Categorical aspects of generating functions (II) : Operations on categories and functors

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Categorical aspects of generating functions (II): Operations on categories and functors.

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Contents.

Abstract. This paper is a sequel to the author's paper [Yos 98]. We construct some operations on categories and functors, for example, summation, multiplication, derivation, partial derivation, exponentiation, substitution. These operations are correspond to those of species. The main results of this paper are two theorems (Theorem 5.8, 6.5) which characterizing strict Krull-Schmidt categories(KS-categories). Especially, Theorem 6.5 states that for a skeletally small and locally finite category $\mathcal{E}$ with finite coproducts and its full subcategory $\mathcal{C}$, if the exponential formula $\mathcal{E}(t) = \exp(\mathcal{C}(t))$ implies, under a technical assumption, that $\mathcal{E}$ is a strict Krull-Schmidt category. Furthermore, we define the substitution $T(S)$ of functors $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$, where $\mathcal{S}$ has finite coproducts. These operations satisfy formulas similar to such operations on the usual power series. There are some applications.

1 Introduction

In Part I ([Yos 98]) of this series of papers, we introduced the generating functions of categories and functors, and then we show that it covers the theory of Joyal's species ([Joy 81], [BerLL 98]) as a special case (faithful functors from groupoids to the category of finite sets). The generating function of a (skeletally small and locally finite) category $\mathcal{E}$ is defined by

$$\mathcal{E}(t) := \sum_{X \in \mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(X)|} t^X. \tag{1.0.a}$$

Furthermore, the generating function of a faithful functor (with locally finite fibers) $F : \mathcal{E} \longrightarrow \mathcal{S}$ is

$$F(t) := \sum_{X \in \mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(X)|} t^{F(X)}. \tag{1.0.b}$$

See Section 4 of Part I.

Both of power series and species are accompanied by some operations (e.g. addition, multiplication, derivation, composition). In this paper (Section 3), we extend several operations related with species or power series to those related to categories and functors. For example, the derivation S^* of a functor $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ is defined to be a functor

$$S^* : \mathsf{Elts}(S) \longrightarrow \mathsf{Set}_f; (X, s) \longmapsto S(X),$$

where $\mathsf{Elts}(S)$ is the category of elements and its object is a pair of object X of $\mathcal{E}$ and an element s of $S(X)$ (see Part I of this series or [Bor 94, I, Definition 1.6.4]). Taking the generating function of S, we know that

$$S^*(t) = t\,\frac{d}{dt}S(t),$$

where we identify a power series with exponents in Set_f as a usual power series in one variable t by the identification $t^N = t^{|N|}$. Furthermore, if $\boldsymbol{A}_S$ is the species associated to S, then the $*$-derivation of $\boldsymbol{A}_S$ is the species $\boldsymbol{A}_{S^*}$ associated to the functor S^*. If $\mathcal{S}$ is a set-like category, then the product law and the composition law:

$$(S \cdot T)^* \cong S^* \cdot T + S \cdot T^*, \; (T(S))^* \cdot S \cong S^* \cdot T^*(S).$$

See (3.9) and

Another important operation on categories and functors is the exponentiations $\mathsf{EXP}(\mathcal{C}), \mathsf{EXP}(S)$ (Section 5). They satisfy the expected relations as follows:

$$(\mathsf{EXP}(\mathcal{C}))(t) = \exp(\mathcal{C}(t)), \ (\mathsf{EXP}(S))(t) = \exp(S(t)).$$

It is interesting that $\mathsf{EXP}(\mathcal{C})$ satisfies the Dress condition, that is, in the following commutative diagram in which the bottom row is a coproduct diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] & B \arrow[l] \arrow[d] \\ X \arrow[r] & X+Y & Z, \arrow[l] \end{tikzcd}$$

the top row is a coproduct diagram if and only if the two squares are both pullback squares. The following theorem characterizes the exponentiation of a category:

Theorem 5.8. *Let $\mathcal{E}$ be a category with finite coproducts and $\mathcal{C}$ a full subcategory of $\mathcal{E}$. Then $\mathcal{E}$ is equivalent to* $\mathsf{EXP}(\mathcal{C})$ *via the functor* $\coprod : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{E}$ (see 5.3.b) *if and only if the following two conditions hold/:*

(a) *Any object of $\mathcal{E}$ is isomorphic to a coproduct of some finitely many objects of $\mathcal{C}$.*
(b) *For each $I \in \mathcal{C}$, the hom-functor*

$$\mathrm{Hom}(I, -) : \mathcal{E} \longrightarrow \mathsf{Set}; X \longmapsto \mathrm{Hom}(I, X)$$

preserves finite coproducts.

The condition (c) can be replaced by the existence of pullbacks and the Dress condition. As is shown in Part I, the exponential formula holds for the generating function of a KS-category $\mathcal{E}$:

$$\mathcal{E}(t) = \exp(\mathsf{Con}(\mathcal{E})(t)).$$

Here a KS-category is a category with finite coproducts satisfying the condition that each object has a unique decomposition into a finite coproduct of connected (indecomposable) objects in the strict sense (Part I, Section 5.5). Using the above theorem, we can prove its inverse holds as follows:

Theorem 6.5. *Let $\mathcal{E}$ be a skeletally small and locally finite category with finite coproducts and $\mathcal{C}$ a full subcategory of $\mathcal{E}$. Then $\mathcal{E}$ is a KS-category* *with* $\mathsf{Con}(\mathcal{E})$ *equivalent to* $\mathcal{C}$ *over* $\mathcal{E}$ *if and only if the following two conditions hold/*:

(a) $\mathcal{E}(t) = \exp(\mathcal{C}(t))$.
(b) *If* $i \circ \sigma = i$ *for any canonical injection* $i : I \longrightarrow I + A$ *with* $I \in \mathcal{C}$ *and* $A \in \mathcal{E}$ *and for an automorphism* σ *of* I, *then* $\sigma = 1$.

In this paper, we could not sufficiently study the composition of two power series in our sense. Let $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T : \mathcal{F} \longrightarrow \mathcal{T}$ be functors where $\mathcal{S}$ and $\mathcal{T}$ are categories with finite coproducts. We want to define the composition $T(S)$ satisfying $T(S)(t) = T(S(t))$. In this paper, we define it only in the case where $\mathcal{T}$ is Set_f (or $\mathsf{Set}_f{}^n$), so that $T(t)$ is viewed as a usual power series in one variable (resp. multivariables. This functor $T(S)$ is constructed by the pullback of $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ and the rank functor $Rank : \mathsf{EXP}(\mathcal{E}) \longrightarrow \mathsf{Set}_f$. This construction is essentially same as in species case. Of course, our construction has broader scope of application.

In Section 8, we give some applications. For example, we answer Stanley's question which asked the generating functions of the numbers $M(n,h,m)$ of solutions $\alpha^{m+h} = \alpha^h$ in the symmetric monoid End$([n])$ ([Sta 78, Example 6.9]). We furthermore study some Cayley type formulas for labeled rooted trees.

It still remains to study some important subjects about generating functions of categories and functors. The first one is the construction of "plethysm composition"of categories and functors. It is difficult to define a kind of plethysm composition for functors (or power series with exponents in categories) at the present stage except for some very special cases. For example, the $(\sum{}' X \in \mathcal{E} c_X\, t^X)^N$ $(N \in \mathcal{E})$ is meanningless in general, but it has a meaning for the category of surjections between finite sets by the plethysm composition ([Nav 87]). To define the plethysm ccomposition for wider classes of categories, we need to take a locally finite toposes as an underlying category and to extend the concept of polynomials and power series to those with coefficients in a Tambara functor (especially the Burnside ring functor) [Tam93]. We will study such generalizations in Part 3 of this series. Other remaining subjects are the categorical versions of weighted species, virtual species, the Lagrange inversion formula ([BerLL 98, Capter 2, 3]), which we will study in other papers.

2 Preliminaries

(2.1) Operations on species. As in the first paper, we denote by Bij_f (resp. Set_f) the category of finite sets and bijections (resp. mappings). Then a *species* is a functor from Bij_f to Set_f and a morphism between two species is a natural transformation. The category $[\mathsf{Bij}_f, \mathsf{Set}_f]$ of species is equivalent to the product category of the categories of finite S_n-sets, $n = 0, 1, 2, \ldots$. Thus the category $[\mathsf{Bij}, \mathsf{Set}_f]$ is a Cartesian closed category, that is, it possesses all finite colimits, finite limits and internal hom-objects Y^X called exponentiations. (Note that $[\mathsf{Bij}, \mathsf{Set}_f]$ is not a topos (4.5) because of the lack of a subobject classifier; and note that it is also not locally finite.) The Cartesian closeness as well as finite colimits and finite limits gives many unary and binary operations on the category of species.

(a) The *sum*(coproduct) $\boldsymbol{A} + \boldsymbol{B}$ of two species $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by

$$(\boldsymbol{A} + \boldsymbol{B})[E] = \boldsymbol{A}[E] + \boldsymbol{B}[E] \tag{2.1.a}$$

where the sum is a disjoint union.
(b) The (tensor or convolutional) *product* $\boldsymbol{A} \cdot \boldsymbol{B}$ of two species $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by

$$(\boldsymbol{A} \cdot \boldsymbol{B})[E] = \sum_{E_1 + E_2 = E} \boldsymbol{A}[E_1] \times \boldsymbol{B}[E_2], \tag{2.1.b}$$

where the summation is taken over all pairs (E_1, E_2) of subsets of E such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$. This product makes $[\mathsf{Bij}, \mathsf{Set}_f]$ a monoidal category, but it is not a categorical product in the categories of species.
(c) The *substitution* (or *composition*) $\boldsymbol{A}(\boldsymbol{B})$ of two species $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\boldsymbol{B}[\emptyset] = \emptyset$ is defined by

$$\boldsymbol{A}(\boldsymbol{B})[E] = \coprod_{\pi \in \Pi(E)} \left(\boldsymbol{A}[\pi] \times \prod_{B \in \pi} \boldsymbol{B}[B] \right), \tag{2.1.c}$$

where $\pi = \{B_1, B_2, \ldots\}$ runs over all partition of the set E.
(d) The *derivation* $\boldsymbol{A}'$ of a species $\boldsymbol{A}$ is defined by

$$\boldsymbol{A}'[E] = \boldsymbol{A}[E \cup \{E\}]. \tag{2.1.d}$$

(e) The *Cartesian product* and the *exponential* of two species $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined by

$$\begin{aligned} (\boldsymbol{A} \times \boldsymbol{B})[E] &= \boldsymbol{A}[E] \times \boldsymbol{B}[E], & \text{(2.1.e)} \\ (\boldsymbol{B}^{\mathbf{A}})[E] &= \mathrm{Map}(\boldsymbol{A}[E], \boldsymbol{B}[E]), & \text{(2.1.f)} \end{aligned}$$

where Map denotes the set of mappings of sets. Then we have an adjoint pair $\boldsymbol{A} \times (-) \dashv (-)^{\mathbf{A}}$.

(f) Assume that a species $\boldsymbol{A}$ is *finite*, that is, $\boldsymbol{A}[n] = \emptyset$ for sufficiently large n. Then the product functor $\boldsymbol{B} \longmapsto \boldsymbol{A} \cdot \boldsymbol{B}$ has a right adjoint functor $\boldsymbol{B} \longmapsto \boldsymbol{B} \uparrow \boldsymbol{A}$:

$$(\boldsymbol{B} \uparrow \boldsymbol{A})[E] := \prod_F{}' \operatorname{Map}(\boldsymbol{A}[F], \boldsymbol{B}[E+F]), \tag{2.1.g}$$

where the product is taken over complete representatives of isomorphism classes of finite sets. Note that the right hand side is a finite product.

(2.2) Remark. The constructions of some operations on species are not categorical. For example, in the definition of products $\boldsymbol{A} \cdot \boldsymbol{B}$, the above summation was taken over the pair (E_1, E_2) of subsets $E_1, E_2 \subseteq E$ with $E_1 + E_2 = E$. However, the category Bij_f is lacking in coproducts and proper subobjects, and so the above definition of products is not categorical; it depends on the embedding of Bij_f into Set_f. Similarly, the definitions of substitutions and derivations are not categorical, too.

Contrarily, the definitions given in this paper are completely categorical as we see in the next section; so they can be extended to more general faithful functors.

(2.3) Generating functions and operations on species. The generating functions of these operations on species have the desired properties.
(a) $(\boldsymbol{A}+\boldsymbol{B})(x) = \boldsymbol{A}(x) + \boldsymbol{B}(x)$.
(b) $(\boldsymbol{A} \cdot \boldsymbol{B})(x) = \boldsymbol{A}(x) \cdot \boldsymbol{B}(x)$.
(c) $\boldsymbol{A}(\boldsymbol{B})(x) = (\boldsymbol{A}(\boldsymbol{B}(x)))$.
(d) $\boldsymbol{A}'(x) = \dfrac{d\boldsymbol{A}(x)}{dx}$.
(e) The generating function of the Cartesian product of species is given by the so-called *Hadamard product*:

$$\begin{aligned} (\boldsymbol{A} \times \boldsymbol{B})(x) &= \sum_{n=0}^{\infty} \frac{|\boldsymbol{A}[n]| \cdot |\boldsymbol{B}[n]|}{n!} x^n, \\ (\boldsymbol{B}^{\mathbf{A}})(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} |\boldsymbol{B}[n]|^{|\mathbf{A}[n]|} x^n. \end{aligned}$$

(f) $(\boldsymbol{B} \uparrow \boldsymbol{A})(x) = \displaystyle\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} |\boldsymbol{B}[m+n]|^{|\mathbf{A}[m]|} x^n$.

(2.4) Species and faithful functors. As is stated in the first paper, the concept of a species is equivalent to the one of a faithful functor with finite fibers from a groupoid to Set_f (PartI, Theorem 3.6). In fact, the faithful functor associated to a species $\boldsymbol{A}$ is given by

$$\Lambda(\boldsymbol{A}) : \mathsf{Elts}(\boldsymbol{A}) \longrightarrow \mathsf{Set}_f; (N, a) \longmapsto N, \tag{2.4.a}$$

where the groupoid $\mathsf{Elts}(\boldsymbol{A})$ is the category of elements of $\boldsymbol{A}$ (Part I, Section 2.3). Remember that an element of $\boldsymbol{A}$ is a pair (N, a) of $N \in \mathsf{Bij}_f$ and $a \in \boldsymbol{A}[N]$.

Let $S : \mathcal{E} \longrightarrow \mathcal{S}$ be a (usually faithful) functor. The functor S *has finite fibers* if

$$|S^{-1}(N)/\cong| = \sharp\{X \in \mathcal{E}/\cong|\ S(X) \cong N\}, \qquad (N \in \mathcal{S}).$$

An $\mathcal{E}$-*structure* on $N \in \mathcal{S}$ (along S) is a pair (X, σ) of an object X of $\mathcal{E}$ and an isomorphism $\sigma : S(X) \xrightarrow{\cong} N$; two $\mathcal{E}$-structures (X, τ) and (X', σ') on N are *isomorphic* if there is an isomorphism $f : X \longrightarrow X'$ such that $\sigma' \circ S(f) = \sigma$.

Now, the species associated to a functor $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ with finite fibers is given by

$$\Gamma(S) : N \longmapsto \mathsf{Str}(\mathcal{E}/N)/\cong, \tag{2.4.b}$$

where $\mathsf{Str}(\mathcal{E}/N)/\cong$ denotes the isomorphism classes of $\mathcal{E}$-structures over N. Remember that a $\mathcal{E}$-*structure* is the pair (X, σ) of an object $X \in \mathcal{E}$ and an isomorphism $\sigma : S(X) \xrightarrow{\cong} N$ called a *labeling* and that such $\mathcal{E}$-structures (X, σ) and (Y, τ) are isomorphic if and only if there exists an isomorphism $f : X \xrightarrow{\cong} Y$ in $\mathcal{E}$ such that $\tau \circ S(f) = \sigma$. Λ and Γ give the required equivalence.

A morphism (F, τ) from $S : \mathcal{E} \longrightarrow \mathcal{S}$ to $T : \mathcal{F} \longrightarrow \mathcal{S}$ is defined to be a morphism in the 2-comma category of skeletally small categories over $\mathcal{S}$. Thus $F : \mathcal{E} \longrightarrow \mathcal{F}$ is a functor and $\theta : S \longrightarrow T \circ F$ is a natural transformation. When τ is identity natural transformation, we denote such a morphism (F, τ) by simply F. In particular, S and T as above are isomorphic if and only if there are an equivalence $F : \mathcal{E} \xrightarrow{\cong} \mathcal{F}$ and a natural isomorphism $\theta : S \xrightarrow{\cong} T \circ F$.

(2.5) Generating functions. When $\mathcal{E}$ is skeletally small and $S : \mathcal{E} \longrightarrow \mathcal{S}$ has finite fibers, the *generating functions* of $\mathcal{E}$ and S can be defined as follows:

$$\mathcal{E}(t) := {\sum_X}' \frac{1}{|\mathrm{Aut}(X)|} t^X, \tag{2.5.a}$$

$$S(t) := {\sum_X}' \frac{1}{|\mathrm{Aut}(X)|} t^{S(X)}, \tag{2.5.b}$$

where the both summations are taken over the isomorphism classes of objects of $\mathcal{E}$. Furthemore, if S is faithful, then

$$S(t) = \sum_N{}' \frac{|\mathsf{Str}(\mathcal{E}/N)/\cong|}{|\mathrm{Aut}(N)|} t^N, \tag{2.5.c}$$

where N runs over all isomorphism classes of $\mathcal{S}$. Equivalent categories (or isomorphic functors) give a same generating functions. If the functor S is corresponding to a species $\boldsymbol{A}$, then $S(t) = \boldsymbol{A}(t)$.

The *ordinary generating functions* of $\mathcal{E}$ and S are the formal series in the following form:

$$\widetilde{\mathcal{E}}(t) := \sum_X{}' t^X, \tag{2.5.d}$$

$$\widetilde{S}(t) := \sum_X{}' t^{S(X)}, \tag{2.5.e}$$

We often write $\mathcal{E}\widetilde{\ }(t)$ for $\widetilde{\mathcal{E}}(t)$. The formal power series $\widetilde{S}(t)$ is equal to the label generating function $\widetilde{\boldsymbol{A}}(t)$ of the corresponding species $\boldsymbol{A}$. However, the label generating function of a category can be presented by the generating function of exponential type (Part I, Corollary 4.7).

Assume that the category $\mathcal{S}$ has any finite coproducts, especially an initial object $\emptyset$ and a coproduct $M+N$ of two objects. Then the R-module $R[\mathcal{S}^{\mathrm{op}}/\cong]$ of polynomials

$$f(t) = \sum_{N\in\mathcal{S}}{}' c_N t^N, \quad c_N \in R \tag{2.5.f}$$

with exponents in $\mathcal{S}/\cong$ becomes a commutative R-algebra(the *polynomial ring* with exponents in $\mathcal{S}$). Here multiplication on the basis is defined by

$$t^M \cdot t^N := t^{M+N}, \; t^{\emptyset} := 1. \tag{2.5.g}$$

In general, this product can not be extend to power series. To do so, we need the following condition:

Condition A: The category $\mathcal{S}$ has any finite coproducts and for any object N there exists only finitely many pairs (A, B) of isomorphism classes of objects of $\mathcal{S}$ such that $N \cong A + B$.

Under this condition, the R-module $R[[\mathcal{S}^{\mathrm{op}}/\cong]]$ of power series with exponents in $\mathcal{S}$ becomes a commutative R-algebra.

In particular, polynomials (resp. power series) with exponents in Set_f are identified with usual polynomials (resp. power series) in one variable by the identification:

$$\sum_N{}' c_N t^N = \sum_N{}' c_N t^{|N|}, \tag{2.5.h}$$

where N runs over all isomorphism classes of finite sets, and so we can write the series as follows:

$$\sum_N{}' c_N t^N = \sum_{n=0}^{\infty} c_{[n]} t^n, \tag{2.5.i}$$

where $[n] = \{1, 2, \ldots, n\}$ and $[0] = \emptyset$.

3 Operations on functors

(3.1) Operations on categories and functors. In our situation, we can rewrite the operations on species by using the language of faithful functors with finite fibers from *groupoids*. We can furthermore define operations on functors; especially taking the identity functors on categories, such operations are viewed as those on categories.

Throughout this section, we take categories $\mathcal{E}, \mathcal{F}, \mathcal{S}$ and functors

$$S : \mathcal{E} \longrightarrow \mathcal{S}, \quad T : \mathcal{F} \longrightarrow \mathcal{S}$$

and we study operations on these categories and functors.

In order to simplify the statements, we further assume that the following conditions hold:

(a) $\mathcal{S}$ has any finite coproducts.
(b) S and T are both faithful and has finite fibers.
(c) $\mathcal{E}$ and $\mathcal{F}$ are skeletally small and locally finite.
(d) $\mathcal{E}$, $\mathcal{F}$, $\mathcal{S}$ satisfy Condition A in (2.5) if they have finte coproducts.

