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Loss of convexity of compact hypersurfaces moved by surface diffusion

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Abstract

We give a way of constructing a compact, closed, strictly convex hypersurface which loses convexity without developing singularities when it moves by its surface diffusion for a short time.

AMS subject classification. 35K99, 80A22

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1 Introduction

We consider the following geometric evolution law of the form

$$\begin{cases} V = -\Delta_{\Gamma(t)} H & \text{on } \Gamma(t),\, t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1}$$

Here t denotes the time variable and $\Gamma(t)$ denotes an unknown evolving hypersurface in $\mathbf{R}^n$ at time $t > 0$ with $n \geq 3$; Γ_0 is a given initial hypersurface in $\mathbf{R}^n$. For each $t > 0$ and $x \in \Gamma(t)$, the quantities $V = V(t, x)$ and $H = H(t, x)$ denote the outward normal velocity and the outward mean curvature of $\Gamma(t)$ at x, respectively. The operator $\Delta_{\Gamma(t)}$ stands for the Laplace-Beltrami operator on $\Gamma(t)$. The law (1) is called the *surface diffusion flow equation.* The purpose of this paper is to give a way of constructing a strictly convex, closed, compact, initial hypersurface Γ_0 such that the solution $\Gamma(t)$ starting from Γ_0 loses its convexity in a finite time interval (t_0, t_1) with $t_0 > 0$. This result is an extension of the result in the two dimensional version obtained by Y. Giga and the author [15] to the higher dimensional version.

Equation (1) was first proposed by Mullins [20] to explain thermal grooving in material sciences. Also Davi and Gurtin [6] derived (1) from a view point of thermodynamics and continuum mechanics. Recently J. W. Cahn, C. M. Elliott and A. Novick-Cohen [3] derived (1) from the Cahn-Hilliard equation with a concentration dependent mobility as a singular limit in a formal basis. A unique local existence result for general dimension was established by J. Escher, U. F. Mayer and G. Simonett [8] by using a sophisticated semi-group theory in small Hölder spases developed by H. Amann [1]. Moreover, when Γ_0 is close to an $(n-1)$-dimensional sphere in some sense, they also obtained a global solution $\Gamma(t)$ with $\Gamma(0) = \Gamma_0$ and proved that $\Gamma(t)$ converges to an $(n-1)$-dimensional sphere with the enclosed volume equal to that of Γ_0 as $t \to \infty$. We should note that such a work was initiated by C. M. Elliott and H. Garcke [7] for $n = 2$ as well as other equations involving the volume preserving mean curvature law

$$V = H - \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} H ds \tag{2}$$

and the intermediate law

$$V = -\Delta_{\Gamma(t)} \left(\frac{1}{D} - \frac{1}{M} \Delta_{\Gamma(t)}\right)^{-1} H,$$

where D and M are called the diffusion and the mobility constants. Recently Escher and Simonett [10, 11] established the higher dimensional version of the result obtained in [7].

The equation (1) is a nonlinear fourth order parabolic equation so there are several phenomena which are differente from those of second order model such as the mean curvature flow equation

$$\begin{cases} V = H \quad \text{on } \Gamma(t),\, t > 0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{3}$$

In particular, several mathematical and numerical studies for (1) show remarkable characteristic phenomena in fourth order model, for example, loss of embeddedness and loss of convexity. The loss of both embeddedness and convexity reflects the fact that a fourth order parabolic equation does not fulfill the maximum or comparison principle. In fact, this principle is satisfied in (3) and it prevents the developments of self-intersections (Grayson [16] for $n = 2$) and preserves convexity (Gage and Hamilton [13] for $n = 2$ and Huisken [17] for $n \geq 3$) of solutions of (3). The loss of embeddedness for (1) was conjectured by Elliott and Garcke [7], numerically established by Escher, Mayer and Simonett [8, 9], and proved by Giga and the author [14] for $n = 2$, and later by Mayer and Simonett [19] for $n \geq 2$. The paper [19] also gave the proof of loss of embeddedness for (2) in general dimension (two dimensional case had been treated by Gage [12]). The loss of convexity for (1) was conjectured by J. Escher in the conference "Nonlinear Evolution Equation" held in the end of June of 1997 in Oberwolfach. This phenomenon was also suggested by numerical studies by B. D. Coleman, R. S. Falk and M. Moakher [4, 5] and recently proved by Giga and the author [15] for $n = 2$.

In this paper we extend the result of [15] to the case $n \geq 3$. Our main result is to give a way to construct a compact, closed, strictly convex initial hypersurface Γ_0 embedded in $\mathbf{R}^n$ such that the solution $\Gamma(t)$ starting from Γ_0 loses its convexity in its smooth evolution, Theorem 10. More precisely, we shall show that for any compact, closed, strictly convex rotationally symmetric hypersurface Γ_0 embedded in $\mathbf{R}^n$ with respect to the x^1-axis, there is a specific deformation from Γ_0 to a rotational hypersurface Γ_0' with locally weaker convexity than that of Γ_0 such that the solution $\Gamma'(t)$ of (1) with $\Gamma'(0) = \Gamma_0'$ loses its convexity in a finite time interval determined by Γ_0. We note that a hypersurface Γ_0'' which is not rotationally symmetric with respect to the x^1-axis but is close to the hypersurface Γ_0' obtained as above also leads to a loss of convexity because of the continuous dependence of the solution on initial data. Hence, in order to prove Theorem 10, it is sufficient and essential to show that there is a closed, compact, strictly convex initial hypersurface of rotation about the x^1-axis and also symmetric with respect to the x^2-axis such that the solution of (1) starting from the above initial hypersurface loses its convexity after a finite time. In particular, our method constructs a hypersurface S' near a sphere S which creats a loss of convexity in motion by surface diffusion.

Basic strategy to prove Theorem 10 may be almost the same as in [15]. However, because of high dimensionality, the work can not be straightforward and the computation becomes tedious and complicated. To simplify computation, we first establish an intrinsic description of the parametrized equation for (1), which is given in (6). The equation (6) is especially useful when we want to obtain detailed qualitative aspects of solutions of (1). We also establish a new unique local existence result in Hölder continuity framework, which is differente from those of [15] (L^2-framework) and [8] (little Hölder continuity framework). Now together with several lemmas which we need to overcome the difficulties arising from high dimensionality the above analytical foundations enable us to carry out the strategy in [15].

This papar is organized as follows. In Section 2 we state a unique local existence theorem for (1) in the framework of the Hölder space. Our parametrization of (1) uses a tubular neighborhood near a fixed convex, compact, closed hypersurface Σ and does not need to restrict the magnitude of the distance between Γ_0 and Σ to be small. This situation is useful to prove Theorem 10 as is seen later. In Section 3 we show that if initial hypersurface is rotationally symmetric with respect to the x^1-axis, then so is the solution of (1). Of course this result is expected but we give its proof because it is not explicitly written in literatures. In Section 4, we introduce a deformation from a given, compact, closed, strictly convex, rotational hypersurface Γ_0 to another such a Γ_0' whose local convexity is sufficiently weak compared with the fourth order derivative of the mean curvature. We also present a fact that the solution $\Gamma'(t)$ starting from Γ_0' exists in a finite time interval $[0, T)$ depending on the original hypersurface Γ_0 and this interval does not shrink by the above deformation. In Section 5, using the results in the previous sections we give the proof our main theorem (Theorem 10). We show that $\Gamma'(t)$ loses its convexity in some time interval which depends on Γ_0 and is contained in $(0, T)$. In the last section and Appendix we sketch a proof of the unique local existence theorem in Section 2 for reader's convenience.

2 Local solutions for motion by surface diffusion near a convex hypersurface

Following [8], we introduce a parametrization for (1). Let Σ be a smooth compact convex closed embedded oriented hypersurface in $\mathbf{R}^n$ and let $\{U_\beta, \psi_\beta\}_{\beta=1}^{\bar{\beta}}$ be an atlas on Σ. For $s \in U_\beta \subset \Sigma$, $\psi_\beta(s) = (u_\beta^1, \cdots, u_\beta^{n-1}) \in U_\beta' := \psi_\beta(U_\beta) \subset \mathbf{R}^{n-1}$ is called the local coordinate of s. Let z be the induced metric on Σ from the Euclidean metric in $\mathbf{R}^n$ and let h be the second fundamental quantity of Σ. The sign convention adopted here is that

$$h[\xi, \xi] \leq 0 \quad \text{for any tangent vector field } \xi \text{ on } \Sigma.$$

In local coordinates they are written as

$$z = \sum_{i,j=1}^{n-1} z_{\beta,ij} du^i_\beta \otimes du^j_\beta, \quad h = \sum_{i,j=1}^{n-1} h_{\beta,ij} du^i_\beta \otimes du^j_\beta$$

at $s \in U_\beta$. Hereafter, if any confusion may not be caused, then using Einstein's convention and omitting the index β we often simply write them as

$$z = z_{ij} du^i \otimes du^j, \quad h = h_{ij} du^i \otimes du^j.$$

Throughout this paper we regard Σ as a Riemannian manifold with the metric z. We call Σ the reference hypersurface.

