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Title	L^∞ -estimate of first-order space derivatives of Stokes flow in a half space
Author(s)	Shimizu, Y.
Citation	Hokkaido University Preprint Series in Mathematics, 434, 1-22
Issue Date	1998-11-01
DOI	https://doi.org/10.14943/83580
Doc URL	https://hdl.handle.net/2115/69184
Type	departmental bulletin paper
File Information	re434.pdf



**L^∞ -ESTIMATE OF FIRST-ORDER
SPACE DERIVATIVES OF STOKES
FLOW IN A HALF SPACE**

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Series #434. November 1998

HOKKAIDO UNIVERSITY
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L^∞ -ESTIMATE OF FIRST-ORDER SPACE DERIVATIVES OF STOKES FLOW IN A HALF SPACE

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§1 Introduction

We consider the Stokes equation

$$(1.1) \quad \begin{aligned} u_t - \Delta u + \nabla p &= 0, \operatorname{div} u = 0 \text{ in } \Omega \times (0, \infty), \\ u &= u_0 \text{ at } t = 0, \\ u &= 0 \text{ on } \partial\Omega \times (0, \infty) \end{aligned}$$

in a domain Ω in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary. Here $u = (u^1, \dots, u^n)$ is unknown velocity field and p is unknown pressure field. Initial data u_0 is assumed to satisfy a *compatibility condition*: $\operatorname{div} u_0 = 0$ in Ω and the normal component of u_0 equals zero on $\partial\Omega$. This system is a typical parabolic equation and it has several properties resembling the heat equation.

Regularity-decay estimates like $L^p - L^q$ estimates are extensively studied especially for $1 < p, q < \infty$ as explained in Giga-Matsui-Shimizu [8]. We reproduce its explanation for reader's convenience. If $\Omega = \mathbb{R}^n$, u is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. In this case regularity-decay estimate

$$(1.2) \quad \|\nabla u(t)\|_p \leq C t^{-1/2} \|u_0\|_p \text{ for } t > 0$$

The author is grateful to the referees for numerous expository suggestions on this paper. He is also grateful to Professor Shin'ya Matsui for several valuable comments.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

holds for all $1 \leq p \leq \infty$ with C independent of t and u_0 , where

$$\|f(t)\|_p := \left(\int_{\Omega} |f(t, x)|^p dx \right)^{1/p}$$

and ∇ denotes the gradient in space variables. If $p = 2$, the estimate (1.2) is still valid for any domain. Indeed, since the Stokes operator A is self-adjoint and nonnegative, the operator A generates an analytic semigroup e^{-tA} . This yields

$$\|A^{1/2}e^{-tA}u_0\|_2 \leq Ct^{-1/2}\|u_0\|_2.$$

Since $u = e^{-tA}u_0$ and $\|A^{1/2}u\|_2 = \|\nabla u\|_2$, (1.2) follows for $p = 2$. (See Borchers and Miyakawa [3] for applications.) For $1 < p < \infty$, (0.2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [14]). The estimate (1.2) is also valid for exterior domain with $n \geq 3$, with extra restriction $1 < p < n$. (See Borchers and Miyakawa [2], Giga and Sohr [9], Iwashita [11].)

However, few results are available for $p = 1$ or $p = \infty$ in the case where the boundary $\partial\Omega$ of Ω is nonempty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 and L^∞ type spaces, because it involves singular integral operators such as Riesz operators. In the case $p = 1$, Giga-Matsui-Shimizu [8] showed (1.2) when Ω is a half-space by estimating Ukai's representation of solutions in Hardy space $\mathcal{H}^1$ of Fefferman and Stein ([6,13]).

In this paper, we prove (1.2) for $p = \infty$ where Ω is a half space $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n); x_n > 0\}$.

Theorem 1.1. *Let u be the solution of the Stokes equation (1.1) in $\Omega = \mathbb{R}_+^n$ with initial data $u_0 \in L^\infty(\mathbb{R}^n)$, which satisfies $\operatorname{div} u_0 = 0$. Then there is a constant C independent of u_0 such that*

$$(1.3) \quad \|\nabla u(t)\|_\infty \leq Ct^{-1/2}\|u_0\|_\infty$$

for all $t > 0$.

Since the Helmholtz projection is not bounded in L^1 and L^∞ , it seems impossible to derive (1.3) from L^1 -estimate $\|\nabla u(t)\|_1 \leq Ct^{-1/2}\|u_0\|_1$ of

[8] by duality argument. Also, the method described in [8] does not directly apply to get (1.3) since many terms in Ukai's formula [14] have singularities on the hyperplane $\{x_n = 0\}$ when we estimate its derivatives in L^∞ spaces.

A key idea of this paper is to rearrange Ukai's formula for ∇u by modifying way of extension so that the terms involving the square root of minus tangential Laplacian Λ have no singularities on $\{x_n = 0\}$. This enables us to estimate all terms in L^∞ following idea of [8]. Of course, we should estimate terms involving singularities. However, since there are no square root of the minus tangential Laplacian, we are able to estimate these terms by estimating its integral kernels directly in L^1 .

The proof of this theorem is divided in three sections. In section 2, we rearrange the solution formula obtained by ukai [14]. We eliminate some of these singularities in Ukai's formula by modifying a way of extending solutions u to $\{x_n < 0\}$, so that the terms involving the square root of minus tangential Laplacian $\Lambda = (-\Delta')^{-1/2}$ have no singularities on $x_n = 0$. In section 3, we estimate the terms which do not include Λ . These terms involve singularities on $\{x_n = 0\}$, but they can be estimated in L^∞ by estimating the corresponding integral kernels directly in the space L^1 . Finally, in section 4 we estimate the terms involving Λ . Since these terms do not have singularities on $\{x_n = 0\}$, one can estimate the corresponding integral kernels in the Hardy space as in [8]. These estimates then provide L^1 -estimates for integral kernels which are needed, as in section 3, in estimating the terms in L^∞ .

§2 Solution formula

In this section, we deduce a new solution formula of (0.1). First, we fix some notation. For an n -dimensional vector a , we denote the tangential component $(a_1, \dots, a_{n-1})$ by $a' \in \mathbb{R}^{n-1}$, so that $a = (a', a_n)$. We set $\partial_j = \partial/\partial x_j$ and let $\nabla' = (\partial_1, \dots, \partial_{n-1})$. Hereafter, C denotes a positive constant which may differ from one occasion to another.