The assumption (b) and (c) is not necessary if we do not consider generating functions. Similarly, (d) is not necessary if we do not consider the product of generating functions. The condition (b) is necessary only for using Part I, Theorem 4.6. All operations on categories and functors can be defined without (b), (c), (d).

(3.2) Summation. The *summation* $\mathcal{E}+\mathcal{F}$ of two categories $\mathcal{E}$ and $\mathcal{F}$ is defined as the disjoint union of categories. Clearly,

$$(\mathcal{E}+\mathcal{F})(t)=\mathcal{E}(t)+\mathcal{F}(t), \quad (\mathcal{E}+\mathcal{F})^{\sim}(t)=\widetilde{\mathcal{E}}(t)+\widetilde{\mathcal{F}}(t) \tag{3.2.a}$$

Next, the *summation* of the functors S and T is defined by

$$(S:\mathcal{E}\longrightarrow\mathcal{S})+(T:\mathcal{F}\longrightarrow\mathcal{S}):=(S+T:\mathcal{E}+\mathcal{F}\longrightarrow\mathcal{S}), \tag{3.2.b}$$

where $\mathcal{E}+\mathcal{F}$ denotes the disjoint union of the categories $\mathcal{E},\mathcal{F}$, and where $S+T$ coincides with S on $\mathcal{E}$ and with T on $\mathcal{F}$. For the generating function of $S+T$, we have

$$(S+T)(t)=S(t)+T(t), \quad (S+T)^{\sim}(t)=\widetilde{S}(t)+\widetilde{T}(t). \tag{3.2.c}$$

Morphisms $(F,\varphi):S_1\longrightarrow S_2$ and $(G,\psi):T_1\longrightarrow T_2$ induces a morphism $(F,\varphi)+(G,\psi):S_1+T_1\longrightarrow S_2+T_2$.

(3.3) Product. The *product* $\mathcal{E}\cdot\mathcal{F}$ of $\mathcal{E}$ and $\mathcal{F}$ is defined as the Cartesian product $\mathcal{E}\times\mathcal{F}$ of categories. Clearly, if $\mathcal{E}$ and $\mathcal{F}$ have well-defined generating functions, then

$$(\mathcal{E}\cdot\mathcal{F})(t)=\mathcal{E}(t)\cdot\mathcal{F}(t), \quad (\mathcal{E}\cdot\mathcal{F})^{\sim}(t)=\widetilde{\mathcal{E}}(t)\cdot\widetilde{\mathcal{F}}(t), \tag{3.3.a}$$

where the multiplication $T^M\cdot t^N$ for $M\in\mathcal{E},N\in\mathcal{F}$ is defined to be $t^{(M,N)}$.

Next, the *product* of the functors S and T is defined by

$$(S:\mathcal{E}\longrightarrow\mathcal{S})\cdot(T:\mathcal{F}\longrightarrow\mathcal{S}):=(S\cdot T:\mathcal{E}\times\mathcal{F}\longrightarrow\mathcal{S}) \tag{3.3.b}$$

where $(S\cdot T)(X,Y):=S(X)+T(Y)$. By an easy calculation, we have

$$(S\cdot T)(t)=S(t)\cdot T(t). \tag{3.3.c}$$

Of course, we have to assume the condition (a), (c), (d) of (3.1) so that the generating functions are well-defined.

Let $S_i:\mathcal{E}_i\longrightarrow\mathcal{S}$ and $T_i:\mathcal{F}\longrightarrow\mathsf{Set}_f$ $(i=1,2)$ be functors, and let $(F,\varphi):S_1\longrightarrow S_2$ and $(G,\psi):T_1\longrightarrow T_2$, so that $F:\mathcal{E}_1\longrightarrow\mathcal{E}_2$ and $G:\mathcal{F}_1\longrightarrow\mathcal{F}_2$ are functors, and $\varphi:S_1\longrightarrow S_2\circ F$ and $\psi:T_1\longrightarrow T_2\circ G$ are natural transformations. Then these morphism (F,φ), (G,ψ) induces a morphism:

$$(F,\varphi)\cdot(G,\psi):=(F\cdot G,\varphi\cdot\psi):S_1\cdot T_1\longrightarrow S_2\cdot T_2,$$

where

$$\begin{aligned}
&F\cdot G:\mathcal{E}_1\times\mathcal{F}_1\longrightarrow\mathcal{E}_2\times\mathcal{F}_2;(X,Y)\longmapsto(F(X),G(Y)),\\
&\varphi\cdot\psi(X,Y):=\varphi(X)+\psi(Y)\\
&\qquad :S_1(X)+T_1(Y)\longrightarrow S_2(F(X))+T_2(G(Y)).
\end{aligned}$$

(3.4) Shift operators $t \cdot S, t^I \cdots S$. Let $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ be a functor. Then the *shift* $(t \cdot S)$ of S is defined by the addition of a base (or a root):

$$t \cdot S : \mathcal{E} \longrightarrow \mathsf{Set}_f ; X \longrightarrow S(X) \cup \{*\} \tag{3.4.a}$$

The generating function is clearly given by $t \cdot S(t)$.

Next let $S : \mathcal{E} \longrightarrow \mathcal{S}$ be a functor into a category $\mathcal{S}$ with finite coproduct and let $I \in \mathcal{S}$. The the shift of S at I is defined by the coproduct with I:

$$t^I \cdot S : \mathcal{E} \longrightarrow \mathcal{S}; X \longrightarrow (X) + I. \tag{3.4.b}$$

The generating function is given by

$$(t^I \cdot S)(t) = t^I \cdot S(t). \tag{3.4.c}$$

$t^I \cdot S$ is also obtained by the product of $S : \mathcal{E} \longrightarrow \mathcal{S}$ and the trivial sucategory $\{I\}$ with embedding into $\mathcal{S}$.

(3.5) $\mathcal{E}^n/n!$ and $S^n/n!$. For a category $\mathcal{E}$ and $n \geq 1$, the category $\mathcal{E}^n/n!$ is defined by the following data:

(i) An object has the form $\{X_1, \ldots, X_n\}$, where $X_i \in \mathcal{E}$. We simply write it as $\{X_i\}$.
(ii) A morphism from $\{X_1, \ldots, X_n\}$ to $\{Y_1, \ldots, Y_n\}$ has the form

$$[f_1, \ldots, f_n; \pi], \quad \pi \in S_n, f_i : X_i \longrightarrow Y_{\pi(i)}.$$

We often write it as $\{f_i; \pi\}$.
(iii) The composition of $\{f_i; \pi\} : \{X_i\} \longrightarrow \{Y_j\}$ and $\{g_j; \rho\} : \{Y_j\} \longrightarrow \{Z_k\}$ is defined by

$$\{g_j; \rho\} \circ \{f_i; \pi\} := \{g_{\pi(i)} \circ f_i; \rho\pi\}.$$

Next, for a functor $S : \mathcal{E} \longrightarrow \mathcal{S}$ into a category $\mathcal{S}$ with finite coproducts, the functor $S^n/n! : \mathcal{E}^n/n! \longrightarrow \mathcal{S}$ is defined by the following data:

(i) $S^n/n!(\{X_i\}) = S(X_1) + \cdots + S(X_n)$.
(ii) $S^n/n!(\{f_i; \pi\} : \{X_i\} \longrightarrow \{Y_j\})$ is defined by

$$\coprod_{i=1}^n S(X_i) \xrightarrow{\coprod_{i=1}^n S(f_i)} \coprod_{i=1}^n S(Y_{\pi(i)}) \xrightarrow{\cong} \coprod_{j=1}^n S(Y_j).$$

By the embedding $(X_i) \longmapsto \{X_i\}$, the category $\mathcal{E}^n$ is viewed as a subcategory of $\mathcal{E}^n/n!$. Clearly, $\mathcal{E}^1/1! = \mathcal{E}$ and $S^1/1! = S$. Furthermre, $\mathcal{E}^0/0!$ is viewed as a trivial category $\mathbf{1}$ with one object and one morphism, and then

$S^0/0!$ is the functor which assigns the initial object $\emptyset$ of $\mathcal{S}$ to the unique object $*$ of $\mathbf{1}$.

A functor $F : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ induces a functor

$$F^n/n! : \mathcal{E}_1^n/n! \longrightarrow \mathcal{E}_2^n/n!; \{X_i\} \longmapsto \{F(X_i)\}. \tag{3.5.a}$$

Furthermore, a morphism $(F, \varphi) : S_1 \longrightarrow S_2$ induces a morphism

$$(F, \varphi)^n/n! : S_1^n/n! \longrightarrow S_2^n/n!,$$

where the associated natural transformation is given by

$$\begin{aligned}(\varphi^n/n!)(\{X_i\}) &:= \coprod \varphi(X_i) \\ &: S_1(X_1) + \cdots + S_1(X_n) \longrightarrow S_2(F(X_1)) + \cdots S_2(F(X_n)).\end{aligned}$$

Remark. Another definition of a morphism $(\{f_i\ \pi\})$ from $\{X_i\}$ to $\{Y_j\}$ is to use the assumption that π is an endo-function on $[n]$ instead of a permutation on $[n]$. The category defined by this way is a full subcategory of $\mathsf{EXP}(\mathcal{C})$ (Section 5). The functor $S^n/n!$ can be extended to a functor from this new category, but it has the same generating function as the old one.

(3.6) Lemma. $(S^n/n!)(t) = S(t)^n/n!$.

PROOF. It is easily proved that $\{X_i\} \cong \{Y_i\}$ if and only if there exist a permutation $\pi \in S_n$ and isomorphisms $f_i : X_i \xrightarrow{\cong} Y_{\pi(i)}$; thus $\mathcal{E}^n/n!$ is bijectively corresponding with multisets on $\mathcal{E}/\cong$ of cardinality n. Take an object $\{X_i\}$ of $\mathcal{E}^n/n!$, and for each $Y \in \mathcal{E}$ let $n(Y)$ be the number of X_i isomorphic to Y. Then we have that

$$\mathrm{Aut}(\{X_i\}) \cong \prod_{Y \in \mathcal{E}}{}' \mathrm{Aut}(Y)\ \mathit{wr}\ S_{n(Y)},$$

where *wr* stands for the wreath product. Hence

$$\begin{aligned}(S^n/n!)(t) &= \sum_{\sum' n(Y) = n} \prod_Y{}' \frac{t^{n(Y)S(Y)}}{|\mathrm{Aut}(Y)|^{n(Y)} n(Y)!} \\ &= \frac{1}{n!}\left(\sum_Y{}' \frac{t^{S(Y)}}{|\mathrm{Aut}(Y)|}\right) \\ &= \frac{1}{n!} S(t)^n.\end{aligned}$$

(3.7) Derivations S', $S^{(n)}$ Assume that $\mathcal{S} = \mathsf{Set}_f$ and that $S(f)$ is a *monomorphism* for each morphism f of $\mathcal{E}$. For example, $\mathcal{E}$ is a groupoid (species case), then this assumption holds. Then we define the *derivation* by

$$(S : \mathcal{E} \longrightarrow \mathsf{Set}_f)' := (S' : \mathsf{Elts}(S) \longrightarrow \mathsf{Set}_f), \tag{3.7.a}$$

where $\mathsf{Elts}(S)$ is the category of elements of S and $S'(A,s) := S(A) - \{s\}$. Note that each $f : (A,s) \longrightarrow (B,t)$ induces $S'(f) : S(A) - \{s\} \longrightarrow S(B) - \{t\}$ because $S(f)$ is assumed to be a monomorphism.

As above, we assume that S is a functor from $\mathcal{E}$ to Set_f and that $S(f)$ is a monomorphism for each morphism f in $\mathcal{E}$. The *n-th category of elements* of $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ consists of objects of the form

$$(A; s_1, s_2, \ldots, s_n), \qquad A \in \mathcal{E}, s_i \in S(A), \, s_i \neq s_j (i \neq j). \tag{3.7.b}$$

A morphism f from $(A; s_1, \ldots, s_n)$ to $(B; t_1, \ldots, t_n)$ is a morphism $f : A \longrightarrow B$ in $\mathcal{E}$ such that $S(f)(s_i) = t_i$ $(i = 1, \ldots, n)$. Now the *n-th derivation* of S is defined by

$$S^{(n)} : \mathsf{Elts}^{(n)} \longrightarrow \mathsf{Set}_f; (A; s_1, \ldots, s_n) \longmapsto A - \{s_1, \ldots, s_n\} \tag{3.7.c}$$

Any morphism $f : (A; s_1, \ldots, s_n) \longrightarrow (B; t_1, \ldots, t_n)$ induces a functor

$$S^{(n)}(f) : S(A) - \{s_1, \ldots, s_N\} \longrightarrow S(B) - \{t_1, \ldots, t_n\}$$

because $S(f)$ is a monomorphism.

Let $S_i : \mathcal{E}_i \longrightarrow \mathsf{Set}_f$ $(i = 1, 2)$ be functors which maps all morphisms to monomorphisms of finite sets. Let $(F, \varphi) : S_1 \longrightarrow S_2$ be a morphism in $\mathsf{Cat}/\mathsf{Set}_f$, the bi-category of (skeletally small) category over Set_f, so that $F : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a functor and $\varphi : S_1 \Rightarrow S_2 \circ F$ is a natural transformation. Then (F, φ) induces a morphism

$$(F, \varphi)' := (F', \varphi') : S_1' \longrightarrow S_2',$$

where $F' : \mathsf{Elts}(\mathcal{E}_1) \longrightarrow \mathsf{Elts}(\mathcal{E}_2)$ is a functor defined by

$$F'(X, s) := (F(X), \varphi(X_1)(s),$$
$$F'((X, s) \xrightarrow{f} (Y, t)) := ((F(X), \varphi(X)(s) \xrightarrow{f} \varphi(Y)(t)),$$

where $X, Y \in \mathcal{E}_1, s \in S_1(X), t \in S_1(Y)$. Similarly, we can define the n-th derivation $(F, \varphi)^{(n)}$.

(3.8) Lemma. *Let $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be two functors and assume that for each morphism f in $\mathcal{E}$ (resp. $\mathcal{F}$) the mapping $S(f)$ (resp. $T(f)$) a monomorphism. Then the following hold:*

(1) $S^{(0)} = S, S^{(1)} = S'$.

(2) $S^{(m+n)} \cong (S^{(m)})^{(n)}$ for $m, n \geq 0$.

(3) $S'(t) = \dfrac{d}{dt}S(t)$ and $S^{(n)}(t) = \dfrac{d^n}{dt^n}S(t)$. *Here, $d^n S(t)/dt^n$ is the n-the derivation of $S(t) \in \boldsymbol{Q}[[\mathsf{Set}_f{}^{op}/\cong]] = \boldsymbol{Q}[[t]]$ viewed as a usual power series in a variable t.*

(4) $S^{(n)}(0) = |\mathsf{Str}(\mathcal{E}/[n])/\cong| = \sum'_{|S(A)|=n} \dfrac{n!}{|\mathrm{Aut}(A)|}$.

(5) *Leibniz's rule holds:*

$$(S \cdot T)^{(n)} \cong \coprod_{r=0}^{n} \binom{n}{r} S^{(r)} \cdot T^{(n-r)}$$

In particular, $(S \cdot T)' \cong S' \cdot T + S \cdot T'$. These isomorphisms are natural over Set_f.

(6) *There is a natural isomorphism over* Set_f:

$$(S^n/n!)' \cong \begin{cases} 0 & (n = 0) \\ (S^{n-1}/(n-1)!) \cdot S' & (n > 0). \end{cases}$$

(7) *Let $\boldsymbol{A}_S$ be the species of the structures of the faithful functor S. Then $(\boldsymbol{A}_S)'$, the derivation of the species, is the species associated to S':*

$$(\boldsymbol{A}_S)' = \boldsymbol{A}_{S'}.$$

PROOF. (1) is trivial; (2) is easily proved.
(3) We have

$$\begin{aligned} S'(t) &= \sum_{(A,s)}' \frac{1}{|\mathrm{Aut}(A,s)|} t^{S(A)-\{s\}} \\ &= \sum_{A \in \mathcal{E}}' \frac{|S(A)|}{|\mathrm{Aut}(A)|} t^{|S(A)|-1} \\ &= \frac{d}{dt}S(t). \end{aligned}$$

Here, of course, we identified t^S as $t^{|S|}$ for any finite set S. Applying this formula n times, we have $S^{(n)}(t) = d^nS(t)/dt^n$, as required.
(4) Since S is faithful and has finite fibers by the condition (b) of (3.1), Theorem 4.6 of Part I implies

$$\begin{aligned} S(t) &= \sum_{A\in\mathcal{E}}{}' \frac{t^{S(A)}}{|\mathrm{Aut}(A)|} = \sum_{n=0}^{\infty}\left(\sum_{|S(A)|=n}{}' \frac{n!}{|\mathrm{Aut}(A)|}\right) t^n \\ &= \sum_{n=0}^{\infty} \frac{|\mathsf{Str}(\mathcal{E}/[n])/\cong|}{n!} t^n. \end{aligned}$$

(5) It will suffice to prove the case $n = 1$. We first construct an isomorphism of functors:

$$H_{ST} : S' \cdot T + S \cdot T' \xrightarrow{\cong} (S \cdot T)'. \tag{3.8.a}$$

Since the functor $S \cdot T : \mathcal{E} \times \mathcal{F} \longrightarrow \mathsf{Set}_f$ is defined by

$$S \cdot T : (X, Y) \longmapsto S(X) + T(Y),$$

an element of $S \cdot T$ has the form (X, Y, u), where $X \in \mathcal{E}, Y \in \mathcal{F}$ and $u \in S(X) + T(Y)$. Thus

$$(S \cdot T)' : \mathsf{Elts}(S \cdot T) \longrightarrow \mathsf{Set}_f; (X, Y, u) \longmapsto (S(X) + T(Y)) - \{u\}.$$

On the other hand, $S' \cdot T$ and $S \cdot T'$ are defined by

$$\begin{aligned} S \times T' : \mathsf{Elts}(S) \times \mathcal{F} &\longrightarrow \mathsf{Set}_f; ((X, s), Y) \longmapsto (S(X) - \{u\}) + T(Y), \\ S' \times T : \mathcal{E} \times \mathsf{Elts}(T) &\longrightarrow \mathsf{Set}_f; (X, (Y, t)) \longmapsto S(X) + (T(Y) - \{t\}. \end{aligned}$$

Thus the required isomorphism H_{ST} in (3.8.a) is now obtained by the summation of

$$\begin{aligned} S' \cdot T : \mathsf{Elts}(S) \times \mathcal{F} &\longrightarrow \mathsf{Elts}(S \cdot T); ((X, s), Y) \longmapsto (X, Y, s), \\ S \cdot T' : \mathcal{E} \times \mathsf{Elts}(T) &\longrightarrow \mathsf{Elts}(S \cdot T); (X, (Y, t)) \longmapsto (X, Y, t). \end{aligned}$$

Clearly, $S' \cdot T$ (resp. $S \cdot T'$) maps canonically and functorially any morphism of $\mathsf{Elts}(S) \times \mathcal{F}$ (resp. $\mathcal{E} \times \mathsf{Elts}(T)$) to a morphism of $\mathsf{Elts}(S \cdot T)$. We can identify $(S' \cdot T + S \cdot T') \circ H_{ST} = (S \cdot T)'$, and so H_{ST} gives an isomorphism.

