf{D}_{\frac{1}{2}}, m \geq 2, \quad (2.8)$$

$$(2) \quad \|A^m\psi\|_{\mathcal{K}} \leq \|B^m\psi\|_{\mathcal{K}}, \psi \in \mathbf{D}_1, m \geq 1. \quad (2.9)$$

Proof: By an induction with respect to m , we shall prove them. (1) In the case where $m = 2$, (2.8) is verified by Proposition 2.3. Assume that (2.8) holds in the cases where $m = 2, 3, \dots, k$. Since $(A + \epsilon)^{\frac{1}{2}}\psi \in \mathbf{D}_{\frac{1}{2}}, 0 \leq \epsilon < \epsilon_0$,

$$\|(A + \epsilon)^{\frac{k+1}{2}}\psi\|_{\mathcal{K}} \leq \|(B + \epsilon)^{\frac{k}{2}}(A + \epsilon)^{\frac{1}{2}}\psi\|_{\mathcal{K}}, 0 \leq \epsilon < \epsilon_0.$$

By (2.6), $(A + \epsilon)^{\frac{1}{2}}$ and $(B + \epsilon)^{\frac{1}{2}}$ commute on $\mathbf{D}_{\frac{1}{2}}$. Then we have

$$\|(A + \epsilon)^{\frac{k+1}{2}}\psi\|_{\mathcal{K}} \leq \|(A + \epsilon)^{\frac{1}{2}}(B + \epsilon)^{\frac{k}{2}}\psi\|_{\mathcal{K}} \leq \|(B + \epsilon)^{\frac{k+1}{2}}\psi\|_{\mathcal{K}}, 0 \leq \epsilon < \epsilon_0. \quad (2.10)$$

Taking $\epsilon \rightarrow 0$ on the both sides of (2.10), we get the desired results. (2) In the case where $m = 1$, it is just assertion of Proposition 2.3. Assume that (2.9) holds in the cases where $m = 1, 2, \dots, k$. Then

$$\|A^{k+1}\psi\|_{\mathcal{K}} \leq \|B^k A\psi\|_{\mathcal{K}}.$$

By (2.3) A and B commute on $\mathbf{D}_1$. Then we have

$$\|A^{k+1}\psi\|_{\mathcal{K}} \leq \|AB^k\psi\|_{\mathcal{K}} \leq \|B^{k+1}\psi\|_{\mathcal{K}}.$$

Thus (2.9) follows. □

The role of Proposition 2.3 is to establish (4.26).

3 DEFINITION OF A HAMILTONIAN

In this section we give a quick sketch of definition of nonrelativistic QED coupled to a radiation field with an ultraviolet cut-off and carefully show some inequalities.

3.1 Quantized Fields

Let A symmetric Fock space $\mathcal{F}_{EM}$ over W is defined by

$$\mathcal{F}_{EM} = \bigoplus_{N=0}^{\infty} [\otimes_s^N W].$$

Here $\otimes_s^N W, N \geq 1$, denotes the N -fold symmetric tensor product of W and $\otimes_s^0 W = \mathbf{C}$, the set of complex numbers. We denote by $\mathcal{F}_{EM}^{\infty}$ the finite particle subspace which is defined by the set of vectors $\{\Phi_n\}_{n=0}^{\infty} \in \mathcal{F}_{EM}$ with at most finitely many non zero elements. The bare vacuum $\Omega_{EM} \in \mathcal{F}_{EM}$ is defined by

$$\Omega_{EM} = \{1, 0, 0, \dots\}.$$

We denote the smeared creation operator and the smeared annihilation operator in $\mathcal{F}_{EM}$ by $a^{\dagger}(F), F \in W$, and $a(G), G \in W$, respectively. In particular, for notational convenience, we

introduce the notations $a^{\dagger r}$ and a^r , $r = 1, 2$, by $a^{\dagger 1}(f) = a^{\dagger}(f \oplus 0)$, $a^{\dagger 2}(f) = a^{\dagger}(0 \oplus f)$, $f \in L^2(\mathbf{R}^3)$, and $a^1(f) = a(f \oplus 0)$, $a^2(f) = a(0 \oplus f)$, $f \in L^2(\mathbf{R}^3)$, respectively. They satisfy canonical commutation relations on $\mathcal{F}_{EM}^{\infty}$:

$$\begin{aligned} [a^r(f), a^{\dagger s}(g)] &= \delta_{rs}(\bar{f}, g)_{L^2(\mathbf{R}^3)}, \\ [a^{\sharp r}(f), a^{\sharp s}(g)] &= 0, f, g \in L^2(\mathbf{R}^3), r, s = 1, 2, \end{aligned}$$

where the symbol $\sharp$ stands for the dagger $\dagger$ or its omission, i.e., $a^{\sharp r}$ is either $a^{\dagger r}$ or a^r , $r = 1, 2$, and $\bar{f}$ is the complex conjugate of f . We formally write

$$a^{\sharp r}(f) = \int a^{\sharp r}(k) f(k) dk.$$

For a contraction operator T on W , the second quantization of T is denoted by $\Gamma(T)$ [13, 22]. In particular, for a nonnegative self-adjoint operator h in W , $\Gamma(e^{-th})$, $t \geq 0$, is a one parameter strongly continuous symmetric semi-group on $\mathcal{F}_{EM}$. Hence we can define a nonnegative self-adjoint operator $d\Gamma(h)$ in $\mathcal{F}_{EM}$ by $\Gamma(e^{-th}) = e^{-td\Gamma(h)}$. Note that $d\Gamma(h)\Omega_{EM} = 0$. The free Hamiltonian in $\mathcal{F}_{EM}$ is defined by $d\Gamma(\omega \oplus \omega)$, where ω is the multiplication operator by $\omega(k) = |k|$ ($|k|$ is the three dimensional Euclidean norm of k). We formally write

$$H_{EM} = \sum_{r=1}^2 \int \omega(k) a^{\dagger r}(k) a^r(k) dk.$$

Besides the number operator N_{EM} is defined by

$$N_{EM} = \sum_{r=1}^2 \int a^{\dagger r}(k) a^r(k) dk.$$

We set

$$\mathcal{M}_n = \left\{ f \mid \|f\|_n = \|(\sqrt{\omega})^n f\|_{L^2(\mathbf{R}^3)} < \infty \right\}.$$

The following relativistic bounds are well known:

$$\|a^{\dagger r}(f)\Psi\|_{\mathcal{F}_{EM}} \leq \|f\|_{-1} \|H_{EM}^{\frac{1}{2}}\Psi\|_{\mathcal{F}_{EM}} + \|f\|_0 \|\Psi\|_{\mathcal{F}_{EM}}, f \in \mathcal{M}_{-1} \cap \mathcal{M}_0, \quad (3.1)$$

$$\|a^r(f)\Psi\|_{\mathcal{F}_{EM}} \leq \|f\|_{-1} \|H_{EM}^{\frac{1}{2}}\Psi\|_{\mathcal{F}_{EM}}, f \in \mathcal{M}_{-1} \cap \mathcal{M}_0, \quad (3.2)$$

$$\|a^{\dagger r}(f)\Phi\|_{\mathcal{F}_{EM}} \leq \|f\|_0 \|(N_{EM} + I)^{\frac{1}{2}}\Phi\|_{\mathcal{F}_{EM}}, f \in \mathcal{M}_0, \quad (3.3)$$

$$\|a^r(f)\Phi\|_{\mathcal{F}_{EM}} \leq \|f\|_0 \|N_{EM}^{\frac{1}{2}}\Phi\|_{\mathcal{F}_{EM}}, f \in \mathcal{M}_0, \quad (3.4)$$

where $\Psi \in D(H_{EM}^{\frac{1}{2}})$ and $\Phi \in D(N_{EM}^{\frac{1}{2}})$. With an ultraviolet cut-off $\hat{\rho}$, we define a quantized radiation field $A_\mu(\hat{\rho}, x)$ and a quantized magnetic field $B_\mu(\hat{\rho}, x)$, $\mu = 1, 2, 3$, by

$$\begin{aligned} A_\mu(\hat{\rho}, x) &= \sum_{r=1}^2 \int \frac{dk}{\sqrt{2\omega(k)}} \left\{ \hat{\rho}(k) e_\mu^r(k) e^{-ikx} a^{\dagger r}(k) + \hat{\rho}(-k) e_\mu^r(k) e^{ikx} a^r(k) \right\}, \\ B_\mu(\hat{\rho}, x) &= \sum_{r=1}^2 \int \frac{dk}{\sqrt{2\omega(k)}} \left\{ \hat{\rho}(k) (-ik \times e^r(k))_\mu e^{-ikx} a^{\dagger r}(k) + \hat{\rho}(-k) ((-i)(-k) \times e^r(k))_\mu e^{ikx} a^r(k) \right\}. \end{aligned}$$

Here $e^r = (e_1^r, e_2^r, e_3^r)$, $r = 1, 2$, are polarization vectors which satisfy

$$e^r(k) e^s(k) = \delta_{rs}, k e^r(k) = 0, \quad a.e. k \in \mathbf{R}^3.$$

For instance and avoiding some confusion, we assume that $e_\mu^r \in C^\infty(\mathbf{R}^3 \setminus \mathcal{P})$, $\mu = 1, 2, 3$, $r = 1, 2$, where $\mathcal{P} = \{k = (0, 0, k_3) \in \mathbf{R}^3 | k_3 \in \mathbf{R}\}$. The operators $A_\mu(\hat{\rho}, x)$, $\hat{\rho} \in \mathcal{M}_{-1}$, and $B_\mu(\hat{\rho}, x)$, $\hat{\rho} \in \mathcal{M}_1$, are well defined and closable on $\mathcal{F}_{EM}^\infty$. We denote their closed extensions by the same symbols.

3.2 Definition of a Hamiltonian

The Hilbert space $\mathcal{H}$ of the system considered in this paper is given by

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}_{EM} = [L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3)] \otimes \mathcal{F}_{EM}.$$

The Hamiltonian of the system $H(\hat{\rho})$ with the ultraviolet cut-off $\hat{\rho}$ is defined by

$$H(\hat{\rho}) = \frac{1}{2} [\sigma \cdot (P \otimes I_{EM} - eA(\hat{\rho}, x))]^2 + V \otimes I_{EM} + I_{el} \otimes H_{EM}.$$

Here $A(\hat{\rho}, x) = (A_1(\hat{\rho}, x), A_2(\hat{\rho}, x), A_3(\hat{\rho}, x))$, $P = (P_1, P_2, P_3) = (-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}, -i\frac{\partial}{\partial x_3})$, V is an external potential and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the three Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote the identity operators on $\mathcal{H}_{el}$ and $\mathcal{F}_{EM}$ by I_{el} and I_{EM} , respectively. We work in units which is $c = \hbar = 1$ and put the mass of the particle one for simplification. We immediately see that

$$H(\hat{\rho}) = H_{el} + eR_I + e^2R_{II} + eR_{III},$$

where

$$\begin{aligned} H_{el} &= -\frac{1}{2}\Delta + V, \\ R_I &= -(P \otimes I_{EM})A(\hat{\rho}, x), \\ R_{II} &= \frac{1}{2}(A(\hat{\rho}, x))^2, \\ R_{III} &= \frac{1}{2}\sigma \cdot B(\hat{\rho}, x). \end{aligned}$$

We introduce Hypothesis 1 and 2:

Hypothesis 1 The ultraviolet cut-off $\hat{\rho}$ satisfies that $\overline{\hat{\rho}(k)} = \hat{\rho}(-k)$, and $\hat{\rho} \in \mathcal{M}_{-2} \cap \mathcal{M}_{-1} \cap \mathcal{M}_0 \cap \mathcal{M}_1$.

Hypothesis 2 The external potential V is such that $-\frac{1}{2}\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^3)$ and constants a and b so that

$$\left\| -\frac{1}{2}\Delta f \right\|_{L^2(\mathbf{R}^3)} \leq a \left\| \left(-\frac{1}{2}\Delta + V \right) f \right\|_{L^2(\mathbf{R}^3)} + b \|f\|_{L^2(\mathbf{R}^3)}^2, f \in C_0^\infty(\mathbf{R}^3) \quad (3.5)$$

exist. Moreover there exists a constant Σ so that

$$(-\infty, \Sigma] \cap \sigma(H_{el}) \subset \sigma_d(H_{el}).$$

Here we subsume $\Sigma = \infty$.

Note that (3.5) implies that, with the same a as in (3.5),

$$\left(f, -\frac{1}{2}\Delta f \right)_{L^2(\mathbf{R}^3)} \leq a \left(f, \left(-\frac{1}{2}\Delta + V \right) f \right)_{L^2(\mathbf{R}^3)} + b' \|f\|_{L^2(\mathbf{R}^3)}, f \in C_0^\infty(\mathbf{R}^3),$$

where b' is a positive constant. We fix the constants a, b, b' and Σ throughout in this paper. We have two standard examples in mind. One choice is (1) $V \in L^2(\mathbf{R}^3) + L^\infty(\mathbf{R}^3)$ with isolated ground state energy, e.g.,

$$V(x) = -\frac{e^2}{|x|}.$$

The other is (2) $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, e.g.,

$$V(x) = |x|^2.$$

For the first case (1), the essential spectrum of H_{el} has continuum edge, i.e., $\Sigma < \infty$. For the latter case (2), H_{el} has purely discrete spectrum, i.e., one can take $\Sigma = \infty$. The case where one can take $\Sigma = \infty$ is studied more easily than the case where $\Sigma < \infty$. Thus, throughout this paper, we assume that $\Sigma < \infty$. (See Remark 5.7.) We have, by (3.1) and (3.2)

$$\|R_I \Psi\|_{\mathcal{H}} \leq A \|I_{el} \otimes H_{EM} \Psi\|_{\mathcal{H}} + B \|H_{el} \otimes I_{EM} \Psi\|_{\mathcal{H}} + C \|\Psi\|_{\mathcal{H}}, \quad (3.6)$$

$$\|R_{II} \Psi\|_{\mathcal{H}} \leq D \|I_{el} \otimes H_{EM} \Psi\|_{\mathcal{H}} + E \|\Psi\|_{\mathcal{H}}, \quad (3.7)$$

$$\|R_{III} \Psi\|_{\mathcal{H}} \leq F \|I_{el} \otimes H_{EM} \Psi\|_{\mathcal{H}} + G \|\Psi\|_{\mathcal{H}}, \quad (3.8)$$

where

$$\begin{aligned} A &= 6\|\hat{\rho}\|_{-2}, B = 6a(2\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1}), C = 6b(2\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1}) + 3\|\hat{\rho}\|_{-1}, \\ D &= E = 6(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})^2 + 3k(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})(\|\hat{\rho}\|_0 + \|\hat{\rho}\|_1), \\ F &= \frac{6}{\sqrt{2}}\|\hat{\rho}\|_0, G = \frac{3}{\sqrt{2}}\|\hat{\rho}\|_1, k = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{2\pi} \int_0^\infty \frac{\sqrt{\lambda}}{\lambda+1} d\lambda. \end{aligned} \quad (3.9)$$

We assume that $0 < e_0$ holds

$$e_0^2 D + e_0(A + B + F) = 1.$$

Hypothesis 3 is as follows:

Hypothesis 3 The coupling constant e is such that $0 \leq e < e_0$.