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and let $\hat{f}$ be the Fourier tranform of f with respect to the tangential

variables x' :

$$\hat{f}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix' \cdot \xi'} f(x', x_n) dx'.$$

The Riesz operators R_j ($j = 1, \dots, n$), and the operator Λ are defined by

$$\begin{aligned} \mathcal{F}(R_j f)(\xi) &= \frac{i\xi_j}{|\xi|} \mathcal{F}f(\xi), \\ \mathcal{F}(\Lambda f)(\xi) &= |\xi'| \mathcal{F}f(\xi) \end{aligned}$$

and we set $R' = (R_1, \dots, R_{n-1})$.

We also define the operator $E(t)$ and $F(t)$ by

$$\begin{aligned} [E(t)f](x) &= \int_{\mathbb{R}_+^n} \{G_t(x - y) - G_t(x' - y', x_n + y_n)\} f(y) dy, \\ [F(t)f](x) &= \int_{\mathbb{R}_+^n} \{G_t(x - y) + G_t(x' - y', x_n + y_n)\} f(y) dy, \end{aligned}$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. Furthermore, we define the operator $E_+(t)$ by

$$[E_+(t)f](x) = \begin{cases} [E(t)f](x) & \text{for } x_n > 0, \\ [E(t)f](x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Note that $z = E(t)f$ (resp. $F(t)f$) solves the heat equation in $\mathbb{R}_+^n$ with zero-Dirichlet (resp. zero-Neumann) boundary condition;

$$\begin{aligned} z_t - \Delta z &= 0 \text{ in } \mathbb{R}_+^n \times (0, T), \\ z|_{t=0} &= f, \\ z|_{x_n=0} &\equiv 0. \text{ (resp. } \partial_n z|_{x_n=0} = 0.) \end{aligned}$$

Now we are ready to show the new formula.

Theorem 2.1. Assume that u_0 is in $L^p(\mathbb{R}_+^n)$, $1 \leq p \leq \infty$ and satisfies $\operatorname{div} u_0 = 0$. Then the function

(2.1a)

$$u^n(t) = -\Lambda(-\Delta)^{-1} \nabla' \cdot E(t) u'_0 + \partial_n(-\Delta)^{-1} \nabla' \cdot E_+(t) u'_0 \\ + (-\Delta)^{-1} \Delta' E_+(t) u_0^n - \partial_n(-\Delta)^{-1} \Lambda E(t) u_0^n,$$

(2.1b)

$$u'(t) = E(t) u'_0 + \Lambda^{-1} \nabla' E(t) u_0^n \\ + \nabla'(-\Delta)^{-1} \{ \nabla' \cdot E_+(t) u'_0 \} - \partial_n(-\Delta)^{-1} \Lambda^{-1} \nabla' \{ \nabla' \cdot E(t) u'_0 \} \\ - \nabla'(-\Delta)^{-1} \{ \Lambda E(t) u_0^n \} + \partial_n(-\Delta)^{-1} \nabla' E_+(t) u_0^n$$

satisfies (1.1) in $\Omega = \mathbb{R}_+^n$.

Proof. From (1.1) we get $\Delta p = 0$ so that

$$(\partial_n^2 - |\xi'|^2) \hat{p} = 0.$$

Solving this under the additional condition that $\hat{p} \rightarrow 0$ as $x_n \rightarrow \infty$, we have $\hat{p}(x_n) = C(\xi') e^{-|\xi'| x_n}$. So $\hat{p}$ satisfies

$$(2.2) \quad (\partial_n + |\xi'|) \hat{p} = 0.$$

We next introduce the function v^n by

$$(2.3) \quad \hat{v}^n = (\partial_n + |\xi'|) \hat{u}^n.$$

Then we have

$$(2.4a) \quad v_t^n - \Delta v^n = 0 \text{ in } (0, \infty) \times \mathbb{R}^n,$$

$$(2.4b) \quad v^n|_{t=0} = (\partial_n + \Lambda) u^n \\ = -\nabla' \cdot u'_0 + \Lambda u_0^n,$$

$$(2.4c.) \quad v^n|_{x_n=0} = 0$$

Solving this gives

$$(2.5) \quad v^n = -\nabla' \cdot E(t) u'_0 + \Lambda E(t) u_0^n \\ \equiv V_1 + V_2.$$

Inserting this in the left-hand side of (2.3) and solving the resulting equation gives

$$\begin{aligned}
 (2.6) \quad \hat{u}^n &= \int_0^{x_n} e^{-|\xi'|(x_n-y_n)} \hat{v}^n(\xi', y_n) dy_n \\
 &= \int_0^{x_n} e^{-|\xi'|(x_n-y_n)} \hat{V}_1(\xi', y_n) dy_n \\
 &\quad + \int_0^{x_n} e^{-|\xi'|(x_n-y_n)} \hat{V}_2(\xi', y_n) dy_n \\
 &\equiv \hat{U}_1 + \hat{U}_2.
 \end{aligned}$$

Here, we consider the following two kinds of extensions of the functions $\hat{U}_k$ ($k = 1, 2$) from $\mathbb{R}_+^n$ to $\mathbb{R}^n$:

$$\begin{aligned}
 \widehat{U_{k+}}(x_n) &= \begin{cases} \hat{U}_k(x_n) & (x_n > 0), \\ \hat{U}_k(-x_n) & (x_n < 0), \end{cases} \\
 \widehat{U_{k-}}(x_n) &= \begin{cases} \hat{U}_k(x_n) & (x_n > 0), \\ -\hat{U}_k(-x_n) & (x_n < 0), \end{cases}
 \end{aligned}$$

and set

$$\begin{aligned}
 \widehat{u_{+-}}^n &= \widehat{U_{1+}} + \widehat{U_{2-}}, \\
 \widehat{u_{-+}}^n &= \widehat{U_{1-}} + \widehat{U_{2+}}.
 \end{aligned}$$