Finally, it remains to prove the naturality of the isomorphism (H_{ST}, θ_{ST}). Let $S_i : \mathcal{E}_i \longrightarrow \mathsf{Set}_f$ and $T_i : \mathcal{F}_i \longrightarrow \mathsf{Set}_f$ $(i = 1, 2)$ be functors which maps all morphisms to monomorphisms of finite sets. Let $(F, \varphi) : S_1 \longrightarrow S_2$ and $(G, \psi) : T_1 \longrightarrow T_2$ be morphisms over Set_f, so that $F : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a functor and $\varphi : S_1 \Rightarrow S_2 \circ F$ is a natural transformation. We need to show the commutativety (in the strict sense) of the following diagram:

$$\begin{tikzcd}
S_1' \cdot T_1 + S_1 \cdot T_1' \arrow[r, "H_{S_1,T_1}"] \arrow[d, "(F,\varphi)'\cdot(G,\psi)+(F,\varphi)\cdot(G,\psi)'"'] & (S_1 \cdot T_1)' \arrow[d, "(F\cdot G,\varphi\cdot\psi)'"] \\
S_2' \cdot T_2 + S_2 \cdot T_2' \arrow[r, "H_{S_2,T_2}"] & (S_2 \cdot T_2)'.
\end{tikzcd}$$

But it is easily checked.
(6) Since $S^0/0!$ is the functor $\{*\} \longrightarrow \mathsf{Set}_f$ which assigns the empty set $\emptyset$ to $*$, its derivation is 0. Let $n \geq 1$. Then the derivation $(S^n/n!)'$ is given by

$$\begin{array}{ccc} \mathsf{Elts}(S^n/n!) & \longrightarrow & \mathsf{Set}_f \\ (\{X_1, \dots, X_n\}, a) & \longmapsto & \coprod_{i=1}^{n} S(X_i) - \{a\}. \end{array}$$

On the other hand, $(S^{n-1}/(n-1)!) \cdot S'$ is given by

$$\begin{array}{ccc} \mathcal{C}^{n-1}/(n-1)! \times \mathsf{Elts}(S) & \longrightarrow & \mathsf{Set}_f, \\ (\{Y_1, \dots, Y_{n-1}\}, (Y, a)) & \longmapsto & \coprod_{j=1}^{n-1} S(Y_j) + (S(Y) - \{a\}). \end{array}$$

(7) By (2.4), the species $\boldsymbol{A}(S), \boldsymbol{A}_{S'} : \mathsf{Bij}_f \longrightarrow \mathsf{Set}_f$ of S, S' are defined by

$$\boldsymbol{A}_S : N \longmapsto \boldsymbol{A}_S[N] := \mathsf{Str}(\mathcal{E}/N)/\cong,$$
$$\boldsymbol{A}_{S'} : N \longmapsto \boldsymbol{A}_S[N] := \mathsf{Str}(\mathsf{Elts}(S)/N)/\cong.$$

Furthermore, by (2.1) the derivation $\boldsymbol{A}(S)'$ of $\boldsymbol{A}_S$ is defined by

$$(\boldsymbol{A}_S)' : N \longmapsto \boldsymbol{A}_S[N \cup \{N\}].$$

Thus we have the required isomorphism

$$\boldsymbol{A}_{S'} \cong (\boldsymbol{A}_S)' : [(X, a), \sigma] \longleftrightarrow [X, \tau],$$

where $X \in \mathcal{E}$, $a \in S(X)$, $\sigma : S(X) - \{a\} \xrightarrow{\cong} N$ and $\tau : S(X) \xrightarrow{\cong} N \cup \{N\}$. Here the isomorphism σ is given by the restriction of τ; the isomorphism τ is given by the unique extension of σ with $\tau(a) := N$.

(3.9) Derivation S^*. Instead of S', we can define a functor

$$S^* : \mathsf{Elts}(S) \longrightarrow \mathsf{Set}_f; (A, s) \longmapsto S(A), \tag{3.9.a}$$

which corresponds to the *pointed species* ([BerLL 98, Section 2.1]). Note that this definition does not need to assume that each induced map $S(f)$ is a monomorphism. This definition S^* of derivation seems to be more natural and easier to treat than the derivation S' defined in (3.7). Its generating function is given by

$$S^*(t) = {\sum_{A \in \mathcal{E}}}' \frac{|S(A)|}{|\mathrm{Aut}(A)|} t^{S(A)} \tag{3.9.b}$$

$$= t \frac{d}{dt} S(t), \tag{3.9.c}$$

where, of course, $S(t)$ is identified with a usual power series by $t^N = t^{|N|}$. Using the existence of finite coproducts in the category $\mathcal{S}$, we have that this operation satisfies Leibniz's rule similarly as in the case of the derivation (3.7):

$$(S \cdot T)^* \cong S^* \cdot T + S \cdot T^*. \tag{3.9.d}$$

In some articles(e.g., [Mcl 71], [MclMo 92]), the symbol $\int S$ is used for the category of elements of S; but this symbol is not suitable in this paper because as we have seen, the category of elements corresponds to the derivation of generating functions !

(3.10) Example. Let Tree, RTree, $\mathsf{RForest}$ be the categories of trees, rooted trees, rooted forests, respectively. Here these graphs are all unlabeled, and trees are all nonempty. Let

$$V_{RT} : \mathsf{RTree} \longrightarrow \mathsf{Set}_f, \; V_T : \mathsf{Tree} \longrightarrow \mathsf{Set}_f, \; V_{RF} : \mathsf{RForest} \longrightarrow \mathsf{Set}_f$$

be the functors which assigns its vertex set to a (rooted) tree or a rooted forest. Then an "element "(T, r) of the functor V_T consist of a tree T and a vertex r, and so $\mathsf{Elts}(V_T)$, the category of elements, is equivalent to RTree. Thus we have

$$V_T^* \cong V_{FT}. \tag{3.10.a}$$

On the other hand, the derivation

$$V_T' : \mathsf{Elts}(V_T) \longrightarrow \mathsf{Set}_f; (T, r) \longmapsto V(T) - \{r\}$$

is isomorphic to

$$V_{RF} : \mathsf{RForest} \longrightarrow \mathsf{Set}_f; F \longmapsto V(F). \tag{3.10.b}$$

In fact, this isomorphism is given by

$$\mathsf{Elts}(V) \longrightarrow \mathsf{RForest}; (T, r) \longmapsto T - \{r\}.$$

The roots of the rooted forest $T-\{r\}$ are defined to be the vertices connected to r. Thus we have

$$V_T' \cong V_{RF}. \tag{3.10.c}$$

The shift of V_{RF} defined in (3.4) is clearly isomorphic to V_{RT}:

$$t \cdot V_{RF} \cong V_{RT}. \tag{3.10.d}$$

See an expression of **RForest** as the exponentiation of RTree

(3.11) Partial derivation $\partial_I^*(S)$. We assume that $\mathcal{S}$ has any finite coproducts. For a functor $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $I \in \mathcal{S}$, *the partial derivation* $\partial_I^*(S)$ is defined by

$$\partial_I^*(S) : I \downarrow S \longrightarrow \mathcal{S}; (\alpha, X) \longmapsto S(X), \tag{3.11.a}$$

where $I \downarrow S$ is the comma category, whose objects has the form (α, X) with $X \in \mathcal{E}$ and $\alpha : I \longrightarrow S(X)$ and whose morphism $f : (\alpha, X) \longrightarrow (\beta, Y)$ is a morphism $f : X \longrightarrow Y$ of $\mathcal{E}$ such that $S(f) \circ \alpha = \beta$.

If the hom-functor $\mathrm{Hom}(I, -)$ on $\mathcal{S}$ preserves coproducts (see Theorem 4.4; Lemma 4.7, Lemma 4.2 (5)), then the partial derivative satisfies Leibniz's rule:

$$\partial_I^*(S \cdot T) \cong \partial_I^*(S) \cdot T + S \cdot \partial_I^*(T). \tag{3.11.b}$$

This isomorphism is give by the following equivalence natural on S and T:

$$\begin{gathered}(I \downarrow S \times \mathcal{F} + \mathcal{E} \times (I \downarrow T) \longrightarrow I \downarrow (\mathcal{E} \times \mathcal{F});\\
(I \xrightarrow{\alpha} S(X), X; Y) \longmapsto (I \xrightarrow{\alpha} S(X) \longrightarrow S(X) + T(Y), (X, Y)),\\
(X; I \xrightarrow{\beta} T(Y), Y) \longmapsto (I \xrightarrow{\beta} T(Y) \longrightarrow S(X) + T(Y), (X, Y)).\end{gathered}$$

For $I, J \in \mathcal{E}$, there are isomorphisms:

$$\partial_I^* \circ \partial_J^* = \partial_J^* \circ \partial_I^* = \partial_{I+J}^*. \tag{3.11.c}$$

The generating function of $\partial_I^*(S)$ is given by

$$\partial_I^*(S)(t) = \sum_{A \in \mathcal{E}} \frac{|\mathrm{Hom}(I, S(A))|}{|\mathrm{Aut}(A)|} t^{S(A)}. \tag{3.11.d}$$

When $\mathcal{S} = \mathsf{Set}_f$ and $I = \mathbf{1}$, this derivation of $S : \mathcal{E} \longrightarrow \mathcal{S}$ satisfies

$$\partial_I^*(S) = S^*, \tag{3.11.e}$$

where S^* is the $*$-derivation defined in (3.9). However, because we can not subtract a number 1 from an object, it is impossible to define a partial derivation corresponding to S' in (c) except for the case where $\mathcal{S} = \mathsf{Set}_f$.

Let K be an object of $\mathcal{E}$. Then the partial derivation $\partial_K^*(\mathcal{E})$ of the category $\mathcal{E}$ is defined by the forgetful functor

$$\partial_K^*(\mathcal{E}) : K \downarrow \mathcal{E} \longrightarrow \mathcal{E}; (\alpha, X) \longrightarrow X, \tag{3.11.f}$$

where $\alpha : K \longrightarrow X$ is a morphism in $\mathcal{E}$. This derivation is same as the partial derivation $\partial_J(\mathrm{Id})$ of the identity functor on $\mathcal{E}$ and its generating function is given by

$$\partial_K^*(\mathcal{E})(t) = \sum_{X \in \mathcal{E}} \frac{|\mathrm{Hom}(K, X)|}{|\mathrm{Aut}(X)|} t^X. \tag{3.11.g}$$

Assume that $\mathcal{E}$ has finite coproducts and $\mathrm{Hom}_{\mathcal{E}}(K, -)$ preserves finite coproducts; let $\coprod : \mathcal{E}^n \longrightarrow \mathcal{E}$ be the funtor which maps $(X_1, \dots, X_n)$ to $X_1 + \cdots + X_n$. Then using (3.11.b) repeatedly, we have

$$\partial_K(\mathcal{E}^n \xrightarrow{\amalg} \mathcal{E}) = n\partial_K(\mathcal{E}) \cdot \coprod .$$

(3.12) Comma category. Further generalizations of $*$-derivations are obtained by using comma categories. Let $\mathcal{A} \xrightarrow{S} \mathcal{S}$ and $\mathcal{B} \xrightarrow{T} \mathcal{S}$ be functors. Let $S \downarrow T$ be the comma category, so that an object has the form (A, λ, B) with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\lambda : S(A) \longrightarrow T(B)$; and a morphism $(f, g) : (A, \lambda, B) \longrightarrow (A', \lambda', B')$ is the pair of $f : A \longrightarrow B$ of $\mathcal{A}$ and $g : B \longrightarrow B'$ of $\mathcal{B}$ such that $\lambda' \circ S(f) = T(g) \circ \lambda$. Then we have a functor

$$U : S \downarrow T \longrightarrow \mathcal{S} \times \mathcal{S}; (A, \lambda, B) \longmapsto (S(A), T(B)). \tag{3.12.a}$$

Note that $(A, \lambda, B) \cong (A, \lambda', B)$ if and only if λ and λ' belongs to a same $(\mathrm{Aut}(A), \mathrm{Aut}(B))$-orbit. Thus if the generating functions are defined (that is, $\mathcal{A}, \mathcal{B}$ are skeletally small and locally finite; S, T are faithful and has finite fibers), then U is also a faithful functor with finite fibers, and so

$$\begin{aligned} U(t) &= \sum_{(A,\lambda,B)}{}' \frac{1}{|\mathrm{Aut}(A,\lambda,B)|} t^{U(A,\lambda,B)} \\ &= \sum_{A\in\mathcal{A}}{}' \sum_{B\in\mathcal{B}}{}' \frac{|\mathrm{Hom}(S(A),T(B))|}{|\mathrm{Aut}(A)|\cdot|\mathrm{Aut}(B)|} t^{[A,B]} \\ &= \sum_{M,N\in\mathcal{S}}{}' \sum_{A:S(A)\cong M}{}' \sum_{B:T(B)\cong N}{}' \frac{|\mathrm{Hom}(S(A),T(B))|}{|\mathrm{Aut}(A)|\cdot|\mathrm{Aut}(B)|} t^{[A,B]} \\ &= \sum_{M,N\in\mathcal{S}}{}' \frac{|\mathsf{Str}(\mathcal{A}/M)/\cong|\cdot|\mathsf{Str}(\mathcal{B}/N)/\cong|}{|\mathrm{Aut}(M)|\cdot|\mathrm{Aut}(B)|} |\mathrm{Hom}(M,N)|\, t^{[M,N]}. \end{aligned}$$

In particular, when $\mathcal{S} = \mathsf{Set}_f$ and S (resp. T) is a functor corresponding to a species $\boldsymbol{A}$ (resp. $\boldsymbol{B}$), the generating function of the species $\boldsymbol{A} \downarrow \boldsymbol{B}$ corresponding to U in two variables is given by

$$(\boldsymbol{A} \downarrow \boldsymbol{B})(s,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|\boldsymbol{A}[m]|\cdot|\boldsymbol{B}[n]|\cdot n^m}{m!\,n!} s^m t^n. \tag{3.12.b}$$

The author does not know what the species $\boldsymbol{A} \downarrow \boldsymbol{B}$ means combinatorically.

Furthermore, we consider the case where $\mathcal{S} = \mathsf{Set}_f$, $\mathcal{A} = \mathbf{1} = \{*\}$ and $F = 1 : \mathbf{1} \longrightarrow \mathsf{Set}_f; * \longmapsto \{*\}$. In this case, $(S \downarrow T)$ is equivalent to the category $\mathsf{Elts}(T)$ of elements, and so

$$U \cong T^*,\ U(t) = T^*(t) = t\, dT(t)/dt.$$

Thus we can view U as a kind of higher order derivations of T.

(3.13) Hadamard product. Let $S : \mathcal{A} \longrightarrow \mathcal{S}$ and $T : \mathcal{B} \longrightarrow \mathcal{S}$ be functors with finite fibers. Make the pullback diagram of S and T in the bi-category Cat of categories:

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\text{P.B.}"] & \mathcal{B} \arrow[d, "T"] \\ \mathcal{A} \arrow[r, "S"] & \mathcal{S}, \end{tikzcd}$$

that is, $\mathcal{P}$ consists of objects of the form (A, λ, B), where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and σ is an isomorphism $\sigma : S(A) \xrightarrow{\cong} T(B)$; an morphism $(f, g) : (A, \lambda, B) \longrightarrow$

(A', λ', B') is the pair of morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ such that $T(g) \circ \lambda = \lambda' \circ S(f)$; thus $\mathcal{P}$ is a subcategory of the comma category $S \downarrow T$. There is a projection functor $U : \mathcal{P} \longrightarrow \mathcal{A}; (A, \lambda, B) \longmapsto A$. Similarly to the case of comma categories, we have

$$\begin{aligned} U(t) &= \sum_{A \in \mathcal{A}}{}' |\mathsf{Str}(\mathcal{B}/S(A))/\cong| \cdot \frac{t^A}{|\mathrm{Aut}(A)|}, \\ (S \circ U)(t) &= \sum_{N \in \mathcal{S}}{}' |\mathsf{Str}(\mathcal{A}/N)/\cong| \cdot |\mathsf{Str}(\mathcal{B}/N)/\cong| \cdot \frac{t^N}{|\mathrm{Aut}(N)|}. \end{aligned}$$

Thus $(S \circ U)(t)$ means the Hadamard product:

$$\left(\sum_{m=0}^{\infty} \frac{a_m}{m!} t^m\right) * \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n\right) = \left(\sum_{n=0}^{\infty} \frac{a_n b_n}{n!} t^n\right) \tag{3.13.a}$$

4 The Dress condition and KS-categories.

(4.1) The Dress condition. We consider the following condition on a category $\mathcal{E}$ with finite coproducts:

The Dress condition: *In the following commutative diagram in which the bottom row is a coproduct diagram:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] & B \arrow[l] \arrow[d] \\ X \arrow[r] & X + Y & Z, \arrow[l] \end{tikzcd}$$

the top row is a coproduct diagram if and only if the two squares are both pullbacks.

The Dress condition above appeared in the definition of the "based category "on which A.Dress [Dre 73] developed an abstract theory of Burnside rings and Mackey functors. His "based category" $\mathcal{B}$ is defined to be a locally finite category satisfying the following conditions:

(BC1) $\mathcal{B}$ has any finite limits and finite coproducts.
(BC2) $\mathcal{B}$ is a KS-category (4.3).
(BC3) The Dress condition holds.
(BC4) An initial object $\emptyset$ is strict, that is, any morphism into $\emptyset$ is an isomorphism.

Note that (BC4) follows from other conditions (Lemma 4.2 (3)). An important example of based categories is a locally finite topos (4.5). Thus our theory of polynomials and power series can be partially developed on a basic category.

(4.2) Lemma. *Let $\mathcal{E}$ be a category with finite coproducts satisfying the Dress condition. Then the following hold/:*

(1) *The canonical injection $X \longrightarrow X+Y$ is a monomorphism.*
(2) *Coproducts are disjoint, that is, the following diagram is a pullback:*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & X \arrow[d] \\ Y \arrow[r] & X+Y. \end{tikzcd}$$

(3) *The initial object $\emptyset$ is strict, that is, any $X \longrightarrow \emptyset$ is an isomorphism.*
(4) *Assume that $\mathcal{E}$ has products. Then the distributive law holds:*

$$(X+Y)\times C \cong X\times C + Y\times C, \quad \emptyset\times C\cong\emptyset. \tag{4.2.a}$$

In particular, the isomorphism classes of objects of $\mathcal{E}$ form a semi-ring.
(5) *Assume that $\mathcal{E}$ has pullbacks. Let $I\in C$ be a connected object of $\mathcal{E}$, that is, I is not initial and $I\cong A+B$ implies that either A or B is an initial object. Then the hom-functor* $\mathrm{Hom}(I,-)$ *preserves finite coproducts.*
(6) *Assume that $\mathcal{E}$ has any pullback diagrams. Then the pullback functors i^*, j^* of canonical injections $X \xrightarrow{i} X+Y \xleftarrow{j} Y$ induce an equivalence:*

$$\langle i^*, j^*\rangle : \mathcal{E}/(X+Y)\cong \mathcal{E}/X\times\mathcal{E}/Y. \tag{4.2.b}$$

(7) *Assume further that $\mathcal{E}$ is a finite complete category with a terminal object $\mathbf{1}$, then there is a bijection:*

$$\mathrm{Sub}_{\neg\neg}(X)\cong \mathrm{Hom}(X,\mathbf{1}+\mathbf{1}),$$

where $\mathrm{Sub}_{\neg\neg}(X)$ *is the set of complementary elements of the poset* $\mathrm{Sub}(X)$ *of subobjects of X.*
(8) *The comma category $\mathcal{E}/Z$ over any object Z satisfies the Dress condition.*

PROOF. (1) (2) Apply the Dress condition to the following commutative diagram whose top row is a coproduct diagram:

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d] & \emptyset \arrow[l] \arrow[d] \\
X \arrow[r] & X+Y & Y. \arrow[l]
\end{tikzcd}$$

(3) Assume that there exists a morphism $X \longrightarrow \emptyset$. Since in the following diagram

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d] & X \arrow[d] & X \arrow[l, "1"'] \arrow[d] \\
\emptyset \arrow[r] & \emptyset & \emptyset, \arrow[l]
\end{tikzcd}$$

the bottom row is a coproduct diagram and the two squares are pullback diagrams, the Dress condition implies that the top row is also a coproduct diagram. By the disjointness (2), we have

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & X \arrow[d, "1"] \\
X \arrow[r, "1"] & X.
\end{tikzcd}$$

is a pullback square, and so we conclude that X is isomorphic to the initial object $\emptyset$.

(4) In the following diagram, the two squares are pullback diagram:

$$\begin{tikzcd}
X \times C \arrow[r, "i\times 1"] \arrow[d] & (X+Y) \times C \arrow[d] & Y \times C \arrow[l, "j\times 1"'] \arrow[d] \\
X \arrow[r, "i"] & X+Y & Y. \arrow[l, "j"']
\end{tikzcd}$$

Thus the Dress condition implies the required isomorphism. The remainder isomorphism $\emptyset \times C \cong \emptyset$ follows from the fact that $\emptyset$ is strictly initial.

(5) Consider the following diagram with two pullback squares:

$$\begin{tikzcd} A \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & B \arrow[d] \\ X & X+Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

Then the Dress condition, the top row is a coproduct diagram. Since I is connected, either A or B is an initial object. Thus any morphism $I \longrightarrow X + Y$ go through either X or Y, and so

$$\mathrm{Hom}(I, X+Y) \cong \mathrm{Hom}(I, X) + \mathrm{Hom}(I, Y).$$

Since an initial object $\emptyset$ is strictly initial, $\mathrm{Hom}(I, \emptyset) = \emptyset$. Hence $\mathrm{Hom}(I, -)$ preserves all finite coproducts.

(6) This easily follows from the Dress condition. An inverse of $\langle i^*, j^* \rangle$ is given by

$$(A \longrightarrow X, B \longrightarrow Y) \longmapsto (A + B \longrightarrow X + Y).$$

(7) This follows from the Dress condition and the injectivity of injections.
(8) This follows from the fact that the localization functor

$$\Sigma_Z : \mathcal{E}/Z \longrightarrow \mathcal{E}; (C \longrightarrow Z) \longmapsto C,$$

which is a left adjoint functor to the pullback functor $Z^* : C \longmapsto C \times Z$, preserves and reflects not only coproducts but also pullbacks.