Let $\rho : [0,T) \times \Sigma \to \mathbf{R}_+ := \{r \in \mathbf{R}; r \geq 0\}$ be a smooth scalar field and we assume that $\Gamma(t)$, $t \in [0,T)$, is written in terms of $\rho(t,s)$ with $\rho(0,s) = \rho_0(s)$ as

$$\Gamma(t) = \{s + \rho(t,s)\nu(s) \in \mathbf{R}^n; s \in \Sigma\},$$

where $\nu(s)$ is the outward normal of Σ at $s \in \Sigma$. We define two geometric quantities of Σ:

$$\begin{aligned} &w(r) = w_{ij}(r) du^i \otimes du^j := (z_{ij} - 2h_{ij}r + z^{kl}h_{ki}h_{lj}r^2) du^i \otimes du^j \quad \text{for } r \geq 0, \\ &\sigma(\rho, d\rho) := w(\rho) + d\rho \otimes d\rho, \end{aligned}$$

where $(z^{ij}) = (z_{ij})^{-1}$ and $d\rho = \frac{\partial \rho}{\partial u^i} du^i$.

Lemma 1 . *Let Σ be a smooth compact convex closed embedded oriented hypersurface in $\mathbf{R}^n$. Then,*
(i) $w(r)$ is a metric on Σ for $r \geq 0$,
(ii) $\sigma(f, df)$ is a metric on Σ for nonnegative (at least) C^1-scalar field $f = f(s)$ on Σ.

Proof. (i): It is sufficient to show the positivity of $w(r)$. Let $\xi = \xi^k \frac{\partial}{\partial u^k}$ be any tangent vector field on Σ. Then $h[\xi,\xi]r \leq 0$ on Σ since Σ is convex and $r \geq 0$. On the other hand, at any $s \in \Sigma$ there is an orthogonal matrix $T = (T^i_j)$ such that

$$z^{kl} = \lambda^a \delta^{ab} T^k_a T^l_b, \quad k,l = 1,2,\cdots,n-1$$

with $\lambda^a > 0$ for $a = 1,2,\cdots,n-1$. Then

$$z^{kl}h_{ki}h_{lj}\xi^i\xi^j = \lambda^a T^k_x T^l_a h_{ki}h_{lj}\xi^i\xi^j = \lambda^a (T^k_a h_{ki}\xi^i)^2 \geq 0.$$

Thus we arrive at

$$\begin{aligned} w(r)[\xi,\xi] &= z[\xi,\xi] - 2h[\xi,\xi]r + z^{kl}h_{ki}h_{lj}\xi^i\xi^j r^2 \\ &\geq z[\xi,\xi] \geq 0. \end{aligned}$$

(ii) immediately follows from (i) and the fact that $df \otimes df \geq 0$. $\square$

By Lemma 1 we can consider the geometric operators acting on scalar fields Ψ on Σ such as $\mathrm{grad}_{w(f)}$, $\mathrm{Hess}_{w(f)}$, $\Delta_{w(f)}$, and $\Delta_{\sigma(f,df)}$ with a nonnegative scalar field f on Σ. They are given in local coordinates by

$$
\begin{aligned}
\mathrm{grad}_{w(f)}\Psi &= w^{ij}(f)\frac{\partial\Psi}{\partial u^j}\frac{\partial}{\partial u^i},\\
\mathrm{Hess}_{w(f)}\Psi &= (\frac{\partial^2\Psi}{\partial u^i\partial u^j} - W^k_{ij}(f,df)\frac{\partial\Psi}{\partial u^k})du^i\otimes du^j,\\
\Delta_{w(f)}\Psi &= w^{ij}(f)(\frac{\partial^2\Psi}{\partial u^i\partial u^j} - W^k_{ij}(f,df)\frac{\partial\Psi}{\partial u^k}),\\
\Delta_{\sigma(f,df)}\Psi &= \sigma^{ij}(f,df)(\frac{\partial^2\Psi}{\partial u^i\partial u^j} - \gamma^k_{ij}(f,df,\nabla df)\frac{\partial\Psi}{\partial u^k}),
\end{aligned}
$$

where $(w^{ij}(f)) := (w_{ij}(f))^{-1}$, $(\sigma^{ij}(f,df)) := (\sigma_{ij}(f,df))^{-1}$; $W^k_{ij}(f,df)$ and $\gamma^k_{ij}(f,df,\nabla df)$ with $i,j,k = 1,2,\cdots,n-1$ are the Christoffel symbols of $w(f)$ and $\sigma(f,df)$, respectively. We also define

$$
\begin{aligned}
L = L(\rho,d\rho) &:= (1 + w(\rho)[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho])^{1/2}, && (4)\\
w'(r) = w'_{ij}(r)du^i\otimes du^j &:= (-2h_{ij} + 2z^{kl}h_{ki}h_{lj}r)du^i\otimes du^j,\\
w'(r)w^*(r) &:= w'_{ij}(r)w^{ij}(r) && (5)
\end{aligned}
$$

for $r \geq 0$.

Then, as computed in [8] we can have the partial differential equation of $\rho(t,s)$ described by local coordinates which parametrizes (1). But this equation can be rewritten in more intrinsic way by making further computations. We present here the equation of $\rho(t,s)$ in the intrinsic manner in the following:

$$
\begin{cases}
\rho_t = -L\Delta_{\sigma(\rho,d\rho)}\big[L^{-3}\{L^2\Delta_{w(\rho)}\rho - \mathrm{Hess}_{w(\rho)}\rho[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] \\
\qquad -\frac{1}{2}w'(\rho)[\mathrm{grad}_{w(\rho)}\rho, \mathrm{grad}_{w(\rho)}\rho] - \frac{1}{2}L^2 w'(\rho)w^*(\rho)\}\big], \quad t>0,\ s\in\Sigma, \qquad (6)\\
\rho(0,s) = \rho_0(s), \quad s\in\Sigma.
\end{cases}
$$

Remark 2 . *The equation (6) is still valid on any smooth compact closed oriented hypersurface Σ (not necessarily convex) as long as $w(\rho)$ and $\sigma(\rho,d\rho)$ are metrics on Σ.*

Next we state the local existence theorem of (6) for positive initial data. The differences between the local existence theorem in [8] and ours is that our theorem uses the convex reference hypersurface Σ, which enable us to treat large magnitude of positive initial data $\rho_0(s)$. This property is needed in the proof of our main theorem (Theorem 10). Another difference is that our local existence theorem is established in a different category from that of [8], that is, the local existence theorem in [8] is established in the framework of t-continuous and s-little Hölder continuous regularity whereas our theorem is in the framework of the type $C^{[m/4]+\alpha, m+4\alpha}(\Sigma)$ for an integer $m \geq 4$ and $0 < \alpha < 1/4$, where the symbol $[q]$ denotes the largest integer less than or equal to q. Our theorem also explicitly gives the information how the existence time depends on initial data, which is also useful in the proof of Theorem 10.

Theorem 3 . *Let $m \geq 4$ be an integer and $0 < \alpha < 1/4$. Let $\rho_0 \in C^{m+4\alpha}(\Sigma)$ with $\rho_0(s) > 0$ for $s \in \Sigma$. Set*

$$m_0 = \min_{s\in\Sigma} \rho_0(s) > 0. \tag{7}$$

Then there are positive constants $T(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}, m_0)$ and $G(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)})$ such that (6) has a unique solution $\rho(t,s)$ satisfying

$$\begin{aligned}
&\rho \in C^{[m/4]+\alpha, m+4\alpha}([0,T_0]\times\Sigma),\\
&\|\rho\|_{C^{[m/4]+\alpha,m+4\alpha}([0,T_0]\times\Sigma)} \leq G(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}),\\
&\min_{(t,s)\in[0,T_0]\times\Sigma} \rho(t,s) \geq m_0/2 > 0,
\end{aligned}$$

where $T_0 := T(\|\rho_0\|_{C^{m+4\alpha}(\Sigma)}, m_0)$.

Remark 4 . *Here $T(M_0, m_0)$ is nonincreasing in M_0 and nondecreasing in m_0; $G(M_0)$ is nondecreasing in M_0.*

For the reader's convenience we state a sketch of the proof of Theorem 3 in Section 6 and Appendix.

3 Local existence of rotationally axisymmetric hypersurfaces for motion by surface diffusion

In this section we consider solutions of (1) which are rotationally axisymmetric hypersurfaces. First of all we specify the reference hypersurface Σ embedded in $\mathbf{R}^n$ with the orthogonal cordinate $(x^1, x^2, \cdots, x^n)$ as a surface of rotation. For this purpose we prepare a fixed smooth convex closed curve M embedded in $\mathbf{R}^2_{x^1,x^2} := \{(x^1, x^2, \cdots, x^n) \in \mathbf{R}^n; x^3 = \cdots = x^n = 0\}$. We assume that M is symmetric with respect to both the x^1 and x^2-axes and also assume that M contains two straight line segments which are parallel to the x^1-axis and intersect the x^2-axis. To be precise we put

$$M = \{\mu = (p(u^1), q(u^1)) \in \mathbf{R}^2_{x^1,x^2}; u^1 \in \mathbf{T}\},$$

where $\mathbf{T} := \mathbf{R}/2l\mathbf{Z}$ and $2l$ is the total length of M; u^1 denotes the arclength parameter of M and is assumed to move clock-wise. We assume that functions p and q are smooth and satisfy

$$p(u^1) = -p(-u^1), \quad q(u^1) = q(-u^1) \quad \text{for } u^1 \in \mathbf{T},$$

$$(p(u^1), q(u^1)) = \begin{cases} (u^1, b) & \text{for } |u^1| \leq a, \\ (l - u^1, -b) & \text{for } |u^1 - l| \leq a, \end{cases}$$

$$\kappa(u^1) < 0 \quad \text{for } a < u^1 < l - a \text{ and } l + a < u^1 < 2l - a$$

for some $a, b \in (0, l/2)$, where $\kappa(u^1)$ is the outward curvature of M.