Proposition 3.1 ([18]) *We assume that Hypothesis 1,2 and 3 hold. Then $H(\hat{\rho})$ is self-adjoint on $D(H_{el} \otimes I_{EM}) \cap D(I_{el} \otimes H_{EM})$, bounded from below and essentially self-adjoint on any core for $H(0) = H_{el} \otimes I_{EM} + I_{el} \otimes H_{EM}$.*

Proof: Note that

$$\|I_{el} \otimes H_{EM}\Psi\|_{\mathcal{H}} \leq \|H(0)\Psi\|_{\mathcal{H}} + |\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}},$$

$$\|H_{el} \otimes I_{EM}\Psi\|_{\mathcal{H}} \leq \|H(0)\Psi\|_{\mathcal{H}} + 2|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}.$$

Then, by (3.6),(3.7) and (3.8), we see that

$$\|R_I\Psi\|_{\mathcal{H}} \leq (A+B)\|H(0)\Psi\|_{\mathcal{H}} + C\|\Psi\|_{\mathcal{H}} + (A+2B)|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}, \quad (3.10)$$

$$\|R_{II}\Psi\|_{\mathcal{H}} \leq D\|H(0)\Psi\|_{\mathcal{H}} + E\|\Psi\|_{\mathcal{H}} + D|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}, \quad (3.11)$$

$$\|R_{III}\Psi\|_{\mathcal{H}} \leq F\|H(0)\Psi\|_{\mathcal{H}} + G\|\Psi\|_{\mathcal{H}} + F|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}. \quad (3.12)$$

Hence Kato-Rellich Theorem [19] yields the desired results. $\square$

We shall estimate the ground state energy of $H(\hat{\rho})$. We have

$$|(\Psi, R_I\Psi)_{\mathcal{H}}| \leq A'(\Psi, I_{el} \otimes H_{EM}\Psi)_{\mathcal{H}} + B'(\Psi, H_{el} \otimes I_{EM}\Psi)_{\mathcal{H}} + C'\|\Psi\|_{\mathcal{H}}^2 \quad (3.13a)$$

$$\leq (A' + B')(\Psi, H(0)\Psi)_{\mathcal{H}} - B'|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}^2 + C'\|\Psi\|_{\mathcal{H}}^2, \quad (3.13b)$$

$$|(\Psi, R_{II}\Psi)_{\mathcal{H}}| \leq D'(\Psi, I_{el} \otimes H_{EM}\Psi)_{\mathcal{H}} + E'\|\Psi\|_{\mathcal{H}}^2 \quad (3.14a)$$

$$\leq D'(\Psi, H(0)\Psi)_{\mathcal{H}} - D'|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}^2 + E'\|\Psi\|_{\mathcal{H}}^2, \quad (3.14b)$$

$$|(\Psi, R_{III}\Psi)_{\mathcal{H}}| \leq F'(\Psi, I_{el} \otimes H_{EM}\Psi)_{\mathcal{H}} + G'\|\Psi\|_{\mathcal{H}}^2 \quad (3.15a)$$

$$\leq F'(\Psi, H(0)\Psi)_{\mathcal{H}} - F'|\mathbf{G}[H_{el}]|\|\Psi\|_{\mathcal{H}}^2 + G'\|\Psi\|_{\mathcal{H}}^2, \quad (3.15b)$$

where

$$A' = \frac{6}{\sqrt{2}}\|\hat{\rho}\|_{-2}, B' = 6a\sqrt{2}\|\hat{\rho}\|_{-2}, C' = 6b\sqrt{2}\|\hat{\rho}\|_{-2},$$

$$D' = 3(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})\|\hat{\rho}\|_{-2}, E' = \frac{3}{2}(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})\|\hat{\rho}\|_{-1}, F' = G' = \frac{6}{2\sqrt{2}}\|\hat{\rho}\|_0.$$

(3.16)

Hence we see that

$$\begin{aligned} & (1 - eA' - eB' - e^2D' - eF')(\Psi, H(0)\Psi)_{\mathcal{H}} + (eA' + e^2D' + eF')\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2 \\ & - (eC' + e^2E' + eG')\|\Psi\|_{\mathcal{H}}^2 \leq (\Psi, H(\hat{\rho})\Psi)_{\mathcal{H}}. \end{aligned} \quad (3.17)$$

Moreover we have

$$\begin{aligned} & (\Psi, H(\hat{\rho})\Psi)_{\mathcal{H}} \leq (\Psi, H(0)\Psi)_{\mathcal{H}} + (eA' + eB' + e^2D' + eF')(\Psi, H(0)\Psi)_{\mathcal{H}} \\ & + (eC' + e^2E' + eG')\|\Psi\|_{\mathcal{H}}^2 - (eA' + e^2D' + eF')\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2. \end{aligned} \quad (3.18)$$

Since $H(\hat{\rho})$ is essentially self-adjoint on any core for H_0 (Proposition 3.1) for coupling constant $0 \leq e < e_0$, in view of (3.17) and (3.18), for $0 \leq e < e_0$, we get the following:

$$|\mathbf{G}[H(\hat{\rho})] - \mathbf{G}[H_{el}]| \leq \Delta(\hat{\rho}), \quad (3.19)$$

where

$$\begin{aligned} \Delta(\hat{\rho}) &= e(B'\mathbf{G}[H_{el}] + C') + e^2E' + eG' \\ &= e6\sqrt{2}\|\hat{\rho}\|_{-2}(a\mathbf{G}[H_{el}] + b') + e^2\frac{3(2\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})\|\hat{\rho}\|_{-1}}{2} + e\frac{3}{\sqrt{2}}\|\hat{\rho}\|_0. \end{aligned}$$

Note that $a\mathbf{G}[H_{el}] + b' \geq 0$. For $e \geq e_0$, it is not known whether core for H_0 is core for $H(\hat{\rho})$ or not. Hence it is not necessarily that $\mathbf{G}[H(\hat{\rho})] = \inf_{\Psi \in D}(\Psi, H_0\Psi)_{\mathcal{H}}$, where D is core for H_0 . Then (3.19) is valid only for $0 \leq e < e_0$.

4 EXISTENCE OF GROUND STATES

General propositions applying in this section are as follows:

Proposition 4.1 I. [24, Lemma 4.6] *Let T_n and T be self-adjoint operators on a Hilbert space and bounded from below, respectively. Suppose that (1) $T_n \rightarrow T$ as $n \rightarrow \infty$ in the norm resolvent sense, (2) T_n has purely discrete spectrum in $[\mathbf{G}[T_n], \mathbf{G}[T_n] + C)$ with a constant $C > 0$. Then, T has purely discrete spectrum in $[\mathbf{G}[T], \mathbf{G}[T] + C)$.*

II. [6, Lemma 4.9] Let T_n and T be self-adjoint operators on a Hilbert space, respectively. Let $T_n, n = 1, 2, \dots$, and T have a common core D such that, for all $\phi \in D$, $T_n \phi \rightarrow T \phi$ as $n \rightarrow \infty$. Let ϕ_n be a normalized ground state of T_n with ground state energy E_n so that (1) $E = \lim_{n \rightarrow \infty} E_n$ and (2) $w - \lim_{n \rightarrow \infty} \phi_n = \phi \neq 0$. Then ϕ is a ground state of T with ground state energy E .

Before we come on to a closer analysis of the existence of the ground states of $H(\hat{\rho})$, let us pause here to look briefly at our strategy. We shall introduce three parameters m, V, L , and define self-adjoint operators $H_V^m(\hat{\rho}_L), H^m(\hat{\rho}_L)$ and $H^m(\hat{\rho})$ with indexes V, m and L , respectively. In addition to this we construct unitary operators U and U_V in $\mathcal{H}$, and define $\widetilde{H}^m(\hat{\rho}_L) = U H^m(\hat{\rho}_L) U^{-1}$ and $\widetilde{H}_V^m(\hat{\rho}_L) = U_V H_V^m(\hat{\rho}_L) U_V^{-1}$, respectively.

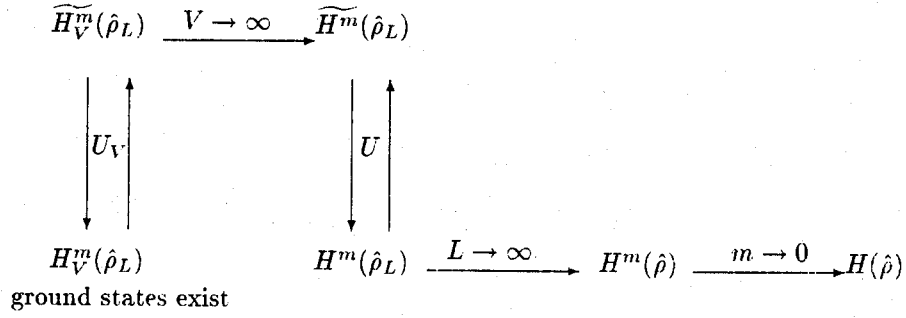


fig.4.1

First we shall show that $H_V^m(\hat{\rho}_L)$ has purely discrete spectrum in the neighborhood of $\mathbf{G}[H_V^m(\hat{\rho}_L)]$. Secondly, we shall show the norm resolvent convergence of $\widetilde{H}_V^m(\hat{\rho}_L)$ to $\widetilde{H}^m(\hat{\rho}_L)$ as $V \rightarrow \infty$. Then by Proposition 4.1 and unitary equivalences between $H_V^m(\hat{\rho}_L)$ and $\widetilde{H}_V^m(\hat{\rho}_L)$, and between $H^m(\hat{\rho}_L)$ and $\widetilde{H}^m(\hat{\rho}_L)$, we can see that $H^m(\hat{\rho}_L)$ has also purely discrete spectrum in the neighborhood of $\mathbf{G}[H^m(\hat{\rho}_L)]$ (See fig.4.1. Here convergences $V \rightarrow \infty$ and $L \rightarrow \infty$ are in the sense of norm resolvent, respectively, and $m \rightarrow 0$ in the sense of strong resolvent.) Finally the similar arguments on $L \rightarrow \infty$ and $m \rightarrow 0$ as those in [16] lead us to the final goal: we show the existence of the ground states of $H(\hat{\rho})$. Remark that it is *difficult* to show the norm resolvent convergence of $H_V^m(\hat{\rho}_L) \rightarrow H^m(\hat{\rho}_L)$ as $V \rightarrow \infty$ directly with potentials V

in Hypothesis 2.

4.1 Periodic Approximation

Periodic approximation is essentially contained in [16]; we include a version of the proof here for the convenience of the reader and to make this paper as self-contained as possible. We reproduce here a variant which suffices for our theorem. Define the set of lattice points by

$$\Gamma_V = \left\{ k = (k_1, k_2, k_3) \mid k_\mu = \frac{2\pi n_\mu}{V}, n_\mu \in \mathbf{Z}, \mu = 1, 2, 3 \right\}.$$

Let $l_2(\Gamma_V)$ be the set of l_2 sequences over Γ_V . We define $\mathcal{F}_V$ as

$$\mathcal{F}_V = \bigoplus_{n=0}^{\infty} \otimes_s^n [l_2(\Gamma_V) \oplus l_2(\Gamma_V)].$$

We identify $l_2(\Gamma_V)$ with a subspace in $L^2(\mathbf{R}^3)$ by

$$l_2(\Gamma_V) \ni \{a_l\}_{l \in \Gamma_V} \mapsto \frac{V}{2\pi} \sum_{l \in \Gamma_V} a_l X_{\Gamma(l,V)}(\cdot) \in L^2(\mathbf{R}^3),$$

where $X_{\Gamma(l,V)}(\cdot), l \in \Gamma_V$, is the characteristic function of the standard cube

$$\Gamma(l, V) = \left[l_1, l_1 + \frac{2\pi}{V} \right) \times \left[l_2, l_2 + \frac{2\pi}{V} \right) \times \left[l_3, l_3 + \frac{2\pi}{V} \right) \subset \mathbf{R}^3.$$

Then we identify $l_2(\Gamma_V) \oplus l_2(\Gamma_V)$ with the subspace of W and we regard $\mathcal{F}_V$ as the closed subspace of $\mathcal{F}_{EM}$. For $0 < m$ and $0 < V$, we define nonnegative self-adjoint operators by

$$\begin{aligned} H_{EM,V}^m &= \sum_{r=1}^{d-1} \int (\omega(k_V) + m) a^{\dagger r}(k) a^r(k) dk, \\ H_{EM}^m &= \sum_{r=1}^{d-1} \int (\omega(k) + m) a^{\dagger r}(k) a^r(k) dk, \end{aligned}$$

where $k_V = (k_{1,V}, k_{2,V}, k_{3,V})$ is the step function as follows:

$$k_{\mu,V} = \frac{2\pi n}{V}, \quad \text{if } k_\mu \in \left[\frac{2\pi n}{V}, \frac{2\pi n}{V} + \frac{2\pi}{V} \right), n \in \mathbf{Z}.$$

The operator $H_{EM,V}^m$ has purely discrete spectrum. We define periodically approximate quantized fields by

$$\begin{aligned} A_{\mu,V}(\hat{\rho}, x) &= \sum_{r=1}^2 \int dk \sum_{l \in \Gamma_V} X_{\Gamma(l,V)}(k) \left\{ \frac{\hat{\rho}(l) e^{-ilx} e_{\mu}^r(l)}{\sqrt{2\omega(l)}} a^{\dagger r}(k) + \frac{\hat{\rho}(-l) e^{ilx} e_{\mu}^r(l)}{\sqrt{2\omega(l)}} a^r(k) \right\}, \\ B_{\mu,V}(\hat{\rho}, x) &= \sum_{r=1}^2 \int dk \sum_{l \in \Gamma_V} X_{\Gamma(l,V)}(k) \\ &\quad \times \left\{ \frac{\hat{\rho}(l) e^{-ilx} (-il \times e^r(l))_{\mu}}{\sqrt{2\omega(l)}} a^{\dagger r}(k) + \frac{\hat{\rho}(-l) e^{ilx} (-i(-l) \times e^r(l))_{\mu}}{\sqrt{2\omega(l)}} a^r(k) \right\}. \end{aligned}$$

Thus $H_V^m(\hat{\rho})$ and $H^m(\hat{\rho})$ are defined by

$$H_V^m(\hat{\rho}) = H_{el} \otimes I + I \otimes H_{EM,V}^m + R_{I,V} + R_{II,V} + R_{III,V},$$

$$H^m(\hat{\rho}) = H_{el} \otimes I + I \otimes H_{EM}^m + R_I + R_{II} + R_{III}.$$

Here $R_{I,V}$, $R_{II,V}$ and $R_{III,V}$ are defined with $A_{\mu}(\hat{\rho}, x)$ and $B_{\mu}(\hat{\rho}, x)$ in R_I , R_{II} and R_{III} replaced by $A_{\mu,V}(\hat{\rho}, x)$ and $B_{\mu,V}(\hat{\rho}, x)$, respectively. Let X_L be the characteristic function of $L = \{k \in \mathbf{R}^3 \mid |k| < L\}$. We put $X_L \hat{\rho} = \hat{\rho}_L$.