(4.3) KS-categories. We here write again the definition and some properties of KS-categories given in the first paper. Let $\mathcal{E}$ be a category with any finite coproducts. An *connected* object J is a non-initial object satisfying the condition that $J \cong A + B$ implies that A or B is an initial object. The full subcategory of connected objects of $\mathcal{E}$ is denoted by $\mathsf{Con}(\mathcal{E})$. The category $\mathcal{E}$ with finite coproducts is a *KS-category* if the following strict Krull-Schmidt type property:

(KS) *Any object X is isomorphic to a coproduct of some finite number of connected objects, and a coproduct decomposition of X is unique in the following sense: if $X = I_1 + \cdots + I_m = J_1 + \cdots + J_n$ are two coproduct decompositions into connected objects with canonical injections $i_\alpha : I_\alpha \longrightarrow X$ and $j_\beta : J_\beta \longrightarrow X$, then $m = n$ and there exist a permutation $\pi \in S_n$ and isomorphisms $f_\alpha : I_\alpha \xrightarrow{\cong} J_{\pi(\alpha)}$ such that $j_{\pi(\alpha)} f_\alpha = i_\alpha$ for all $\alpha = 1, \ldots, n$.*

For example, the category of finite sets is a KS-category. But the category of finite dimensional vector spaces is not a KS-category; the category does not satisfy the it strict Krull-Schmidt property.

A skeletally small and locally finite KS-category $\mathcal{E}$ satisfies the following exponential formula (the main theorem of the first paper of this series):

$$\mathcal{E}(t) = \exp\left(\mathsf{Con}(\mathcal{E})(t)\right). \tag{4.3.a}$$

(4.4) Theorem. *Assume that a skeletally small category $\mathcal{E}$ with finite coproducts.and its full subcategory $\mathcal{C}$ satisfy the following conditions:*

(a) *Any object of $\mathcal{E}$ is isomorphic to a coproduct of some finitely many objects of $\mathcal{C}$.*
(b) *For each $I \in \mathcal{C}$, the hom-functor*

$$\mathrm{Hom}(I, -) : \mathcal{E} \longrightarrow \mathsf{Set}; X \longmapsto \mathrm{Hom}(I, X) \tag{4.4.a}$$

preserves finite coproducts. In particular, $\mathrm{Hom}(I, \emptyset) = \emptyset$.

Then $\mathcal{E}$ is a KS-category with $\mathsf{Con}(\mathcal{E})$ *equivalent to $\mathcal{C}$ satisfying the Dress condition.*

Proof. We divide the proof into three parts.
Step 1. Let Φ be the product of the hom-functors $\mathrm{Hom}(I, -)$ on $\mathcal{E}$ with $I \in \mathcal{C}$:

$$\begin{aligned} \Phi = \prod \mathrm{Hom}(I, -) \quad &: \quad \mathcal{E} \longrightarrow {\prod_{I \in \mathcal{C}}}' \mathsf{Set} \\ &; \quad X \longmapsto (\mathrm{Hom}(I, X))_I, \end{aligned} \tag{4.4.b}$$

where I runs over all complete representatives of isomorphism classes of objects of $\mathcal{C}$, We first show the following three facts:

(A1) *The functor Φ reflects isomorphisms.*

(A2) *The functor Φ preserves and reflects all existing finite limits.*

(A3) *The functor Φ preserves and reflects all finite coproducts.*

To prove the statement (A1), let $f : X \longrightarrow Y$ be a morphism of $\mathcal{E}$ such that

$$\widehat{f}_I : \mathrm{Hom}(I, X) \xrightarrow{\cong} \mathrm{Hom}(I, Y); \lambda \longmapsto f \circ \lambda$$

is a bijection for any $I \in \mathcal{C}$. Let A be any object of $\mathcal{E}$ and $A = \coprod_\alpha I_\alpha$ a decomposition of A into a finite coproduct of objects I_α of $\mathcal{C}$. Then we have a commutative diagram of sets and maps:

$$\begin{tikzcd}
\widehat{f}: \quad \mathrm{Hom}(A,X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(A,Y) \arrow[d, "\cong"] \\
\prod_\alpha \widehat{f}_{I_\alpha} : \prod_\alpha \mathrm{Hom}(I_\alpha, X) \arrow[r, "\cong"] & \prod_\alpha \mathrm{Hom}(I_\alpha, Y).
\end{tikzcd}$$

Thus f is mapped to an isomorphism $\widehat{f}$ of the functor category $[\mathcal{E}^{\mathrm{op}}, \mathsf{Set}]$ by the Yoneda embedding, and so f is an isomorphism.

We next prove the statement (A2). As is well-known, a representable functor preserves any finite limits, so is Φ. Conversely, in order to prove that Φ reflects finite limits, let $(p_k : L \longrightarrow X_k)_k$ be a cone in $\mathcal{E}$ on a finite diagram $X : \mathcal{K} \longrightarrow \mathcal{E}$ such that

$$(\mathrm{Hom}(I,L) \xrightarrow{\widehat{p}_k} \mathrm{Hom}(I,X_k) : \lambda \longmapsto p_k\lambda)_k \tag{4.4.c}$$

is a universal cone for any $I \in \mathcal{C}$. We have to show that the original cone $(p_k : L \longrightarrow X_k)_k$ is also a universal cone. Let $(f_k : A \longrightarrow X_k)_k$ be any family of morphisms of $\mathcal{E}$ such that $X(\lambda) \circ f_k = f_l$ for any $\lambda : k \longrightarrow l$ in $\mathcal{K}$, By the universality of (4.4.c), we have that *when A is an object of $\mathcal{C}$*, there exists a unique $g : A \longrightarrow L$ satisfying the condition $f_k = p_k \circ g$ for all k.

$$\begin{tikzcd}
A \arrow[r, "f_k"] \arrow[d, dotted, "g"'] \arrow[dr, "f_l"' near start] & X_k \arrow[d, "X(\lambda)"] & k \arrow[d, "\lambda"] \\
L \arrow[r, "p_l"'] \arrow[ur, "p_k"' near end] & X_l & l
\end{tikzcd}$$

We need to show the existence of such a g for any object A of $\mathcal{E}$; so let

$$A = \coprod_\alpha I_\alpha, \quad I_\alpha \in \mathcal{C}$$

be a decomposition of A into a finite coproduct of some objects of $\mathcal{C}$ and

$$i_\alpha : I_\alpha \longrightarrow A$$

a canonical injection. Then for each I_α, since

$$(I_\alpha \xrightarrow{i_\alpha} A \longrightarrow X_k)_k$$

is a cone with vertex I_α and the diagram X, we have that there exists a unique morphism g_α from I_α to L which makes the following diagram commutative for all k:

$$\begin{tikzcd} I_\alpha \arrow[r, "i_\alpha"] \arrow[d, "\exists g_\alpha"'] & A = \coprod I_\alpha \arrow[d, "f_k"] \\ L \arrow[r, "p_k"] & X_k. \end{tikzcd}$$

Take a unique morphism $g : A \longrightarrow L$ such that $g_\alpha = g \circ i_\alpha$ for each α. Then this morphism satisfies $p_k \circ g = f_k$ for each $k \in \mathcal{K}$ by using the universality of coproducts. The uniqueness of such a morphism $g : A \longrightarrow L$ easily follows from the universality of limits. Hence (A2) is proved.

Finally, we prove (A3). Since $\mathrm{Hom}(I, -)$ $(I \in \mathcal{C})$ preserves finite coproducts by the assumption (b), Φ also preserves coproducts. The initial object $\emptyset$ of $\mathcal{E}$ is strictly initial in $\mathcal{E}$ by the special case $\mathrm{Hom}(I, \emptyset) = \emptyset$ for $I \in \mathcal{C}$ of the assumption (b). We have to prove that Φ reflects finite coproducts. By the assumption (a), if $\mathrm{Hom}(I, X) = \emptyset$ for any $I \in \mathcal{C}$, then X is an initial object. Thus Φ reflect initial objects. Next, take a pair of morphisms

$$X \xrightarrow{f} S \xleftarrow{g} Y$$

such that

$$\langle \widehat{f}, \widehat{g} \rangle : \mathrm{Hom}(I, X) + \mathrm{Hom}(I, Y) \xrightarrow{\cong} \mathrm{Hom}(I, S)$$

is isomorphic for any $I \in \mathcal{C}$. Then since $\mathrm{Hom}(I, X + Y) \cong \mathrm{Hom}(I, X) + \mathrm{Hom}(I, Y)$ by (b), the functor Φ maps the morphism $\langle f, g \rangle : X + Y \longrightarrow S$ to an isomorphism, and so $\langle f, g \rangle$ is an isomorphism by (A.1).

STEP 2. In order to prove the Dress condition, take a commutative diagram in which the bottom row is a coproduct diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] & B \arrow[l] \arrow[d] \\ X \arrow[r] & X + Y & Z. \arrow[l] \end{tikzcd}$$

We have to show that the top row is a coproduct diagram if and only if the two squares are both pullbacks. Applying the functor Φ to this diagram, we have the following commutative diagram:

$$\begin{tikzcd}
\Phi(A) \arrow[r] \arrow[d] & \Phi(C) \arrow[d] & \Phi(B) \arrow[l] \arrow[d] \\
\Phi(X) \arrow[r] & \Phi(X+Y) & \Phi(Z). \arrow[l]
\end{tikzcd}$$

Since the Dress condition holds in the product category of **Set** and *Phi* preserves coproducts by (A3), the top row of this diagram is a coproduct diagram if and only if the two squares are pullback diagrams. Thus the Dress condition for $\mathcal{E}$ follows from the fact that Φ preserves and reflects pullbacks and finite coproducts (Step 1).

STEP 3. It remains to prove that $\mathcal{E}$ is a KS-category. Suppose X has two finite coproduct decompositions as follows:

$$X \cong \coprod_{\alpha \in M} I_\alpha \cong \coprod_{\beta \in N} J_\beta, \quad I_\alpha, J_\beta \in \mathcal{C},$$

with canonical injections

$$i_\alpha : I_\alpha \longrightarrow X, \quad j_\beta : J_\beta \longrightarrow X.$$

Since by (b) there is a bijection

$$\begin{array}{ccc}
\coprod_{\beta \in N} \mathrm{Hom}(I_\alpha, J_\beta) & \xrightarrow{\cong} & \mathrm{Hom}(I_\alpha, X) \\
(f : I_\alpha \longrightarrow J_\beta) & \longmapsto & j_\beta \circ f
\end{array}$$

for each $\alpha \in M$, we have that $i_\alpha = j_{\alpha'} \circ f_\alpha$ for a unique $\alpha' \in N$ and a unique $f_\alpha : I_\alpha \longrightarrow J_{\alpha'}$. Similarly, for each $\beta \in N$, there exist a unique $\beta' \in M$ and a unique $g_\beta : J_\beta \longrightarrow I_{\beta'}$ such that $j_\beta = i_{\beta'} \circ g_\beta$:

$$\begin{tikzcd}
I_\alpha \arrow[r, "f_\alpha"] \arrow[d, "i_\alpha"'] & J_{\alpha'} \arrow[dl, "j_{\alpha'}"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
J_\beta \arrow[r, "g_\beta"] \arrow[d, "j_\beta"'] & I_{\beta'} \arrow[dl, "i_{\beta'}"] \\
X. &
\end{tikzcd}$$

If $\alpha'' \neq \alpha$, then by Lemma 4.2 (2) the following diagram is a pullback:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & I_{\alpha''} \arrow[d, "i_{\alpha''}"] \\ I_\alpha \arrow[r, "i_\alpha"] & X. \end{tikzcd}$$

However there is the morphism $g_{\alpha'} \circ f_\alpha : I_\alpha \longrightarrow I_{\alpha''}$ over X, and so $I_\alpha = \emptyset$ by Lemma 4.2 (3). This contradicts the assumption (c) of this lemma. Thus

$$\alpha'' = \alpha, \quad \beta'' = \beta, \quad \text{for all } \alpha \in M \text{ and } \beta \in N,$$

that is, M and N are bijectively correspondent each other via $\alpha \leftrightarrow \alpha'$. We now have a commutative diagram as follows:

$$\begin{tikzcd} I_\alpha \arrow[r, shift left, "g_{\alpha'} f_\alpha"] \arrow[r, shift right, "1"'] & I_\alpha \arrow[r, "i_\alpha"] & X, \end{tikzcd}$$

and so $g_{\alpha'} \circ f_\alpha = 1$ by the Dress condition and Lemma 4.2 (1); similarly we have $f_{\beta'} \circ g_\beta = 1$, proving the uniqueness of the coproduct decomposition of X. This also implies that objects of $\mathcal{C}$ is connected in $\mathcal{E}$. Hence we complete the proof of the theorem.

(4.5) Toposes. Topos theory is an important source of the Krull-Schmidt property and the Dress condition. A complete and cocomplete category $\mathcal{E}$ is a *topos* if it is Cartesian closed and has a *subobject classifier* $\mathbf{1} \xrightarrow{t} \Omega$. Thus for any $Y, Z \in \mathcal{E}$, there is an *exponentiation* $Z^Y \in \mathcal{E}$ with natural bijection:

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y), \tag{4.5.a}$$

and there is a natural bijection

$$\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega). \tag{4.5.b}$$

Pulling back the truth morphism $t : \mathbf{1} \longrightarrow \Omega$ by χ gives the subobject of X corresponding to a morphism $\chi : X \longrightarrow \Omega$. Thus the class of subobjects of X is really a set. For topos theory, see the books [Joh 77], [MclMo 92].

When $\mathcal{E}$ is a topos, $\mathrm{Sub}_{\neg\neg}(X)$ appeared in Lemma 4.2(7) is the Boolean subalgebra of the distributive lattice $\mathrm{Sub}(X)$ consisting of $\neg\neg$-closed subobjects of X with respect to the topology $\neg\neg : \Omega \longrightarrow \Omega$ [Joh 77, Theorem 5.17]. The category **Set** of all sets and maps is a topos; in fact, an exponentiation Y^X is the set of maps from X to Y; a subobject classifier Ω is a two-point sets $\{0, 1\}$ with injection $t : \{1\} \longrightarrow \{0, 1\}$ as the truth value; (4.5.b) is given by the bijective correspondence between subset $A \subseteq X$ and its characteristic map. A topos can be viewed as the category of sets under intuitional logic, and so we can treat toposes like the category of sets.

(4.6) Locally finite toposes. Toposes we will mainly concern in this series are locally finite (and skeletally small) toposes, that is, toposes with geometric morphism into Set_f ([MclMo 92, Chapter VI]). By the existence of subobject classifier, each object of a locally finite topos has a finite number of subobjects:

$$|\mathrm{Sub}(X)| = |\mathrm{Hom}(X, \Omega)| < \infty. \tag{4.6.a}$$

The finite poset $\mathrm{Sub}(X)$ of subobjects of $X \in \mathcal{E}$ has the structure of a Heyting algebra ([Joh 77, Definiton 5.12], [MclMo 92, Section I.8, Theorem 1 of Section IV.8]). Note that a finite Heyting algebra H is nothing but a distributive lattice with negation operator $\neg$; here for an element $x \in H$ the negation $\neg x$ is an element satisfying

$$x \wedge y = 0 \iff y \leq \neg x.$$

We list some examples of locally finite toposes which will play some important parts in this serial papers. They are all KS-categories satisfying the Dress condition.

(1) Set_f, the category of finite sets and mappings.

(2) ${\mathsf{Set}_f}^G$, the category of finite G-sets and G-maps on a (finite or infinite) group G. In this topos, Y^X is the set of all maps from X to Y with diagonal G-action; and the subobject classifier $\Omega = \{0, 1\}$ is the two elements set. This topos is *Boolean*, that is, any subobject is complementary ([Joh 77, Proposition 5.14]).

(3) ${\mathsf{Set}_f}^S$, the category of finite S-sets and S-maps on a *finite* monoid S. This category is not Boolean unless S is a group.

(4) $\mathsf{RForest}_{\leq h}$, the category of (finite unlabeled) rooted forests of height at most h for a fixed h. Unfortunately, the category of all rooted forests of unlimited height is not a topos because of the lack of subobject classifier.

(5) DGraph, the category of (finite) digraphs.

(6) Graph, the category of (finite undirected) graphs possibly with loops and multi-edges.

(7) $\mathsf{BiPGraph}$, the category of (finite) bi-partite graphs possibly with multi-edges.

(8) $[\Gamma, \mathsf{Set}_f]$, the functor category on a finite category Γ. This example contains all of the above examples (1)–(7) except for ${\mathsf{Set}_f}^G$ for an infinite group

G. In this category, an exponentiation Y^X and the subobject classifier Ω are easily defined on objects by using Yoneda's lemma as follows:

$$\begin{aligned} Y^X(i) &= \mathrm{Hom}(H^i, Y^X) \cong \mathrm{Hom}(H^i \times X, Y), & (4.6.\mathrm{b}) \\ \Omega(i) &= \mathrm{Hom}(H^i, \Omega) = \mathrm{Sub}(H^i), & (4.6.\mathrm{c}) \end{aligned}$$

where $H^i := \mathrm{Hom}(i, -) \in [\Gamma, \mathsf{Set}]$ is the hom-functor.

(9) Surj, the category of surjections between finite sets. A morphism between two surjections $(X \overset{p}{\twoheadrightarrow} X')$ and $(q : Y \overset{q}{\twoheadrightarrow} Y')$ is a pair (f, f') of maps $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ such that $f' \circ p = q \circ f$. The exponentiation q^p is defined by

$$q^p := (\mathrm{Hom}(p, q) \twoheadrightarrow \mathrm{Map}(X', Y')). \qquad (4.6.\mathrm{d})$$

the subobject classifier Ω is defined by

$$(\{0, 1/2, 1\} \twoheadrightarrow \{0, 1\} : 0 \mapsto 0, 1/2 \mapsto 1, 1 \mapsto 1).$$

Note that we used the axiom of choice for finite sets in order to prove the exponentiation q^p is an object of Surj; thus for example, surjective G-maps of finite G-sets does not form a topos if G is non-trivial.

This category Surj is equivalent to the category ${\mathsf{Set}_f}^E$ of finite E-sets and E-maps, where $E = \{1, e\}$ $(e^2 = e)$ is a monoid with an idempotent e; in fact, for any surjection $X \overset{p}{\twoheadrightarrow} X'$, we have a E-set $X \cup X'$ with E-action defined by $e \cdot x := p(x), e \cdot x' := x'$ for $x \in X$ and $x' \in X'$.

(4.7) Lemma. *Any topos satisfies the Dress condition, and so the conclusions of* Lemma 4.2 *hold.*

PROOF. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d, "h"'] & B \arrow[l] \arrow[d] \\ X \arrow[r] & X + Y & Z, \arrow[l] \end{tikzcd}$$

where the lower row is a coproduct diagram. If both squares are pullback diagrams, then the top row is a coproduct diagram; in fact, this follows, for example, from the fact that the pullback functor h^* has a right adjoint functor Π_h. See [Joh 77, Corollary 1.43] or [MclMo 92, Section IV.7, Theorem 2]. Conversely, assume that the top row in the above diagram is a coproduct diagram, and so $C = A + B$. Taking the pullback, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & A+B \arrow[d, "1"] & B \arrow[l, tail] \arrow[d] \\
A' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & A+B \arrow[d, "h"] \arrow[dr, phantom, "\text{P.B.}"] & B' \arrow[l, tail] \arrow[d] \\
X \arrow[r] & X+Y & Z. \arrow[l]
\end{tikzcd}$$

The middle row is a coproduct diagram as is already proved. Thus in the Heyting algebra Sub(A+B), we have

$$A' + B' = A + B, A \cap B = \emptyset, A' \cap B' = \emptyset,\ A \subseteq A',\ B \subseteq B',$$

and so $A = A'$ and $B = B'$ as subobjects. This implies that the both squares in the above diagram is pullback diagrams.

(4.8) Lemma. *A locally finite topos is a KS-category satisfying the Dress condition.*

PROOF. Since a topos has finite coproducts (by the definition of a topos or by the existence theorem), it will suffice to show that a locally topos $\mathcal{E}$ satisfies the two conditions (a), (b) of Theorem 4.4. Let X be any object of $\mathcal{E}$. Then $\mathrm{Sub}_{\neg\neg}(X)$, the finite poset of complementary subobjects of X, is a Boolean algebra, and so its largest element X is a disjoint union of minimal elements of $\mathrm{Sub}_{\neg\neg}(X)$, which prove X is a finite coproduct of some connected objects, and so (a) holds. The second condition (b) follows from Lemma 4.2 and Lemma 4.7. (b) follows also from the fact that a pullback functor preserves a coproducts ([Joh 77, Theorem 1.42, Corollary 1.37]).