We define a reference hypersurface Σ as the surface of rotation of M about the x^1-axis. This is given by

$$\Sigma = \{s = (p(u^1), q(u^1)\omega) \in \mathbf{R}^n; u^1 \in [-l/2, l/2], \omega \in S^{n-2}\},$$

where S^{n-2} is the unit sphere in $\mathbf{R}^{n-1}_{x^2,\cdots,x^{n-1}}$ centered at $(x^2,\cdots,x^{n-1}) = (0,\cdots,0)$. We introduce a specified atlas of Σ as follows. Let $\{V_\gamma, \theta_\gamma\}_{\gamma=1}^{\bar\gamma}$ be an atlas of S^{n-2}. Here we assume that the coordinate functions θ_γ

$$\begin{array}{rccl} \theta_\gamma : & V_\gamma & \to & V'_\gamma \subset \mathbf{R}^{n-1} \\ & \omega & \mapsto & (u^2_\gamma, \cdots, u^{n-1}_\gamma) \end{array}$$

gives polar coordinates for V_γ for $\gamma = 1,\cdots,\bar\gamma$. Set $U'_\gamma := (-l/2, l/2) \times V'_\gamma$ for $\gamma = 1,\cdots,\bar\gamma$. Define the mappings φ_γ $(\gamma = 1,\cdots,\bar\gamma)$ by

$$\begin{array}{rccl} \varphi_\gamma : & U'_\gamma & \to & \Sigma \\ & (u^1, u^2_\gamma, \cdots, u^{n-1}_\gamma) & \mapsto & (p(u^1), q(u^1)\theta_\gamma^{-1}(u^2_\gamma, \cdots, u^{n-1}_\gamma)). \end{array}$$

Let $U'_\gamma := \varphi_\gamma(U'_\gamma)$ and define the mappings $\psi_\gamma : U_\gamma \to U'_\gamma$ by $\psi_\gamma := \varphi_\gamma^{-1}$ for $\gamma = 1,\cdots,\bar\gamma$. Moreover set

$$\begin{aligned} U_+ &= \{s = (p(u^1), q(u^1)\omega) \in \Sigma; a < u^1 \le l/2, \omega \in S^{n-2}\}, \\ U_- &= \{s = (p(u^1), q(u^1)\omega) \in \Sigma; -l/2 \le u^1 < -a, \omega \in S^{n-2}\}. \end{aligned}$$

Define the mappings $\psi_\pm$ by

$$\begin{array}{rccl} \psi_\pm : & U_\pm & \to & \mathbf{R}^{n-1}_{x^2,\cdots,x^n} \\ & (x^1, x^2, \cdots, x^n) & \mapsto & (u^1_\pm, \cdots, u^{n-1}_\pm) := (x^2, \cdots, x^n). \end{array}$$

Then $\{U_\beta, \psi_\beta\}_{\beta=\pm,1,\cdots,\bar\gamma}$ gives an atlas of Σ.

Next we show the explicit way to give the initial hypersurface of rotation Γ_0. For an integer $m \ge 4$ and $0 < \alpha < 1/4$, let $E^{m+4\alpha}(M)$ be

$$E^{m+4\alpha}(M) = \{\lambda_0 \in C^{m+4\alpha}(M); \lambda_0(-u^1) = \lambda_0(u^1) = \lambda_0(l - u^1) > 0 \quad \text{for } u^1 \in \mathbf{T}\}.$$

Here we regard $\lambda_0(u^1)$ as $\lambda_0(p(u^1), q(u^1))$. Hereafter, if any confusion may not be caused, we will always identify a scalar field f on a manifold $\mathcal{M}$ as a function of the (local) coordinates of $\mathcal{M}$. For $\lambda_0 \in E^{m+4\alpha}(M)$ we associate a curve

$$\gamma[\lambda_0] := \{\mu + \lambda_0(\mu)N(\mu) \in \mathbf{R}^2; \mu \in M\},$$

where $N(\mu)$ is the outward normal of M at $\mu \in M$. By symmetry of M and $\lambda_0 \in E^{m+4\alpha}(M)$, $\gamma[\lambda_0]$ is simple and closed. Let $\mathcal{C}^{m+4\alpha}(M)$ be the set of all $\gamma[\lambda_0]$ with $\lambda_0 \in E^{m+4\alpha}(M)$ such that the outward curvature κ_0 of $\gamma[\lambda_0]$ is negative everywhere. Of course, for each $\gamma_0 \in \mathcal{C}^{m+4\alpha}(M)$ there is a unique $\lambda_0 \in E^{m+4\alpha}(M)$ such that $\gamma_0 = \gamma[\lambda_0]$. For $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ let $\Gamma[\lambda_0]$ be the hypersurface of rotation of $\gamma_0 \cap \{(x^1, x^2); x^2 \ge 0\}$ which can be written as

$$\Gamma[\lambda_0] = \{s + \rho_0(s)\nu(s) \in \mathbf{R}^n; s \in \Sigma\}.$$

Here $\rho_0(s) := \lambda_0(\zeta(s))$ and $\zeta(s)$ is a mapping from Σ to M defined by

$$\begin{array}{rccl} \zeta : & \Sigma & \to & M \\ & (p(u^1), q(u^1)\omega) & \mapsto & (p(u^1), q(u^1)) \end{array} \tag{8}$$

and $\nu(s)$ is the outward normal of $\Gamma[\lambda_0]$ given by

$$\nu(s) = (-q'(u^1), p'(u^1)\omega).$$

We call γ_0 the generator of $\Gamma[\lambda_0]$. Note that the followings hold

$$\min_{s\in\Sigma} \rho_0(s) = \min_{\mu\in M} \lambda_0(\mu) > 0, \tag{9}$$

$$\|\rho_0\|_{C^{m+4\alpha}(\Sigma)} \leq K_0 \|\lambda_0\|_{C^{m+4\alpha}(M)} \tag{10}$$

with some constant $K_0 > 0$. Let $\mathcal{S}^{m+4\alpha}(\Sigma)$ be the set of all $\Gamma[\lambda_0]$ with $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$. Then every element belonging to $\mathcal{S}^{m+4\alpha}(\Sigma)$ is strictly convex closed compact hypersurface of rotation about the x^1-axis with a generator symmetric with respect to the x^2-axis. It is clear that for each $\Gamma_0 \in \mathcal{S}^{m+4\alpha}(\Sigma)$ there is a unique generator $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ such that $\Gamma_0 = \Gamma[\lambda_0]$. Then for $\Gamma_0 \in \mathcal{S}^{m+4\alpha}(\Sigma)$ as the initial hypersurface we have the following local existence result for (1), which assures that the solution is also rotationally axisymmetric about the x^1-axis with a generator symmetric with respect to the x^2-axis.

Theorem 5 . *Let $m \geq 4$ be an integer and $0 < \alpha < 1/4$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$ with $\rho_0(s) = \lambda_0(\zeta(s))$ and $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$. Then:*
(i) There is a positive constant $T_1 := T(\|\lambda_0\|_{C^{m+4\alpha}(M)}, \min_{\mu\in M}\lambda_0(\mu))$ such that (1) has a unique solution

$$\Gamma(t) = \{s + \rho(t,s)\nu(s); s \in \Sigma\} \quad \textit{for } t \in [0, T_1]$$

with $\rho(t,s)$ established in Theorem 3.
(ii) There is a unique $\lambda \in C^{[m/4]+\alpha, m+4\alpha}([0,T_1] \times M)$ such that $\rho(t,s) = \lambda(t, \zeta(s))$ with $\lambda(0,\zeta(s)) = \lambda_0(\zeta(s))$.
(iii) $\rho(t,s)$ is even in the u^1-coordinate.

Proof. The first assertion follows from Theorem 3.