Lemma 4.2 *We assume that Hypothesis 1,2 and 3 hold. Then (1) the operator $H^m(\hat{\rho})$ is self-adjoint on $\mathcal{D}_m = D(H_{el} \otimes I_{EM}) \cap D(I_{el} \otimes H_{EM}^m)$ and bounded from below. moreover it is essentially self-adjoint on any core for*

$$H^m(0) = H_{el} \otimes I_{EM} + I_{el} \otimes H_{EM}^m;$$

(2) for sufficiently large V , the operator $H_V^m(\hat{\rho}_L)$ is self-adjoint on $\mathcal{D}_m$ and bounded from below, moreover it is essentially self-adjoint on any core for

$$H_V^m(0) = H_{el} \otimes I_{EM} + I_{el} \otimes H_{EM,V}^m.$$

Proof: See [16, Proposition 3.3]. □

Let notations A', B', C', D', E', F' and G' be in (3.18), and $A'_{LV}, B'_{LV}, C'_{LV}, D'_{LV}, E'_{LV}, F'_{LV}$ and G'_{LV} are defined with $\hat{\rho}$ in $A', B', C', D', E', F', G'$ replaced by $\hat{\rho}_{LV} = \sum_{l \in \Gamma_l} X_{\Gamma(l,V)}(\cdot) \hat{\rho}_L(l)$, respectively.

Lemma 4.3 *Assume that Hypothesis 1,2 and 3 hold. Then the following inequalities hold:*

$$(1) \quad (1 - eA' - eB' - e^2D' - eF')(\Psi, H^m(0)\Psi)_{\mathcal{H}} + (eA' + e^2D' + eF')\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2 - (eC' + e^2E' + eG')\|\Psi\|_{\mathcal{H}}^2 \leq (\Psi, H^m(\hat{\rho})\Psi)_{\mathcal{H}}, \quad (4.1)$$

$$(2) \quad (\Psi, H^m(\hat{\rho})\Psi)_{\mathcal{H}} \leq (\Psi, H^m(0)\Psi)_{\mathcal{H}} + (eA' + eB' + e^2D' + eF')(\Psi, H^m(0)\Psi)_{\mathcal{H}} + (eC' + e^2E' + eG')\|\Psi\|_{\mathcal{H}}^2 - (eA' + e^2D' + eF')\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2, \quad (4.2)$$

$$(3) \quad (1 - eA'_{LV} - eB'_{LV} - e^2D'_{LV} - eF'_{LV})(\Psi, H_V^m(0)\Psi)_{\mathcal{H}} + (eA'_{LV} + e^2D'_{LV} + eF'_{LV})\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2 - (eC'_{LV} + e^2E'_{LV} + eG'_{LV})\|\Psi\|_{\mathcal{H}}^2 \leq (\Psi, H_V^m(\hat{\rho}_L)\Psi)_{\mathcal{H}}, \quad (4.3)$$

$$(4) \quad (\Psi, H_V^m(\hat{\rho})\Psi)_{\mathcal{H}} \leq (\Psi, H_V^m(0)\Psi)_{\mathcal{H}} + (eA_{LV} + eB_{LV} + e^2D_{LV} + eF_{LV})(\Psi, H_V^m(0)\Psi)_{\mathcal{H}} + (eC_{LV} + e^2E_{LV} + eG_{LV})\|\Psi\|_{\mathcal{H}}^2 - (eA_{LV} + e^2D_{LV} + eF_{LV})\mathbf{G}[H_{el}]\|\Psi\|_{\mathcal{H}}^2. \quad (4.4)$$

In particular

$$|\mathbf{G}[H^m(\hat{\rho})] - \mathbf{G}[H_{el}]| \leq \Delta(\hat{\rho}), \quad (4.5)$$

$$|\mathbf{G}[H_{el}] - \mathbf{G}[H_V^m(\hat{\rho}_L)]| \leq \Delta(\hat{\rho}_{LV}). \quad (4.6)$$

Proof: Note that for sufficient large V ,

$$(\Psi, H(0)\Psi)_{\mathcal{H}} \leq (\Psi, H_V^m(0)\Psi)_{\mathcal{H}}. \quad (4.7)$$

By (3.13b), (3.14b), (3.15b) and (4.7), one sees (4.3) and (4.4). (4.1) and (4.2) can be proved quite similarly. $\square$

We introduce Hypothesis 4:

Hypothesis 4 The coupling constant e is such that ¹

$$\frac{2\Delta(\hat{\rho})}{1 - eB'} < \Sigma - \mathbf{G}[H_{el}]. \quad (4.8)$$

Lemma 4.4 We assume that Hypothesis 1,2,3 and 4 hold, and V is sufficiently large and m satisfies that

$$\frac{2\Delta(\hat{\rho})}{1 - eB'} + \frac{m}{1 - eB'} < \Sigma - \mathbf{G}[H_{el}]. \quad (4.9)$$

Then $H_V^m(\hat{\rho}_L)$ has purely discrete spectrum in the interval $[\mathbf{G}[H_V^m(\hat{\rho}_L)], \mathbf{G}[H_V^m(\hat{\rho}_L)] + m)$.

Remark 4.5 By (4.6) and a limiting argument, for V sufficiently large, (4.9) implies that

$$(1 - eB'_{LV})(\Sigma - \mathbf{G}[H_{el}]) + (\mathbf{G}[H_{el}] - \mathbf{G}[H_V^m(\hat{\rho}_L)]) - \Delta(\hat{\rho}_{LV}) - m > 0. \quad (4.10)$$

Proof: For notational brevity, we put $H_V^m(\hat{\rho}_L) = H_V$, $\mathbf{G}[H_V^m(\hat{\rho}_L)] = \mathbf{G}_V$ and $\mathbf{G}[H_{el}] = \mathbf{G}$, respectively. Moreover set $\overline{H_{el}} = H_{el} - \mathbf{G}$. Let $E_T(\mathcal{B})$ be the spectral projection of an operator T on a Borel set $\mathcal{B}$ and $\dim \text{Ran} E_T(\mathcal{B})$ the dimension of the range of $E_T(\mathcal{B})$. Put

$$\mathcal{H} = \mathcal{H}_{el} \otimes [\mathcal{F}_V \oplus \mathcal{F}_V^\perp] = [\mathcal{H}_{el} \otimes \mathcal{F}_V] \oplus [\mathcal{H}_{el} \otimes \mathcal{F}_V^\perp] \equiv \mathcal{H}_1 \oplus \mathcal{H}_2.$$

We see that H_V is reduced by $\mathcal{H}_1$ ([16, Lemma 3.7]). By (3.13a), (3.14a) and (3.15a), we have

$$(\Psi, H_V \Psi)_{\mathcal{H}} \geq (\Psi, (\beta I_{el} \otimes H_{EM} + \alpha \overline{H_{el}} \otimes I_{EM} - \Delta(\hat{\rho}_{LV}) + \mathbf{G}) \Psi)_{\mathcal{H}}, \quad (4.11)$$

¹Concretely we have

$$\frac{2 \left\{ 6e\sqrt{2} \|\hat{\rho}\|_{-2} (a\mathbf{G}[H_{el}] + b') + \frac{3e^2(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})\|\hat{\rho}\|_{-1}}{2} + \frac{3e\|\hat{\rho}\|_0}{\sqrt{2}} \right\}}{1 - 6ea\sqrt{2}\|\hat{\rho}\|_{-2}} < \Sigma - \mathbf{G}[H_{el}].$$

where

$$\alpha = 1 - eB'_{LV}, \beta = 1 - (eA'_{LV} + e^2D'_{LV} + eF'_{LV}).$$

Note that by Hypothesis 3, $\alpha > 0$ and $\beta > 0$. We have, for $\Psi_1 \in \mathcal{H}_1 \cap D(H_V)$, by (4.11),

$$\begin{aligned} & ((H_V - m - \mathbf{G}_V)\Psi_1, \Psi_1)_{\mathcal{H}} \\ & \geq (E_{\overline{H}_{cl}}([0, \Sigma - \mathbf{G}]) \otimes [\beta H_{EM,V}^m - (m + \mathbf{G}_V - \mathbf{G} + \Delta(\hat{\rho}_{LV}))] \Psi_1, \Psi_1)_{\mathcal{H}} \\ & + [\alpha(\Sigma - \mathbf{G}) - (m + \mathbf{G}_V - \mathbf{G} + \Delta(\hat{\rho}_{LV}))] (E_{\overline{H}_{cl}}([\Sigma - \mathbf{G}, \infty)) \otimes I \Psi_1, \Psi_1)_{\mathcal{H}} \end{aligned} \quad (4.12)$$

$$+ (E_{\overline{H}_{cl}}([\Sigma - \mathbf{G}, \infty)) \otimes \beta H_{EM,V}^m \Psi_1, \Psi_1)_{\mathcal{H}}. \quad (4.13)$$

By virtue of (4.10), (4.12) is nonnegative, and (4.13) is also nonnegative. Therefor we have

$$\begin{aligned} & \text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) \upharpoonright_{\mathcal{H}_1} \\ & \subset \text{Ran} E_{\overline{H}_{cl}}([0, \Sigma - \mathbf{G}]) \otimes E_{H_{EM,V}^m} \left(\left[0, \frac{m + \mathbf{G}_V - \mathbf{G} + \Delta(\hat{\rho}_{LV})}{\beta} \right] \right) \upharpoonright_{\mathcal{H}_1}. \end{aligned} \quad (4.14)$$

Note that $\mathbf{G}_V + \Delta(\hat{\rho}_{LV}) \geq \mathbf{G}$. (4.14) implies that

$$\begin{aligned} & \dim \text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) \upharpoonright_{\mathcal{H}_1} \\ & \leq \dim \text{Ran} E_{\overline{H}_{cl}}([0, \Sigma - \mathbf{G}]) \times \dim \text{Ran} E_{H_{EM,V}^m} \left(\left[0, \frac{m + \mathbf{G}_V - \mathbf{G} + \Delta(\hat{\rho}_{LV})}{\beta} \right] \right) \\ & < \infty. \end{aligned} \quad (4.15)$$

On the other hand, we see that [16, Lemma 3.7]

$$(\Psi_2, H_V \Psi_2)_{\mathcal{H}} \geq (\mathbf{G}_V + m) \|\Psi_2\|_{\mathcal{H}}^2.$$

Thus we have

$$\text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) = \text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) \upharpoonright_{\mathcal{H}_1}. \quad (4.16)$$

It follows from (4.15) and (4.16) that

$$\dim \text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) \leq \dim \text{Ran} E_{H_V}([\mathbf{G}_V, \mathbf{G}_V + m]) \upharpoonright_{\mathcal{H}_1} < \infty.$$

Thus proof is complete. $\square$

4.2 Unitary Transformations

We define unitary transformations U and U_V by

$$\begin{aligned} U &= \exp \left(i \sum_{\mu=1}^3 x_{\mu} \otimes d\Gamma(k_{\mu}) \right), \\ U_V &= \exp \left(i \sum_{\mu=1}^3 x_{\mu} \otimes d\Gamma(k_{\mu,V}) \right), V > 0. \end{aligned}$$

Define

$$\mathcal{O} = \mathcal{L} \left\{ F(x) a^{\dagger r_1}(e^{-i\theta_1(k)x} f_1) \dots a^{\dagger r_n}(e^{-i\theta_n(k)x} f_n) \Omega_{EM}, \Omega_{EM} \mid F \in C_0^\infty(\mathbf{R}^3), f_j \in L_{comp}^2(\mathbf{R}^3), \right.$$

$$\left. \theta_j(k) \in \mathcal{L}\{k, k_V, 0\}, r_j = 1, 2, j = 1, 2, \dots, n, n \in \mathbf{N} \right\} \subset \mathcal{H},$$

$$\mathcal{D}_0 = \mathcal{L} \left\{ a^{\dagger r_1}(f_1) \dots a^{\dagger r_n}(f_n) \Omega_{EM}, \Omega_{EM} \mid f_j \in L_{comp}^2(\mathbf{R}^3), r_j = 1, 2, j = 1, \dots, n, n \in \mathbf{N} \right\} \subset \mathcal{F}_{EM},$$

where $\mathcal{L}\{\dots\}$ denotes the finite linear hull of the vectors in $\{\dots\}$ and $L_{comp}^2(\mathbf{R}^3)$ is the set of functions in $L^2(\mathbf{R}^3)$ with compact support. The set $\mathcal{D}_0$ is a core for both $H_{EM,V}^m$ and H_{EM}^m , and $C_0^\infty(\mathbf{R}^3)$ for H_{el} . Thus since $C_0^\infty(\mathbf{R}^3) \hat{\otimes} \mathcal{D}_0 \subset \mathcal{O}$, the set $\mathcal{O}$ is a core for both $H^m(0)$ and $H_V^m(0)$. Then by Lemma 4.2, $\mathcal{O}$ is a core for both $H^m(\hat{\rho}_L)$ and $H_V^m(\hat{\rho}_L)$. Operators $U, U_V, d\Gamma(k_{\mu}), d\Gamma(k_{\mu,V}), H_{EM}^m, H_{EM,V}^m$ and $a^{\dagger r}(e^{-i\theta(k)x} f), \theta(k) \in \mathcal{L}\{k, k_V, 0\}$, leave $\mathcal{O}$ invariant.