Since $[\mathcal{F}f](\xi) = \int_{-\infty}^{+\infty} e^{-i\xi_n x_n} \hat{f}(\xi', x_n) dx_n$, we have

$$\begin{aligned}
 \mathcal{F}U_{k+}(\xi) &= \int_{-\infty}^{+\infty} e^{-i\xi_n x_n} \widehat{U_{k+}}(\xi', x_n) dx_n \\
 &= \int_0^{+\infty} e^{i\xi_n x_n} \int_0^{x_n} e^{-|\xi'| (x_n - y_n)} \hat{V}_k(\xi', y_n) dy_n dx_n \\
 &\quad + \int_0^{+\infty} e^{-i\xi_n x_n} \int_0^{x_n} e^{-|\xi'| (x_n - y_n)} \hat{V}_k(\xi', y_n) dy_n dx_n \\
 &= \int_0^{+\infty} e^{|\xi'| y_n} \hat{V}_k(\xi', y_n) \int_{y_n}^{+\infty} e^{(-|\xi'| + i\xi_n) x_n} dx_n dy_n \\
 &\quad + \int_0^{+\infty} e^{|\xi'| y_n} \hat{V}_k(\xi', y_n) \int_{y_n}^{+\infty} e^{(-|\xi'| - i\xi_n) x_n} dx_n dy_n \\
 &= \frac{|\xi'| + i\xi_n}{|\xi|^2} \int_0^{+\infty} e^{i\xi_n y_n} \hat{V}_k(\xi', y_n) dy_n \\
 &\quad + \frac{|\xi'| - i\xi_n}{|\xi|^2} \int_0^{+\infty} e^{-i\xi_n y_n} \hat{V}_k(\xi', y_n) dy_n \\
 &= \frac{|\xi'|}{|\xi|} \mathcal{F}V_{k+}(\xi) - \frac{i\xi_n}{|\xi|} \mathcal{F}V_{k-}(\xi),
 \end{aligned}$$

and, similary,

$$\mathcal{F}U_{k-}(\xi) = \frac{|\xi'|}{|\xi|} \mathcal{F}V_{k-}(\xi) - \frac{i\xi_n}{|\xi|} \mathcal{F}V_{k+}(\xi),$$

where

$$\begin{aligned}
 V_{k+}(x_n) &= \begin{cases} V_k(x_n) & \text{for } x_n > 0, \\ V_k(-x_n) & \text{for } x_n < 0, \end{cases} \\
 V_{k-}(x_n) &= \begin{cases} V_k(x_n) & \text{for } x_n > 0, \\ -V_k(-x_n) & \text{for } x_n < 0. \end{cases}
 \end{aligned}$$

We note that $E(t)f$ is odd in x_n . Direct calculation gives

(2.7a)

$$\begin{aligned} u_{+-}^n &= -\Lambda(-\Delta)^{-1}\nabla' \cdot E_+(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla' \cdot E(t)u'_0 \\ &\quad + (-\Delta)^{-1}\Delta' E(t)u_0^n - \partial_n(-\Delta)^{-1}\Lambda E_+(t)u_0^n, \end{aligned}$$

(2.7b)

$$\begin{aligned} u_{-+}^n &= -\Lambda(-\Delta)^{-1}\nabla' \cdot E(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla' \cdot E_+(t)u'_0 \\ &\quad + (-\Delta)^{-1}\Delta' E_+(t)u_0^n - \partial_n(-\Delta)^{-1}\Lambda E(t)u_0^n. \end{aligned}$$

The formula (2.7b) proves (2.1a). We shall use u_{+-} later on. To get the desired formula (2.1b) for u' , we set $\hat{v}' = \hat{u}' + \frac{i\xi'}{|\xi'|}\hat{u}^n$. Then

$$\begin{aligned} \hat{v}'_t &= \hat{u}'_t + \frac{i\xi'}{|\xi'|}\hat{u}^n_t \\ &= (-|\xi'|^2 + \partial_n^2)\hat{v}' + \frac{i\xi'}{|\xi'|}(|\xi'| + \partial_n)\hat{p} \\ &= (-|\xi'|^2 + \partial_n^2)\hat{v}', \end{aligned}$$

hence $v'_t - \Delta v' = 0$ in $\mathbb{R}^n$. We also have the initial condition $v'|_{t=0} = u'_0 + \nabla'\Lambda^{-1}u_0^n$ and the boundary condition $v'|_{x_n=0} = 0$. Solving these gives $v' = E(t)u'_0 + \nabla'\Lambda^{-1}u_0^n$, and

$$(2.8) \quad u' = E(t)u'_0 + \nabla'\Lambda^{-1}u_0^n - \nabla'\Lambda^{-1}u^n.$$

Putting $u^n = u_{+-}^n$ in (2.7a) and computing operators, we obtain the desired formula. $\square$

Remark. Ukai[14] used the zero-extension operator e , that is, $ef = f$ for $x_n > 0$ and $ef = 0$ for $x_n < 0$ when extending the function $\hat{u}_0$ in (2.6) from $\mathbb{R}_+^n$ over $\mathbb{R}^n$. Using this extension, we have the following formula:

(2.9a)

$$\begin{aligned} u^n(t) &= -\Lambda(-\Delta)^{-1}\nabla' \cdot eE(t)u'_0 + \partial_n(-\Delta)^{-1}\nabla' \cdot eE(t)u'_0 \\ &\quad + (-\Delta)^{-1}\Delta' eE(t)u_0^n - \partial_n(-\Delta)^{-1}\Lambda eE(t)u_0^n, \end{aligned}$$

(2.9b)

$$\begin{aligned} u'(t) &= E(t)u'_0 + \Lambda^{-1}\nabla' E(t)u_0^n \\ &\quad + \nabla'(-\Delta)^{-1}\{\nabla' \cdot eE(t)u'_0\} - \partial_n(-\Delta)^{-1}\Lambda^{-1}\nabla'\{\nabla' \cdot eE(t)u'_0\} \\ &\quad - \nabla'(-\Delta)^{-1}\{\Lambda eE(t)u_0^n\} + \partial_n(-\Delta)^{-1}\nabla' eE(t)u_0^n \end{aligned}$$

In this formula, the terms involving Λ contain singularities on the boundary which will cause difficulties in estimating the solutions in L^∞ . See section 4 for more details.