In order to prove the second assertion, it is sufficient to prove that $\rho(t,s)$ does not depend on the $(u^2_\gamma, \cdots, u^{n-1}_\gamma)$-coordinate for $s \in U_\gamma$ with $\gamma \in \{1, \cdots, \bar{\gamma}\}$. Let $\mathcal{T}$ be any smooth mapping from S^{n-2} to itself such that $\mathcal{T}$ maps V_γ into itself for $\gamma \in \{1, \cdots, \bar{\gamma}\}$. Then for $\omega = \theta^{-1}_\gamma(u^2_\gamma, \cdots, u^{n-1}_\gamma) \in V_\gamma$, $\mathcal{T}(\omega)$ can be written as

$$\mathcal{T}(\omega) = \theta^{-1}_\gamma(u^2_\gamma + a^2_\gamma, \cdots, u^{n-1}_\gamma + a^{n-1}_\gamma)$$

with some $a^2_\gamma, \cdots, a^{n-1}_\gamma$. Define the mapping $\mathcal{R}$ from Σ to itself by

$$\begin{array}{rccl} \mathcal{R}: & \Sigma & \to & \Sigma \\ & (p(u^1), q(u^1)\omega) & \mapsto & (p(u^1), q(u^1)\mathcal{T}(\omega)). \end{array}$$

Set $\bar{\rho}(t,s) := \rho(t, \mathcal{R}(s))$ for $t \in [0, T_1]$, $s \in \Sigma$.

Claim: $\bar{\rho}(t,s)$ is also a solution of (6) with $\bar{\rho}(0,s) = \rho_0(s)$.

Once *Claim* is verified, then the uniqueness of solutions to (6) implies

$$\bar{\rho}(t,s) = \rho(t,s) \quad \text{for } (t,s) \in [0,T_1] \times \Sigma.$$

This means

$$\rho_\gamma(t,u^1,u^2_\gamma+a^2_\gamma,\cdots,u^{n-1}_\gamma+a^{n-1}_\gamma)=\rho_\gamma(t,u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma)$$

for $(u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma)\in U'_\gamma$ with $\gamma\in 1,\cdots,\bar\gamma$, which shows that $\rho_\gamma(t,u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma)$ does not depend on the $(u^2_\gamma,\cdots,u^{n-1}_\gamma)$-coordinate in U'_γ for $\gamma\in 1,\cdots,\bar\gamma$. Thus (ii) is proved.

It remains to prove *Claim*. Since $\zeta(\mathcal{R}(s))=\zeta(s)$, we see $\lambda_0(\zeta(\mathcal{R}(s)))=\lambda_0(\zeta(s))$. Then it follows from the definition of ρ_0 that

$$\rho(0,\mathcal{R}(s))=\rho_0(\mathcal{R}(s))=\lambda_0(\zeta(\mathcal{R}(s)))=\lambda_0(\zeta(s))=\rho_0(s),$$

which shows $\bar\rho(0,s)=\rho_0(s)$.

Next we check that $\bar\rho(t,s)$ also satisfies the equation (6). Set

$$\bar\rho_\gamma(t,u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma):=\bar\rho(t,\psi_\gamma^{-1}(u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma))$$

in U'_γ for $\gamma\in\{1,\cdots,\bar\gamma\}$. Then

$$\begin{aligned}&\partial_t\bar\rho_\gamma-F_\gamma(u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma,\bar\rho_\gamma,\cdots,D^4\bar\rho_\gamma)\\&=[\partial_t\rho_\gamma-F_\gamma(\cdot,\rho_\gamma,\cdots,D^4\rho_\gamma)]|_{(t,u^1,u^2_\gamma+a^2_\gamma,\cdots,u^{n-1}_\gamma+a^{n-1}_\gamma)}=0,\end{aligned}$$

where in the first equality we have used the fact that z and h do not depend on the $(u^2_\gamma,\cdots,u^{n-1}_\gamma)$-coordinate. Thus we see that $\bar\rho(t,s)$ satisfies (6) in at least $\bigcup_{\gamma=1}^{\bar\gamma}U_\gamma$. Changing the variable $(u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma)$ to $(u^1,u^2_\pm,\cdots,u^{n-1}_\pm)$, we see that $\bar\rho(t,s)$ also satisfies

$$\partial_t\bar\rho_\pm=F_\pm(u^1_\pm,\cdots,u^{n-1}_\pm,\bar\rho_\pm,\cdots,D^4\bar\rho_\pm)$$

in $\{(u^1_\pm,\cdots,u^{n-1}_\pm)\in U'_\pm;(u^1_\pm,\cdots,u^{n-1}_\pm)\neq(0,\cdots,0)\}$, where

$$\bar\rho_\pm(t,u^1_\pm,\cdots,u^{n-1}_\pm):=\bar\rho(t,\psi_\pm^{-1}(u^1_\pm,\cdots,u^{n-1}_\pm)).$$

Letting $s\to s_\pm$, we get

$$\partial_t\bar\rho_\pm=F_\pm(0,\cdots,0,\bar\rho_\pm,\cdots,D^4\bar\rho_\pm)$$

at $(u^1_\pm,\cdots,u^{n-1}_\pm)=(0,\cdots,0)$, which shows that $\bar\rho(t,s)$ also solves (6) in $U_\pm$. Thus we conclude that $\bar\rho(t,s)$ solves (6) in whole Σ, which proves *Claim*.

Similarly we can prove (iii) and we omit the details of its proof. □

Remark 6 . *Let $\rho(t,s)$ be the unique solution of (6) obtained in Theorem 5 with the initial data $\rho_0(s)=\lambda_0(\zeta(s))$. Since $\rho(t,s)$ depends only on t and the u^1-coordinate in $\Sigma\backslash\{s_\pm\}$, we simply write it as $\rho(t,u^1)$ instead of $\rho(t,\psi_\gamma^{-1}(u^1,u^2_\gamma,\cdots,u^{n-1}_\gamma))$ for $\gamma\in\{1,\cdots,\bar\gamma\}$. Set $\Sigma_c=\{(u^1,b\omega)\in\Sigma;|u^1|<a,\omega\in S^{n-1}\}$. Then, when s is in Σ_c, (6) is simply written as*

$$\begin{cases}\rho_t=-L^{-1}\Big[\dfrac{\partial^2H}{\partial u^1\partial u^1}+\Big(\dfrac{n-2}{b+\rho}-L^{-2}\dfrac{\partial^2\rho}{\partial u^1\partial u^1}\Big)\dfrac{\partial\rho}{\partial u^1}\dfrac{\partial H}{\partial u^1}\Big], & t\in[0,T_1],\ |u^1|<a,\\ \rho(0,u^1)=\lambda_0(u^1),\quad |u^1|<a,\end{cases}\tag{11}$$

where $L=(1+(\frac{\partial\rho}{\partial u^1})^2)^{1/2}$ and H is the mean curvature which is given by

$$H=(1+(\frac{\partial\rho}{\partial u^1})^2)^{-3/2}\Big\{\frac{\partial^2\rho}{\partial u^1\partial u^1}-\frac{n-2}{b+\rho}(1+(\frac{\partial\rho}{\partial u^1})^2)\Big\}.$$

This fact will be used in Section 5.

4 Deformation of strictly convex compact rotationally axisymmetric hypersurfaces

For a given strictly convex compact smooth rotationally axisymmetric hypersurface, using the results of [15], we shall give an explicit way of its deformation which weaken the local convexity of the original surface. First we summarize the results [15] for a deformation operator for concave functions and for strictly convex closed curve. Next we give a deformation of strictly convex compact hypersurface of rotation by deforming its generator.

Deformation of concave functions. Let f be smooth concave functions defined on the interval $(-1,1)$. For parameter $\delta \in (0,1/8)$, let χ_δ be a smooth cut-off function such as

$$\chi_\delta(y) = \begin{cases} 1 & \text{for } |y| < \delta, \\ 0 & \text{for } 2\delta < |y| < 1 \end{cases}$$

and $0 \leq \chi_\delta(y) \leq 1$ for $y \in (-1,1)$. Then, for parameters $\varepsilon \in (0,1]$ and $\delta \in (0,1/8)$ we define a deformation of f by

$$M^{\varepsilon,\delta}(y) = f(\frac{1}{2}) + \int_0^y v^{\varepsilon,\delta}(\xi)d\xi \quad \text{for } y \in (-1,1),$$

where

$$v^{\varepsilon,\delta}(y) = \int_0^y w^{\varepsilon,\delta}(\xi)d\xi\chi_{1/4}(y) + f'(y)(1-\chi_{1/4}(y)),$$

$$w^{\varepsilon,\delta}(y) = (-\varepsilon - \frac{y^4}{4!})\chi_\delta(y) + f''(y)(1-\chi_\delta(y))$$

for $y \in (-1,1)$.

Next we deform smooth concave functions f defined on the interval $(-\alpha_0,\alpha_0)$ with $\alpha_0 > 0$. For $c > 0$ set

$$(L_c f)(y) = f(cy) \quad \text{for } |y| < \alpha_0/c.$$

This L_c gives the dilation operator for f and $L_c f$ becomes smooth concave functions from $(-\alpha_0/c, \alpha_0/c)$ to $\mathbf{R}$. Then we define the deformation of f by

$$(M^{\varepsilon,\delta}_{\alpha_0} f)(y) = ((L_{1/\alpha_0} M^{\varepsilon,\delta} L_{\alpha_0})f)(y) \quad \text{for } y \in (-\alpha_0,\alpha_0).$$

Deformation of strictly convex closed curve. We shall deform $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ with an integer $m \geq 4$ and $\alpha \in (0,1/4)$. Let $r_a\lambda_0$ be the restriction of λ_0 on the interval $(-a,a)$. Set

$$\xi^{\varepsilon,\delta}(u^1) = \begin{cases} ((M^{\varepsilon,\delta}_a r_a)\lambda_0)(u^1) & \text{for } |u^1| < a, \\ \lambda_0(u^1) & \text{for } a \leq |u^1| \leq l/2, \end{cases}$$

and

$$\lambda^{\varepsilon,\delta}_0(u^1) = \begin{cases} \xi^{\varepsilon,\delta}(u^1), & \text{for } |u^1| \leq l/2, \\ \xi^{\varepsilon,\delta}(l-u^1), & \text{for } l/2 \leq u^1 \leq 3l/2. \end{cases} \tag{12}$$

We define a deformation of γ_0 by $\gamma^{\varepsilon,\delta}_0 := \gamma[\lambda^{\varepsilon,\delta}_0]$.