5 Exponentials of categories.

(5.1) Exponentials. For an arbitrary category $\mathcal{C}$, its *exponential* $\mathsf{EXP}(\mathcal{C})$ is the category whose object has the form (N, Y), where N is a finite set and $Y : N \longrightarrow \mathcal{C}$ is a functor from N viewed as a discrete category (and so Y is an N-indexed objects $(Y_\alpha)_{\alpha\in N}$ of $\mathcal{C}$), and whose morphism has the form

$(f,\tau) : (N,Y) \longrightarrow (N',Y')$, where $f : N \longrightarrow N'$ is a map and $\tau : Y \longrightarrow Y' \circ f$ is a natural transformation; thus τ is a family of maps

$$\tau_\alpha : Y_\alpha \longrightarrow Y'_{f(\alpha)}, \quad (\alpha \in N).$$

The composition is defined by

$$(f',\tau') \circ (f,\tau) := (f' \circ f, (\tau' * f) \circ \tau), \tag{5.1.a}$$

that is,

$$((\tau' * f) \circ \tau)_\alpha := \tau'_{f(\alpha)} \circ \tau_\alpha : Y_\alpha \xrightarrow{\tau_\alpha} Y'_{f(\alpha)} \xrightarrow{\tau'_{f(\alpha)}} Y''_{f'f(\alpha)}. \tag{5.1.b}$$

For the sake of convenience, we often denote an object (N,Y) by the following symbol:

$$\bigoplus_{\alpha \in N} Y_\alpha$$

if there is no confusion between this notation and the notation of biproducts inside $\mathcal{C}$.

Each functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ induces a functor

$$\begin{aligned} \bigoplus F \quad &: \quad \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathsf{EXP}(\mathcal{D}) \\ &: \quad \bigoplus_{\alpha \in N} Y_\alpha \longmapsto \bigoplus_{\alpha \in N} F(Y_\alpha), (f,\tau) \longmapsto (f, F * \tau), \end{aligned}$$

where $F * \tau = (F(\tau_\alpha) : F(Y_\alpha) \longrightarrow F(Y'_{f(\alpha)})_{\alpha \in N}$ is the horizontal composition of functors. Furthermore, if $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors and $\theta : F \Longrightarrow G$ a natural transformation, then a natural transformation $\bigoplus \theta$ from $\bigoplus F$ to $\bigoplus G$ is defined by $(\bigoplus \theta)_{(N,Y)} := (1_N, \theta * Y)_{N,Y}$. Here we used the symbol $\bigoplus F$ instead of $\mathsf{EXP}(F)$ in order to avoid the confusion with that of the exponentials of functors into a category with finite coproducts. See (5.3).

(5.2) The embedding functor and the rank functor. There is an *embedding functor*:

$$Emb : \mathcal{C} \longrightarrow \mathsf{EXP}(\mathcal{C}); A \longmapsto ([1], 1 \longmapsto A). \tag{5.2.a}$$

This functor gives a full embedding of $\mathcal{C}$ into $\mathsf{EXP}(\mathcal{C})$ and we usually identify $\mathcal{C}$ as a subcategory of $\mathsf{EXP}(\mathcal{C})$.

There is a *rank* functor

$$Rank : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathsf{Set}_f; \bigoplus_{\alpha \in N} Y_\alpha \longmapsto N. \tag{5.2.b}$$

Thus in the viewpoint of the following lemma, $N = Rank(\bigoplus_{\alpha \in N} Y_\alpha)$ is the set of connected components of $\bigoplus_{\alpha \in N} Y_\alpha$.

(5.3) Exponentials of functors. Let $\mathcal{S}$ be a category with finite coproducts and $S : \mathcal{C} \longrightarrow \mathcal{S}$ a functor. Then the *exponential* $\mathsf{EXP}(S)$ of S is defined to be a functor

$$\mathsf{EXP}(S) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{S};\ \bigoplus_{\alpha\in N} Y_\alpha \longmapsto \coprod_{\alpha\in N} S(Y_\alpha). \tag{5.3.a}$$

Here, for a morphism $(f, \sigma) : \bigoplus_{\alpha\in M} X_\alpha \longrightarrow \bigoplus_{\alpha\in N} Y_\alpha$ in $\mathsf{EXP}(\mathcal{C})$, the morphism $\mathsf{EXP}(S)(f,\sigma)$ in $\mathcal{S}$ is defined by the following commutative diagram:

$$\begin{tikzcd}
S(X_\alpha) \arrow[r, "inj"] \arrow[d, "S(\sigma_\alpha)"'] & \coprod_{\alpha\in M} S(X_\alpha) \arrow[d, "\mathsf{EXP}(S)(f,\sigma)"] \\
S(Y_{f(\alpha)}) \arrow[r, "inj"'] & \coprod_{\beta\in N} S(Y_\beta).
\end{tikzcd}$$

The functor $\mathsf{EXP}(S) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{S}$ is an extension of S unique up to isomorphisms of functors:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "Emb"] \arrow[d, "S"'] & \mathsf{EXP}(\mathcal{C}) \arrow[dl, "\mathsf{EXP}(S)"] \\
\mathcal{S}. &
\end{tikzcd}$$

This functor $\mathsf{EXP}(S)$ preserves any finite coproducts (see Lemma 5.4 below). In particular, for the inclusion functor $Incl : \mathcal{C} \longrightarrow \mathcal{S}$ into a category $\mathcal{S}$ with finite coproducts, we denote $\mathsf{EXP}(Incl)$ simply by $\coprod$:

$$\coprod := \mathsf{EXP}(Incl) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{S};\ \bigoplus_{\alpha\in N} Y_\alpha \longrightarrow \coprod_{\alpha\in N} Y_\alpha. \tag{5.3.b}$$

(5.4) Lemma. (1) *The category* $\mathsf{EXP}(\mathcal{C})$ *is a KS-category satisfying the Dress condition. In particular, the category* $\mathsf{EXP}(\mathcal{C})$ *has finite coproducts. Furthermore, the embedding functor Emb induces the following equivalence:*

$$\mathsf{Con}(\mathsf{EXP}(\mathcal{C})) \cong \mathcal{C}. \tag{5.4.a}$$

(2) *The functor* $\bigoplus F : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathsf{EXP}(\mathcal{D})$ *induced from a functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *preserves finite coproducts.*

(3) *The category $\mathcal{C}$ has finite coproducts if and only if the embedding Emb has a left adjoint functor. If $\mathcal{C}$ has finite coproducts, then such an adjoint functor $\coprod$ is defined by*

$$\coprod : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{C};\ \bigoplus_{\alpha\in N} Y_\alpha \longmapsto \coprod_{\alpha\in N} Y_\alpha. \tag{5.4.b}$$

PROOF. (1) The coproduct is given by

$$\bigoplus_{\alpha\in N} Y_\alpha + \bigoplus_{\alpha\in N'} Y'_\alpha = \bigoplus_{\alpha\in N+N'} Y''_\alpha,\quad Y''_\alpha = \begin{cases} Y_i & \text{if } \alpha\in N \\ Y'_\alpha & \text{if } \alpha \in N'. \end{cases} \tag{5.4.c}$$

An initial object of $\mathsf{EXP}(\mathcal{C})$ has the form $(\emptyset,\emptyset)$, that is, the pair of the emptyset and the empty functor $\emptyset \longrightarrow \mathcal{C}$ from the empty category. By Theorem 4.4, it will suffice to prove that for each $A \in \mathcal{C}$ the hom-functor

$$\mathrm{Hom}(Emb(A), -) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathsf{Set}$$

preserves finite coproducts. In fact, any morphism in $\mathsf{EXP}(\mathcal{C})$

$$(f,\tau) : Emb(A) = ([1], A) \longrightarrow (N, Y)$$

decides a unique $\alpha_1 = f(1)$ and a morphism $\tau_1 : A \longrightarrow X_{\alpha_1}$; conversely, any $\alpha_1 \in N$ and $\tau_1 : A \longrightarrow X_{\alpha_1}$ decides a morphism of $\mathsf{EXP}(\mathcal{C})$ as above. This implies directly that the above hom-functor preserves finite coproducts, as required.
(2) This follows easily from (5.4.c).
(3) It is easily proved that if $\mathcal{C}$ has any finite coproducts, then the functor $\coprod$: $\mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{C}$ is a left adjoint functor to $Emb : \mathcal{C} \hookrightarrow \mathsf{EXP}(\mathcal{C})$. Conversely, assume that there exists a left adjoint functor F to Emb. Let $\widetilde{Y} := \bigoplus_{\alpha\in N} Y_\alpha$ be any object of $\mathsf{EXP}(\mathcal{C})$ and put $B := F(\bigoplus Y_\alpha)$. Note that for any object A of $\mathcal{C}$, a morphism $(f,\tau) : \widetilde{Y} \longrightarrow Emb(A)$ is bijectively corresponding to a family of morphisms $\tau_\alpha : Y_\alpha \longrightarrow A$ $(\alpha \in N)$. In particular, applying this to the unit of the adjunction, we have a family of morphisms $\eta_\alpha : Y_\alpha \longrightarrow B$. Then it is easily follows from the properties of adoint functors that $B = F(\widetilde{Y})$ together with injections $\eta_\alpha : Y_\alpha \longrightarrow B$ is a coproduct of Y_α $(\alpha \in N)$.

(5.5) Remark. (1) The embedding functor Emb does not preserves finite coproducts, and $Emb(\emptyset)$ is not an initial object even if $\mathcal{C}$ has an initial object $\emptyset$. The object $\bigoplus_{\alpha\in N}\emptyset$, where $\emptyset$ is an initial object of $\mathcal{C}$, is not an initial object of $\mathsf{EXP}(\mathcal{C})$ when N is not empty..

(2) Let CAT (resp. CAT_+) be the bicategory of all skeletally small categories (resp. all skeletally small categories with any finite coproducts and functors

preserving finite coproducts). Let $\mathcal{S}$ be a category with finite coproducts. Then we have a 2-adjunction between 2-comma categories:

$$\begin{tikzcd} \mathsf{CAT}/\mathcal{S} \arrow[r, shift left, "\mathsf{EXP}"] \arrow[r, phantom, "\perp"] & \mathsf{CAT}_+/\mathcal{S}. \arrow[l, shift left, "forgetful"] \end{tikzcd}$$

The universality of $\mathsf{EXP}(S)$ follows from this adjunction.

(3) Even if $\mathcal{E}$ is a KS-category, $\mathsf{EXP}(\mathsf{Con}(\mathcal{E}))$ is not always equivalent to $\mathcal{E}$; KS-categories of the form $\mathsf{EXP}(\mathcal{C})$ are characterized by the following theorem.

(5.6) An expansion of $\mathsf{EXP}(\mathcal{C})$. Let $S : \mathcal{C} \longrightarrow \mathcal{S}$ be a functor into a KS-category $\mathcal{S}$. Then the category $\mathcal{C}^n/n!$ defined in (3.5) is a subcategory of $\mathsf{EXP}(\mathcal{C})$ and so we have an embedding of categories:

$$E : \sum_{n\geq 0} \mathcal{C}^n/n! \hookrightarrow \mathsf{EXP}(\mathcal{C}); \{X_i\} \longmapsto \bigoplus_{i=1}^{n} X_i.. \tag{5.6.a}$$

This functor is surjective on objects and maps each automorphism groups isomorphically, and so E gives an equivalence of categories of isomorphisms:

$$E : \sum_{n\geq 0} (\mathcal{C}_{\text{iso}})^n/n! \cong (\mathsf{EXP}(\mathcal{C}))_{\text{iso}} . \tag{5.6.b}$$

Furthermore, the following diagram is commutative:

$$\begin{tikzcd} \sum_{n\geq 0} \mathcal{C}^n/n! \arrow[r, hook, "E"] \arrow[d, "\sum S^n/n!"'] & \mathsf{EXP}(\mathcal{C}) \arrow[dl, "\mathsf{EXP}(S)"] \\ \mathcal{S}. & \end{tikzcd}$$

This means that if $S(A) \neq \emptyset$, then the generating functions of $\mathsf{EXP}(\mathcal{C})$ is well-defined and

$$\sum_{n\geq 0} (S^n/n!)(t) = \mathsf{EXP}(S)(t). \tag{5.6.c}$$

By Lemma 3.6, we have that

$$\mathsf{EXP}(S)(t) = \sum_{n\geq 0} S(t)^n/n! = \exp(S(t)). \tag{5.6.d}$$

Another proof of this formula will be found in the proof of Lemma 6.2.

(5.7) Example. (1) If $\mathbf{1}$ is the trivial category with only one object $*$ and only one morphism, then $\mathsf{EXP}(\mathbf{1})$ is equivalent to Set_f by the identification $\bigoplus_{\alpha \in N} *$ with N. Note that the generating function of $\mathbf{1}$ (resp. Set_f) is t (resp. $\exp(t)$).

(2) Let E a set viewed as a discrete category. Then an object of $\mathsf{EXP}(E)$ is a map from a finite set M to E, and so it can be written as $(x_i)_{i \in M}$. A morphism $f : (x_i)_{i \in M} \longrightarrow (y_j)_{j \in N}$ is a map $f : M \longrightarrow N$ such that $x_i = y_{f(i)}$ for each $i \in M$. The isomorphism class of an object $(x_i)_{i \in N}$ is identified with the multiset $\{x_i \mid i \in N\}$.

(3) Let Gp be the category of finite groups. Then $\mathsf{EXP}(\mathsf{Gp})$ is equivalent to the category of finite groupoids, where two groupoids are viewed to be isomorphic if they are equivalent as categories:

$$\mathsf{EXP}(\mathsf{Gp}) \cong \mathsf{Gpd}.$$

(4) Let RTree (resp. $\mathsf{RForest}$) be the category of finite rooted trees (resp. forests); and let $V_{RT} : \mathsf{RTree} \longrightarrow \mathsf{Set}_f$ and $V_{RF} : \mathsf{RForest} \longrightarrow \mathsf{Set}_f$ be the forgetful functors which assigns the underlying vertex sets. Then the vategory $\mathsf{EXP}(\mathsf{RTree})$ is equivalent to $\mathsf{RForests}$ and the functor $\mathsf{EXP}(V_{RT})$ is isomorphic to V_{RF}:

$$\mathsf{EXP}(\mathsf{RTree}) \cong \mathsf{RForest}, \; \mathsf{EXP}(V_{RT}) \cong V_{RF}.$$

(5) Let $\mathrm{Sub}(G)$ be the subgroup category of a (finite or infinite) group G, that is, its object is a subgroup of G, a morphism $xH : H \longrightarrow K$ is a coset $xH \in G/H$ such that $xHx^{-1} \subseteq K$, and the composition is defined by

$$(H \xrightarrow{xH} K \xrightarrow{yK} L) := (H \xrightarrow{yxH} L).$$

Then $\mathsf{EXP}(\mathrm{Sub}(G))$ is equivalent to ${\mathsf{Set}_f}^G$, the category of finite G-sets.

(6) Let G be a group (or semi-group) viewed as a category. Then $\mathsf{EXP}(G)$ is equivalent to the groupoid that is equivalent to the disjoint union of the direct product groups G^n $(n = 0, 1, 2, \dots)$.

(5.8) Theorem. *Let $\mathcal{E}$ be a category with finite coproducts and $\mathcal{C}$ a subcategory of $\mathcal{E}$. Then $\mathcal{E}$ is equivalent to* $\mathsf{EXP}(\mathcal{C})$ *via the functor*

$$\coprod := \mathsf{EXP}(\mathit{Incl}) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{E}; \; \bigoplus_{\alpha} I_\alpha \longmapsto \coprod_{\alpha} I_\alpha.$$

(see 5.3) *if and only if the following two conditions hold*:

(a) *Any object of $\mathcal{E}$ is isomorphic to a coproduct of some finitely many objects of $\mathcal{C}$.*
(b) *For each $I \in \mathcal{C}$, the hom-functor*

$$\mathrm{Hom}(I,-) : \mathcal{E} \longrightarrow \mathsf{Set}; X \longmapsto \mathrm{Hom}(I,X) \tag{5.8.a}$$

preserves finite coproducts.

PROOF. When $\coprod$ gives an equivalence between $\mathcal{E}$ and $\mathsf{EXP}(\mathcal{C})$, (a) and (b) easily follows from the definition of $\mathsf{EXP}(\mathcal{C})$. See also Lemma 5.4(1), (2) and (5.4.c). Conversely, assume that the conditions (a) and (b) hold. We have to prove that $\coprod$ is an equivalence. Note that by Theorem 4.4, $\mathcal{E}$ is a KS-category with $\mathsf{Con}(\mathsf{EXP}(\mathcal{C})) \cong \mathcal{C}$.

STEP 1. Any object of $\mathcal{E}$ is isomorphic to the image of an object by the functor $\coprod$. This directly follows from the condition (a).

STEP 2. We prove that the functor $\coprod$ reflects the isomorphism of objects. To prove it, assume that $\coprod(\bigoplus_{\alpha\in M} I_\alpha) \cong \coprod(\bigoplus_{\beta\in N} J_\beta)$, so that

$$\coprod_{\alpha\in M} I_\alpha \cong \coprod_{\beta\in N} J_\beta.$$

Since $\mathcal{E}$ is a KS-category with $\mathsf{Con}(\mathsf{EXP}(\mathcal{C})) \cong \mathcal{C}$ (Theorem 4.4), there is a bijection $f : M \longrightarrow N$ and isomorphisms $\sigma : I_\alpha \longrightarrow J_{f(\alpha)}$ for each $\alpha \in M$; this means that $(f,\sigma) : \bigoplus_{\alpha\in M} I_\alpha \longrightarrow \bigoplus_{\beta\in N} J_\beta$ gives an isomorphism in $\mathsf{EXP}(\mathcal{C})$. Hence the functor $\coprod$ reflects the isomorphism of objects.

STEP 3. We prove the faithfulness of the functor $\coprod$. Assume that $\coprod(f,\sigma) = \coprod(f',\sigma')$ for

$$(f,\sigma),(f',\sigma') : \bigoplus_{\alpha\in M} I_\alpha \longrightarrow \bigoplus_{\beta\in N} J_\beta.$$

Then the following diagram is commutative:

$$\begin{tikzcd} I_\alpha \arrow[r, "\sigma_\alpha"] \arrow[d, "\sigma'_\alpha"'] & J_{f(\alpha)} \arrow[d, "inj"] \\ J_{f'(\alpha)} \arrow[r, "inj"'] & \coprod J_\beta. \end{tikzcd}$$

Since I_α is not an initial object, it follows from Lemma 4.2 (2) and Theorem 4.4 that $f(\alpha) = f'(\alpha)$ for any $\alpha \in M$, and so $f = f'$. Furthermore, since the

injections $inj : J_{f(\alpha)} \longrightarrow \coprod J_\beta$ is a monomorphism by Lemma 4.2(1), we have that $\sigma_\alpha = \sigma'_\alpha$, proving the faithfulness of the functor $\coprod$.
STEP 4. Finally, we prove that $\coprod$ is full. Let

$$h : \coprod\left(\bigoplus_{\alpha\in M} I_\alpha\right) \longrightarrow \coprod\left(\bigoplus_{\beta\in N} J_\beta\right)$$

be a morphism of $\mathcal{E}$, that is,

$$h : \coprod_{\alpha\in M} I_\alpha \longrightarrow \coprod_{\beta\in N} J_\beta,$$

and let

$$\begin{aligned} \tau_\alpha &: I_\alpha \longrightarrow X := \coprod_{\alpha\in M} I_\alpha \\ \rho_\beta &: J_\beta \longrightarrow Y := \coprod_{\beta\in N} J_\beta \end{aligned}$$

be canonical injections. By the assumption (b), we have that for each $\alpha \in M$ there exists a unique $f(\alpha) \in N$ which makes the following diagram commutative:

$$\begin{tikzcd} I_\alpha \arrow[r, "\tau_\alpha"] \arrow[d, "\sigma_\alpha"'] & X \arrow[d, "h"] \\ J_{f(\alpha)} \arrow[r, "\rho_{f(\alpha)}"] & Y, \end{tikzcd}$$

and so $h = \coprod(f, \sigma)$, that is, the functor $\coprod$ is full.

Now, the theorem follows from Step 1 – 4,

(5.9) Corollary. *In the above theorem, the condition (c) can be replaced by the condition that $\mathcal{E}$ has pullbacks and satisfies the Dress condition.*

PROOF. This directly follows from the theorem and Lemma 4.2(5).