We now deduce our desired formula for the function ∇u . Noting that $\operatorname{div} u_0 = 0$ and $R_j = \partial_j(-\Delta)^{-1}$, we have the following theorem:

Theorem 2.2. *Let u is the function in Theorem 2.1. Then*

(2.10a)

$$\begin{aligned} \partial_j u^n = & -R_j R' \cdot \Lambda E(t) u'_0 + R_j R_n \nabla' \cdot E_+(t) u'_0 \\ & + R_j R' \cdot \nabla' E_+(t) u_0^n - R_j R_n \Lambda E(t) u_0^n, \end{aligned}$$

(2.10b)

$$\begin{aligned} \partial_j u' = & \partial_j E(t) u'_0 + w_j \\ & + R_j R' \{ \nabla' \cdot E_+(t) u'_0 \} - R_j R_n \Lambda^{-1} \nabla' \{ \nabla' \cdot E(t) u'_0 \} \\ & - R_j R' \{ \Lambda E(t) u_0^n \} + R_j R_n \nabla' E_+(t) u_0^n, \end{aligned}$$

where

$$(2.10c) \quad w_j = \begin{cases} \Lambda^{-1} \partial_j \nabla' E(t) u_0^n & \text{for } j < n, \\ \Lambda^{-1} \nabla' \{ \nabla' \cdot E(t) u'_0 \} & \text{for } j = n. \end{cases}$$

Note that the terms containing Λ do not contain E_+ (which has singularities at $x_n = 0$).

§3 Estimates of solutions

In this section, we estimate the term without Λ . Note that $\partial_j E(t) u'_0 = (\partial_j G_t) * \bar{u}'_0$, where $\bar{f}$ is the odd extension of f with respect to x_n , that is,

$$\bar{f}(x) = \begin{cases} f(x', x_n) & \text{for } x_n > 0, \\ -f(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

So the estimate of this term is easily obtained by Young's theorem:

(3.1)

$$\|\partial_j E(t) u'_0\|_{L^\infty(\mathbb{R}_+^n)} \leq \|\partial_j G_t\|_{L^1(\mathbb{R}^n)} \|\bar{u}'_0\|_{L^\infty(\mathbb{R}_+^n)} \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Next we estimate the term $R_j R_k \partial_i E_+(t) u_0$. By the definition of $E_+(t)$, we have $E_+(t) u_0(x) = \chi_+(x) [E(t) u_0](x) - \chi_-(x) [E(t) f](x)$, where $\chi_\pm$ is

defined as

$$(3.2) \quad \begin{aligned} \chi_+(x_n) &= \begin{cases} 1 & \text{for } x_n > 0, \\ 0 & \text{for } x_n < 0, \end{cases} \\ \chi_-(x_n) &= 1 - \chi_+(x_n). \end{aligned}$$

So we have

$$(3.3) \quad \begin{aligned} R_j R_k \partial_i E_+(t) u_0(x) &= \int_{\mathbb{R}^n} \bar{u}_0(y) R_{x_j} R_{x_k} \partial_{x_i} (\chi_+(x_n) G_t(x, y)) dy \\ &\quad - \int_{\mathbb{R}^n} \bar{u}_0(y) R_{x_j} R_{x_k} \partial_{x_i} (\chi_-(x_n) G_t(x, y)) dy, \end{aligned}$$

where $G_t(x, y) = G_t(x - y)$.

To estimate (3.3) in L^∞ , we have only to estimate the integral kernel $R_{x_j} R_{x_k} \partial_{x_i} (\chi_\pm(x_n) G_t(x, y))$ in the space L^1 of function of y .

Lemma 3.1. *Assume that $1 \leq i \leq n-1$, $1 \leq j, k \leq n$, and the function f is in $L^\infty(\mathbb{R})$. Then*

$$(3.4) \quad \|R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_t(x, \cdot))\|_{L^1(\mathbb{R}^n)} \leq C t^{-1/2} \|f\|_{L^\infty(\mathbb{R})}$$

with C independent of x .

Proof. It suffices to show (3.4) for $t = 1$. Indeed, setting $x = t^{1/2} X$ and $y = t^{1/2} Y$, we then obtain

$$(3.5) \quad R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_t(x, y)) = t^{-(n+1)/2} R_{X_j} R_{X_k} \partial_{X_i} (f(t^{1/2} X_n) G_1(X, Y))$$

and $\|f(t^{1/2} \cdot)\|_{L^\infty} = \|f\|_{L^\infty}$. Moreover we may assume $j \neq n$, because if $j = k = n$, then we can reduce the problem to the $j \neq n$, using $\sum_{\alpha=1}^n R_\alpha^2 = -I$.

Note that $R_j R_k = \partial_j \partial_k (-\Delta)^{-1}$ and the integral kernel of $\partial_k (-\Delta)^{-1}$ is $c_n x_k |x|^{-n}$. So we have

$$(3.6) \quad R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_1(x, y)) = c_n \partial_{x_i} \partial_{x_j} \int_{\mathbb{R}^n} \frac{x_k - z_k}{|x - z|^n} f(z_n) G_1(z, y) dz.$$

Let ψ_1 be a smooth function in $\mathbb{R}^{n-1}$ such that $0 \leq \psi_1 \leq 1$, $\text{supp } \psi \subset \{|x| \leq 1\}$, and $\psi_1|_{|x| < 1/2} \equiv 1$. Set $\psi_2 = 1 - \psi_1$. Then

$$\begin{aligned}
 (3.7) \quad & c_n^{-1} R_{x_j} R_{x_k} \partial_{x_i} (f(x_n) G_1(x, y)) \\
 &= \partial_{x_i} \partial_{x_j} \int_{\mathbb{R}^n} \frac{(x_k - z_k) \psi_1(x - z)}{|x - z|^n} f(z_n) G_1(z, y) dz \\
 &+ \partial_{x_i} \partial_{x_j} \int_{\mathbb{R}^n} \frac{(x_k - z_k) \psi_2(x - z)}{|x - z|^n} f(z_n) G_1(z, y) dz \\
 &= I_1(x, y) + I_2(x, y).
 \end{aligned}$$

The estimate of I_1 : Recalling that $i, j \neq n$, we have

$$\begin{aligned}
 I_1(x, y) = \int_{\mathbb{R}^n} & \left\{ \frac{w_k \psi_1(w)}{|w|^n} \right\} f(x_n - w_n) \\
 & \left\{ \frac{(x_i - y_i - w_i)(x_j - y_j - w_j)}{4} - \frac{\delta_{ij}}{2} \right\} G_1(x - w, y) dw,
 \end{aligned}$$

where δ_{ij} is Kronecker's delta.