We state several properties for the deformed curve $\gamma^{\varepsilon,\delta}_0$.

Lemma 7 . *Let $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$ with an integer $m \geq 4$ and $\alpha \in (0, 1/4)$. let $\varepsilon \in (0,1]$ and $\delta \in (0, 1/8)$ and let $\lambda_0^{\varepsilon,\delta}$ be given in (12).*
(i) $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}]$ becomes a closed curve and is symmetric with respect to the both x^1 and x^2-axes.
(ii) $\lambda_0^{\varepsilon,\delta} \in E^{m+4\alpha}(M)$ and there is a constant $\Lambda^\delta > 0$ (which is also depends on λ_0, m, α and a but independent of ε) such that

$$\|\lambda_0^{\varepsilon,\delta}\|_{C^{m+4\alpha}(M)} \leq \Lambda^\delta \quad \text{for all } \varepsilon \in (0,1] \tag{13}$$

and Λ^δ is unbounded when $\delta \to 0$.
(iii) Outside the set $\{(x^1, x^2); |x^1| < \delta a, x^2 > 0\}$, the set $\gamma[\lambda_0^{\varepsilon,\delta}]$ agrees with $\gamma[\lambda_0]$.
(iv) $((\frac{d}{du^1})^2\lambda_0^{\varepsilon,\delta})(0) = -\varepsilon/a^2$, $((\frac{d}{du^1})^4\lambda_0^{\varepsilon,\delta})(0) = 0$, $((\frac{d}{du^1})^6\lambda_0^{\varepsilon,\delta})(0) = -1/a^6$.
(v) There exists a $\delta_0 > 0$ depending on λ_0 such that

$$\sup_{|u^1|\leq a} ((\frac{d}{du^1})^2\lambda_0^{\varepsilon,\delta})(u^1) < 0,$$

$$\inf_{|u^1|\leq a} \lambda_0^{\varepsilon,\delta}(u^1) \geq \lambda_0(\pm a) > 0$$

for all $\varepsilon \in (0,1]$, $\delta \in (0, \delta_0)$. In particular, $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}] \in \mathcal{C}^{m+4\alpha}(M)$ with

$$\inf_{\mu\in M} \lambda_0^{\varepsilon,\delta}(\mu) \geq \inf_{\mu\in M} \lambda_0(\mu) > 0 \quad \text{for all } \varepsilon \in (0,1],\ \delta \in (0,\delta_0). \tag{14}$$

This lemma is almost directly follows from Theorem 3 in [15].

Deformation of strictly convex compact closed hypersurface of rotation. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$. For $\delta \in (0, \delta_0)$ with δ_0 established in Lemma 7 (v), we define a deformation of Γ_0 by $\Gamma_0^{\varepsilon,\delta} := \Gamma[\lambda_0^{\varepsilon,\delta}]$. By virtue of Lemma 7, $\Gamma_0^{\varepsilon,\delta}$ is a strictly convex closed compact hypersurface of rotation about the x^1-axis with the generator $\gamma_0^{\varepsilon,\delta} = \gamma[\lambda_0^{\varepsilon,\delta}]$. We summarize the properties of $\Gamma_0^{\varepsilon,\delta}$ in the next lemma which follows from Lemma 7, (9), and (10).

Lemma 8 . *If $\delta \in (0, \delta_0)$, then $\Gamma_0^{\varepsilon,\delta}$ belongs to $\mathcal{S}^{m+4\alpha}(\Sigma)$. Moreover, the following uniform estimates hold:*

$$\begin{aligned} &\|\rho_0^{\varepsilon,\delta}\|_{C^{m+4\alpha}(\Sigma)} \leq K_0\Lambda^\delta, \\ &\min_{s\in\Sigma} \rho_0^{\varepsilon,\delta}(s) \geq \min_{\mu\in M} \lambda_0(\mu) > 0 \end{aligned} \tag{15}$$

for all $\varepsilon \in (0,1]$ and $\delta \in (0,\delta_0)$, where $\rho_0^{\varepsilon,\delta}(s) := \lambda_0^{\varepsilon,\delta}(\zeta(s))$ and $\zeta(s)$ is in (8), and K_0 is in (10).

Combining Theorems 3 and 5 and Lemma 8, we have the following uniform local existence result in $\varepsilon \in (0,1]$, which assures that the existence time of the solution does not shrink to 0 and the magnitude of the solution remains bounded when $\varepsilon \to 0$.

Proposition 9 . *Let $m \geq 4$ be an integer and $\alpha \in (0, 1/4)$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$ with $\gamma_0 = \gamma[\lambda_0] \in \mathcal{C}^{m+4\alpha}(M)$. Let δ_0 be positive constant determined by λ_0 in Lemma 7 (v) and let $\delta \in (0, \delta_0)$. Set $T^\delta := T(K_0\Lambda^\delta, \min_{\mu\in M}\lambda_0(\mu)) > 0$, where $K_0\Lambda^\delta$ is in (15). Then the followings hold.*

(i) For any $\varepsilon \in (0,1]$ there exists a unique solution $\rho^{\varepsilon,\delta}(t,s)$ of (6) with initial data $\rho_0^{\varepsilon,\delta}(s)$ (defined in Lemma 8) which satisfies

$$\rho^{\varepsilon,\delta} \in C^{[m/4]+\alpha,m+4\alpha}([0,T^\delta]\times\Sigma),$$

$$\|\rho^{\varepsilon,\delta}\|_{C^{[m/4]+\alpha,m+4\alpha}([0,T^\delta]\times\Sigma)} \le G(K_0\Lambda^\delta),$$

$$\min_{(t,s)\in[0,T^\delta]\times\Sigma} \rho^{\varepsilon,\delta}(t,s) \ge \frac{1}{2}\min_{\mu\in M}\lambda_0(\mu) > 0$$

for all $\varepsilon \in (0,1]$ and $\delta \in (0,\delta_0)$.
(ii) There is a unique $\lambda^{\varepsilon,\delta} \in C^{m+4\alpha}([0,T^\delta]\times M)$ such that $\rho^{\varepsilon,\delta}(t,s) = \lambda^{\varepsilon,\delta}(t,\zeta(s))$ with $\lambda^{\varepsilon,\delta}(0,\mu) = \lambda_0^{\varepsilon,\delta}(\mu)$ for $\mu \in M \cap \{x^2 \ge 0\}$.
(iii) $\rho^{\varepsilon,\delta}(t,s)$ is even in the u^1-coordinate.

5 Loss of convexity

In this section we prove our main theorem in the following.

Theorem 10 . *(Loss of convexity). Let $m \ge 10$ be an integer and let $\alpha \in (0,1/4)$. Let $\Gamma_0 = \Gamma[\lambda_0] \in \mathcal{S}^{m+4\alpha}(\Sigma)$. Let δ_0 be a positive constant established in Lemma 7 and let δ be in $(0,\delta_0)$. Let $\lambda_0^{\varepsilon,\delta}$ be the deformed function of λ_0 (defined in (12)) for $\varepsilon \in (0,1]$. Let T^δ be the time (established in Proposition 9) such that there is a unique solution $\rho^{\varepsilon,\delta}(t,s) = \lambda^{\varepsilon,\delta}(t,\zeta(s))$ of (6) in $[0,T^\delta]\times\Sigma$ with initial data $\rho_0^{\varepsilon,\delta}(s) = \lambda_0^{\varepsilon,\delta}(\zeta(s))$, where $\zeta(s)$ is in (8). Then there is an $\varepsilon_0^\delta > 0$ such that for any $\varepsilon \in (0,\varepsilon_0^\delta)$ there are positive times $t_0^{\varepsilon,\delta}$ and $t_1^{\varepsilon,\delta}$ with $t_0^{\varepsilon,\delta} < \min(T^\delta, t_1^{\varepsilon,\delta})$ having the property that the solution $\Gamma^{\varepsilon,\delta}(t) := \Gamma[\lambda^{\varepsilon,\delta}(t,\cdot)]$ of (1) starting from $\Gamma[\lambda_0^{\varepsilon,\delta}] \in \mathcal{S}^{m+4\alpha}(\Sigma)$ loses its convexity for at least $t \in (t_0^{\varepsilon,\delta}, \min(T^\delta, t_1^{\varepsilon,\delta}))$. Moreover, $t_0^{\varepsilon,\delta} \to 0$ as $\varepsilon \to 0$ for any $\delta \in (0,\delta_0)$.*

Proof. Recall that by Proposition 9 $\rho^{\varepsilon,\delta}(t,s)$ does not depend on the $(u_\gamma^2,\cdots,u_\gamma^{n-1})$-coordinate for $s \in U_\gamma$ with $\gamma \in \{1,\cdots,\bar{\gamma}\}$. So we simply write $\rho^{\varepsilon,\delta}(t,u^1)$ instead of $\rho^{\varepsilon,\delta}(t,\psi_\gamma^{-1}(u^1,u_\gamma^2,\cdots,u_\gamma^{n-1}))$. Our purpose is to show that $((\partial/\partial u^1)^2\rho^{\varepsilon,\delta})(t,0)$ becomes positive after a finite time in its smooth evolution by (6). We devide the proof into two steps.