Lemma 4.6 *The unitary operators U and U_V satisfy that, on $\mathcal{O}$,*

$$U H^m(\hat{\rho}_L) U^{-1} = \frac{1}{2} \{ \sigma(P_{tot} - eA(\hat{\rho}_L, 0)) \}^2 + I_{el} \otimes H_{EM}^m + V \otimes I_{EM}, \quad (4.17)$$

$$U_V H_V^m(\hat{\rho}_L) U_V^{-1} = \frac{1}{2} \{ \sigma(P_{tot,V} - eA_V(\hat{\rho}_L, 0)) \}^2 + I_{el} \otimes H_{EM,V}^m + V \otimes I_{EM}. \quad (4.18)$$

where $P_{tot} = (P_{tot,1}, P_{tot,2}, P_{tot,3})$ and $P_{tot,V} = (P_{tot,V,1}, P_{tot,V,2}, P_{tot,V,3})$,

$$P_{tot,\mu} = P_{\mu} \otimes I_{EM} - I_{el} \otimes d\Gamma(k_{\mu}),$$

$$P_{tot,V,\mu} = P_{\mu} \otimes I_{EM} - I_{el} \otimes d\Gamma(k_{\mu,V}), \mu = 1, 2, 3.$$

Moreover U_V leaves $\mathcal{H}_1$ invariant.

Proof: It is easily seen that, on $\mathcal{O}$,

$$\begin{aligned} U(P \otimes I_{EM})U^{-1} &= P_{tot}, \quad U_V(P \otimes I_{EM})U_V^{-1} = P_{tot,V}, \\ Ua^r(e^{ikx}f)U^{-1} &= I_{el} \otimes a^r(f), \quad Ua^{\dagger r}(e^{-ikx}f)U^{-1} = I_{el} \otimes a^{\dagger r}(f), \\ U_Va^r(e^{ik_Vx}f)U_V^{-1} &= I_{el} \otimes a^r(f), \quad U_Va^{\dagger r}(e^{-ik_Vx}f)U_V^{-1} = I_{el} \otimes a^{\dagger r}(f), \\ U(I_{el} \otimes H_{EM}^m)U^{-1} &= I_{el} \otimes H_{EM}^m, \quad U_V(I_{el} \otimes H_{EM,V}^m)U_V^{-1} = I_{el} \otimes H_{EM,V}^m. \end{aligned}$$

Thus (4.17) and (4.18) follow. Let $\Psi = F \otimes a^{\dagger r_1}(f_1) \dots a^{\dagger r_n}(f_n) \Omega_{EM}$, $F \in C_0^\infty(\mathbf{R}^3)$, $f_j \in l_2(\Gamma_V) \oplus l_2(\Gamma_V)$, $r_j = 1, 2, j = 1, \dots, n$. The finite linear hull of the form of the vectors Ψ is dense in $\mathcal{H}_1$. Since $e^{-ik_Vx}f_j \in l_2(\Gamma_V) \oplus l_2(\Gamma_V)$, $j = 1, \dots, n$, for each $x \in \mathbf{R}^3$, we have

$$UF \otimes a^{\dagger r_1}(f_1) \dots a^{\dagger r_n}(f_n) \Omega_{EM} = F a^{\dagger r_1}(e^{-ik_Vx}f_1) \dots a^{\dagger r_n}(e^{-ik_Vx}f_n) \Omega_{EM} \in \mathcal{H}_1.$$

Thus U leaves $\mathcal{H}_1$ invariant. □

We write the right hand sides of (4.17) and (4.18) by $\widetilde{H}^m(\hat{\rho}_L)$ and $\widetilde{H}_V^m(\hat{\rho}_L)$, respectively.

Lemma 4.7 *The operators $\widetilde{H}^m(\hat{\rho}_L)$ and $\widetilde{H}_V^m(\hat{\rho}_L)$ are essentially self-adjoint on $\mathcal{O}$. The unitary operator U maps $D(\widetilde{H}^m(\hat{\rho}_L))$ onto $D(H^m(\hat{\rho}_L))$ with (4.17). The unitary operators U_V map $D(\widetilde{H}_V^m(\hat{\rho}_L))$ onto $D(H_V^m(\hat{\rho}_L))$ with (4.18).*

Proof: Since $\mathcal{O}$ is a core for $H^m(\hat{\rho}_L)$ and U maps $\mathcal{O}$ onto itself, $UH^m(\hat{\rho}_L)U^{-1}|_{\mathcal{O}}$ is essentially self-adjoint, which implies that $\widetilde{H}^m(\hat{\rho}_L)|_{\mathcal{O}}$ is essentially self-adjoint. Thus we obtain that U maps $D(\widetilde{H}^m(\hat{\rho}_L))$ onto $D(H^m(\hat{\rho}_L))$ with (4.17). The case of $\widetilde{H}_V^m(\hat{\rho}_L)$, $H_V^m(\hat{\rho}_L)$ and U_V is similarly proved. □

We define

$$\begin{aligned} H_{tot} &= P_{tot}^2 + V \otimes I_{EM} + I_{el} \otimes H_{EM}^m, \\ H_{tot}(V) &= P_{tot,V}^2 + V \otimes I_{EM} + I_{el} \otimes H_{EM,V}^m. \end{aligned}$$

Then

$$\begin{aligned}\widetilde{H}_V^m(\hat{\rho}_L) &= H_{tot}(V) + \mathbf{R}_V, \\ \widetilde{H}^m(\hat{\rho}_L) &= H_{tot} + \mathbf{R},\end{aligned}$$

where

$$\begin{aligned}\mathbf{R}_V &= -e(P_{tot,V})A_V(\hat{\rho}, 0) + e^2 \frac{1}{2} A_V(\hat{\rho}, 0)^2 + e \frac{1}{2} \sigma \cdot B_V(\hat{\rho}, 0), \\ \mathbf{R} &= -e(P_{tot})A(\hat{\rho}, 0) + e^2 \frac{1}{2} A(\hat{\rho}, 0)^2 + e \frac{1}{2} \sigma \cdot B(\hat{\rho}, 0).\end{aligned}$$

Lemma 4.8 *Assume that Hypothesis 1, 2 and 3 hold. Then, for sufficiently large V , there exist positive constant γ which is independent of sufficiently large V so that, for $\Psi \in \mathcal{O}$,*

$$\|H_{tot}\Psi\|_{\mathcal{H}} \leq \gamma(\|\widetilde{H}^m(\hat{\rho}_L)\Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \quad (4.19)$$

$$\|H_{tot}(V)\Psi\|_{\mathcal{H}} \leq \gamma(\|\widetilde{H}_V^m(\hat{\rho}_L)\Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}). \quad (4.20)$$

Proof: One sees that there exist constants $\alpha < 1$ and $\beta \geq 0$ which are independent of large V so that

$$\|(eR_I(V) + e^2 R_{II}(V) + eR_{III}(V))\Psi\|_{\mathcal{H}} \leq \alpha \|H_V^m(0)\Psi\|_{\mathcal{H}} + \beta \|\Psi\|_{\mathcal{H}}, \Psi \in \mathcal{O}.$$

Since U_V leaves $\mathcal{O}$ invariant,

$$\|\mathbf{R}_V\Psi\|_{\mathcal{H}} \leq \alpha \|H_{tot}(V)\Psi\|_{\mathcal{H}} + \beta \|\Psi\|_{\mathcal{H}}, \Psi \in \mathcal{O}.$$

Thus it follows that

$$\|H_{tot}(V)\Psi\|_{\mathcal{H}} \leq \frac{1}{1-\alpha} \|\widetilde{H}_V^m(\hat{\rho}_L)\Psi\|_{\mathcal{H}} + \frac{\beta}{1-\alpha} \|\Psi\|_{\mathcal{H}}. \quad (4.21)$$

(4.21) implies (4.19). (4.20) is proved similarly. $\square$

4.3 Existence of Ground States with Artificial Mass

The purpose of this subsection is to obtain the following lemma:

Lemma 4.9 *We assume that Hypothesis 1,2 and 3 hold. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\lim_{V \rightarrow \infty} (\widetilde{H}_V^m(\hat{\rho}_L) - z)^{-1} = (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \quad (4.22)$$

in the uniform norm.

Before proving Lemma 4.9, we prepare some lemmas. We have

$$0 \leq k_\mu - k_{\mu,V} \leq \frac{2\pi}{V}, \mu = 1, 2, 3. \quad (4.23)$$

Then

$$\begin{aligned} 0 \leq (\Psi, (d\Gamma(k_\mu) - d\Gamma(k_{\mu,V}))\Psi)_{\mathcal{H}} &\leq \frac{2\pi}{mV} (\Psi, mN_{EM}\Psi)_{\mathcal{H}} \\ &\leq \frac{2\pi}{mV} (\Psi, H_{EM}^m\Psi)_{\mathcal{H}}, \end{aligned} \quad (4.24)$$

where $\Psi \in \mathcal{D}_0$. Since the set $\mathcal{D}_0$ is a core for H_{EM}^m , (4.24) extends to

$$\|(d\Gamma(k_\mu) - d\Gamma(k_{\mu,V}))^{\frac{1}{2}}\Psi\|_{\mathcal{H}} \leq \sqrt{\frac{2\pi}{mV}} \|(H_{EM}^m)^{\frac{1}{2}}\Psi\|_{\mathcal{H}}, \Psi \in D((H_{EM}^m)^{\frac{1}{2}}). \quad (4.25)$$

On the other hand we see that $d\Gamma(k_\mu) - d\Gamma(k_{\mu,V})$ and H_{EM}^m strongly commute. Since $\mathcal{D}_0$ is invariant under both H_{EM}^m and $d\Gamma(k_\mu) - d\Gamma(k_{\mu,V})$, (4.25) and Proposition 2.3 lead us to the following lemma:

Lemma 4.10 *Assume that $\Psi \in D(H_{EM}^m)$. Then*

$$\|(d\Gamma(k_\mu) - d\Gamma(k_{\mu,V}))\Psi\|_{\mathcal{H}} \leq \frac{2\pi}{mV} \|H_{EM}^m\Psi\|_{\mathcal{H}}, \mu = 1, 2, 3. \quad (4.26)$$

Lemma 4.11 *Let $\Psi \in \mathcal{O}$. Then there exists a positive constant α so that*

$$\|P_{tot,V,\mu}\Psi\|_{\mathcal{H}} \leq \alpha(\|H_{tot}(V)\Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \quad (4.27)$$

$$\begin{aligned}
\|I_{el} \otimes H_{EM}^{\frac{1}{2}} \Psi\|_{\mathcal{H}} &\leq \alpha(\|H_{tot} \Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \\
\|I_{el} \otimes H_{EM}^{\frac{1}{2}} \Psi\|_{\mathcal{H}} &\leq \alpha(\|H_{tot}(V) \Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \\
\|I_{el} \otimes H_{EM}^m \Psi\|_{\mathcal{H}} &\leq \alpha(\|H_{tot} \Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \\
\|I_{el} \otimes H_{EM}^m \Psi\|_{\mathcal{H}} &\leq \alpha(\|H_{tot}(V) \Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}), \\
\mu &= 1, 2, 3.
\end{aligned}$$

Proof: One sees easily that

$$\|(P_{\mu} \otimes I) \Psi\|_{\mathcal{H}} \leq \alpha(\|H_V^m(0) \Psi\|_{\mathcal{H}} + \|\Psi\|_{\mathcal{H}}).$$

Since U_V leaves $\mathcal{O}$ invariant, (4.27) holds. The other inequalities are handled similarly. $\square$

Proof of Lemma 4.9

We have

$$(\widetilde{H^m}(\hat{\rho}_L) - z)^{-1} - (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} = I(V) + eII(V) + e^2III(V) + eIV(V) + R(V),$$

where

$$\begin{aligned}
I(V) &= \sum_{\mu=1}^3 (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} \left((P_{tot,V,\mu})^2 - (P_{tot,\mu})^2 \right) (\widetilde{H^m}(\hat{\rho}_L) - z)^{-1}, \\
II(V) &= \sum_{\mu=1}^3 (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} (P_{tot,V,\mu} A_{\mu,V}(\hat{\rho}_L, 0) - P_{tot,\mu} A_{\mu}(\hat{\rho}_L, 0)) (\widetilde{H^m}(\hat{\rho}_L) - z)^{-1}, \\
III(V) &= \sum_{\mu=1}^3 (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} (A_{\mu,V}(\hat{\rho}_L, 0)^2 - A_{\mu}(\hat{\rho}_L, 0)^2) (\widetilde{H^m}(\hat{\rho}_L) - z)^{-1}, \\
IV(V) &= \sum_{\mu=1}^3 (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} (\sigma_{\mu} B_{\mu,V}(\hat{\rho}_L, 0) - \sigma_{\mu} B_{\mu}(\hat{\rho}_L, 0)) (\widetilde{H^m}(\hat{\rho}_L) - z)^{-1}, \\
R(V) &= (\widetilde{H_V^m}(\hat{\rho}_L) - z)^{-1} (H_{EM,V}^m - H_{EM}^m) (\widetilde{H^m}(\hat{\rho}_L) - z)^{-1}.
\end{aligned}$$

We shall check that $I(V)$, $II(V)$, $III(V)$, $IV(V)$ and $R(V)$ go to zero as $V \rightarrow \infty$, respectively.