6 A characterization of KS-categories.

(6.1) Lemma. *Let $\mathcal{S}$ be a category satisfying the following conditions:*

(a) *$\mathcal{S}$ has finite coproducts.*

(b) *Any coproduct is disjoint, that is, the following diagram is a pullback:*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & A \arrow[d, "i"] \\ B \arrow[r, "j"] & A+B. \end{tikzcd}$$

(c) *The canonical injection $A \longrightarrow A+B$ of any coproduct is a monomorphism.*
(d) *The initial object $\emptyset$ is strict, that is, a morphism $A \longrightarrow \emptyset$ is an isomorphism.*
(e) *Each object has a finitely many complementary subobjects.*

(*Note that these conditions* (a) – (d) *follows from the Dress condition and that* (a) – (e) *hold for any locally finite toposes.*) *Let $F : \mathcal{C} \longrightarrow \mathcal{S}$ be a faithful functor with finite fibers such that $F(X) \neq \emptyset$ for every $X \in \mathcal{C}$. Then $\mathsf{EXP}(F) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{S}$ is also a faithful functor with finite fibers.*

PROOF: STEP 1. In order to prove that the faithfulness of $\mathsf{EXP}(F)$, let

$$(f,\tau),(f',\tau') : \bigoplus_{\alpha\in M} X_\alpha \longrightarrow \bigoplus_{\beta\in N} Y_\beta$$

be morphisms such that $\mathsf{EXP}(F)(f,\tau) = \mathsf{EXP}(F)(f',\tau')$, so that the following diagrams are commutative:

$$\begin{tikzcd} F(X_i) \arrow[r, "u_i"] \arrow[d, "F(\tau_i)"] & \coprod_{a\in M} F(X_a) \arrow[d, "\mathsf{EXP}(F)(f,\tau)"] \\ F(Y_{f(i)}) \arrow[r, "v_{f(i)}"] & \coprod_{b\in N} F(Y_b), \end{tikzcd} \qquad \begin{tikzcd} F(X_i) \arrow[r, "u_i"] \arrow[d, "F(\tau'_i)"] & \coprod_{a\in M} F(X_a) \arrow[d, "\mathsf{EXP}(F)(f',\tau')"] \\ F(Y_{f'(i)}) \arrow[r, "v_{f'(i)}"] & \coprod_{b\in N} F(Y_b), \end{tikzcd}$$

where u_i and v_j are canonical injections into coproducts. Thus we have the following commutative diagram:

$$\begin{tikzcd} F(X_i) \arrow[r, "F(\tau_i)"] \arrow[d, "F(\tau'_i)"] & F(Y_{f(i)}) \arrow[d, "v_{f(i)}"] \\ F(Y_{f'(i)}) \arrow[r, "v_{f'(i)}"] & \coprod_{b\in N} F(Y_b). \end{tikzcd}$$

Suppose $f(i) \neq f'(i)$ for a number $i \in M$; then the disjointness of coproducts of $\mathcal{S}$ implies that there is a morphism $F(X_i) \longrightarrow \emptyset$, and so $F(X_i) = \emptyset$ by (d), which contradicts the assumption about F. Thus $f(i) = f'(i)$ for all $i \in M$. The commutativity of the above diagram and the assumption (c) implies that $F(\tau_i) = F(\tau_i')$, and so $\tau_i = \tau_i'$ because F is faithful. Hence $(f, \tau) = (f', \tau')$, as required.

STEP 2. Next, we show that $\mathsf{EXP}(F)$ has finite fibers. Let S be an object of $\mathcal{S}$. If

$$\mathsf{EXP}(F)\left(\bigoplus_{\alpha \in N} Y_\alpha\right) = \coprod_{i \in N} F(Y_i) \cong S,$$

then each $F(Y_i)$ is isomorphic to a complementary and non-initial subobject of S; by (b), (c) and (e), S has only finitely many coproduct factorizations, and the number of components $|N|$ is bounded by a function of $|\mathrm{Sub}_{\neg\neg}(S)| < \infty$, the number of complementary subobjects of S. Furthermore, since F has finite fibers, for any factorization $S = \coprod_{i \in N} S_i$, there are finitely many $Y_i \in \mathcal{C}$ up to isomorphisms such that $F(Y_i) \cong S_i$. Hence for any $S \in \mathcal{S}$, there are finitely many isomorphism classes of $\bigoplus_{i \in N} Y_i \in \mathsf{EXP}(\mathcal{C})$ such that $\coprod F(N, Y) \cong S$. The lemma is proved.

(6.2) Lemma. *Let $\mathcal{C}$ be a skeletally small category with finite automorphism groups. Furthermore, let S be a functor with finite fibers from such a category $\mathcal{C}$ to a KS-category $\mathcal{S}$. Then the generating function $\mathsf{EXP}(S)(t)$ is well-defined and*

$$\mathsf{EXP}(\mathcal{C})(t) = \exp(\mathcal{C}(t)), \ \mathsf{EXP}(S)(t) = \exp(S(t)). \tag{6.2.a}$$

PROOF. Since $\mathsf{EXP}(\mathcal{C})$ is a KS-category with $\mathsf{Con}(\mathsf{EXP}(\mathcal{C})) \cong \mathcal{C}$ (see Lemma 5.4). The first formula directly follows from Lemma 5.8 of the first paper [Yos 98]. See (5.6) for another proof. Next, by the assumption, the fiber of $\mathsf{EXP}(S)$ at each $N \in \mathcal{S}$:

$$\left\{ \bigoplus_{\alpha} X_\alpha \in \mathsf{EXP}(\mathcal{C})/\cong \;\middle|\; \coprod_{\alpha \in N} S(X_\alpha) \cong N \right\}$$

is a finite set, and so the generating function $\mathsf{EXP}(S)(t)$ is well-defined.

(6.3) Exponential formula, another form. Since a skeletally small category $\mathcal{C}$ is equivalent to the full subcategory of connected objects of the KS-category $\mathsf{EXP}(\mathcal{C})$, the exponential formula

$$\mathcal{E}(t) = \exp(\mathsf{Con}(\mathcal{E})) \tag{6.3.a}$$

for a KS-category $\mathcal{E}$ (see Part I of this series) can be rewritten as follows:

$$\mathsf{EXP}(\mathcal{C})(t) = \exp(\mathcal{C} \xrightarrow{Emb} \mathsf{EXP}(\mathcal{C}))(t)). \tag{6.3.b}$$

Let $\mathcal{S}$ be a skeletally small category satisfying the conditions of Lemma 6.1 and $F : \mathcal{C} \longrightarrow \mathcal{S}$ be a faithful with finite fibers such that $F(X) \neq \emptyset$ for every $X \in \mathcal{C}$. Then applying the functor $\mathsf{EXP}(F) : \mathsf{EXP}(\mathcal{C}) \longrightarrow \mathcal{S}$ to this formula, we have

$$\mathsf{EXP}(F)(t) = \exp(F(t)), \tag{6.3.c}$$

Both sides are well-defined by Lemma 6.1.

(6.4) Example. (1) $\mathsf{EXP}(\mathsf{Set}_f)$ is equivalent to $\mathsf{Set}_f^{\cdot \rightarrow \cdot}$, the category of maps between two finite sets. This equivalence is given by

$$\bigoplus_{\alpha \in N} X_\alpha \longmapsto \left(\coprod_{\alpha \in N} X_\alpha \longrightarrow N\right),$$
$$(X \xrightarrow{f} N) \longmapsto \bigoplus_{\alpha \in N} f^{-1}(\alpha).$$

Let Id : $\mathsf{Set}_f \longrightarrow \mathsf{Set}_f$ be the identity functor. Then we have the functor

$$\mathsf{EXP}(\mathrm{Id}) : \mathsf{EXP}(\mathsf{Set}_f) \longrightarrow \mathsf{Set}_f; \bigoplus_{\alpha \in N} X_\alpha \longmapsto \coprod_{\alpha \in N} X_\alpha.$$

This functor does not have finite fibers, and so the generating function $\mathsf{EXP}(\mathrm{Id})(t)$ is not well-defined. Since $\mathrm{Id}(\emptyset) = \emptyset$, the assumption of Lemma 6.1 does not hold.

(2) Let $\mathsf{Set}_{f\geq 1}$ be the category of non-empty finite sets and maps. Then we have an equivalence

$$\mathsf{EXP}(\mathsf{Set}_{f\geq 1}) \cong \mathsf{Surj} \tag{6.4.a}$$

defined similarly as in the first example. Here Surj is the category of surjections between finite sets (4.6(9)). Contrarily to (1), the generating function $F(t)$ of the inclusion functor $F : \mathsf{Set}_{f\geq 1} \longrightarrow \mathsf{Set}_f$ is well-defined by Lemma 6.1.

(6.5) Theorem. *Let $\mathcal{E}$ be a skeletally small and locally finite category with finite coproducts and $\mathcal{C}$ a full subcategory of $\mathcal{E}$. Then $\mathcal{E}$ is a KS-category with $\mathsf{Con}(\mathcal{E})$ equivalent to $\mathcal{C}$ over $\mathcal{E}$ if and only if the following two conditions hold:*

(a) $\mathcal{E}(t) = \exp(\mathcal{C}(t))$ *in* $\boldsymbol{Q}[[\mathcal{E}^{op}/\cong]]$, *the ring of formal power series.*
(b) *If* $i \circ \sigma = i$ *for any canonical injection* $i : I \longrightarrow I + A$ *with* $I \in \mathcal{C}$ *and* $A \in \mathcal{E}$ *and for an automorphism* σ *of* I, *then* $\sigma = 1$.

PROOF. STEP 1. Assume first that $\mathcal{E}$ is a KS-category with $\mathsf{Con}(\mathcal{E}) \cong \mathcal{C}$ over $\mathcal{E}$. Then (a) follows from the exponential formula given in Part I([Yos 98, Theorem 5.8]); and (b) follows directly from the definition of a KS-category.

STEP 2. Conversely, assume that (a) and (b) hold. Expanding the right hand side of (a), we have that for any $X \in \mathcal{E}$,

$$\sum_{(n_I):\coprod n_I I \cong X} \frac{|\mathrm{Aut}(X)|}{\prod'_{I \in \mathcal{C}} |\mathrm{Aut}(I)|^{n_I} n_I!} = 1, \tag{6.5.a}$$

where the summation is taken over indexed nonnegative integers $(n_I) \in \boldsymbol{N}^{\mathcal{C}/\cong}$ such that $\sum n_I < \infty$. Since the right hand side of (a) is well-defined, the summation (6.5.a) is a finite sum, and so there exist only finitely many indexed non-negative integers $\boldsymbol{n} = (n_I) \in \boldsymbol{N}^{\mathcal{C}/\cong}$ such that

$$X \cong \coprod_{I \in \mathcal{C}}{}' n_I I \quad \text{(finite coproduct)}. \tag{6.5.b}$$

In particular, $\emptyset \notin \mathcal{C}$ and

$$X + Y \cong X \quad \Longrightarrow \quad Y \cong \emptyset. \tag{6.5.c}$$

Now for such an $\boldsymbol{n} = (n_I)$, there is a canonical group homomorphism

$$\varphi_n : \prod_I{}' \mathrm{Aut}(I) \text{ wr } S_{n_I} \longrightarrow \mathrm{Aut}\left(\coprod_{I \in \mathcal{C}}{}' n_I I\right) \tag{6.5.d}$$

Thus by the equation (6.5.a), in order to prove that $\mathcal{E}$ is a KS-category for which $\mathcal{C}$ is equivalent to $\mathsf{Con}(\mathcal{E})$ over $\mathcal{E}$, it is suffice to show that φ_n is a monomorphism for any $\boldsymbol{n}$.

Let $(\pi_I, (\alpha_{I,j})_j)_I$ be an element of $\mathrm{Ker}(\varphi_n)$; put $X := \coprod_I n_I I$ and let $\tau_{I,j} : I \longrightarrow X$, $j = 1, \ldots, n_I$ be the canonical injections. Then $\pi_I \in S_{n_I}$ and the diagram

$$\begin{tikzcd} I \arrow[r, "\alpha_{Ij}"] \arrow[d, "\tau_{Ij}"'] & I \arrow[dl, "\tau_{I,\pi_I(j)}"] \\ X & \end{tikzcd}$$

is commutative.

Step 3. We prove that $\pi_I = 1$ for any $I \in \mathcal{C}$. Suppose false; we may assume that $\pi_K(1) = 2$ for some $K \in \mathcal{C}$. We first prove the following statement:

$$\mathrm{Hom}(X, Y) \neq \emptyset \quad \Longrightarrow \quad |\mathrm{Hom}(K, Y)| = 1 \tag{6.5.e}$$

for any $Y \in \mathcal{E}$; in particular, $\tau_{K,1} = \tau_{K,2} = \cdots$. Let $f : X \longrightarrow Y$ and put

$$\rho_{I,j} := f \circ \tau_{I,j} : I \longrightarrow Y \quad \text{for } (I, j) \neq (K, 1);$$

we take as $\rho_{K,1} : K \longrightarrow Y$ an arbitrary morphism. Then there is a unique morphism $g : X \longrightarrow Y$ such that $\rho_{I,j} = g\tau_{I,j}$, and so we have

$$\begin{aligned} \rho_{K,1} &= g \circ \tau_{K,1} = g \circ \tau_{K,2} \circ \alpha_{K,1} = \rho_{K,2} \circ \alpha_{K,1} \\ &= f \circ \tau_{K,2} \circ \alpha_{K,1} = f \circ \tau_{K,1}, \end{aligned}$$

as required. See the following diagram:

$$\begin{tikzcd}
K \arrow[rr, "\tau_{K1}"] \arrow[dd, equal] \arrow[dr, "\alpha_{K1}"'] & & X \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\
& K \arrow[ur, "\tau_{K2}"] \arrow[dr, "\rho_{K2}"] & \\
K \arrow[rr, "\rho_{K1}"'] & & Y
\end{tikzcd}$$

We can now derive a contradiction from $\pi_K \neq 1$. By the statement (6.5.e), we have that

$$K \xrightarrow{\tau_{K1}} X \xleftarrow{1} X$$

is a coproduct diagram, that is, $K + X \cong X$ which contradicts (6.5.c). Thus $\pi_I = 1$ for all $I \in \mathcal{C}$.

Step 4. We can now complete the proof. Since $\pi_I = 1$ for all I, we have

$$\tau_{I,j} = \tau_{I,j} \circ \alpha_{I,j}$$

for all (I, j). Thus by the assumption (b), we have that $\alpha_{I,j} = 1$ for all (I, j), proving the injectivity of φ_n. The theorem is proved.

7 Subtitution.

(7.1) Substitution $T(\mathcal{E})$. Except for very special cases (e.g. plethysm composition [BerLL 98, Section 1.4, Definition 3]), there is no satisfactory definition of substitution (composition) $T(S)$ of two functors $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T : \mathcal{F} \longrightarrow \mathcal{T}$. In fact, if such a substitution $T(S)$ was defined, then its generating function should naturally satisfy the formula $T(S)(t) = T(S(t))$; however, we can not compose and make a series $T(S(t))$ an element $S(t) \in R[[\mathcal{S}^{\mathrm{op}}/\cong]]$ and an element $T(t) \in R[[\mathcal{T}^{\mathrm{op}}/\cong]]$ in general. Our definition of polynomials and power series is inconvenient for substitution. In order to do more freely substitution of any functors, we need to extend the notion of polynomials and power series of categories as is studied in another paper of this series.

Here we give the definition of the substitution in the case where T is a functor to Set_f. In this case, since the generating function of T is simply viewed as a usual rational power series in one variable, there is no difficulty in the definition of substitutions if $\mathcal{S}$ has finite coproducts.

Let $\mathcal{E}$ be a category and $T : \mathcal{F} \longrightarrow \mathcal{S}$ a functors. Then the *substitution* $T(\mathcal{E})$ is defined by the pullback diagram of categories:

$$\begin{tikzcd} T(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & \mathcal{F} \arrow[d, "T"] \\ \mathsf{EXP}(\mathcal{E}) \arrow[r, "Rank"] & \mathsf{Set}_f, \end{tikzcd}$$

where $Rank : (N, X) \longmapsto N$ is the rank functor defined in (5.2). Thus an object of $T(\mathcal{E})$ has the form (B, X), where B is an object of $\mathcal{F}$ and $X : T(B) \longrightarrow \mathcal{E}$ is a functor from the discrete category $T(B)$ to $\mathcal{E}$, that is, X is an $T(B)$-indexed object $(X_j)_{j\in T(B)} \in \mathcal{E}^{T(B)}$. A morphism from (B, X) to (B', X') has the form (g, τ), where $g : B \longrightarrow B'$ and $\tau : X \longrightarrow X' \circ T(g)$, that is, τ is a family of morphisms $\tau_j : X_j \longrightarrow X'_{T(g)(j)}$ $(j \in T(B))$. The composition of (g, τ) and (g', τ') is defined by

$$(g', \tau') \circ (g, \tau) := (g' \circ g, (\tau' * T(g)) \circ \tau),$$

where

$$((\tau' * g) \circ \tau)_j = \tau'_{T(g)(j)} \circ \tau_j : X_j \xrightarrow{\tau_j} X'_{T(g)(j)} \xrightarrow{\tau'_{T(g)(j)}} X''_{T(g')T(g)(j)}.$$

(7.2) Substitution $T(S)$. Next, let $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be functors. We only assume that $\mathcal{S}$ has finite coproduct. The required functor $T(S)$ is now defined by the composition of functors:

$$T(S) : T(\mathcal{E}) \longrightarrow \mathsf{EXP}(\mathcal{E}) \xrightarrow{\mathsf{EXP}\ (S)} \mathcal{S};\ (B, X) \longmapsto \coprod_{j \in T(B)} S(X_j). \qquad (7.2.a)$$

The substitutions $T(\mathcal{E}), T(S)$ are often called *compositions* and denoted by $T \circ \mathcal{E}, T \circ S$. But we do not use this notation and this terminology in this paper in order to avoid the confusion of them with those of compositions of functors,

(7.3) Example. (1) As a special case, consider the identity functor $T = \mathrm{Id} : \mathcal{F} = \mathsf{Set}_f \longrightarrow \mathsf{Set}_f$. Then the composition of Id and $\mathcal{E}$ (resp. S) is given by the exponentials of $\mathcal{E}$ (resp. S):

$$\mathrm{Id}(\mathcal{E}) = \mathsf{EXP}(\mathcal{E}), \quad \mathrm{Id}(S) = \mathsf{EXP}(S).$$

Thus we can write EXP for the identity functor Id.

(2) Sometimes the substitution $T(\mathcal{E})$ is called a *wreath product* of categories with action T and is denoted by $\mathcal{E}\mathrm{wr}^T\mathcal{F}$. See Wells [We 80], [Wel 88], Kelly [Kel 74]. However, in these papers, the wreath products only of set-valued (and category-valued) functors are defined.

(3) Let G and H be groups (or more generally semi-groups with identity elements) and T be an H-set. We view a group as categories with only one object as usual; furthermore, we view H-set T as a functor from H to Set_f. Then the substitution $T(G)$ is nothing but the wreath product $G\mathrm{wr}^T H$ of groups. Next, take a G-set S which we view as a functor from G to Set_f. Then the functor $T(S)$ is the product $S \times T$ as a $G\mathrm{wr}^T H$-set. Note that $\mathsf{EXP}(G)$ is equivalent to the disjoint union of the direct products G^n $(n \geq 0)$ and that its rank functor maps the subcategory G^n to $[n]$. See [Mel 95]. The pullback diagram in (7.1) is same as the following pullback diagram of groups:

$$\begin{tikzcd} G\mathrm{wr}^T H \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & H \arrow[d] \\ G\mathrm{wr}S_n \arrow[r] & S_n, \end{tikzcd}$$

where $n := |T|$.

(4) Let **Chain** be the category of finite chains, that is, finite totally ordered sets, and $T : \mathsf{Chain} \longrightarrow \mathsf{Set}_f$ the forgetful functor. Let $\mathcal{E}$ be a category. Then since any finite chain is isomorphic to some $\{1 < \cdots < m\}$, the composition $T(\mathcal{E})$ is, up to equivalence, the category of ordered objects $(X_1, \ldots, X_m)$ $(m \geq 0,\ X_i \in \mathcal{E})$. Any morphism from $(X_1, \ldots, X_m)$ to $(Y_1, \ldots, Y_n)$ has the form $(f, (\tau_i))$, where f is an order-preserving map from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ and $\tau_i : X_i \longrightarrow Y_{f(i)}$ $(1 \leq i \leq m)$ are morphisms in $\mathcal{E}$.