First step. We shall prove that for any $\delta \in (0,\delta_0)$ there is a positive constant $C_{\varepsilon,\delta}$ such that

$$(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0) = a^{-6} + C_{\varepsilon,\delta}, \quad C_{\varepsilon,\delta} = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \tag{16}$$

for each $\delta \in (0,\delta_0)$. To show (16) we recall that $\rho^{\varepsilon,\delta}(t,u^1)$ satisfies (11) for $t \in [0,T^\delta]$ and $|u^1| < a$ with $\rho^{\varepsilon,\delta}(0,u^1) = \lambda_0^{\varepsilon,\delta}(u^1)$ for $|u^1| < a$. Differentiating (11) in u^1 twice and using the property $\rho^{\varepsilon,\delta}(t,u^1) = \rho^{\varepsilon,\delta}(t,-u^1)$ for $|u^1| < a$ of Proposition 9 (iii), we get

$$(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)$$
$$= \Big[- D_1^6\rho^{\varepsilon,\delta} - \{45(D_1^2\rho^{\varepsilon,\delta})^4(-\frac{n-2}{b+\rho^{\varepsilon,\delta}} + D_1^2\rho^{\varepsilon,\delta})$$

$$
\begin{aligned}
&-18(D_1^2\rho^{\varepsilon,\delta})^2\Big(\frac{(n-2)D_1^2\rho^{\varepsilon,\delta}}{(b+\rho^{\varepsilon,\delta})^2}-\frac{2(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{b+\rho^{\varepsilon,\delta}}\Big)\\
&-\frac{6(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{(b+\rho^{\varepsilon,\delta})^3}+\frac{12(n-2)(D_1^2\rho^{\varepsilon,\delta})^3}{(b+\rho^{\varepsilon,\delta})^2}\Big\}\\
&+\{(D_1^2\rho^{\varepsilon,\delta})^2-2D_1^2\rho^{\varepsilon,\delta}(\frac{n-2}{b+\rho^{\varepsilon,\delta}}-D_1^2\rho^{\varepsilon,\delta})\}\cdot\\
&\cdot\{3(D_1^2\rho^{\varepsilon,\delta})^2(\frac{n-2}{b+\rho^{\varepsilon,\delta}}-D_1^2\rho^{\varepsilon,\delta})+\frac{(n-2)D_1^2\rho^{\varepsilon,\delta}}{(b+\rho^{\varepsilon,\delta})^2}-\frac{2(n-2)(D_1^2\rho^{\varepsilon,\delta})^2}{b+\rho^{\varepsilon,\delta}}\}\Big](0,0),
\end{aligned}
$$

where $D_1 := \partial/\partial u^1$. We use values of derivatives of $\lambda_0^{\varepsilon,\delta}$ in Lemma 7 (iv) and use the inequality in Lemma 7 (v) to get (16).

(16) guarantees that for any $\delta \in (0,\delta_0)$ there is an $\varepsilon_1^\delta > 0$ such that

$$
(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0) \geq 1/2a^6 \quad \text{for } \varepsilon \in (0,\varepsilon_1^\delta]. \tag{17}
$$

We take $\varepsilon \in (0,\varepsilon_1^\delta]$ in the following.

Second step. We shall complete the proof of Theorem 10. We note that it follows from Proposition 9 that there is a constant $B^\delta > 0$ such that

$$
\sup_{t\in[0,T^\delta]} |((\frac{\partial}{\partial t})^2(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0)| \leq B^\delta. \tag{18}
$$

By Taylor's expansion, Lemma 7, (17), and (18), we obtain

$$
\begin{aligned}
((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0) &= ((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)+(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)t\\
&\quad +\int_0^t\int_0^\sigma((\frac{\partial}{\partial t})^2(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(\tau,0)d\tau d\sigma\\
&\geq ((\frac{d}{du^1})^2\rho_0^{\varepsilon,\delta})(0)+(\frac{\partial}{\partial t}(\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(0,0)t-B^\delta t^2\\
&\geq -a^{-2}\varepsilon+\frac{1}{2}a^{-6}t-B^\delta t^2 \qquad (19)
\end{aligned}
$$

for $0 \leq t \leq T^\delta$. We can take ε small so that the quadratic polynomial $B^\delta t^2 - \frac{1}{2}a^{-6}t + a^{-2}\varepsilon$ has the smallest positive zero less than T^δ. In fact, we can take $\varepsilon_2^\delta > 0$ small so that

$$
0 < \frac{\frac{1}{2}a^{-6}-(\frac{1}{4}a^{-12}-4B^\delta a^{-2}\varepsilon_2^\delta)^{1/2}}{2B^\delta} < T^\delta.
$$

Set $\varepsilon_0^\delta = \min(\varepsilon_1^\delta, \varepsilon_2^\delta)$. Then for each $\varepsilon \in (0,\varepsilon_0^\delta)$ the polynomial $B^\delta t^2 - \frac{1}{2}a^{-6}t + a^{-2}\varepsilon$ has two positive zeros $t_0^{\varepsilon,\delta} < t_1^{\varepsilon,\delta}$ with $t_0^{\varepsilon,\delta} < T^\delta$. By (19),

$$
((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0) \geq -(t-t_0^{\varepsilon,\delta})(t-t_1^{\varepsilon,\delta}) \quad \text{for } t \in [0,T^\delta].
$$

This implies

$$
((\frac{\partial}{\partial u^1})^2\rho^{\varepsilon,\delta})(t,0) > 0 \quad \text{for } t \in (t_0^{\varepsilon,\delta}, \min(T^\delta, t_1^{\varepsilon,\delta})).
$$

This shows that $\Gamma^{\varepsilon,\delta}(t) = \Gamma[\lambda^{\varepsilon,\delta}(t,\cdot)]$ loses its convexity for at least $t \in (t_0^{\varepsilon,\delta}, \min(T^\delta, t_1^{\varepsilon,\delta}))$. The assertion $t_0^{\varepsilon,\delta} \to 0$ as $\varepsilon \to 0$ follows from its definition. □

6 Sketch of the proof of Theorem 3

In this section we sketch the proof of Theorem 3 only for the case $m = 4$. General idea owes to the book of Lunardi [18]. The case $m \geq 4$ can be treated by a standard method using difference quotients of the derivatives of ρ. This technique is found in Proposition 8.5.6 in [18].

To prove Theorem 3, we shall linearize (6) at initial data $\rho_0(s)$ and apply the Banach fixed point theorem to the linearized equation. First we rewrite (6) in local coordinates. Let $\beta \in \{1, \cdots, \bar{\beta}\}$ be fixed and let $s = \psi_\beta^{-1}(u_\beta)$ be in $U_\beta \subset \Sigma$, where $u_\beta = (u_\beta^1, \cdots, u_\beta^{n-1})$. Set $\rho_\beta(t, u_\beta) := \rho(t, \psi_\beta^{-1}(u_\beta))$. We denote by $D_\beta^k \rho_\beta$ any k-th order derivatives (not co-variant derivatives) of ρ_β with respect to the u_β-coordinate. Then (6) can be written as

$$\partial_t \rho_\beta = F_\beta(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta, \cdots, D_\beta^4 \rho_\beta), \quad t > 0, \ u_\beta \in U_\beta' = \psi_\beta(U_\beta). \tag{20}$$

Here F_β has the following form:

$$\begin{aligned} F_\beta(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta, \cdots, D_\beta^4 \rho_\beta) &= -A_\beta^{ijkl}(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta) \frac{\partial^4 \rho_\beta}{\partial u_\beta^i \partial u_\beta^j \partial u_\beta^k \partial u_\beta^l} \\ &\quad + B_\beta^{ijk}(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta, D_\beta^2 \rho_\beta) \frac{\partial^3 \rho_\beta}{\partial u_\beta^i \partial u_\beta^j \partial u_\beta^k} \\ &\quad + C_\beta(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta, D_\beta^2 \rho_\beta) \quad \text{in } U_\beta', \end{aligned}$$

and A_β^{ijkl}, B_β^{ijk}, C_β are smooth in their arguments if $\rho > 0$. In particular, the coefficient of the fourth order derivative is explicitly given by

$$A_\beta^{ijkl}(u_\beta, \rho_\beta, D_\beta^1 \rho_\beta) = L^{-2} \sigma_\beta^{ij}(\rho_\beta, d\rho)(L^2 w_\beta^{kl}(\rho_\beta) - w_\beta^{ka}(\rho_\beta) w_\beta^{lb}(\rho_\beta) \frac{\partial \rho_\beta}{\partial u_\beta^a} \frac{\partial \rho_\beta}{\partial u_\beta^b}).$$