We denote the uniform norm of operators in $\mathcal{H}$ by $\|\cdot\|$. For $f \in \mathcal{M}_n$, we define $R_n(f)$ by

$$R_n(f) = \left\| f - \sum_{l \in \Gamma_V} f(l) X_{\Gamma(l,V)} \right\|_n.$$

Note that, if f has a compact support, then $R_n(f)$ goes to zero as $V \rightarrow \infty$. We put

$$S_n = \max_{\mu=1,2,3,r=1,2} R_n \left(\frac{\hat{\rho}_L e_\mu^r}{\sqrt{2\omega}} \right),$$

$$Y_n = \max_{\mu=1,2,3,r=1,2} R_n \left(\frac{\hat{\rho}_L(k \times e^r)_\mu}{\sqrt{2\omega}} \right).$$

In addition to these, for notational simplification, we set

$$a_n = \max_{\mu=1,2,3,r=1,2} \left\| \frac{\hat{\rho}_L e_\mu^r}{\sqrt{2\omega}} \right\|_n.$$

Define

$$\alpha = \max \left\{ \begin{array}{ll} \|(I_{el} \otimes H_{EM}^m)(H_{tot}(V) - z^*)^{-1}\|, & \|(I_{el} \otimes H_{EM}^m)(H_{tot} - z^*)^{-1}\| \\ \|(I_{el} \otimes H_{EM}^{\frac{1}{2}})(H_{tot}(V) - z^*)^{-1}\|, & \|(I_{el} \otimes H_{EM}^{\frac{1}{2}})(H_{tot} - z^*)^{-1}\| \\ \|P_{tot,V,\mu}(H_{tot}(V) - z^*)^{-1}\| & \end{array} \right\}$$

and

$$A = \max \left\{ \|H_{tot}(V)(\widetilde{H}_V^m(\hat{\rho}_L) - z^*)^{-1}\|, \|H_{tot}(\widetilde{H}^m(\hat{\rho}_L) - z^*)^{-1}\| \right\},$$

where $z^* = \bar{z}$ or z . The existence of α and A is ensured by Lemmas 4.8 and 4.11.

[$R(V)$] Note that, for $\Psi \in D(H_{EM}^m)$,

$$\|(H_{EM}^m - H_{EM,V}^m)\Psi\|_{\mathcal{F}_{EM}} \leq \frac{2C_V}{1 - C_V} \|H_{EM}^m \Psi\|_{\mathcal{F}_{EM}},$$

where $C_V = \sqrt{3} \left(\frac{\pi}{V} \right) \left(\frac{1}{2m} + 1 \right)$ [6, Lemmas 3.1 and 3.6]. Thus we have

$$\|R(V)\| \leq \frac{1}{|\Im z|} \frac{2C_V}{1 - C_V} A\alpha,$$

which deduces that $R(V) \rightarrow 0$ as $V \rightarrow \infty$.

[$I(V)$] We see that, by Lemma 4.10,

$$\begin{aligned} & |(\Psi, I(V)\Phi)_{\mathcal{H}}| \\ & \leq \sum_{\mu=1}^3 \left\{ |(P_{tot,V,\mu}(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi, (P_{tot,V,\mu} - P_{tot,\mu})(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi)_{\mathcal{H}}| \right. \end{aligned}$$

$$\begin{aligned}
& + |((P_{tot,V,\mu} - P_{tot,\mu})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi, P_{tot,\mu}(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi)_{\mathcal{H}}| \Big\} \\
& \leq \sum_{\mu=1}^3 \frac{2 \cdot 2\pi}{mV} \left\{ \|P_{tot,V,\mu}(H_{tot}(V) - \bar{z})^{-1}(H_{tot}(V) - \bar{z})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \right. \\
& \quad \times \|(I_{el} \otimes H_{EM}^m)(H_{tot} - z)^{-1}(H_{tot} - z)(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \\
& \quad + \|(I_{el} \otimes H_{EM}^m)(H_{tot}(V) - \bar{z})^{-1}(H_{tot}(V) - \bar{z})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \\
& \quad \times \|P_{tot,\mu}(H_{tot} - z)^{-1}(H_{tot} - z)(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \Big\} \\
& \leq \frac{3 \cdot 2 \cdot 2\pi \cdot 2}{mV} (\alpha A)^2 \|\Psi\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}}.
\end{aligned}$$

Thus we have $I(V) \rightarrow 0$ as $V \rightarrow \infty$. $[II(V)]$ We see that, by Lemma 4.10,

$$\begin{aligned}
& |(\Psi, II(V)\Phi)| \\
& \leq \sum_{\mu=1}^3 \left\{ \|P_{tot,V,\mu}(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \|(A_{\mu,V}(\hat{\rho}_L, 0) - A_{\mu}(\hat{\rho}_L, 0))(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \right. \\
& \quad + \|(P_{tot,V,\mu} - P_{tot,\mu})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \|A_{\mu}(\hat{\rho}_L, 0)(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \Big\} \\
& \leq 2 \sum_{\mu=1}^3 \left[\|P_{tot,V,\mu}(H_{tot}(V) - \bar{z})^{-1}(H_{tot}(V) - \bar{z})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \right. \\
& \quad \times \left\{ 2S_{-1} \|(I_{el} \otimes H_{EM}^{\frac{1}{2}})(H_{tot} - z)^{-1}(H_{tot} - z)(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \right. \\
& \quad + S_0 \|(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \Big\} \\
& \quad + \left(\frac{2\pi}{mV} \right) \|(I_{el} \otimes H_{EM}^m)(H_{tot}(V) - \bar{z})^{-1}(H_{tot}(V) - \bar{z})(\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1}\Psi\|_{\mathcal{H}} \\
& \quad \times \left\{ 2a_{-1} \|(I_{el} \otimes H_{EM}^{\frac{1}{2}})(H_{tot} - z)^{-1}(H_{tot} - z)(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \right. \\
& \quad + a_0 \|(\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}\Phi\|_{\mathcal{H}} \Big\} \Big] \\
& \leq 3 \cdot 2 \left\{ \alpha A \left(2\alpha A S_{-1} + \frac{S_0}{|\Im z|} \right) + \frac{2\pi}{mV} \alpha A \left(2\alpha A a_{-1} + \frac{a_0}{|\Im z|} \right) \right\} \|\Psi\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}}.
\end{aligned}$$

Thus we see that $II(V) \rightarrow 0$ as $V \rightarrow \infty$.

$[III(V)]$ We have

$$\begin{aligned}
III(V) &= \sum_{\mu=1}^3 \frac{1}{2} (\widetilde{H}_V^m(\hat{\rho}_L) - z)^{-1} A_{\mu,V}(\hat{\rho}_L, 0) \\
& \quad \times [A_{\mu,V}(\hat{\rho}_L, 0) - A_{\mu}(\hat{\rho}_L, 0)] (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\mu=1}^3 \frac{1}{2} (\widetilde{H}_V^m(\hat{\rho}_L) - z)^{-1} [A_{\mu,V}(\hat{\rho}_L, 0) - A_{\mu}(\hat{\rho}_L, 0)] \\
& \quad \times A_{\mu}(\hat{\rho}_L, 0) (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1}.
\end{aligned}$$

Thus the second terms in $III(V)$ can be estimated as follows: for $\mu = 1, 2, 3$,

$$\begin{aligned}
& \left| ([A_{\mu,V}(\hat{\rho}_L, 0) - A_{\mu}(\hat{\rho}_L, 0)] (\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1} \Psi, A_{\mu}(\hat{\rho}_L, 0) (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \Phi)_{\mathcal{H}} \right| \\
& \leq 2 \left\{ 2S_{-1} \| (I_{el} \otimes H_{EM}^{\frac{1}{2}}) (H_{tot}(V) - \bar{z})^{-1} (H_{tot}(V) - \bar{z}) (\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1} \Psi \|_{\mathcal{H}} \right. \\
& \quad \left. + S_0 \| (\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1} \Psi \|_{\mathcal{H}} \right\} \\
& \quad \times \left\{ 2a_{-1} \| (I_{el} \otimes H_{EM}^{\frac{1}{2}}) (H_{tot} - z)^{-1} (H_{tot} - z) (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \Phi \|_{\mathcal{H}} \right. \\
& \quad \left. + a_0 \| (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \Phi \|_{\mathcal{H}} \right\} \\
& \leq 2 \left(2\alpha AS_{-1} + \frac{S_0}{|\Im z|} \right) \left(2\alpha Aa_{-1} + \frac{a_0}{|\Im z|} \right) \|\Psi\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}}.
\end{aligned}$$

The first terms in $III(V)$ are similarly treated. Then $III(V) \rightarrow 0$ as $V \rightarrow \infty$.

$[IV(V)]$ We see that

$$\begin{aligned}
|(\Psi, IV(V)\Phi)| & \leq 2 \sum_{\mu=1}^3 \| (\widetilde{H}_V^m(\hat{\rho}_L) - \bar{z})^{-1} \Psi \|_{\mathcal{H}} \\
& \quad \times \left\{ 2Y_{-1} \| (I_{el} \otimes H_{EM}^{\frac{1}{2}}) (H_{tot} - z)^{-1} (H_{tot} - z) (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \Phi \|_{\mathcal{H}} \right. \\
& \quad \left. + Y_0 \| (\widetilde{H}^m(\hat{\rho}_L) - z)^{-1} \Phi \|_{\mathcal{H}} \right\} \\
& \leq \frac{3 \cdot 2}{|\Im z|} \left(2\alpha AY_{-1} + \frac{Y_0}{|\Im z|} \right) \|\Psi\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}}.
\end{aligned}$$

Then $IV(V) \rightarrow 0$ as $V \rightarrow \infty$. Thus, the terms $R(V)$, $I(V)$, $II(V)$, $III(V)$ and $IV(V)$ go to zero, as $V \rightarrow \infty$, respectively. (4.22) follows.

Lemma 4.12 *We assume that Hypothesis 1, 2, 3 and 4 hold and m satisfies (4.9). Then $H^m(\hat{\rho})$ has purely discrete spectrum in the interval*

$$[\mathbf{G}[H^m(\hat{\rho})], \mathbf{G}[H^m(\hat{\rho})] + m). \quad (4.28)$$

In particular, the ground states of $H^m(\hat{\rho})$ exist.

Proof: By Lemma 4.4 and (4.18), $\widetilde{H}_V^m(\hat{\rho}_L)$ has purely discrete spectrum in the interval

$$[\mathbf{G}[\widetilde{H}_V^m(\hat{\rho}_L)], \mathbf{G}[\widetilde{H}_V^m(\hat{\rho}_L)] + m).$$

By Proposition 4.1 (I) and Lemma 4.9, we see that $\widetilde{H}^m(\hat{\rho}_L)$ has purely discrete spectrum in the interval

$$[\mathbf{G}[\widetilde{H}^m(\hat{\rho}_L)], \mathbf{G}[\widetilde{H}^m(\hat{\rho}_L)] + m),$$

which implies that, by (4.17), $H^m(\hat{\rho}_L)$ has purely discrete spectrum in the interval

$$[\mathbf{G}[H^m(\hat{\rho}_L)], \mathbf{G}[H^m(\hat{\rho}_L)] + m).$$

Since $\|\hat{\rho}_L - \hat{\rho}\|_n \rightarrow 0$ as $L \rightarrow \infty$ for $n = -2, -1$, we can easily see that, for $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\lim_{L \rightarrow \infty} (H^m(\hat{\rho}_L) - z)^{-1} = (H^m(\hat{\rho}) - z)^{-1} \quad (4.29)$$

in the uniform norm. Then again by Proposition 4.1 (I), we see that $H^m(\hat{\rho})$ has purely discrete spectrum in the interval (4.28). $\square$

4.4 Binding

Let a ground state of $H^m(\hat{\rho})$ be Ω_m . Define $T = K \otimes E_{H_{EM}}(\{0\})$, where K is a projection operator on $L^2(\mathbf{R}^3)$. In this subsection, we estimate both $\|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}}$ and $|(T\Omega_m, (eR_I + e^2R_{II} + eR_{III})\Omega_m)_{\mathcal{H}}|$. Notations A, B, C, D, E, F, G in this subsection are in (3.9). Define

$$A'' = \frac{1}{2}A, B'' = 6a\|\hat{\rho}\|_{-2}, C'' = 6b\|\hat{\rho}\|_{-2}, D'' = E'' = \frac{1}{2}D = \frac{1}{2}E, F'' = G'' = \frac{1}{2}F = \frac{1}{2}G.$$

Let

$$P = e(A + B + F) + e^2D, Q_1 = e(C + G) + e^2E, Q_2 = e(A + 2B + F) + e^2D,$$

$$P'' = e(A'' + B'' + F'') + e^2D'', Q_1'' = e(C'' + G'') + e^2E'',$$

$$Q_2'' = e(A'' + 2B'' + F'') + e^2D''.$$

We define

$$\mathcal{B} = \max \{ |\mathbf{G}[H_{el}] - \Delta(\hat{\rho})|, |\mathbf{G}[H_{el}] + \Delta(\hat{\rho})| \}.$$

By (4.5)

$$|\mathbf{G}[H^m(\hat{\rho})]| \leq \mathcal{B}.$$

We shall estimate $\|P_\mu^2 \otimes I_{EM} \Omega_m\|_{\mathcal{H}}$. We have

$$\begin{aligned} & \|P_\mu^2 \otimes I_{EM} \Omega_m\|_{\mathcal{H}} \\ & \leq \|P_\mu^2 \otimes I_{EM} (H(0) - i)^{-1}\| \| (H(0) - i)(H^m(\hat{\rho}) - i)^{-1} \| \| (H^m(\hat{\rho}) - i) \Omega_m \|_{\mathcal{H}} \\ & \leq \|P_\mu^2 \otimes I_{EM} (H(0) - i)^{-1}\| \| (H(0) - i)(H^m(\hat{\rho}) - i)^{-1} \| (\mathcal{B} + 1) \| \Omega_m \|_{\mathcal{H}}. \end{aligned}$$