(5) Let P and Q be posets. We view a poset as a category by the standard way. As a functor $T : Q \longrightarrow \mathsf{Set}_f$, we take the hom-functor $\mathrm{Hom}_Q(q, -)$:

$$T(x) \;:=\; \begin{cases} \{*_{q,x}\} & \text{if } q \leq x \\ \emptyset & \text{else,} \end{cases}$$

$$T(x \leq y) : T(x) \longrightarrow T(y);\ *_{q,x} \longmapsto *_{q,y}\,.$$

Then the composition $T(P)$ is the following subposet of the product poset $Q \times P$:

$$T(P) = \{(y, x) \mid Q \times P, y \geq q\}.$$

(7.4) Proposition. *Let $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be faithful functors with finite fibers. Assume that $\mathcal{S}$ satisfies the conditions* (a)–(e) *of Lemma* 6.1 *and that $S(A) \neq \emptyset$ for every $A \in \mathcal{E}$. Then the following holds*:

(1) *$T(S)$ is a faithful functor with finite fibers.*

(2) *$T(S(t)) = (T(S))(t)$. (Note that since $\mathcal{S}$ has finite coproducts and $T(t)$ is viewed as a rational power series in one variable as usual, we can substitute $S(t)$ for $T(t)$.)*

(3) *Let I be an object of $\mathcal{S}$ such that the hom-functor* $\mathrm{Hom}(I, -)$ *preserves coproducts. Assume that $T(g)$ is a monomorphism for any morphism g in $\mathcal{F}$. Then there is a natural isomorphism $\partial_I^*(T(S)) \cong T'(S) \cdot \partial_I^*(S)$.*

(4) *Let I be an object of $\mathcal{S}$ such that the hom-functor* $\mathrm{Hom}(I, -)$ *preserves coproducts. Then there is a natural isomorphism $\partial_I^*(T(S)) \cdot S \cong T^*(S) \cdot \partial_I^*(S)$.*

PROOF. (1) Since S is faithful and has finite fibers, so is the functor $S(\mathcal{F}) \longrightarrow \mathsf{EXP}(\mathcal{F})$. Furthermore, by Lemma 6.1, $\mathsf{EXP}(T)$ is also faithful with finite fibers. Hence so is $S(T)$.

(2) By the definitions of exponential generating functions and the composition $T(S)$, we have

$$\begin{aligned} T(S(t)) &= \sum_{B \in \mathcal{F}}{}' \frac{1}{|\mathrm{Aut}(B)|} (S(t))^{|T(B)|} \\ &= \sum_{B \in \mathcal{F}}{}' \frac{1}{|\mathrm{Aut}(B)|} \left(\sum_{X \in \mathcal{E}}{}' \frac{1}{|\mathrm{Aut}(X)|} t^{S(X)} \right)^{|T(B)|} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{B\in\mathcal{F}}{}' \frac{1}{|\mathrm{Aut}(B)|} \sum_{(X_i)\in\mathcal{E}^{T(B)}}{}' \frac{1}{\prod_i |\mathrm{Aut}(X_i)|} t^{\coprod_i S(X_i)} \\
&= \sum_{(B,X)\in T(\mathcal{E})}{}' \frac{1}{|\mathrm{Aut}(B,X)|} t^{T(S)(B,X)} \\
&= (T(S))(t).
\end{aligned}
$$

(3) We first remember the constructions of $\partial_I^*(T(S))$, $T'(S)$ and $\partial_I^*(S)$ for functors $S : \mathcal{E} \longrightarrow \mathcal{S}$ and $T.\mathcal{F} \longrightarrow \mathbf{Set}_f$ satisfying the assumption of (3).
(i) The I-th derivation $\partial_I^*(T(S)) : I \downarrow T(\mathcal{E}) \longrightarrow \mathcal{S}$ is defined as follows. An object of $I \downarrow (T(S))$ has the form

$$(\alpha, B, X), \quad B \in \mathcal{F}, X \in \mathcal{E}^{T(B)}, \alpha \in \mathrm{Hom}_{\mathcal{S}}(I, \coprod_{b\in T(B)} S(X_b)),$$

and a morphism from (α, B, X) to (α', B', X') has the form

$$
\begin{aligned}
&(g, \sigma) : (\alpha, B, X) \longrightarrow (\alpha', B', X'), \\
&\text{where } g : B \longrightarrow B', \sigma : X \longrightarrow X' \circ T(g) \text{ with } (\textstyle\coprod S)(g,\sigma)\circ\alpha = \alpha',
\end{aligned}
$$

where $(\coprod S)(g, \sigma)$ is defined by the following commutative diagram:

$$
\begin{array}{ccc}
S(X_b) & \xrightarrow{\mathrm{inj}} & \coprod_{b\in T(B)} S(X_b) \\
{\scriptstyle S(\sigma_b)}\downarrow & & \downarrow{\scriptstyle (\coprod S)(g,\sigma)} \\
S(X'_{T(g)(b)}) & \xrightarrow{\mathrm{inj}} & \coprod_{b'\in T(B')} S(X'_{b'})
\end{array}
$$

Now, the functor $\partial_I^* T(S)$ is defined as follows:

$$\partial_I^*(T(S)) : I \downarrow (T(S)) \longrightarrow \mathcal{S}; (\alpha, B, X) \longmapsto \coprod_{b\in T(B)} S(X_b).$$

(ii) The functor $T'(S) : T'(\mathcal{E}) \longrightarrow \mathcal{S}$ is defined as follows. An object of $T'(\mathcal{E})$ has the form

$$(B, t, X), \quad B \in \mathcal{F}, t \in T(B), X : T(B) - \{t\} \longrightarrow \mathcal{E},$$

and a morphism from $(B, t, X) \longrightarrow (B', t', X')$ has the form

$$
\begin{aligned}
&(g, \sigma) : (B, t, X) \longrightarrow (B', t', X'), \\
&\text{where } g : B \longrightarrow B', \sigma : X \longrightarrow X' \circ T(g) \text{ with } T(g)(t) = t',
\end{aligned}
$$

and so we have

$$T'(S) : T'(\mathcal{E}) \longrightarrow \mathcal{S}; (B,t,X) \longmapsto \coprod_{b \in T(B)-\{t\}} S(X_b).$$

(iii) The functor $\partial_I^*(S) : I \downarrow S \longrightarrow \mathcal{S}$ is defined as follows. Remember that an object of the comma category $I \downarrow S$ has the form

$$(\beta, A), \quad A \in \mathcal{E}, \beta \in \mathrm{Hom}_{\mathcal{S}}(I, S(A)),$$

and a morphism has the form

$$f : (\beta, A) \longrightarrow (\beta', A'), \quad \text{where } f : A \longrightarrow A' \text{ with } S(f) \circ \beta = \beta'.$$

Then $\partial_I^*(S)$ is defined by

$$\partial_I^*(S) : I \downarrow S \longrightarrow \mathcal{S}; (\beta, A) \longmapsto S(A).$$

Now, the required isomorphism is given by the functor $\Phi : (I \downarrow S) \times T'(\mathcal{E}) \longrightarrow I \downarrow (T(S))$ defined by

$$\begin{aligned}
&\Phi : ((\beta, A), (B,t,X)) \longmapsto (\alpha, B, \widetilde{X}), \quad \text{where} \\
&\widetilde{X}_b := \begin{cases} X_b & \text{if } b \neq t \\ A & \text{if } b = t, \end{cases} \\
&\alpha : I \xrightarrow{\beta} S(A) = T(\widetilde{X}_t) \longrightarrow \coprod_{b \in T(B)} S(\widetilde{X}_b); \\
&\Phi \left(((\beta, A), (B,t,X)) \xrightarrow{(f,(g,\sigma))} ((\beta, A), (B,t,X)) \right) \\
&\qquad := \left((\alpha, B, \widetilde{X}) \xrightarrow{(g,\widetilde{\sigma})} (\alpha', B', \widetilde{X}') \right), \quad \text{where} \\
&\qquad \widetilde{\sigma} := (\widetilde{\sigma}_b)_{b \in T(B)} : \widetilde{X}_b \longrightarrow \widetilde{X}'_{T(g)(b)}, \quad \widetilde{\sigma}_b := \begin{cases} \sigma_b & \text{if } b \neq t \\ f & \text{if } b = t. \end{cases}
\end{aligned}$$

The inverse $\Psi : I \downarrow (T(S)) \longrightarrow (I \downarrow S) \times T'(\mathcal{E})$ of Φ is given by the following:

$$\Psi : (\alpha, B, \widetilde{X}) \longmapsto ((\beta, A), (B,t,X)),$$

where $t \in T(B)$ and β is a morphism such that α is factorized as follows:

$$\alpha : I \xrightarrow{\beta} S(\widetilde{X}_t) \xrightarrow{inj} \coprod_{b \in T(B)} S(\widetilde{X}_b),$$

and then $A := \widetilde{X}_t$, and $X_b := \widetilde{X}_b$ for all $b \in T(B) - \{t\}$. Note that such $t \in T(B)$ uniquely exists because

$$\coprod_{b \in T(B)} \mathrm{Hom}(I, S(\widetilde{X}_b)) \cong \mathrm{Hom}(I, \coprod_{b \in T(B)} S(\widetilde{X}_b))$$

by the assumption of this lemma. It is easily checked that Ψ is a morphism of functors.

Furthermore, Φ gives an isomorphis from $\partial_I^*(S)\cdot T'(S)$ to $\partial_I^*(T(S))$. In fact, for any $(\beta, A)\in I\downarrow S$ and $(B,t,X)\in T'(\mathcal{E})$, we have

$$\begin{aligned}\partial_I^*(S)\cdot T'(S) \ &: \ ((\beta,A),(B,t,X)) \longmapsto S(A)+\coprod_{b\in T(B)-\{t\}} S(X_b),\\ \partial_I^*(T(S))\circ\Phi \ &: \ ((\beta,A),(B,t,X)) \longmapsto \coprod_{b\in T(B)} S(\widetilde{X}_b),\end{aligned}$$

and so $\partial_I^*(T(S))\circ\Phi = \partial_I^*(S)\cdot T'(S)$ on objects. We can also check that this equation holds on morphisms. Hence Φ gives the isomorphism between $\partial_I^*(T(S))\cong\partial_I^*(S)\cdot T'(S)$. We omit the proof of the naturality of Φ on S,T.
(4) This isomorphism is constructed by the similar way to (3). The functors $\partial_I^*(T(S)) : I\downarrow T(\mathcal{E}) \longrightarrow \mathcal{S}$ and $\partial_I^*(S) : I\downarrow S \longrightarrow \mathcal{S}$ have been given in the proof of (3). We first remember the definiton of the functor $T^*(S) : T^*(\mathcal{E}) \longrightarrow \mathcal{S}$. An object of the category $T^*(\mathcal{E})$ has the form

$$(B,b,X), \quad B\in\mathcal{F}, b\in T(B), X\in\mathcal{E}^{T(B)}.$$

Then the functor $T^*(S) : T^*(\mathcal{E}) \longrightarrow \mathbf{Set}_f$ is defined by

$$T^*(S) : (B,b,X) \longmapsto \coprod_{b\in T(B)} S(X_b).$$

Next, we construct the following functors which give an isomorphism of $T^*(S)\cdot\partial_I^*(S)$ and $\partial_I^*(T(S))\cdot S$:

$$(I\downarrow S)\times T^*(\mathcal{E}) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} (I\downarrow T(\mathcal{E}))\times E.$$

We define a functor Φ by

$$\Phi : ((\alpha,A),(B,c,X)) \longmapsto ((\beta,\widetilde{X},B),\widetilde{A}),$$

where $\widetilde{X}\in\mathcal{E}^{T(B)}, \beta, \widetilde{A}$ are given as follows:

$$\begin{aligned}&\widetilde{X}_b := \begin{cases} X_b & \text{if } b\neq c\\ A & \text{if } b=c\end{cases}\\ &\beta : I \longrightarrow S(X_c) \xrightarrow{inc} \coprod_{b\in T(B)} S(\widetilde{X}_b),\\ &\widetilde{A} := X_c.\end{aligned}$$

Then Φ give the required isomorphism. We omit the naturality of Φ on S,T.

(7.5) Corollary. *Let $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be faithful functors with finite fibers. Assume that $S(A) \neq \emptyset$ for every $A \in \mathcal{E}$. Then the following holds:*

(1) *$T(S)$ is a faithful functor with finite fibers.*

(2) *$T(S(t)) = (T(S))(t)$. (Note that $S(t), T(t)$ are power series in one variable t.)*

(3) *Assume that $T(g)$ is a monomorphism for any morphism g in $\mathcal{F}$. Then there is a natural isomorphism $(T(S))^* \cong T'(S) \cdot S^*$.*

(4) *Assume that S, T maps all morphism to monomorphisms between finite sets.. Then there is a natural isomorphism $(T(S))' \cong T'(S) \cdot S'$.*

(5) *There is a natural isomorphism $(T(S))^* \cdot S \cong T^*(S) \cdot S^*$.*

PROOF. It remains to prove (4). The functors $S', T'(S), S' \cdot T'(S)$ are constructed as folows:

$$\begin{aligned}
S' \; &: \; \mathsf{Elts}(S) \longrightarrow \mathsf{Set}_f, \\
&: \; (A, u) \longmapsto S(A) - \{u\} \quad (A \in \mathcal{E}, u \in S(A)); \\
T'(S) \; &: \; T'(\mathcal{E}) \longrightarrow \mathsf{Set}_f, \\
&: \; (B, v, X) \longmapsto \coprod_{b \in T(B) - \{v\}} S(X_b) \\
&\quad\; (B \in \mathcal{F}, v \in T(B), X \in \mathcal{E}^{T(B) - \{v\}}); \\
S' \cdot T(S) \; &: \; \mathsf{Elts}(S) \times T'(\mathcal{E}) \longrightarrow \mathsf{Set}_f, \\
&: \; ((A, u), (B, v, X)) \longmapsto (S(A) - \{u\}) + \coprod_{b \in T(B) - \{v\}} S(X_b) \\
&\quad\; (A \in \mathcal{E}, u \in S(A), B \in \mathcal{F}, v \in T(B), X \in \mathcal{E}^{T(B) - \{v\}}).
\end{aligned}$$

Next, the functors $T(S)'$ are constructed as folows:

$$\begin{aligned}
T(S)' \; &: \; \mathsf{Elts}(T(S)) \longrightarrow \mathsf{Set}_f, \\
&: \; (B, X, u) \longmapsto \coprod_{b \in T(B)} S(X_b) - \{u\} \\
&\quad\; (B \in \mathcal{F}, X \in \mathcal{E}^{T(B)}, u \in \coprod_{b \in T(B)} S(X_b)).
\end{aligned}$$

Then the isomorphism $\Phi : S' \times T'(S) \longrightarrow T(S)'$ is defined as follows:

$$\begin{aligned}
\Phi \;&:\; \mathsf{Elts}(S) \times T'(\mathcal{E}) \longrightarrow \mathsf{Elts}(T(S)),\\
&:\; ((A,u),(B,v,X)) \longmapsto (B,\widetilde{X},u),\\
&\qquad (A \in \mathcal{E}, u \in S(A), B \in \mathcal{F}, v \in T(B), X \in \mathcal{E}^{T(B)-\{v\}});\\
&\qquad \widetilde{X}_b := \begin{cases} A & \text{if } b = v \\ X_b & b \in T(B) - \{v\}. \end{cases}
\end{aligned}$$

Furthermore, any morphism

$$\begin{aligned}
\Phi \;:\; &(((A,u),(B,v,X)) \xrightarrow{(f,(g,\tau))} ((A',u'),(B',v',X')))\\
&\longmapsto (B,\widetilde{X},u) \xrightarrow{(f,\widetilde{\tau})} (B',\widetilde{X'},u')),\\
&\widetilde{\tau}_b := \begin{cases} f & \text{if } b = v \\ \tau_b & \text{else.} \end{cases}
\end{aligned}$$

Then we have

$$T(S)' \circ \Phi = S' \cdot T'(S).$$

Hence Φ gives a required isomorphism. We omit the proof of the naturality of $\Phi : S' \times T'(S) \longrightarrow T(S)'$ on S and T.

(7.6) Lemma. *Let $S : \mathcal{E} \longrightarrow \mathsf{Set}_f$ and $T : \mathcal{F} \longrightarrow \mathsf{Set}_f$ be faithful functors such that $S(A) \neq \emptyset$ for any $A \in \mathcal{E}$. Then the substitution of functors corresponds to that of corresponding species*:

$$\boldsymbol{A}_T(\boldsymbol{A}_S) \cong \boldsymbol{A}_{T(S)}. \tag{7.6.a}$$

PROOF. Note that the species $\boldsymbol{A}_S$ is defined by $\boldsymbol{A}_S[N] = \mathsf{Str}(\mathcal{E}/N)/\cong$ (see Part I of the series). Let (B,X,σ) be any $T(\mathcal{E})$-structure over $N \in \mathsf{Set}_f$ along $T(S)$, so that B is an object of $\mathcal{F}$, $X : T(B) \longrightarrow \mathcal{E}$ is a functor from the set $T(B)$ viewed as a discrete category to $\mathcal{E}$ and

$$\sigma : \coprod_{j \in T(B)} S(X_j) \xrightarrow{\cong} N$$

is a bijection of sets. Put $N_j := \sigma(S(X_j))$ for each $j \in T(B)$. By the assumption, N_j is not empty, and so $\pi := \{N_j \mid j \in T(B)\}$ is a partition. The restriction σ_j of σ to $S(X_j)$ gives a bijection $\sigma_j : S(X_j) \xrightarrow{\cong} N_j$, and so we have an $\mathcal{E}$-structure (X_j, σ_j) on N_j along S. Furthermore, since the map

$$\tau : T(B) \longrightarrow \pi; j \longmapsto N_j$$

is a bijection, (B,τ) is an $\mathcal{F}$-structure over π along T. Thus the assignment

$$f:(B,X,\sigma)\longmapsto((B,\tau),(S(X_j),\sigma)_{j\in T(B)})\in\coprod_{\pi\in\Pi(N)}\boldsymbol{A}_T[\pi]\times\prod_{A\in\pi}\boldsymbol{A}_S[A]$$

gives a map $\boldsymbol{A}_{T(S)}[N]\longrightarrow\boldsymbol{A}_T(\boldsymbol{A}_S)[N]$. This map f is commutative with isomorphisms of $T(\mathcal{E})$-structures. Conversely, given $((B,\tau),(Y_C,\sigma_C)_{C\in\pi})$, where (B,τ) is a $\mathcal{F}$-structure over $\pi\in\Pi(N)$ along T and (Y_C,σ_C) is an $\mathcal{E}$-structure over π along S, we have a $T(\mathcal{E})$-structure (B,X,σ) over N along $T(S)$, where

$$X:T(B)\xrightarrow{\cong}\mathcal{E};j\longmapsto Y_{\tau(j)}$$
$$\sigma:\coprod_{j\in T(B)}S(X_j)\xrightarrow{\cong}N;(a\in S(X_j)\longmapsto\sigma_{\tau(j)}(a).$$

These maps give the required equivalence of species.

8 Applications.

(8.1) G-surjections. Let G be a finite group. Take the functors

$$S:\mathcal{E}=\mathsf{Con}(\mathsf{Set}_f{}^G)\xrightarrow{Emb}\mathcal{S}=\mathsf{Set}_f{}^G;\ G/H\longmapsto G/H,$$
$$T:\mathcal{F}=\mathsf{Surj}\longrightarrow\mathsf{Set}_f;\ (X\twoheadrightarrow X')\longmapsto X.$$

Then $\mathsf{EXP}(\mathcal{E})$ is equivalent to $\mathsf{Set}_f{}^G=\mathcal{S}$. Thus an object of $T(\mathcal{E})$ has the form $(X\overset{p}{\twoheadrightarrow}X')$, where X is a finite G-set and π is a G-surjection to X' with trivial G-action, that is, $T(\mathcal{E})$ is the category of G-surjections:

$$T(\mathcal{E})\cong\mathsf{Surj}^G. \tag{8.1.a}$$

The substitution $T(S):T(\mathcal{E})\longrightarrow\mathcal{S}$ is then defined by

$$T(S):T(\mathcal{E})\longrightarrow\mathsf{Set}_f{}^G;(X\overset{p}{\twoheadrightarrow}X')\longmapsto X. \tag{8.1.b}$$

Furthermore, $T(\mathcal{E})$ is a locallly finite topos (and so a KS-category) with $\mathsf{Con}(T(\mathcal{E}))$ equivalent to the category of nonempty finite G-sets, that is,

$$T(\mathcal{E})\cong\mathsf{EXP}(\mathsf{Set}_{f\geq 1}^G), \tag{8.1.c}$$

where $\mathsf{Set}_{f\geq 1}^G$ is the category of non-empty finite G-sets, and so

$$(T(S))(t)=\exp\left(\exp\left(\sum_{H\leq G}\frac{1}{(G:H)}t^{G/H}\right)-1\right) \tag{8.1.d}$$

Substituting $t^{(G:H)}$ for $t^{G/H}$, we have the following formula:

$$\sum_{n=0}^{\infty} \frac{b_G(n)}{n!} t^n = \exp\left(\exp\left(\sum_{H \le G} \frac{1}{(G:H)} t^{(G:H)}\right) - 1\right), \tag{8.1.e}$$

where

$$b_G(n) := \sharp\left\{(f, R) \;\middle|\; \begin{array}{l} \pi : G \longrightarrow S_n, R \text{ is an equivalence relation} \\ iRj, g \in G \;\Rightarrow\; \pi(g)(i)\, R\, \pi(g)(j) \end{array}\right\}. \tag{8.1.f}$$

(8.2) Arrow sets. An *arrow set* (X, α) is a finite set X together with an endomorphism $\alpha : X \longrightarrow X$; a morphism $f : (X, \alpha) \longrightarrow (Y, \beta)$ of arrow sets is a map $f : X \longrightarrow Y$ such that $\beta \circ f = f \circ \alpha$. The category of arrow sets is denoted by ASet_f, which is the functor category from a free monoid with one generator.