Recalling $|x - y - w| \leq |x - y| + 1$ and $|x - y - w|^2 \geq |x - y|^2/2 - 1$ hold for $|w| \leq 1$, we get

$$\begin{aligned}
 (3.9) \quad & |I_1(x, y)| \leq C \int_{|w| \leq 1} |w|^{-(n-1)} |f(x_n - w_n)| \{ (|x - y| + 1)^2 + 1 \} \\
 & e^{-(|x-y|^2-2)/8} dw \\
 & \leq C(|x - y|^2 + 1) e^{-|x-y|^2/8} \|f\|_{L^\infty(\mathbb{R})}
 \end{aligned}$$

and obtain $\|I_1(x, \cdot)\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(\mathbb{R})}$.

The estimate of I_2 : We have

$$\begin{aligned}
 (3.10) \quad & I_2(x, y) = \sum_{\alpha=0}^2 \int_{\mathbb{R}^n} K_\alpha(x - z) f(z_n) G_1(z, y) dz \\
 & = \sum_{\alpha=0}^2 J_\alpha(x, y),
 \end{aligned}$$

where

(3.11a)

$$K_0(x) = \frac{x_k(\partial_i \partial_j \phi_2)(x)}{|x|^n},$$

(3.11b)

$$K_1(x) = \left(\frac{\delta_{jk}}{|x|^n} - n \frac{x_j x_k}{|x|^{n+2}} \right) (\partial_i \phi_2)(x) + \left(\frac{\delta_{ik}}{|x|^n} - n \frac{x_i x_k}{|x|^{n+2}} \right) (\partial_j \phi_2)(x)$$

(3.11c)

$$K_2(x) = n \left\{ (n+2) \frac{x_j x_k x_k}{|x|^{n+4}} - \frac{\delta_{jk} x_i + \delta_{ki} x_j + \delta_{ij} x_k}{|x|^{n+2}} \right\} \phi_2(x).$$

Noting that $\text{supp } \nabla \phi_2 \subset 1/2 < |x| < 1$, we can estimate J_α ($\alpha = 0, 1$) as in the case of I_1 , to obtain

(3.12)

$$\begin{aligned} |J_\alpha(x, y)| &\leq C \int_{1/2 \leq |x-z| \leq 1} |x-z|^{-n-\alpha+1} |f(z_n)| G_1(z, y) dz \\ &= C \int_{1/2 \leq |w| \leq 1} |w|^{-n-\alpha+1} |f(x_n - w_n)| G_1(x - w, y) dw \\ &\leq C(|x-y|^2 + 1) e^{-|x-y|^2/8} \|f\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

To estimate the term J_2 , we use the inequality $|x-y|^l \leq C_l(|x-z|^l + |z-y|^l)$ for $l \geq 0$. Assuming that $0 \leq l \leq n+1$, we have

(3.13)

$$\begin{aligned} |J_2(x, y)| &\leq C \int_{|x-z| \geq 1/2} |x-z|^{-n-1} |f(z_n)| G_1(z, y) dz \\ &\leq C \int_{|x-z| \geq 1/2} |x-y|^{-l} (|x-z|^{l-n-1} + |x-z|^{-n-1} |z-y|^l) \\ &\quad |f(z_n)| G_1(z, y) dz \\ &\leq C |x-y|^{-l} \|f\|_{L^\infty(\mathbb{R})} \\ &\quad \int_{|x-z| \geq 1/2} (2^{n-l+1} + 2^{n+1} |z-y|^l) G_1(z, y) dz \\ &\leq C |x-y|^{-l} \|f\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

hence

$$(3.14) \quad |J_2(x, y)| \leq \frac{C}{|x - y|^{n+1} + 1} \|f\|_{L^\infty(\mathbb{R})}.$$

Combining the estimates for J_α ($\alpha = 0, 1, 2$), we obtain $\|I_2(x, \cdot)\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(\mathbb{R})}$. Collecting the estimates of I_1 and I_2 completes the proof of Lemma 3.1. $\square$

Using Lemma 3.1, we obtain the estimate of (3.1):

$$(3.15) \quad \begin{aligned} & \|R_j R_k \partial_i E_+(t) u_0\|_{L^\infty(\mathbb{R}^n)} \\ & \leq \|\bar{u}_0\|_{L^\infty(\mathbb{R}^n)} \|R_{x_j} R_{x_k} \partial_{x_i} (\chi_+(x_n) G_t(x, \cdot))\|_{L^1(\mathbb{R}^n)} \\ & \quad + \|\bar{u}_0\|_{L^\infty(\mathbb{R}^n)} \|R_{x_j} R_{x_k} \partial_{x_i} (\chi_-(x_n) G_t(x, \cdot))\|_{L^1(\mathbb{R}^n)} \\ & \leq \|u_0\|_{L^\infty(\mathbb{R}_+^n)} C t^{-1/2} \|\chi_+\|_{L^\infty(\mathbb{R})} \\ & \quad + \|u_0\|_{L^\infty(\mathbb{R}_+^n)} C t^{-1/2} \|\chi_-\|_{L^\infty(\mathbb{R})} \\ & \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}_+^n)}. \end{aligned}$$

§4 Application of Hardy space theory

Finally, we estimate the term $R_j R_k \partial_h \partial_i \Lambda^{-1} E(t) u_0$ ($1 \leq h, i \leq n-1$, $1 \leq j, k \leq n$). Since we have

$$(4.1) \quad R_j R_k \partial_h \partial_i \Lambda^{-1} E(t) u_0(x) = \int_{\mathbb{R}^n} \bar{u}_0(y) R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} G_t(x, y) dy,$$

we need to show

$$(4.2) \quad \|R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} G_t(x, \cdot)\|_{L^1(\mathbb{R}^n)} \leq C t^{-1/2}.$$

In this section we prove (4.2) with the aid of the theory of Hardy space $\mathcal{H}^1$. First, we recall the definition of $\mathcal{H}^1$.

Definition 4.1. A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$ if

$$(4.3) \quad f^+(x) = \sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n),$$

where the symbol $*$ denotes the convolution with respect to the space variable x . The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$(4.4) \quad \|f\|_{\mathcal{H}^1} := \|f^+\|_{L^1(\mathbb{R}^n)}$$

It is known ([6,13]) that an L^1 -function f is in $\mathcal{H}^1$ if and only if all its Riesz transforms $R_j f$ are in $L^1(\mathbb{R}^n)$ and that

$$(4.5) \quad \|f\|_{\mathcal{H}^1} \cong \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j f\|_{L^1(\mathbb{R}^n)} \text{ (equivalent norm)}.$$

We denote by $|||\cdot|||_{\mathcal{H}^1}$ the operator-norm of R_j in $\mathcal{H}^1$.