Since

$$\begin{aligned} \|P_\mu^2 \otimes I_{EM} \Psi\|_{\mathcal{H}} & \leq 2a \|H_{el} \otimes I_{EM} \Psi\|_{\mathcal{H}} + 2b \|\Psi\|_{\mathcal{H}} \\ & \leq 2a \|H(0) \Psi\|_{\mathcal{H}} + (4a |\mathbf{G}[H_{el}]| + 2b) \|\Psi\|_{\mathcal{H}}, \end{aligned}$$

we have

$$\|P_\mu^2 \otimes I_{EM} (H(0) - i)^{-1}\| \leq 2a + 2b + 4a |\mathbf{G}[H_{el}]|. \quad (4.30)$$

Since

$$\|H(0) \Psi\|_{\mathcal{H}} \leq \|H^m(\hat{\rho}) \Psi\|_{\mathcal{H}} + c \|R_I \Psi\|_{\mathcal{H}} + c^2 \|R_{II} \Psi\|_{\mathcal{H}} + c \|R_{III} \Psi\|_{\mathcal{H}} + 2 |\mathbf{G}[H_{el}]| \|\Psi\|_{\mathcal{H}}, \quad (4.31)$$

we have

$$\|H(0) \Psi\|_{\mathcal{H}} \leq \frac{1}{1-P} \{ \|H^m(\hat{\rho}) \Psi\|_{\mathcal{H}} + (Q_1 + (Q_2 + 2) |\mathbf{G}[H_{el}]|) \|\Psi\|_{\mathcal{H}} \}. \quad (4.32)$$

From (4.32) it follows that

$$\begin{aligned} \| (H(0) - i)(H^m(\hat{\rho}) - i)^{-1} \| & \leq \frac{1}{1-P} + \frac{Q_1 + (Q_2 + 2) |\mathbf{G}[H_{el}]|}{1-P} + 1 \\ & = \frac{2 - P + Q_1 + (Q_2 + 2) |\mathbf{G}[H_{el}]|}{1-P}. \end{aligned} \quad (4.33)$$

We define

$$\mathcal{A} = \left(\frac{2 - P + Q_1 + (Q_2 + 2)|\mathbf{G}[H_{el}]|}{1 - P} \right) (2a + 2b + 4a|\mathbf{G}[H_{el}]|).$$

Thus, by (4.30) and (4.33) we have

$$\|P_\mu^2 \otimes I_{EM} \Omega_m\|_{\mathcal{H}} \leq \mathcal{A}(\mathcal{B} + 1) \|\Omega_m\|_{\mathcal{H}}.$$

In particular,

$$\|P_\mu \otimes I_{EM} \Omega_m\|_{\mathcal{H}} \leq \sqrt{\mathcal{A}(\mathcal{B} + 1)} \|\Omega_m\|_{\mathcal{H}} \quad (4.34)$$

We introduce Hypothesis 5:

Hypothesis 5 The ultraviolet cut-off holds $\|\hat{\rho}\|_{-3} < \infty$ and the coupling constant e satisfies that

$$\|\hat{\rho}\|_{-3} \leq \frac{1 - \frac{3e}{\sqrt{2}} \|\hat{\rho}\|_{-1}}{6e \left(\sqrt{\frac{\mathcal{A}(\mathcal{B}+1)}{2}} + \frac{3e \|\hat{\rho}\|_{-1}}{2} \right)}. \quad (4.35)$$

Remark 4.13 *Physically it should be that*

$$\hat{\rho}(0) = \int_{\mathbf{R}^3} \rho(x) dx = e \neq 0.$$

However Hypothesis 5 breaks above condition, i.e., Hypothesis 5 implies that $\hat{\rho}(0) = 0$. See [27].

Lemma 4.14 *We assume that Hypothesis 1,2,3,4 and 5 hold. Then*

$$\|N_{EM}^{\frac{1}{2}} \Omega_m\|_{\mathcal{H}} \leq \delta(e) < 1, \quad (4.36)$$

where $\delta(e)$ is a positive constant.

Proof: By (3.3), (3.4) and (4.34), we have [16, Lemma 3.11]

$$\begin{aligned} \|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}}^2 &\leq \frac{3 \cdot 2e}{\sqrt{2}} \|\hat{\rho}\|_{-3} \|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}} \sqrt{\mathcal{A}(\mathcal{B}+1)} \|\Omega_m\|_{\mathcal{H}} \\ &\quad + \frac{3 \cdot 2e^2}{2} \|\hat{\rho}\|_{-3} \|\hat{\rho}\|_{-1} \|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}} \left(\|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}} + \|(N_{EM} + I)^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}} \right) \\ &\quad + \frac{3 \cdot 2e}{2\sqrt{2}} \|\hat{\rho}\|_{-1} \|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}} \|\Omega_m\|_{\mathcal{H}}. \end{aligned}$$

Hence from Hypothesis 5 it follows that

$$\begin{aligned} \frac{\|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}}}{\|\Omega_m\|_{\mathcal{H}}} &\leq \frac{6e\|\hat{\rho}\|_{-3} \left(\sqrt{\frac{\mathcal{A}(\mathcal{B}+1)}{2}} + \frac{e\|\hat{\rho}\|_{-1}}{2} \right) + \frac{3e}{\sqrt{2}}\|\hat{\rho}\|_{-1}}{1 - 6e^2\|\hat{\rho}\|_{-3}\|\hat{\rho}\|_{-1}} \\ &\equiv \delta(e) < 1. \end{aligned}$$

Thus (4.36) follows. $\square$

Next we shall estimate $|(T\Omega_m, (eR_I + e^2R_{II} + eR_{III})\Omega_m)_{\mathcal{H}}|$. Note that $a^r(f)Q\Psi = 0, f \in L^2(\mathbf{R}^3)$. We have

$$\begin{aligned} \frac{|(T\Omega_m, R_I\Omega_m)_{\mathcal{H}}|}{\|T\Omega_m\|_{\mathcal{H}}} &\leq (A'' + B'')\|H(0)\Omega_m\|_{\mathcal{H}} + C'\|\Omega_m\|_{\mathcal{H}} + (A'' + 2B'')|\mathbf{G}[H_{el}]|\|\Omega_m\|_{\mathcal{H}}, \\ \frac{|(T\Omega_m, R_{II}\Omega_m)_{\mathcal{H}}|}{\|T\Omega_m\|_{\mathcal{H}}} &\leq D''\|H(0)\Omega_m\|_{\mathcal{H}} + E''\|\Omega_m\|_{\mathcal{H}} + D''|\mathbf{G}[H_{el}]|\|\Omega_m\|_{\mathcal{H}}, \\ \frac{|(T\Omega_m, R_{III}\Omega_m)_{\mathcal{H}}|}{\|T\Omega_m\|_{\mathcal{H}}} &\leq F''\|H(0)\Omega_m\|_{\mathcal{H}} + G''\|\Omega_m\|_{\mathcal{H}} + F''|\mathbf{G}[H_{el}]|\|\Omega_m\|_{\mathcal{H}}. \end{aligned}$$

From (4.32) it follows that

$$\begin{aligned} &\frac{|(T\Omega_m, (eR_I + e^2R_{II} + eR_{III})\Omega_m)_{\mathcal{H}}|}{\|\Omega_m\|_{\mathcal{H}}} \\ &\leq \frac{P''}{1-P} \{ \mathcal{B} + Q_1 + (Q_2 + 2)|\mathbf{G}[H_{el}]| \} + Q_1'' + Q_2''|\mathbf{G}[H_{el}]|. \end{aligned} \quad (4.37)$$

We put the right hand side of (4.37) by $\mathcal{C}$.

4.5 Overlap and Existence of Ground States

Without loss of generality, we can assume that $\|\Omega_m\|_{\mathcal{H}} = 1$. Then we can find a subsequence $\{m_j\}_{j=1}^{\infty}$ so that $\Omega_0 = w - \lim_{j \rightarrow \infty} \Omega_{m_j}$ exists.

Lemma 4.15 *We assume that Hypothesis 1, 2 and 3 hold. Then*

$$\lim_{m \rightarrow 0} \mathbf{G}[H^m(\hat{\rho})] = \mathbf{G}[H(\hat{\rho})].$$

Proof: See [16, Lemma 3.13]. □

We introduce Hypothesis 6:

Hypothesis 6 The coupling constant e is such that²

$$\Sigma - \mathbf{G}[H_{el}] > \Delta(\hat{\rho}) + \frac{C}{\sqrt{1 - \delta(e)^2}}. \quad (4.38)$$

Note that Hypothesis 6 implies that $\Sigma - \mathbf{G}[H^m(\hat{\rho})] > 0$.

Lemma 4.16 *We assume that Hypothesis 1, 2, 3, 4, 5 and 6 hold. Then*

$$\Omega_0 \neq 0. \quad (4.39)$$

Proof: Put $P = E_{H_{el}}([\mathbf{G}[H_{el}], \Sigma)) \otimes E_{H_{EM}}(\{0\})$, $Q = E_{H_{el}}([\Sigma, \infty)) \otimes E_{H_{EM}}(\{0\})$. Then we have

$$(\Omega_m, P\Omega_m)_{\mathcal{H}} \geq 1 - \|N_{EM}^{\frac{1}{2}}\Omega_m\|_{\mathcal{H}}^2 - (\Omega_m, Q\Omega_m)_{\mathcal{H}}.$$

Since

$$(\Omega_m, Q(H_{el} \otimes I_{EM} - \mathbf{G}[H^m(\hat{\rho})])\Omega_m)_{\mathcal{H}} = -(\Omega_m, Q(eR_I + e^2R_{II} + eR_{III})\Omega_m)_{\mathcal{H}},$$

by (4.37) we have

$$(\Sigma - \mathbf{G}[H^m(\hat{\rho})])(\Omega_m, Q\Omega_m)_{\mathcal{H}} \leq C\|Q\Omega_m\|_{\mathcal{H}}.$$

²Concretely we have

$$6e\sqrt{2}\|\hat{\rho}\|_{-2}(a\mathbf{G}[H_{el}] + b') + \frac{3e^2(\|\hat{\rho}\|_{-2} + \|\hat{\rho}\|_{-1})\|\hat{\rho}\|_{-1}}{2} + \frac{3e\|\hat{\rho}\|_0}{\sqrt{2}} + \frac{C}{\sqrt{1 - \delta(e)^2}} < \Sigma - \mathbf{G}[H_{el}].$$

Hence

$$(\Omega_m, P\Omega_m)_\mathcal{H} \geq 1 - \delta(e)^2 - \left(\frac{C}{\Sigma - \mathbf{G}[H^m(\hat{\rho})]} \right)^2. \quad (4.40)$$

Taking the subsequence $\{m_j\}_{j=1}^\infty$, we see that both sides of (4.40) converge with

$$(\Omega_0, P\Omega_0)_\mathcal{H} \geq 1 - \delta(e)^2 - \left(\frac{C}{\Sigma - \mathbf{G}[H(\hat{\rho})]} \right)^2. \quad (4.41)$$

The right hand side of (4.41) is positive by Hypothesis 5 and 6. Thus 4.39 follows, $\square$

We state the main theorem in this section.

Theorem 4.17 *We assume that Hypothesis 1,2,3,4,5 and 6 hold. Then Ω_0 is a ground state of $H(\hat{\rho})$. In addition to this, we assume that $-\frac{1}{2}\Delta + V$ has a unique ground state ϕ . Then*

$$\lim_{e \rightarrow 0} \|\phi \otimes \Omega_{EM} - \Omega_0\|_\mathcal{H} = 0.$$

Proof: By Lemmas 4.14 and 4.16 and Proposition 4.1(II), theorem follows. $\square$

4.6 Convergence of Ground States

Define the Hamiltonian with “ g -factor” as follows:

$$H_g(\hat{\rho}) = \frac{1}{2}(P \otimes I_{EM} + A(\hat{\rho}, x))^2 + V \otimes I_{EM} + I_{el} \otimes H_{EM} + g \frac{1}{2} \sigma B(\hat{\rho}, x).$$

Since $0 \leq g \leq 1$, under Hypothesis³ 1,2,3,4,5 and 6, their ground states exist. We denote a normalized ground state of $H_g(\hat{\rho})$ by $\Omega(g)$; $\|\Omega(g)\|_\mathcal{H} = 1$. Then there exists a subsequence $\{g_j\}_{j=1}^\infty$ so that $w - \lim_{j \rightarrow \infty} \Omega(g_j) \equiv \tilde{\Omega}$ exists.

Theorem 4.18 *We assume that Hypothesis 1,2,3,4,5 and 6 hold, and the ground state $\Omega(0)$ is a unique ground state of $H_0(\hat{\rho})$. Then*

$$\tilde{\Omega} = \theta \Omega(0), \quad (4.42)$$

where $0 < |\theta| \leq 1$.

³In the case of $\Sigma = \infty$, under Hypothesis 1,2,3 and 5, their ground states exist. In the case $g > 1$, taking the coupling constant e sufficiently small, similarly we can see that their ground states exist.

Proof: It is easily seen that $H_g(\hat{\rho})$ converges to $H_0(\hat{\rho})$ as $g \rightarrow 0$ in the sense of the norm resolvent. Thus $\mathbf{G}(H_g(\hat{\rho})) \rightarrow \mathbf{G}(H_0(\hat{\rho}))$ as $g \rightarrow 0$. We define $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ with F, G, F', G', F'' and G'' in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ replaced by zero, respectively. For $P = E_{H_{cl}}([\mathbf{G}[H_{cl}], \Sigma]) \otimes E_{H_{EM}}(\{0\})$, from (4.41) it follows that

$$(\tilde{\Omega}, P\tilde{\Omega})_{\mathcal{H}} \geq 1 - \left\{ \frac{6e\|\hat{\rho}\|_{-3} \left(\sqrt{\frac{\tilde{\mathcal{A}}(\tilde{\mathcal{B}}+1)}{2}} + \frac{e\|\hat{\rho}\|_{-1}}{2} \right)}{1 - 6e^2\|\hat{\rho}\|_{-3}\|\hat{\rho}\|_{-1}} \right\}^2 - \left(\frac{\tilde{\mathcal{C}}}{\Sigma - \mathbf{G}(H_0(\hat{\rho}))} \right)^2 > 0.$$

Hence $\tilde{\Omega} \neq 0$. Then it follows that $\tilde{\Omega}$ is a ground state of $H_0(\hat{\rho})$. Thus (4.42) follows. $\square$

Remark 4.19 *The uniqueness of the ground state of $H_0(\hat{\rho})$ is studied in [17].*

5 GROUND STATE ENERGY

In this section, we shall study the ground state energy of $H_0(\hat{\rho})$, $\inf \sigma(H_0(\hat{\rho}))$, where $H_0(\hat{\rho}) = H_{g=0}(\hat{\rho})$. In order to do so, we apply functional integral representation.