Let (X, α) be an arrow set. An element $x \in X$ is called a *root* if $x = \alpha^i(x)$ for some $i \ge 1$; we denote by $R(X, \alpha)$ the set of roots. The root $\rho(x)$ of an element $x \in X$ is an element $\alpha^i(x) \in R(X, \alpha)$ with $\alpha^j(x) \notin R(X, \alpha)$ for all $0 \le j < i$; when x is a root element, we put $\rho(x) = x$; there exist a unique root for every element. The set $R(X, \alpha)$ is a cyclic set with permutation α; and an arrow set (X, α) is connected if and only if $R(X, \alpha)$ is a connected cyclic set. Thus a connected object of ASet_f can be viewed as a directed circuit on which trees grow. See [MuRot 70, p.210] for functional digraphs.

The assignment $(X, \alpha) \longmapsto R(X, \alpha)$ is functorial, and so we have a functor $R : \mathsf{Aset}_f \longrightarrow \mathsf{Set}_f{}^C$. Next, cutting off all edges connecting roots, we have a faithful functor

$$F : \mathsf{ASet}_f \longrightarrow \mathsf{RForest} \tag{8.2.a}$$

into the category of (unlabeled) rooted forests.

Consider the following diagram of categories:

$$\begin{tikzcd}
 & \mathsf{ASet}_f \arrow[r, "R"] \arrow[d, "F"] \arrow[dr, phantom, "\text{P.B.}"] & \mathsf{Set}_f{}^C \arrow[d, "S"] \\
\mathsf{RTree} \arrow[r, "Incl"] \arrow[dr, "U"'] & \mathsf{RForest} \arrow[r, "Root"] \arrow[d, "\mathsf{EXP}(U)"] & \mathsf{Set}_f \\
 & \mathsf{Set}_f &
\end{tikzcd}$$

where *Root* is the functor which assigns the set of roots to a rooted forest, and S is the forgetful functor; furthermore $U : \mathsf{RTree} \longrightarrow \mathsf{Set}_f$ denotes the forgetful functor from the cateogry of (unlabeled) rooted trees which assigns the underlying vertex-set to a rooted tree. We can easily check that the square in this diagram is a pullback diagram. Since $\mathsf{RForest} \cong \mathsf{EXP}(\mathsf{RTree})$, we have that $S(\mathsf{RTree}) \cong \mathsf{ASet}_f$ with the substitution $S(U)$ is given by

$$S(U) : \mathsf{ASet}_f \xrightarrow{F} \mathsf{RForest} \xrightarrow{\mathsf{EXP}(U)} \mathsf{Set}_f, \tag{8.2.b}$$

but clearly this functor coincides with the forgetful functor, and so

$$(S(U))(t) = \sum_{n=0}^{\infty} \frac{n^n}{n!} t^n \tag{8.2.c}$$

by using the bijection

$$\mathsf{Str}(\mathsf{ASet}_f/[n])/\cong \quad \longleftrightarrow \quad \mathrm{Map}([n],[n]).$$

By Wohlfahrt's formula (Example 6.4 of Part I, [Woh 77]) or the fact that $|\mathsf{Str}(\mathsf{Set}_f{}^C/[n])/\cong| = n!$, we have

$$S(t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n}\right) = \frac{1}{1-t}.$$

On the other hand, as we already studied, we have

$$u(t) := U(t) = \sum_{n=1}^{\infty} \frac{u_n}{n!} t^n, \tag{8.2.d}$$

where u_n is the number of labeled rooted trees on $[n]$.

Hence

$$(S(U))(t) = \sum_{n=0}^{\infty} \frac{n^n}{n!} t^n = \frac{1}{1-u(t)}, \tag{8.2.e}$$

that is,

$$u(t) = \sum_{n=1}^{\infty} \frac{u_n}{n!} t^n = 1 - \frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!} t^n}. \tag{8.2.f}$$

(8.3) Remark. (1) The equation corresponding to (8.2.f) is

$$\mathrm{End} = \mathcal{S} \circ \mathcal{A}, \tag{8.3.a}$$

where End, $\mathcal{S}$ and $\mathcal{A}$ are the species of endofunctions, permutations and rooted trees, respectively. See [BerLL 98, Section 1.4 Example 4(a)].
(2) the formula (8.2.f) can be proved also by using Cayley's formulas $u_n = n^{n-1}$ and Abel's binomial formula (e.g.[Aig 79, Proposition 5.12, Exercise III.3.2], [Rio 68]):

$$x^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1}(y+n-k)^{n-k}. \tag{8.3.b}$$

As is well-known, $x(x+k)^{k-1}$ is the number of labeled rooted forests with $x+k$ vertices and with x roots.

(8.4) Arrowsets, again. As is stated in the preceding paragraph, the results of 8.2 follows also from the theory of species, However, by a little extension of the method of (8.2), we can prove a multi-varable version of (8.2.f) and deduce many formula from it. Now, take the inclusion functor *Incl* instead of U in (8.2). Then we have the following natural isomorphism of functors by the same way as in 8.2:

$$F \cong S(\mathit{Incl}), \tag{8.4.a}$$

where $F : \mathrm{ASet}_f \longrightarrow \mathsf{RForest}$ and $S : {\mathsf{Set}_f}^C \longrightarrow \mathsf{Set}_f$ were the functors defined in (8.2). Thus

$$F(t) = \sum_{(X,\alpha)}{}' \frac{t^{F(X,\alpha)}}{|\mathrm{Aut}(X,\alpha)|} = \frac{1}{1 - \mathrm{RTree(t)}}. \tag{8.4.b}$$

Substituting suitable weight functions for (8.4.b), we have some interesting formulas. First take the following weight function ρ:

$$\rho(t^F) := t^{|V(F)|} s^{|Root(F)|}, \tag{8.4.c}$$

where $Root(F)$ denotes the set of roots of a rooted forest F. Define the set of periodic elements for any endo-map $\alpha : X \longrightarrow X$ by

$$Per(\alpha) := \{i \in X | \alpha^m(i) = i \text{ (for some } m \geq 1)\}.$$

Then for a cyclic set (X, α), an periodic element for α is a root of the rooted forest associated to (X, α):

$$Per(\alpha) = R(F(X, \alpha)).$$

Thus (8.4.b) yields the following formula:

$$\sum_{n=0}^{\infty} \frac{\pi_n(s)}{n!} t^n = \frac{1}{1 - s \sum_{n=1}^{\infty} \frac{u_n}{n!} t^n}, \tag{8.4.d}$$

where $u_n = n^{n-1}$ is the number of labeled rooted trees over $[n]$ and

$$\pi_n(s) := \sum_{\alpha:[n]\to[n]} s^{|\mathrm{Per}(\alpha)|} = \sum_{m=0}^{n} \pi_{n,m}\, s^m, \tag{8.4.e}$$

so that $\pi_{n,m}$ is the number of endo-maps on $[n]$ with m periodic elements. In particular, we have (8.2.e). Applying the Lagrange inversion formula (for example, see [HarPa 73, Section 1.7], [Wil 94, Chapter 5]), we have

$$\pi_{n,m} = \frac{m(n-1)!n^{n-m}}{(n-m)!} \tag{8.4.f}$$

Note that

$$\exp\left(s \sum_{n=1}^{\infty} \frac{u_n}{n!} t^n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{v_{nk}}{n!} s^k t^n, \tag{8.4.g}$$

where $v_{n,k} = \binom{n-1}{k-1} n^{n-k}$ is the number of labeled rooted forest with n vertices and with k roots.

As another easy application of (8.4.b), we enumerate the number $A(n,h,d)$ of endo-functions $\alpha : [n] \longrightarrow [n]$ such that for fixed $h \geq 0$ and $d \geq 2$ the condition

$$\alpha^{h+m} = \alpha^h, \ |\alpha^{-1}(x)| \leq d \quad \text{for all } x \in X \tag{8.4.h}$$

holds for some $m \geq 1$. The condition $\alpha^{h+m} = \alpha^h$ for some $m \geq 1$ is equivalent to the condition that the associated rooted forest has the height at most h; and since $d \geq 2$, the condition $|\alpha^{-1}(x)| \leq d$ is equivalent to the condition that the associated rooted forest has the degree at most d. Substituting

$$w(t^F) := \begin{cases} t^{|V(F)|} & \text{if } ht(F) \leq h \text{ and } deg(F) \leq d \\ 0 & \text{else} \end{cases} \tag{8.4.i}$$

for t^F in the equation (8.4.b), we have

$$\sum_{n=0}^{\infty} \frac{A(n,h,d)}{n!} t^n = \left(1 - \sideset{}{'}\sum_{\substack{ht(T) \\ deg(T)\leq d}} \frac{t^{|V(T)|}}{|\mathrm{Aut}(T)|}\right)^{-1}, \tag{8.4.j}$$

where T runs over isomorphism classes of rooted trees T of height $\leq h$ and of degree at most d.

In particular, we have

$$\sum_{n=0}^{\infty} \frac{A(n,1,d)}{n!} t^n = \left(1 - \sum_{i=0}^{d} \frac{t^{i+1}}{i!}\right)^{-1}, \tag{8.4.k}$$

$$\sum_{n=0}^{\infty} \frac{A(n,1,\infty)}{n!} t^n = \left(1 - te^t\right)^{-1}, \tag{8.4.l}$$

where $A(n,1,\infty)$ denotes the number of endo-functions $\alpha : [n] \longrightarrow [n]$ such that $\alpha^{1+m} = \alpha$ for some $m \geq 1$.

(8.5) The number of solutions of $\alpha^{h+m} = \alpha^h$. Let $h(n)$ be the number of idempotent fnctions $\beta : [n] \longrightarrow [n]$, that is, $\beta(\beta(i)) = \beta(i)$ for all $i \in [n] := \{1, 2, \dots, n\}$. Then its exponential generating function of $h(n)$ is given by

$$\sum_{n=0}^{\infty} \frac{h(n)}{n!} t^n = \exp(te^t). \tag{8.5.a}$$

This formula is found in [HarSc 67]. R.Stanley asked a generalization of this formula; for example, for given $1 \leq i < j$, how many functions $\beta : [n] \longrightarrow [n]$ satisfy $\beta^i = \beta^j$ ([Sta 78, Example 6.9]). In this paragraph, we determine the exponential generating function of such numbers.

Let $\mathsf{ASet}_f^{h,m}$ be the full subcategory of ASet_f consisting arrow set (X, α) with $\alpha^{h+m} = \alpha^h$. Let C_m be a cyclic group of order m and $\mathsf{Set}_f{}^{C_m}$ the category of C_m-sets. Let $\mathsf{RForest}_{\leq h}$ (resp. $\mathsf{RTree}_{\leq h}$) be the category of rooted forests (resp. rooted trees) of height at most h. Then $\mathsf{RForest}_{\leq h}$ is equivalent to $\mathsf{RTree}_{\leq h+1}$ as is well-known. Similarly as in the preceding example, we have the following diagram:

$$\begin{tikzcd}
 & \mathsf{ASet}_f^{h,m} \arrow[r, "R"] \arrow[d, "F"] \arrow[dr, phantom, "\text{P.B.}"] & \mathsf{Set}_f{}^{C_m} \arrow[d, "S"] \\
\mathsf{RTree}_{\leq h} \arrow[r, "Incl"] \arrow[dr, "U"'] & \mathsf{RForest}_{\leq h} \arrow[r, "Root"] \arrow[d, "\mathsf{EXP}(U)"] & \mathsf{Set}_f \\
 & \mathsf{Set}_f &
\end{tikzcd}$$

where *Root* is, as before, the functor which assigns the set of roots to a rooted forest, and S is the forgetful functor; furthermore U denotes the forgetful functor.

Let $B(n, h, m)$ be the number of endo-functions $\alpha : [n] \longrightarrow [n]$ such that $\alpha^h = \alpha^{h+m}$. By Wohlfahrt's formula (Example 6.4 of Part I, [Woh 77]), we have

$$S(t) = \exp\left(\sum_{d|m} \frac{t^d}{d}\right). \tag{8.5.b}$$

Furthermore, we have

$$U(t) = u_h(t) = \sum_{ht(T)\le h} \frac{t^{|V(T)|}}{|\mathrm{Aut}(T)|}. \tag{8.5.c}$$

This series can be calculated by the following recursion formula:

$$u_0(t) = t, u_{h+1}(t) = t\exp(u_h(t)). \tag{8.5.d}$$

(See Part I, 6.8.) Then the above diagram gives the following formula

$$(S(U))(t) = \sum_{n=0}^{\infty} \frac{B(n,h,m)}{n!} t^n = \exp\left(\sum_{d|m} \frac{u_h(t)^d}{d}\right). \tag{8.5.e}$$

In particular, when $h = m = 1$, this formula gives (8.5.a),

(8.6) Some chains of subsets. This example is taken form Stanley's paper ([Sta 78, Example 3.8]). For a finite set S with $|S| = n$, let $M(n)$ denote the number of chains

$$\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_k = S \quad (k \ge 0) \tag{8.6.a}$$

such that $|S_{i+1} - S_i| \ge 2$ $(0 \le i < k)$. Then

$$M(t) := \sum_{n=0}^{\infty} \frac{M(n)}{n!} t^n = \frac{1}{1-(e^t-1-t)} = \frac{1}{2+t-e^t}. \tag{8.6.b}$$

From the viewpoint of the theory of generating functions of categories, we can prove this formula as follows. In this paragraph, the symbol $[k]$ denotes the chain $(1 < 2 < \cdots < k)$ of length k. Let $\mathcal{P}$ be the category with objects the finite sequences of finite sets as follows:

$$\emptyset = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_k = S,\ k \ge 0,$$

such that $|S_{i+1} - S_i| \geq 2, 0 \leq i < k$. A morphism (f, g) from $(S_0 \subset S_1 \subset \cdots \subset S_k = S)$ to $(T_0 \subset T_1 \subset \cdot \subset T_n = T)$ is defined to be the pair of an order-preserving function $f : [k] \longrightarrow [l]$ and a map $S \longrightarrow T$ satisfying the condition

$$g(S_i) \subseteq T_{f(i)},\ g^{-1}(T_{f(i)}) \subseteq S_i \quad (0 \leq i \leq k).$$

There is a faithful functor

$$M : \mathcal{P} \longrightarrow \mathsf{Set}_f; (\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S) \longmapsto S.$$

Then the generating function of M is exactly the required series $M(t)$.

Let $\mathcal{F} = \mathsf{Chain}$ be the category of finite totally ordered sets and maps preserving orders and let $T : \mathsf{Chain} \longrightarrow \mathsf{Set}_f$ the forgetful functor. Let $\mathcal{E} = \mathsf{Set}_{f\geq 2}$ be the full subcategory of Set_f consisting of finite sets of cardinality ≥ 2 accompanied by the inclusion functor $S : \mathsf{Set}_{f\geq 2} \hookrightarrow \mathsf{Set}_f$. Then $\mathcal{P}$ is equivalent to the substitution $T(\mathcal{E})$:

$$\begin{tikzcd}
 & \mathcal{P} \arrow[r, "R'"] \arrow[d, "S'"] \arrow[dr, phantom, "\text{P.B.}"] & \mathsf{Chain} \arrow[d, "S"] \\
\mathsf{Set}_{f\geq 2} \arrow[r, "incl."] \arrow[dr, "T"] & \mathsf{EXP}(\mathsf{Set}_{f\geq 2}) \arrow[r, "Rank"] \arrow[d, "\mathsf{EXP}(T)"] & \mathsf{Set}_f \\
 & \mathsf{Set}_f, &
\end{tikzcd}$$

where the functor S' and R' are defined by

$$\begin{aligned}
S' &: (\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S) \longmapsto \bigoplus_{i=1}^{m} (S_i - S_{i-1}), \\
R' &: (\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S) \longmapsto [k].
\end{aligned}$$

Furthermore, we have $\mathsf{EXP}(T) \circ F \cong M$, and so $M \cong T(S)$. Note that

$$S(t) = \frac{1}{1-t},\ T(t) = \sum_{n=2}^{\infty} \frac{1}{n!} t^n = e^t - 1 - t.$$

Hence

$$M(t) = T(S)(t) = T(S(t)) = \frac{1}{1 - (e^t - 1 - t)}, \tag{8.6.c}$$

as required.

(8.7) r-partitions. Many convolution formulas and exponential formulas can be extended to exponential formulas on the level of categories. As one of such examples, we consider Stanley's exponential formula for r-partitions (see Stanley [Sta 78, Corollary 6.2]). For a fixed positive integer r, an *r-partition* of a finite set S is defined to be a set

$$\pi = \{(B_{11}, B_{12}, \dots, B_{1r}), (B_{21}, B_{22}, \dots, B_{2r}), (B_{1k}, B_{k2}, \dots, B_{rk})\}$$

satisfying

(i) For each $1 \le j \le r$, the set $\pi_j = \{B_{1j}, B_{2j}, \dots, B_{kj}\}$ forms a partition of S into k blocks, that is, each B_{ij} is a non-empty subset of S, the k subsets $B_{1j}, B_{2j}, \dots, B_{kj}$ are pairwise disjoint, and $\bigcup_i B_{ij} = S$.
(ii) For fixed i, $|B_{i1}| = |B_{i2}| = \cdots = |B_{ir}|$.

The r partitions $\pi_1, \dots, \pi_r$ all have the same type $(a_1, \dots, a_n)$ called the *type* of π, where a_j is the number of the B_{i1}'s with $|B_{i1}| = j$.

Given series $f(1), f(2), \dots$, define a new series $h(0) = 1, h(1), h(2), \dots$ by

$$h(n) = \sum_{\pi} f(1)^{a_1} f(2)^{a_2} \cdots f(n)^{a_n}, \tag{8.7.a}$$

where π runs over all r-partitions of $[n]$, and where π has type $(a_1, a_2, \dots, a_n)$. Then

$$\sum_{n=0}^{\infty} \frac{h(n)}{n!^r} t^n = \exp\left(\sum_{n=1}^{\infty} \frac{f(n)}{n!^r} t^n\right) \tag{8.7.b}$$

In order to give a categorical version of this formula, let $\mathsf{Set}_{f\geq 1}^{[r]}$ be the category of finite sets with morphisms r-tuples of maps:

$$\mathrm{Hom}(X, Y) := \{(f_1, \dots, f_r) \mid f_i : X \longrightarrow Y\}.$$

The composition is defined by the usual composition of maps: $(g_j) \circ (f_j) := (g_j \circ f_j)$. Furthermore, let $\mathsf{Surj}^{[r]}$ be the category of surjections between finite sets in which a morphism from $(X \overset{p}{\twoheadrightarrow} X')$ to $(Y \overset{q}{\twoheadrightarrow} Y')$ is defined to be a $r+1$-tuples $(f_1, \dots, f_r; f')$, where $f_j : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$, such that $f' \circ p = q \circ f_j$ for each j.

Then we have that an "exponential formula "as follows:

$$\mathsf{EXP}(\mathsf{Set}_{f\geq 1}^{[r]}) \cong \mathsf{Surj}^{[r]}; \ \bigoplus_{\alpha \in N} X_\alpha \longleftrightarrow \left(\coprod_\alpha X_\alpha \longrightarrow N\right).$$

Furtheremore, let $F : \mathsf{Set}_{f\geq 1}^{[r]} \longrightarrow \mathsf{Set}_f$ is a functor, and define a functor $H : \mathsf{Surj}^{[r]} \longrightarrow \mathsf{Set}_f$ by

$$H(X \xrightarrow{p} X') := \coprod_{x' \in X'} \{x'\} \times F(p^{-1}(x')),$$

and

$$H(f_1, \dots, f_r; f')(x', a) := (f'(x'), F(f_1, \dots, f_r)(a))$$

for any morphism $(f_1, \dots, f_r; f') : (X \xrightarrow{p} X') \longrightarrow (Y \xrightarrow{q} Y')$ in $\mathsf{Set}_{f\geq 1}^{[n]}$ and $x' \in X'$, $a \in F(p^{-1}(x'))$.

Then we have another "exponential formula":

$$\mathsf{EXP}(F) \cong H. \tag{8.7.c}$$

Let $f(n) := |\mathsf{Str}(\mathsf{Set}_{f\geq 1}^{[r]}/[n])/\cong|$ and $h(n) := |\mathsf{Str}(\mathsf{Surj}^{[r]}/[n])/\cong|$. Then $f(n)$ and $h(n)$ are the coefficients of $t^n/n!$ in $F(t)$ and $H(t)$, respectively, and $h(n)$ is presented by the formula (8.7.a).

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