Since $G_t(x, y) = G_t(y, x)$, we have

$$(4.6) \quad R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} G_t(x, y) = R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, y)$$

and

$$(4.7) \quad \begin{aligned} & \|R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, \cdot)\|_{\mathcal{H}^1} \\ & \leq |||R_j|||_{\mathcal{H}^1} |||R_j|||_{\mathcal{H}^1} \|\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, \cdot)\|_{\mathcal{H}^1}. \end{aligned}$$

So we have only to show that $\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x - y)$ is in $\mathcal{H}^1$ as a function of y .

Remark. Using the formula obtained by Ukai[14], we have

$$(4.8) \quad \begin{aligned} & R_j R_k \partial_h \partial_i \Lambda^{-1} eE(t) u_0(x) \\ & = \int_{\mathbb{R}^n} \bar{u}_0(y) R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} \Lambda_x^{-1} \{\theta(x_n) G_t(x, y)\} dy, \end{aligned}$$

where θ is the Heaviside function, that is, $\theta = 1$ for $x_n > 0$ and $\theta = 0$ for $x_n < 0$. Since $\theta(x_n) G_t(x - y)$ is not symmetric with the respect to x and y , we cannot change the operator R_{x_j} to R_{y_j} and so we have to estimate (4.8) directly.

The following is our key lemma for estimating $\partial_j \partial_k \Lambda^{-1} G_t$ in the space $\mathcal{H}^1$.

Lemma 4.1. [8] Assume that real parameters l and m satisfy $0 \leq l \leq n$ and $m \geq 0$. Then there exists a constant $C = C_{l,m}$ which does not depend on $x \in \mathbb{R}^n$ and $t \geq 0$ such that

$$(4.9) \quad |\partial_j \partial_k \Lambda^{-1} G_t(x)| \leq C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \\ \text{for } 1 \leq j, k \leq n-1 \text{ with } n \geq 3,$$

$$(4.10) \quad |\Lambda G_t(x)| \leq C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m} \\ \text{with } n \geq 2.$$

Proof. The proof is given in [8], but we will reproduce the proof for reader's convenience. Note that the operator Λ^{-1} is equal to $(-\Delta')^{-1/2} = \left(-\sum_{k=1}^{n-1} \partial_k^2\right)^{-1/2}$, so the integral kernel of Λ^{-1} is $c_n |x'|^{-n+2}$ for $n \geq 3$, where c_n is some positive constant. Therefore we have

$$(4.11) \quad \partial_j \partial_k \Lambda^{-1} G_t(x) = c_n \partial_j \partial_k \int_{\mathbb{R}^{n-1}} |x' - y'|^{-n+2} G_t(y', x_n) dy'.$$

Set $x = t^{1/2} z$ to get

$$\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_t(x) = t^{-(n+1)/2} \partial_{z_j} \partial_{z_k} \Lambda^{-1} G_1(z).$$

So it is sufficient to show (4.9) for $t = 1$, i.e.

$$(4.12) \quad |\partial_j \partial_k \Lambda^{-1} G_1(z)| \leq C |z'|^{-l} |z_n|^{-m}.$$

In fact, if (4.12) is valid, then we have

$$|\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_t(x)| = t^{-(n+1)/2} |\partial_{z_j} \partial_{z_k} \Lambda^{-1} G_1(z)| \\ \leq C t^{-(n+1)/2} |z'|^{-l} |z_n|^{-m} \\ = C t^{(l+m-n-1)/2} |x'|^{-l} |x_n|^{-m}$$

for any $t > 0$.

Let ψ_1 be a smooth function in $\mathbb{R}^{n-1}$ such that $0 \leq \psi_1 \leq 1$, $\text{supp } \psi \subset \{|z'| \leq 1\}$, and $\psi_1|_{|z'| < 1/2} \equiv 1$. Set $\psi_2 = 1 - \psi_1$. Then

$$(4.13) \quad \partial_j \partial_k \Lambda^{-1} G_1(z) = \frac{C}{(4\pi)^{n/2}} e^{-z_n^2/4} \left\{ \partial_j \partial_k \int_{\mathbb{R}^{n-1}} \frac{\psi_1(z' - y')}{|z' - y'|^{n-2}} e^{-|y'|^2/4} dy' \right. \\ \left. + \partial_j \partial_k \int_{\mathbb{R}^{n-1}} \frac{\psi_2(z' - y')}{|z' - y'|^{n-2}} e^{-|y'|^2/4} dy' \right\} \\ = C e^{-z_n^2/4} \{I_1(z') + I_2(z')\}.$$

We first estimate I_1 . We have

$$(4.14) \quad \begin{aligned} I_1(z') &= \partial_j \partial_k \int_{|y'| \leq 1} \frac{\psi_1(y')}{|y'|^{n-2}} e^{-|z'-y'|^2/4} dy' \\ &= \int_{|y'| \leq 1} \frac{\psi_1(y')}{|y'|^{n-2}} K_{j,k}(z' - y') dy', \end{aligned}$$

where

$$K_{j,k}(z') = \left(\frac{z_j z_k}{4} - \frac{\delta_{j,k}}{2} \right) e^{-|z'|^2/4}$$

and $\delta_{j,k}$ is Kronecker's delta. since $|z' - y'| \leq |z'| + 1$ and $|z' - y'|^2 \geq |z'|^2/2 - 1$ if $|y'| \leq 1$, we get

$$\begin{aligned} |K_{j,k}(z' - y')| &\leq \left\{ \frac{(|z'| + 1)^2}{4} + \frac{1}{2} \right\} e^{-(|z'|^2 - 2)/8} \\ &= \frac{e^{1/4}}{4} \{ (|z'| + 1)^2 + 2 \} e^{-|z'|^2/8} \\ &\leq C|z'|^{-l}. \end{aligned}$$

Hence we have

$$\begin{aligned} |I_1(z')| &\leq C \int_{|y'| \leq 1} \frac{1}{|y'|^{n-2}} |z'|^{-l} dy' \\ &\leq C|z'|^{-l} \end{aligned}$$