5.1 Functional Integral Representation

It is too involved a subject to be treated here in detail. Only a bare general sketch is given of functional integrations. See [15]. We take Schrödinger representation [10,15,21] of $\mathcal{F}_{EM}$. Define

$$h = \left\{ f \in \mathcal{S}'_{real}(\mathbf{R}^3) \left| \int_{\mathbf{R}^3} \frac{|\hat{f}(k)|^2}{|k|} dk < \infty \right. \right\},$$

$$h_E = \left\{ f \in \mathcal{S}'_{real}(\mathbf{R}^3 \times \mathbf{R}) \left| 2 \int_{\mathbf{R}^3 \times \mathbf{R}} \frac{|\hat{f}(k, k_0)|^2}{|k|^2 + |k_0|^2} dk dk_0 < \infty \right. \right\}.$$

Here $\mathcal{S}'_{real}(\mathbf{R}^n)$, $n = 3, 4$, denote the sets of real tempered distributions on $\mathbf{R}^n$. Put

$$\tilde{h} = h \oplus h \oplus h,$$

$$\tilde{h}_E = h_E \oplus h_E \oplus h_E.$$

which are equipped with the bilinear forms

$$\begin{aligned}(f, g)_{\tilde{h}} &= \sum_{\mu, \nu=1}^3 \int_{\mathbf{R}^3} \frac{d_{\mu\nu}(k) \overline{\hat{f}_\mu(k)} \hat{g}_\nu(k)}{|k|} dk, \\ (F, G)_{\tilde{h}_E} &= 2 \sum_{\mu, \nu=1}^3 \int_{\mathbf{R}^3 \times \mathbf{R}} \frac{d_{\mu\nu}(k) \overline{\hat{F}_\mu(k, k_0)} \hat{G}_\nu(k, k_0)}{|k|^2 + |k_0|^2} dk dk_0,\end{aligned}$$

where $f = f_1 \oplus f_2 \oplus f_3$, $g = g_1 \oplus g_2 \oplus g_3$, $f_\mu, g_\mu \in h$ and $F = F_1 \oplus F_2 \oplus F_3$, $G = G_1 \oplus G_2 \oplus G_3$, $F_\mu, G_\mu \in h_E$, $\mu = 1, 2, 3$, respectively, and

$$d_{\mu\nu}(k) \equiv \sum_{r=1}^3 e_\mu^r(k) e_\nu^r(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}, \mu, \nu = 1, 2, 3.$$

Then we define

$$\begin{aligned}\tilde{F} &= \tilde{h}/\mathcal{K}, \\ \tilde{E} &= \tilde{h}_E/\mathcal{K}_E.\end{aligned}$$

Here the sets $\mathcal{K}$ and $\mathcal{K}_E$ are the kernels:

$$\begin{aligned}\mathcal{K} &= \{f \in \tilde{h} | (f, f)_{\tilde{h}} = 0\}, \\ \mathcal{K}_E &= \{F \in \tilde{h}_E | (F, F)_{\tilde{h}_E} = 0\}.\end{aligned}$$

We denote the equivalence classes of $f \in \tilde{h}$ and $F \in \tilde{h}_E$ by $[f]$ and $[F]_E$, respectively. The completions of $\tilde{F}$ and $\tilde{E}$ with respect to the scalar products

$$\begin{aligned}([f], [g])_{\tilde{F}} &\equiv (f, g)_{\tilde{h}}, \\ ([F]_E, [G]_E)_{\tilde{E}} &\equiv (F, G)_{\tilde{h}_E}\end{aligned}$$

are written as the same symbols, respectively. Let $\{\phi([f]) | f \in \tilde{h}\}$ be the Gaussian mean zero random process indexed by $\tilde{F}$ so that the characteristic function is given by

$$\int_{\tilde{Q}} e^{i\phi([f])} d\tilde{\mu} = e^{-\frac{1}{4}\|f\|_{\tilde{h}}^2}, \quad f \in \tilde{h},$$

where $(\tilde{Q}, d\tilde{\mu})$ denotes an underlying probability measure space.

Let $\{\psi([F]_E) | F \in \tilde{h}_E\}$ be the Gaussian mean zero random process indexed by $\tilde{E}$ so that the characteristic function is given by

$$\int_{\tilde{Q}_E} e^{i\psi([F]_E)} d\tilde{\mu}_E = e^{-\frac{1}{4}\|F\|_{\tilde{h}_E}^2}, \quad F \in \tilde{h}_E,$$

where $(\tilde{Q}_E, d\tilde{\mu}_E)$ denotes an underlying probability measure space. Let $F = L^2(\tilde{Q}, d\tilde{\mu})$ and $E = L^2(\tilde{Q}_E, d\tilde{\mu}_E)$.

Remark 5.1 *Actually, we can take $\tilde{Q}$ and $\tilde{Q}_E$ concretely. However we do not mention about it. See [10,13].*

We see that

$$\begin{aligned} F &= \bigoplus_{n=0}^{\infty} \Gamma_n, \\ \Gamma_n &= \overline{\mathcal{L}\{:\phi([f_1])\dots\phi([f_n]): | f_j \in \tilde{h}, j=1, \dots, n\}}, n \geq 1, \Gamma_0 = \mathbf{C}, \\ E &= \bigoplus_{n=0}^{\infty} \Gamma_{En}, \\ \Gamma_{En} &= \overline{\mathcal{L}\{:\psi([F_1]_E)\dots\psi([F_n]_E):_E | F_j \in \tilde{h}_E, j=1, \dots, n\}}, n \geq 1, \Gamma_{E0} = \mathbf{C}, \end{aligned}$$

where $::$ and $::_E$ denote the Wick products [22,p.9] in F and E , respectively. We set $\Omega_F = 1$ in F and $\Omega_E = 1$ in E . Put

$$F^\infty = \bigcup_{N=0}^{\infty} \{\oplus_{n=0}^N \Gamma_n\}.$$

We define an operator $\tilde{\omega} : h \rightarrow h$ by

$$\widehat{\tilde{\omega}f}(k) = \omega(k)\hat{f}(k),$$

with

$$D(\tilde{\omega}) = \left\{ f \in h \left| \int_{\mathbf{R}^3} \frac{|\omega(k)\hat{f}(k)|^2}{|k|} dk < \infty \right. \right\}$$

and ω_F by

$$\omega_F[f] = [(\tilde{\omega} \oplus \tilde{\omega} \oplus \tilde{\omega})f], f \in D(\tilde{\omega}) \oplus D(\tilde{\omega}) \oplus D(\tilde{\omega}).$$

Note that ω_F leaves $[D(\tilde{\omega}) \oplus D(\tilde{\omega}) \oplus D(\tilde{\omega})] \cap \mathcal{K}$ invariant. We define an operator H_F by

$$H_F : \phi([f_1]) \dots \phi([f_n]) := \sum_{j=1}^n : \phi([f_1]) \dots \phi(\omega_F[f_j]) \dots \phi([f_n]) :, \\ f_j \in D(\tilde{\omega}) \oplus D(\tilde{\omega}) \oplus D(\tilde{\omega}), j = 1, \dots, n.$$

We extend H_F by linearity. Then H_F is closable and denote its closed extension by the same symbol. We put, for $\rho \in h$, and $x \in \mathbf{R}^3$,

$$\rho(\cdot - x) = (e^{-i(\cdot, x)} \hat{\rho}(\cdot))^{\vee},$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product and $f^{\vee}$ the inverse Fourier transform of f in the sense of distribution. Moreover we put

$$\rho_1(x) = \rho(\cdot - x) \oplus 0 \oplus 0,$$

$$\rho_2(x) = 0 \oplus \rho(\cdot - x) \oplus 0,$$

$$\rho_3(x) = 0 \oplus 0 \oplus \rho(\cdot - x).$$

Set

$$A_{\mu}^F(\hat{\rho}, x) \equiv \phi([\rho_{\mu}(x)]).$$

The operator $A_{\mu}^F(\hat{\rho}, x)$ is closable on F^{∞} . We denote its closed extension by the same symbol.

We define U by

$$U\Omega_E = \Omega_F,$$

$$UA_{\mu_1}(\hat{\rho}_1, 0) \dots A_{\mu_n}(\hat{\rho}_n, 0)\Omega_{EM} \equiv A_{\mu_1}^F(\hat{\rho}_1, 0) \dots A_{\mu_n}^F(\hat{\rho}_n, 0),$$

where $\rho_j \in h, \mu_j = 1, 2, 3, j = 1, \dots, n$, and U extends by linearity. Then the operator U extends to a unitary operator and denote its unitary extension by the same symbol. We have the following lemma:

Lemma 5.2 ([15, Theorem 3.1]) *The unitary operator U satisfies that (1) U maps $\otimes_s^n W$ to Γ_n , $n \geq 1$. (2) U maps $D(A_\mu(\hat{\rho}, x))$ to $D(A_\mu^F(\hat{\rho}, x))$ with $A_\mu(\hat{\rho}, x) = U^{-1}A_\mu^F(\hat{\rho}, x)U$, $\rho \in h, x \in \mathbf{R}^3$, $\mu = 1, 2, 3$, (3) U maps $D(H_{EM})$ to $D(H_F)$ with $H_{EM} = U^{-1}H_FU$.*

Set

$$\mathcal{H}_F = L^2(\mathbf{R}^3) \otimes F.$$

An operator $H_F(\hat{\rho})$ in $\mathcal{H}_F$ is defined by

$$H_F(\hat{\rho}) = \frac{1}{2} \left(P \otimes I_F - eA_\mu^F(\hat{\rho}, x) \right)^2 + V \otimes I_F + I_{el} \otimes H_F,$$

where I_F denotes the identity operator in F and $A^F(\hat{\rho}, x) = (A_1^F(\hat{\rho}, x), A_2^F(\hat{\rho}, x), A_3^F(\hat{\rho}, x))$.

Lemma 5.2 implies the following lemma:

Lemma 5.3 *We assume that Hypothesis 1, 2 and 3 hold. Then $H_F(\hat{\rho})$ is self-adjoint on $D(H_{el} \otimes I_F) \cap D(I_{el} \otimes H_F)$, bounded from below and essentially self-adjoint on any core for $H_{el} \otimes I_F + I_{el} \otimes H_F$.*

Proof: By Lemma 5.2, the inequalities (3.10), (3.11) and (3.12) with $A_\mu(\hat{\rho}, x)$ and H_{EM} replaced by $A_\mu^F(\hat{\rho}, x)$ and H_F hold. Thus lemma follows. $\square$

Remark 5.4 *In the case where particle has no spin, we can take e_0 in Hypothesis 3 larger than that of the case where particle has spin. However, avoiding confusion, we do not mention of it explicitly.*

It is straightforward to see that U maps $\mathcal{F}_{EM}^\infty \cap D(H_{EM})$ onto $F^\infty \cap D(H_F)$ with

$$H_0(\hat{\rho})|_{C_0^\infty(\mathbf{R}^3) \widehat{\otimes} \mathcal{F}_{EM}^\infty \cap D(H_{EM})} = \mathcal{U}^{-1} H_F(\hat{\rho}) \mathcal{U}|_{C_0^\infty(\mathbf{R}^3) \widehat{\otimes} \mathcal{F}_{EM}^\infty \cap D(H_{EM})},$$

where

$$\mathcal{U} = \int_{\mathbf{R}^3}^\oplus U dx.$$

Hence Lemma 5.2 yields that $\mathcal{U}$ maps $D(H_0(\hat{\rho}))$ onto $D(H_F(\hat{\rho}))$ with

$$H_0(\hat{\rho}) = \mathcal{U}^{-1} H_F(\hat{\rho}) \mathcal{U}. \quad (5.1)$$

We define a family of operators $\{j_t\}_{t \geq 0}$ from h to h_E as follows:

$$j_t f = \delta_t \otimes f,$$

where δ_t is the delta function with mass at $t \in \mathbf{R}$. Let

$$\mathbf{j}_t[f] = [(j_t \oplus j_t \oplus j_t)f]_E, f \in \tilde{h}.$$

It is seen that $\mathbf{j}_t, t \geq 0$, are isometries from $\tilde{F}$ to $\tilde{E}$ and

$$\mathbf{j}_t^* \mathbf{j}_s = e^{-|t-s|\omega_F}, t, s \geq 0. \quad (5.2)$$

Then the family of isometries $\{J_t\}_{t \geq 0}$ from F to E is defined by

$$J_t : \phi([F_1]) \dots \phi([F_n]) :=: \psi(\mathbf{j}_t[F_1]) \dots \psi(\mathbf{j}_t[F_n]) :_E,$$

$$J_t \Omega_F = \Omega_E.$$

From (5.2) it follows that

$$J_t^* J_s = e^{-|t-s|H_F}, t, s \geq 0. \quad (5.3)$$

We shall prepare probabilistic notations. Let (M, db) be a probability space and $\mathbf{b}(b, t) = \mathbf{b}(t) = (b_\mu(t))_{\mu=1,2,3}$ the three dimensional Brownian motion on (M, db) starting at the origin.

We introduce Hypothesis 7 as follows:

Hypothesis 7 External potential $V = V_+ - V_-$, where V_+ denotes the positive part and V_- the nonpositive part, satisfies that $V_+ \in L^1_{loc}(\mathbf{R}^3)$ and V_- is infinitesimally small with respect to the Laplacian Δ in $L^2(\mathbf{R}^3)$.