Hereafter, for simplicity of notatin we omit the index β and do not distinguish between $\rho(t, s)$ and $\rho_\beta(t, u_\beta)$. In order to linearize (20) at initial data ρ_0 we set

$$\begin{aligned} \mathcal{A} &= -A^{ijkl}(u, \rho_0, D^1 \rho_0) \frac{\partial^4}{\partial u^i \partial u^j \partial u^k \partial u^l} + B^{ijk}(u, \rho_0, D^1 \rho_0, D^2 \rho_0) \frac{\partial^3}{\partial u^i \partial u^j \partial u^k} \\ &\quad + \frac{\partial F}{\partial Y^{ij}}(u, \rho_0, D^1 \rho_0, \cdots, D^4 \rho_0) \frac{\partial^2}{\partial u^i \partial u^j} + \frac{\partial F}{\partial X^i}(u, \rho_0, D^1 \rho_0, \cdots, D^4 \rho_0) \frac{\partial}{\partial u^i} \\ &\quad + \frac{\partial F}{\partial p}(u, \rho_0, D^1 \rho_0, \cdots, D^4 \rho_0) I, \end{aligned}$$

where (p, X, Y) with $p \in \mathbf{R}$, $X = (X^i)_{1 \leq i \leq n-1}$, $Y = (Y^{ij})_{1 \leq i,j, \leq n-1}$ denotes the arguments corresponding to $(\rho, D^1 \rho, D^2 \rho)$ and I is the identity operator. Then it is straightforward to check that $\mathcal{A}$ is a well-defined differential operator acting on scalar fields on Σ. For positive parameters T and M and positive function ρ_0 in $C^{4+4\alpha}(\Sigma)$, we set

$$\begin{aligned} R_{T,M} &= \{ r \in C^{1+\alpha, 4+4\alpha}([0, T] \times \Sigma); \|r\|_T \leq 2M, \\ &\qquad \min_{(t,s) \in [0,T] \times \Sigma} r(t, s) \geq m_0/2, \quad r(0, s) = \rho_0(s) > 0 \ (s \in \Sigma) \} \end{aligned}$$

where $m_0 = \min_{s \in \Sigma} \rho_0(s)$ and $\|r\|_T = \|r\|_{C^{1+\alpha, 4+4\alpha}([0,T] \times \Sigma)}$. It is clear that $R_{T,M}$ is a complete metric space endowed with the norm $\| \cdot \|_T$. For a given $r \in R_{T,M}$ we consider the inhomogeneous linear equation of an unknown function $\rho : [0,T] \times \Sigma \to \mathbf{R}$ of the form:

$$\begin{cases} \rho_t = \mathcal{A}\rho + [F(u, r, D^1 r, \cdots, D^4 r) - \mathcal{A}r], & t \in [0,T],\ s \in \Sigma, \\ \rho(0,s) = \rho_0(s), \quad s \in \Sigma. \end{cases} \tag{21}$$

Then we have the following existence result for (21).

Lemma 11 . *Let T and M be positive constants and let $r \in R_{T,M}$. Then for any positive function ρ_0 in $C^{4+4\alpha}(\Sigma)$, (21) has a unique solution ρ in $C^{1+\alpha,4+4\alpha}([0,T] \times \Sigma)$ with $\rho(0,s) = \rho_0(s)$ for $s \in \Sigma$ satisfying*

$$\|\rho\|_T \leq C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)} + \|F(\cdot, r, \cdots, D^4 r) - \mathcal{A}r\|_T) \tag{22}$$

with some positive constant $C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})$.

Remark 12 . *$C_*(M_0)$ is nondecreasing in M_0.*

For the proof of Lemma 11 it is sufficient to show that a realization of $\mathcal{A}$ in $C(\Sigma)$ is sectorial in $C(\Sigma)$. Once this is verified, we can derive Lemma 11 from the standard theory for analytic semi-groups. We give the flow of the proof of Lemma 11 in Appendix.

Making use of Lemma 11, we prove Theorem 3.

Proof of Theorem 3. For $r \in R_{T,M}$ we define $\Phi(r) = \rho$, where ρ is the unique solution of (21) establishd in Lemma 11. We shall show that Φ has a unique fixed point in $R_{T,M}$ for suitablly chosen T and M depending on ρ_0. It is clear that every fixed point of Φ in $R_{T,M}$ is a solution of (6).

Claim: If r, $\bar{r} \in R_{T,M}$, then

$$\begin{aligned} \|F(\cdot, r, \cdots, D^4 r) - \mathcal{A}r\|_T \ \leq\ & \|F(\cdot, \rho_0, \cdots, D^4 \rho_0) - \mathcal{A}\rho_0\|_{C^{4\alpha}(\Sigma)} \\ & + MK(M, \|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha, \end{aligned} \tag{23}$$

$$\begin{aligned} & \|F(\cdot, r, \cdots, D^4 r) - F(\cdot, \bar{r}, \cdots, D^4 \bar{r}) - \mathcal{A}(r - \bar{r})\|_T \\ & \leq K(M, \|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha \|r - \bar{r}\|_T \end{aligned} \tag{24}$$

with some positive constant $K(M, \|\rho_0\|_{C^{4+4\alpha}(\Sigma)})$ depending also on up to second derivatives of F and $K(M, M_0)$ is nondecreasing in M and M_0.

We show (23). Using the mean value theorem, we see

$$\begin{aligned} & \|F(\cdot, r, \cdots, D^4 r) - \mathcal{A}r\|_T \\ & = \|F(\cdot, \rho_0, \cdots, D^4 \rho_0) + \frac{\partial F}{\partial p}(\cdot, (1-\theta)\rho_0 + \theta r, \cdots, (1-\theta)D^4 \rho_0 + \theta D^4 r)(r - \rho_0) \\ & + \cdots - A^{ijkl}(\cdot, (1-\theta)\rho_0 + \theta r, (1-\theta)D^1 \rho_0 + \theta D^1 r)\frac{\partial^4 (r - \rho_0)}{\partial u^i \partial u^j \partial u^k \partial u^l} \\ & -(\mathcal{A}\rho_0 + \mathcal{A}(r - \rho_0))\|_T \\ & \leq \|F(\cdot, \rho_0, \cdots, D^4 \rho_0) - \mathcal{A}\rho_0\|_{C^{4\alpha}(\Sigma)} \\ & + \|\{\frac{\partial F}{\partial p}(\cdot, (1-\theta)\rho_0 + \theta r, \cdots, (1-\theta)D^4 \rho_0 + \theta D^4 r) - \frac{\partial F}{\partial p}(\cdot, \rho_0, \cdots, D^4 \rho_0)\}(r - \rho_0)\|_T \\ & + \cdots + \|\{A^{ijkl}(\cdot, (1-\theta)\rho_0 + \theta r, (1-\theta)D^1 \rho_0 + \theta D^1 r) - A^{ijkl}(\cdot, \rho_0, D^1 \rho_0)\} \cdot \\ & \cdot \frac{\partial^4 (r - \rho_0)}{\partial u^i \partial u^j \partial u^k \partial u^l}\|_T \end{aligned}$$

with some $\theta \in [0,1]$. We assume that T is sufficiently small and we only give the estimates of the $C^{\alpha,4\alpha}$-seminorm of the fourth order derivatives (the rest terms can be estimated more easily):

$$\sup_{s\in\Sigma}[\tilde{A}^{ijkl}\frac{\partial^4(r-\rho_0)}{\partial u^i\partial u^j\partial u^k\partial u^l}]_{C^\alpha([0,T])} \le \sup_{r\in R_{T,M}} |D^1 A^{ijkl}| \cdot M^2 T^\alpha,$$

$$\sup_{t\in[0,T]}[\tilde{A}^{ijkl}\frac{\partial^4(r-\rho_0)}{\partial u^i\partial u^j\partial u^k\partial u^l}]_{C^{4\alpha}(\Sigma)} \le \sup_{r\in R_{T,M}} |D^1 A^{ijkl}| \cdot M(M+\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha,$$

where

$$\tilde{A}^{ijkl} = A^{ijkl}(u,(1-\theta)\rho_0+\theta r,(1-\theta)D^1\rho_0+\theta D^1 r) - A^{ijkl}(u,\rho_0,D^1\rho_0).$$

Also it is straightforward to check that

$$\sup_{r\in R_{T,M}} |D^1 A^{ijkl}| \le \bar{K}(M,\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})$$

with some constant $\bar{K}(M,\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})$ nondecreasing in the two arguments. Thus we have proved (23). One can show (24) in a similar way and *Claim* is proved.