We next estimate I_2 . We have

$$(4.15) \quad \begin{aligned} I_2(z') &= \int_{\mathbb{R}^{n-1}} \frac{(\partial_j \partial_k \psi_2)(z' - y')}{|z' - y'|^{n-2}} e^{-|y'|^2/4} dy' \\ &\quad - (n-2) \left\{ \int_{\mathbb{R}^{n-1}} (\partial_j \psi_2)(z' - y') \frac{z_k - y_k}{|z' - y'|^n} e^{-|y'|^2/4} dy' \right. \\ &\quad \left. + \int_{\mathbb{R}^{n-1}} (\partial_k \psi_2)(z' - y') \frac{z_j - y_j}{|z' - y'|^n} e^{-|y'|^2/4} dy' \right\} \\ &\quad + \int_{\mathbb{R}^{n-1}} \psi_2(z' - y') L_{j,k}(z' - y') e^{-|y'|^2/4} dy' \\ &= J_1(z') - (n-1)J_2(z') + J_3(z'), \end{aligned}$$

where

$$L_{j,k}(z') = (n-2) \left\{ n \frac{x_j x_k}{|z'|^{n+2}} - \frac{\delta_{j,k}}{|z'|^n} \right\}.$$

Since the support of the derivatives of ψ_2 are included in $1/2 \leq |z| \leq 1$, we can estimate J_1 and J_2 as in the case of I_1 :

(4.16)

$$\begin{aligned} |J_1(z')| &= \left| \int_{1/2 \leq |y'| \leq 1} \frac{(\partial_j \partial_k \psi_2)(y')}{|y'|^{n-2}} e^{-|z'-y'|^2/4} dy' \right| \\ &\leq \|\nabla^2 \psi_2\|_{L^\infty} \int_{1/2 \leq |y'| \leq 1} \frac{1}{|y'|^{n-2}} e^{-(|z'|^2-2)/8} dy' \\ &\leq C|z'|^{-l}, \end{aligned}$$

(4.17)

$$\begin{aligned} |J_2(z')| &\leq \|\nabla \psi_2\|_{L^\infty} \int_{1/2 \leq |y'| \leq 1} \frac{1}{|y'|^{n-1}} e^{-(|z'|^2-2)/8} dy' \\ &\leq C|z'|^{-l}. \end{aligned}$$

To estimate the term J_3 , we use the inequality $|z'|^l \leq C_l(|z'-y'|^l + |y'|^l)$. Since $|L_{j,k}(z')| \leq \frac{C}{|z'-y'|^{n+1}}$, we get

(4.18)

$$\begin{aligned} |J_3(z')| &\leq C|z'|^{-l} \int_{|z'-y'| \geq 1/2} \left(\frac{|z'-y'|^l}{|z'-y'|^n} + \frac{|y'|^l}{|z'-y'|^n} \right) e^{-|y'|^2/4t} dy' \\ &\leq C|z'|^{-l} \int_{|z'-y'| \geq 1/2} (2^{l-n} + 2^n |y'|^l) e^{-|y'|^2/4} dy' \\ &= C|z'|^{-l}. \end{aligned}$$

Combining (4.16)-(4.18) gives $|I_2(z')| \leq C|z'|^{-l}$ and

$$\begin{aligned} (4.19) \quad |\partial_j \partial_k \Lambda^{-1} G_1(z)| &\leq C e^{-x_n^2/4} |z'|^{-l} \\ &\leq C_{l,m} |z'|^{-l} |z_n|^{-m}. \end{aligned}$$

This proves (4.12) for $n \geq 3$.

The estimate (4.10) for $n \geq 3$ is easily obtained by applying (4.9), since

$$\Lambda = (-\Delta')\Lambda^{-1} = -(\partial_1^2 + \cdots \partial_{n-1}^2)\Lambda^{-1}.$$

Finally, we show (4.10) for $n = 2$. Note that Λ is equal to $|\partial_1| = \partial_1 S_1$. So we have

$$\begin{aligned} (4.20) \quad \Lambda G_t(x) &= \partial_1 S_1 G_t(x) \\ &= \partial_1 \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y_1| > \epsilon} \frac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1. \end{aligned}$$

(See Torchinsky [13], p.266.) Integrating by parts, we get

$$\begin{aligned} \int_{|y_1| > \epsilon} \frac{1}{y_1} G_t(x_1 - y_1, x_2) dy_1 &= \left[\log |y_1| G_t(x_1 - y_1, x_2) \right]_{\epsilon}^{\infty} \\ &\quad + \left[\log |y_1| G_t(x_1 - y_1, x_2) \right]_{-\infty}^{-\epsilon} \\ &\quad - \int_{|y_1| > \epsilon} \log |y_1| \partial_{y_1} G_t(x_1 - y_1, x_2) dy_1 \\ &= \log \epsilon (G_t(x_1 + \epsilon, x_2) - G_t(x_1 - \epsilon, x_2)) \\ &\quad + \int_{|y_1| > \epsilon} \log |y_1| \frac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1. \end{aligned}$$

Letting $\epsilon \downarrow 0$, we get

$$(4.21) \quad \Lambda G_t(x) = \frac{1}{\pi} \partial_1 \int_{-\infty}^{\infty} \log |y_1| \frac{x_1 - y_1}{2t} G_t(x_1 - y_1, x_2) dy_1.$$

Set $x = t^{1/2}z$ and $y = t^{1/2}w$. Then we have

$$\begin{aligned} (\Lambda G_t)(x) &= \frac{1}{\pi} t^{-1/2} \partial_{z_1} \int_{-\infty}^{\infty} (\log |w_1| + \log t^{1/2}) \\ &\quad \frac{z_1 - w_1}{2t^{1/2}} t^{-1} G_1(z_1 - w_1, w_2) t^{1/2} dw_1 \\ &= t^{-3/2} (\Lambda G_1)(z). \end{aligned}$$

So it is sufficient to show (4.10) for $t = 1$. Letting $t = 1$ in (4.21), we have

(4.22)

$$\begin{aligned}\Lambda G_1(z) &= \frac{1}{\pi} \frac{1}{4\pi} e^{-z_2^2/4} \partial_1 \left\{ \int_{|y_1| < 1} \log |y_1| \frac{z_1 - y_1}{2} e^{-(z_1 - y_1)^2/4} dy_1 \right. \\ &\quad \left. + \int_{|y_1| > 1} \log |y_1| \frac{z_1 - y_1}{2} e^{-(z_1 - y_1)^2/4} dy_1 \right\} \\ &= \frac{1}{4\pi^2} e^{-z_2^2/4} (I_1(z_1) + I_2(z_1)).\end{aligned}$$