By functional integral representation [23], Hypothesis 7 implies that the semigroup $e^{-tH_{el}}$, $H_{el} = -\frac{1}{2}\Delta + V_+ - V_-$, where $\pm$ denotes form sum, is positivity improving [20, XIII.12]. Hence, if H_{el} has a ground state, then the ground state is unique and strictly positive [20, Theorem XIII.44]. We set, for $x \in \mathbf{R}^3$,

$$J_n^\mu(\rho, x, t) = \sum_{k=1}^{2^n} \mathbf{j}_{\frac{k-1}{2^n}t} \left[\rho_\mu \left(x + \mathbf{b} \left(\frac{k-1}{2^n}t \right) \right) \right] \left\{ b_\mu \left(\frac{k}{2^n}t \right) - b_\mu \left(\frac{k-1}{2^n}t \right) \right\}, \mu = 1, 2, 3.$$

The condition $\hat{\rho} \in \mathcal{M}_{-1} \cap \mathcal{M}_1$ implies that $\mathbf{j}_s[\rho_\mu(x+u)] = f(s, u) = f \in C_b^1(\mathbf{R} \times \mathbf{R}^3; \tilde{h}_E)$, the set of $\tilde{h}_E$ -valued strongly continuously differentiable bounded (up to the differential functions) functions on $\mathbf{R} \times \mathbf{R}^3$. Thus one can see that, for $x \in \mathbf{R}^3$ and $\hat{\rho} \in \mathcal{M}_{-1} \cap \mathcal{M}_1$, $J_n^\mu(\rho, x, t)$ is a Cauchy sequence in $L^2(M, \tilde{h}_E)$, the $\tilde{h}_E$ -valued L^2 -space over M . We write the strong limit of $J_n^\mu(\rho, x, t)$ as $n \rightarrow \infty$ in $L^2(M, \tilde{h}_E)$ as follows:

$$s - \lim_{n \rightarrow \infty} J_n^\mu(\rho, x, t) \equiv \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s).$$

Proposition 5.5 *We assume that Hypothesis 1, 2, 3 and 7 hold. Then*

$$\begin{aligned} & (F, e^{-tH_F(\hat{\rho})}G)_{\mathcal{H}_F} \\ &= \int_{\mathbf{R}^3 \times M} e^{-\int_0^t V(x+\mathbf{b}(s))ds} \left(J_t F(x + \mathbf{b}(t)), e^{ie\psi \left(\sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_\mu(x+\mathbf{b}(s))] db_\mu(s) \right)} J_0 G(x) \right)_E dx db. \end{aligned} \quad (5.4)$$

Proof: In addition to Hypothesis 1, 2, 3 and 7, we assume that $\hat{\rho} \in \mathcal{M}_3$. Then $\mathbf{j}_s[\rho_\mu(x+u)] = f(s, u) = f \in C_b^2(\mathbf{R} \times \mathbf{R}^3; \tilde{h}_E)$, the set of $\tilde{h}_E$ -valued two times strongly continuously differentiable bounded (up to the two times differential functions) functions on $\mathbf{R} \times \mathbf{R}^3$. Then, by [15, Theorem 4.10], (5.4) holds. For $\hat{\rho} \notin \mathcal{M}_3$, we take a sequence $\hat{\rho}_N, N = 1, 2, \dots$, so that $\|\hat{\rho}_N - \hat{\rho}\|_n \rightarrow 0$ as $N \rightarrow \infty, n = -2, -1, 0, 1, 3$. Then we have

$$\begin{aligned} & (F, e^{-tH_F(\hat{\rho}_N)}G)_{\mathcal{H}_F} \\ &= \int_{\mathbf{R}^3 \times M} e^{-\int_0^t V(x+\mathbf{b}(s))ds} \left(J_t F(x + \mathbf{b}(t)), e^{ie\psi \left(\sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_{N\mu}(x+\mathbf{b}(s))] db_\mu(s) \right)} J_0 G(x) \right)_E dx db. \end{aligned} \quad (5.5)$$

Since

$$\begin{aligned}
& \left\| \sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) - \sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_{N_\mu}(x + \mathbf{b}(s))] db_\mu(s) \right\|_{L^2(M; \tilde{h}_E)}^2 \\
&= \sum_{\mu=1}^3 \int_M \left(\int_0^t ds \|\mathbf{j}_s([\rho_\mu(x + \mathbf{b}(s))] - [\rho_{N_\mu}(x + \mathbf{b}(s))])\|_{\tilde{h}_E}^2 \right) db \\
&\leq \sum_{\mu=1}^3 \int_M \left(\int_0^t ds 3 \|\hat{\rho} - \hat{\rho}_N\|_{-1}^2 \right) db \\
&\leq 9t \|\hat{\rho} - \hat{\rho}_N\|_{-1}^2,
\end{aligned}$$

we can take a subsequence N_k so that, for almost everywhere $(x, b) \in \mathbf{R}^3 \times M$,

$$\sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) \rightarrow \sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_{N_k\mu}(x + \mathbf{b}(s))] db_\mu(s),$$

as $k \rightarrow \infty$ strongly in $\tilde{h}_E$. Then by the Lebesgue dominated convergence theorem, the right hand side of (5.5) with N replace by N_k converges to the right hand side of (5.4) as $k \rightarrow \infty$. On the other hand, since $H_F(\hat{\rho}_{N_k})$ has common core, e.g., $C_0^\infty(\mathbf{R}^3) \hat{\otimes} [F^\infty \cap D(H_F)]$, we see that the left hand side of (5.5) with N replaced by N_k converges to the left hand side of (5.4) as $k \rightarrow \infty$. Thus (5.4) follows. $\square$

5.2 Ground State Energy

In this subsection we shall give an expression of $\inf \sigma(H_0(\hat{\rho}))$ by the functional integral representation given in the previous subsection. We have the following theorem:

Theorem 5.6 *We assume that Hypothesis 1,2,3,4,5,6 and 7 hold. Let Φ be the strictly positive unique ground state of H_{el} . Then*

$$\begin{aligned}
\inf \sigma(H_0(\hat{\rho})) &= \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} \log \int_{\mathbf{R}^3 \times M} dx db \Phi(x + \mathbf{b}(\beta)) \Phi(x) e^{-\int_0^\beta V(x + \mathbf{b}(s)) ds} \\
&\quad \times e^{-\frac{c^2}{4} \left\| \sum_{\mu=1}^3 \int_0^\beta \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) \right\|_{\tilde{h}_E}^2}.
\end{aligned} \tag{5.6}$$

Proof: By (4.41), there exists a ground state of $H_0(\hat{\rho})$, $\Omega(0)$, so that $(\Phi \otimes \Omega_{EM}, \Omega(0))_{\mathcal{H}_F} \neq 0$.

Hence

$$\inf \sigma(H_0(\hat{\rho})) = \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} \log \left(\Phi \otimes \Omega_{EM}, e^{-\beta H_0(\hat{\rho})} \Phi \otimes \Omega_{EM} \right)_{\mathcal{H}}.$$

Owing to (5.1) and (5.4), we see that

$$\begin{aligned} & \left(\Phi \otimes \Omega_{EM}, e^{-\beta H_0(\hat{\rho})} \Phi \otimes \Omega_{EM} \right)_{\mathcal{H}} \\ &= \left(\Phi \otimes \Omega_F, e^{-\beta H_F(\hat{\rho})} \Phi \otimes \Omega_F \right)_{\mathcal{H}_F} \\ &= \int_{\mathbf{R}^3 \times M} dx db e^{-\int_0^\beta V(x+\mathbf{b}(s)) ds} \Phi(x + \mathbf{b}(\beta)) \Phi(x) \\ & \quad \times \left(J_\beta \Omega_F, e^{ie\Psi\left(\sum_{\mu=1}^3 \int_0^\beta \mathbf{j}_s[\rho_\mu(x+\mathbf{b}(s))] db_\mu(s)\right)} J_0 \Omega_F \right)_E \\ &= \int_{\mathbf{R}^3 \times M} dx db \Phi(x + \mathbf{b}(\beta)) \Phi(x) e^{-\int_0^\beta V(x+\mathbf{b}(s)) ds} \int_{\tilde{Q}_E} e^{ie\Psi\left(\sum_{\mu=1}^3 \int_0^\beta \mathbf{j}_s[\rho_\mu(x+\mathbf{b}(s))] db_\mu(s)\right)} d\tilde{\mu}_E \\ &= \int_{\mathbf{R}^3 \times M} dx db \Phi(x + \mathbf{b}(\beta)) \Phi(x) e^{-\int_0^\beta V(x+\mathbf{b}(s)) ds} e^{-\frac{\epsilon^2}{4} \left\| \sum_{\mu=1}^3 \int_0^\beta \mathbf{j}_s[\rho_\mu(x+\mathbf{b}(s))] db_\mu(s) \right\|_{\tilde{h}_E}^2}. \end{aligned}$$

Thus theorem follows. $\square$

We shall give a remark on the term $\left\| \sum_{\mu=1}^3 \int_0^\beta \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) \right\|_{\tilde{h}_E}^2$ in (5.6). It follows from (5.3) that

$$\begin{aligned} \left(J_t A_\mu^F(\hat{\rho}, x), J_s A_\nu^F(\hat{\rho}, y) \right)_{\tilde{Q}_E} &= \frac{1}{2} \int_{\mathbf{R}^3} \frac{d_{\mu\nu}(k) |\hat{\rho}(k)|^2}{|k|} e^{-ik(x-y)} e^{-|t-s||k|} dk \\ &\equiv D_{\mu\nu}(x-y, t-s). \end{aligned}$$

By the definition of $\int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s)$, taking a subsequence n' in the set of natural numbers, for almost surely $b \in M$, we see that

$$\begin{aligned} & \sum_{k=1}^{2^{n'}} \mathbf{j}_{\frac{k-1}{2^{n'}} t} \left[\rho_\mu \left(x + \mathbf{b} \left(\frac{k-1}{2^{n'}} t \right) \right) \right] \left\{ b_\mu \left(\frac{k}{2^{n'}} t \right) - b_\mu \left(\frac{k-1}{2^{n'}} t \right) \right\} \\ & \longrightarrow \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s), \end{aligned}$$

as $n' \rightarrow \infty$ strongly in $\tilde{h}_E$. Then it follows that for almost surely $b \in M$, taking subsequences n' and m' in the set of natural numbers,

we see that

$$\begin{aligned}
& \left\| \sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) \right\|_{\tilde{h}_E}^2 \\
&= \lim_{m' \rightarrow \infty} \lim_{n' \rightarrow \infty} \sum_{\mu, \nu=1}^3 \sum_{k=1}^{2^{n'}} \sum_{l=1}^{2^{m'}} \\
&\quad \times \left(\mathbf{j}_{\frac{k-1}{2^{n'}}t} \left[\rho_\mu \left(x + \mathbf{b} \left(\frac{k-1}{2^{n'}}t \right) \right) \right], \mathbf{j}_{\frac{l-1}{2^{m'}}t} \left[\rho_\nu \left(x + \mathbf{b} \left(\frac{l-1}{2^{m'}}t \right) \right) \right] \right)_{\tilde{h}_E} \\
&\quad \times \left\{ b_\nu \left(\frac{l}{2^{m'}}t \right) - b_\nu \left(\frac{l-1}{2^{m'}}t \right) \right\} \left\{ b_\mu \left(\frac{k}{2^{n'}}t \right) - b_\mu \left(\frac{k-1}{2^{n'}}t \right) \right\} \\
&= \lim_{m' \rightarrow \infty} \lim_{n' \rightarrow \infty} \sum_{\mu, \nu=1}^3 \sum_{k=1}^{2^{n'}} \sum_{l=1}^{2^{m'}} \\
&\quad \times \int_{\mathbf{R}^3} \frac{d_{\mu\nu}(k) |\hat{\rho}(k)|^2 \exp(-ik(\mathbf{b}(\frac{l-1}{2^{m'}}t) - \mathbf{b}(\frac{k-1}{2^{n'}}t))) \exp(-|\frac{l-1}{2^{m'}}t - \frac{k-1}{2^{n'}}t| |k|)}{|k|} dk \\
&\quad \times \left\{ b_\nu \left(\frac{l}{2^{m'}}t \right) - b_\nu \left(\frac{l-1}{2^{m'}}t \right) \right\} \left\{ b_\mu \left(\frac{k}{2^{n'}}t \right) - b_\mu \left(\frac{k-1}{2^{n'}}t \right) \right\} \\
&= \lim_{m' \rightarrow \infty} \lim_{n' \rightarrow \infty} \sum_{\mu, \nu=1}^3 \sum_{k=1}^{2^{n'}} \sum_{l=1}^{2^{m'}} D_{\mu\nu} \left(\mathbf{b} \left(\frac{l-1}{2^{m'}}t \right) - \mathbf{b} \left(\frac{k-1}{2^{n'}}t \right), \frac{l-1}{2^{m'}}t - \frac{k-1}{2^{n'}}t \right) \\
&\quad \times \left\{ b_\nu \left(\frac{l}{2^{m'}}t \right) - b_\nu \left(\frac{l-1}{2^{m'}}t \right) \right\} \left\{ b_\mu \left(\frac{k}{2^{n'}}t \right) - b_\mu \left(\frac{k-1}{2^{n'}}t \right) \right\}.
\end{aligned}$$

Thus we may “formally” put as follows:

$$\begin{aligned}
& \left\| \sum_{\mu=1}^3 \int_0^t \mathbf{j}_s[\rho_\mu(x + \mathbf{b}(s))] db_\mu(s) \right\|_{\tilde{h}_E}^2 \\
&= 2 \sum_{\mu, \nu=1}^3 \int_0^t \left(\int_0^t D_{\mu\nu}(\mathbf{b}(s) - \mathbf{b}(s'), s - s') db_\mu(s') \right) db_\nu(s). \quad (5.7)
\end{aligned}$$

Remark 5.7 (1) Double stochastic integral in the right hand side of (5.7) is pointed out in [25] and studied in [14]. (2) In the case of $\Sigma = \infty$, we do not need Hypothesis 4 and 6, i.e., in Lemmas 4.4 and 4.12, we assume that Hypothesis 1,2 and 3 hold, in Lemmas 4.14, 4.16, Theorems 4.17 and 4.18, we assume that Hypothesis 1,2,3 and 5 hold, and in Theorem 5.6, we assume that Hypothesis 1,2,3,5 and 7 hold.

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