It then follows from (22) and *Claim* that

$$\begin{aligned}\|\rho\|_T \;\le\; & C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)} + \|F(\cdot,\rho_0,\cdots,D^4\rho_0) - \mathcal{A}\rho_0\|_{C^{4\alpha}(\Sigma)}) \\ & + C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})MK(M,\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha,\end{aligned}$$

$$\|\rho-\bar{\rho}\|_T \le C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)},m_0)K(M,\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha\|r-\bar{r}\|_T.$$

Now we set

$$M = 2C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)} + \|F(\cdot,\rho_0,\cdots,D^4\rho_0) - \mathcal{A}\rho_0\|_{C^{4\alpha}(\Sigma)})$$

and take T be small so that

$$C_*(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})K(M,\|\rho_0\|_{C^{4+4\alpha}(\Sigma)})T^\alpha \le \frac{1}{2} \quad \text{and} \quad T \le \frac{m_0}{4M}.$$

Then there hold

$$\|\rho\|_T \le 2M, \quad \|\rho-\bar{\rho}\|_T \le \frac{1}{2}\|r-\bar{r}\|_T.$$

Moreover,

$$\begin{aligned}\rho(t,s) \;&=\; \rho_0(s) + \int_0^t \rho_\tau(\tau,s)d\tau \\ &\ge\; \min_{s\in\Sigma}\rho_0(s) - T\|\rho_t\|_{L^\infty([0,T]\times\Sigma)} \\ &\ge\; m_0 - \frac{m_0}{4M}\cdot 2M \ge \frac{1}{2}m_0\end{aligned}$$

for $t \in [0,T]$ and $s \in \Sigma$. These ienqualities show that Φ is a contraction mapping from $R_{T,M}$ into itself. Thus, by the Banach fixed point theorem, we conclude that there exists a unique fixed point $\rho \in R_{T,M}$ of Φ and this ρ is the desired solution of (6). The proof of Theroem 3 is complete. $\square$

Appendix

Here we show the *outline of the proof of Lemma 11.*

Our purpose is to show that the realization of $\mathcal{A}$ in $C(\Sigma)$, A_c, is sectorial. Once this is verified, the standard theory for analytic semi-groups (for instance [18]) assures that A_c generates an analytic semi-group e^{tA_c} in $C(\Sigma)$ for $t \geq 0$ by means of the Dunford integral and (21) can be written to the integral equation

$$\rho(t) = e^{tA_c}\rho_0 + \int_0^t e^{(t-\tau)A_c}[F(u,r,D^1r,\cdots,D^4r) - \mathcal{A}r](\tau)d\tau \quad \text{in } C(\Sigma)$$

for $t \in [0,T]$. This expression and the knowledges in [18] enables us to obtain the desired results in Lemma 11. The dependence of C_* on initial data ρ_0 can be clarified by examining the definition of e^{tA_c} and the resolvent estimates shown in the proof of Lemma 13 stated below.

In order to show that the realization of $\mathcal{A}$ in $C(\Sigma)$ is sectorial, we set

$$\begin{aligned} X_c &:= C(\Sigma), \\ D_c &:= \{\varphi \in \bigcap_{1<q<\infty} W^{4,q}(\Sigma); \mathcal{A}\varphi \in X_c\} \subset X_c \end{aligned}$$

and define the operator A_c by

$$\begin{array}{rccc} A_c: & D_c & \to & X_c \\ & \varphi & \mapsto & \mathcal{A}\varphi. \end{array}$$

Lemma 13 . *The following two statements hold.*

(i) Let $p > n-1$ be arbitrarily fixed. Then there is an $\omega = \omega(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that for any $f \in X_c$ and $Re\lambda \geq \omega$ the equation

$$\lambda\varphi - \mathcal{A}\varphi = f$$

has a unique solution $\varphi \in D_c$.

(ii) There is a $K_p = K_p(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that

$$\sum_{j=0}^{3} |\lambda|^{1-j/4}\|\varphi\|_{W^{j,\infty}(\Sigma)} + |\lambda|^{(n-1)/4p} \sup_{s\in\Sigma} \|\varphi\|_{W^{4,p}(B(s,|\lambda|^{-1/4}))} \leq K_p\|\lambda\varphi - \mathcal{A}\varphi\|_{L^\infty(\Sigma)}$$

for $Re\lambda \geq \omega$ and $\varphi \in D_c$. Here $B(s,|\lambda|^{-1/4})$ is a ball on Σ centered at s with radius $|\lambda|^{-1/4}$.

Thus the set $\{\lambda \in \mathbf{C}; Re\lambda \geq \omega\}$ is in the resolvent set of A_c and consequently A_c is sectorial in X_c.

Outline of the proof of Lemma 13. The proof proceeds via three steps.

First step. The purpose of this step is to prove that the realization of $\mathcal{A}$ in $L^p(\Sigma)$ for any fixed $p \in (1, \infty)$ is sectorial.

To do this we set

$$X_p := L^p(\Sigma), \quad D_p := W^{4,p}(\Sigma)$$

and define the operator A_p by

$$\begin{array}{rccc} A_p : & D_p & \to & X_p \\ & \varphi & \mapsto & \mathcal{A}\varphi. \end{array}$$

Claim 1: The following two statements hold.

(i) There is an $\omega_p = \omega_p(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that for any $f \in X_p$ and $\mathrm{Re}\lambda \geq \omega_p$ the equation

$$\lambda\varphi - \mathcal{A}\varphi = f$$

has a unique solution $\varphi \in D_p$.

(ii) There is an $M_p = M_p(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that

$$\sum_{j=0}^{4} |\lambda|^{1-j/4} \|\varphi\|_{W^{j,p}(\Sigma)} \leq M_p \|\lambda\varphi - \mathcal{A}\varphi\|_{L^p(\Sigma)}$$

for $\mathrm{Re}\lambda \geq \omega_p$ and $\varphi \in D_p$.

Thus $\{\lambda \in \mathbf{C}; \mathrm{Re}\lambda \geq \omega_p\}$ is in the resolvent set of A_p and consequently A_p is sectorial in X_p.

The proofs of (i) and (ii) are quite long but standard so they are omitted.

Seconf step. Next purpose is to show that the realization of $\mathcal{A}$ in $L^\infty(\Sigma)$ is sectorial. Set

$$\begin{aligned} X_\infty &:= L^\infty(\Sigma), \\ D_\infty &:= \{\varphi \in \bigcap_{1<q<\infty} W^{4,q}(\Sigma); \mathcal{A}\varphi \in X_\infty\} \subset X_\infty \end{aligned}$$

and define the operator A_∞ by

$$\begin{array}{rccc} A_\infty : & D_\infty & \to & X_\infty \\ & \varphi & \mapsto & \mathcal{A}\varphi. \end{array}$$

Claim 2: The following two statements hold.

(i) Let $p > n - 1$ be arbitrarily fixed. Then there is an $\omega_\infty = \omega_\infty(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that for any $f \in X_\infty$ and $\mathrm{Re}\lambda \geq \omega_\infty$ the equation

$$\lambda\varphi - \mathcal{A}\varphi = f$$

has a unique solution $\varphi \in D_\infty$.

(ii) There is a $K_p = K_p(\|\rho_0\|_{C^{4+4\alpha}(\Sigma)}) > 0$, which is nondecreasing in its argument, such that

$$\sum_{j=0}^{3} |\lambda|^{1-j/4}\|\varphi\|_{W^{j,\infty}(\Sigma)} + |\lambda|^{(n-1)/4p} \sup_{s\in\Sigma} \|\varphi\|_{W^{4,p}(B(s,|\lambda|^{-1/4}))} \leq K_p\|\lambda\varphi - \mathcal{A}\varphi\|_{L^\infty(\Sigma)}$$

for $\mathrm{Re}\lambda \geq \omega_\infty$ and $\varphi \in D_\infty$.

Thus the set $\{\lambda \in \mathbf{C}; \mathrm{Re}\lambda \geq \omega_\infty\}$ is in the resolvent set of A_∞ and consequently A_∞ is sectorial in X_∞.

Claim 2 can be derived from *Claim 1* by virtue of Sobolev embedding theorem

$$W^{1,p}(\Sigma) \subset C^{1-(n-1)/p} \quad \text{for } p > n-1. \tag{25}$$

We refer the reader to Chapter 3 in [18].

Step 3.

Claim 3: The following holds:

$$(\text{the resolvent set of } A_\infty) \subset (\text{the resolvent set of } A_c). \tag{26}$$

Once this is verified, then (26) together with *Claim 2* as $\omega = \omega_\infty$ shows that A_c is sectorial in X_c.

We show (26). Take any λ in the resolvent set of A_∞. Then for any $f \in C(\Sigma) \subset L^\infty(\Sigma)$, $\lambda\varphi - \mathcal{A}\varphi = f$ has a unique solution $\varphi \in D_\infty$. From the definition of D_∞ it follows that $\varphi \in W^{1,p}(\Sigma)$ for $p > n-1$, and (25) implies $\varphi \in C(\Sigma)$. Thus $\mathcal{A}\varphi = \lambda\varphi - f \in C(\Sigma)$. This shows that $\varphi \in D_c$, which together with *Claim 2* (ii) implies that λ is in the resolvent set of A_c. Thus (26) is proved. The proof of Lemma 13 is complete. □

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