We first estimate I_1 . We have

$$I_1(z_1) = \int_{-1}^1 \log |y_1| \frac{1}{2} \left(1 - \frac{|z_1 - y_1|^2}{2} \right) e^{-(z_1 - y_1)^2/4} dy_1.$$

In the same way as in the case $n \geq 3$, we obtain

(4.23)

$$\begin{aligned}|I_1(z_1)| &\leq \frac{1}{2} \int_{-1}^1 |\log |y_1|| \left(1 + \frac{(|z_1| + 1)^2}{4} \right) e^{-\frac{|z_1|^2}{8} + \frac{1}{4}} dy_1 \\ &\leq C(1 + |z_1|^2) e^{-|z_1|^2/8}.\end{aligned}$$

To estimate I_2 , we integrate it by parts and get

$$\begin{aligned}I_2(z_1) &= \partial_1 \left\{ \left[\log |y_1| e^{-(z_1 - y_1)^2/4} \right]_1^{+\infty} \right. \\ &\quad \left. + \left[\log |y_1| e^{-(z_1 - y_1)^2/4} \right]_{-\infty}^{-1} \right. \\ &\quad \left. - \int_{|y_1| > 1} \frac{1}{y_1} e^{-(z_1 - y_1)^2/4} dy_1 \right\} \\ &= \int_{|y_1| > 1} \frac{1}{y_1} \frac{z_1 - y_1}{2} e^{-(z_1 - y_1)^2/4} dy_1 \\ &= e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|y_1| > 1} \frac{1}{y_1^2} e^{-(z_1 - y_1)^2/4} dy_1.\end{aligned}$$

We set $w_1 = z_1 - y_1$ and obtain

$$I_2(z_1) = e^{-(z_1+1)^2/4} - e^{-(z_1-1)^2/4} + \int_{|z_1 - w_1| > 1} \frac{1}{(z_1 - w_1)^2} e^{-w_1^2/4} dw_1.$$

Using $|z_1|^l \leq C(|z_1 - w_1|^l + |w_1|^l)$, we obtain

(4.24)

$$\begin{aligned} |I_2(z_1)| &\leq |e^{-(z_1+1)^2/4}| + |e^{-(z_1-1)^2/4}| \\ &\quad + C \int_{|z_1-w_1|>1} \frac{1}{|z_1|^l} \left(|z_1 - w_1|^{l-2} + \frac{|w_1|^l}{|z_1 - w_1|^2} \right) e^{-|w_1|^2/4} dw_1 \\ &\leq C|z|^{-l} \end{aligned}$$

since $l \leq 2$ and so $|z_1 - w_1|^{l-2} \leq 1$. Combining (4.23) and (4.24) gives (4.10) for $n = 2$. for $n = 2$. $\square$

Now we are ready to show the key lemma for the main theorem.

Lemma 4.2. *Assume that $1 \leq j, k \leq n - 1$. Then*

$$(4.25) \quad \|\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, \cdot)\|_{\mathcal{H}^1} \leq C t^{-1/2}.$$

Proof. Note that $G_s * (\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, y))(y) = \partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_{s+t}(x, y)$. Using Lemma 4.1, we have

(4.26)

$$\begin{aligned} |\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_{s+t}(x, y)| &\leq C(s+t)^{(l+m-n-1)/2} |x' - y'|^{-l} |x_n - y_n|^{-m} \\ &\leq C t^{(l+m-n-1)/2} |x' - y'|^{-l} |x_n - y_n|^{-m}, \end{aligned}$$

where l and m satisfy the assumption in Lemma 4.1. Therefore we obtain

(4.27)

$$\begin{aligned} &\|\partial_{y_h} \partial_{y_i} \Lambda_y^{-1} G_t(x, y)\|_{\mathcal{H}_y^1} \\ &\leq \sum_{k=1}^4 C_{l,m} t^{(l+m-n-1)/2} \int_{\Omega_k} |x' - y'|^{-l} |x_n - y_n|^{-m} dy, \end{aligned}$$

where

$$\begin{aligned} \Omega_1 &= \{|x' - y'| \leq t^{1/2}, |x_n - y_n| \leq t^{1/2}\}, \\ \Omega_2 &= \{|x' - y'| > t^{1/2}, |x_n - y_n| \leq t^{1/2}\}, \\ \Omega_3 &= \{|x' - y'| \leq t^{1/2}, |x_n - y_n| > t^{1/2}\}, \\ \Omega_4 &= \{|x' - y'| > t^{1/2}, |x_n - y_n| > t^{1/2}\}. \end{aligned}$$

We estimate the integrals on the right-hand side of (4.27), taking $l = m = 0$ for $k = 1$, $l = n$, $m = 0$ for $k = 2$, $l = 0$, $m = 2$ for $k = 3$ and $l = n - 1/2$, $m = 3/2$ for $k = 4$, to find that the integrals of (4.27) are all bounded above by a constant multiple of $t^{-1/2}$. This proves (4.25). $\square$

Recalling $\|f\|_{L^1} \leq \|f\|_{\mathcal{H}^1}$ and using Lemma 4.2, we have

(4.28)

$$\begin{aligned} & \|R_j R_k \partial_h \partial_i \Lambda^{-1} E(t) u_0\|_{L^\infty(\mathbb{R}_+^n)} \\ & \leq \|\bar{u}_0\|_{L^\infty(\mathbb{R}^n)} \|R_{x_j} R_{x_k} \partial_{x_h} \partial_{x_i} G_t(x, \cdot)\|_{L^1(\mathbb{R}^n)} \\ & \leq \|u_0\|_{L^\infty(\mathbb{R}_+^n)} \|R_{y_j} R_{y_k} \partial_{y_h} \partial_{y_i} G_t(x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ & \leq \|u_0\|_{L^\infty(\mathbb{R}_+^n)} \|R_j\|_{\mathcal{H}^1} \|R_k\|_{\mathcal{H}^1} \|\partial_{y_h} \partial_{y_i} G_t(x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ & \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}_+^n)}. \end{aligned}$$

Combining the estimate in (3.15) and (4.28), we obtain the desired estimate:

$$\|\nabla u\|_{L^\infty(\mathbb{R}_+^n)} \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}_+^n)}.$$

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