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On the Absence of Eigenvectors of Hamiltonians in a Class of Massless Quantum Field Models Without Infrared Cutoff

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Abstract

A class of models of quantized, *massless* Bose fields, called the generalized spin-boson model (A. Arai and M. Hirokawa, J. Funct. Anal. **151** (1997), 455–503) is considered. Theorems on the absence of ground states and the other eigenvectors of the model *without infrared cutoff* (but with ultraviolet cutoff) are established with conditions in terms of correlation functions for some operators.

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1 Introduction

It is well known that, in *massless* Bose quantum field theories, one encounters the so-called "infrared catastrophe", a situation where the total energy of bosons emitted at low frequency is finite, but the number of such bosons ("soft bosons") blows up (e.g., [31, Chapter 4, §4-1-2], cf. also [16]). Conventionally or heuristically the infrared catastrophe may be interpreted as an indication of absence of ground states (or other eigenvectors) of the model under consideration in the Hilbert space of state vectors where the "bare" boson number operator is defined. An elementary method to avoid the infrared catastrophe is to

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introduce an infrared cutoff for boson momenta so that the bosons with small momenta (frequencies) do not interact or to make bosons massive. In such a case the existence of ground states with finite numbers of bosons is shown for some models (but with ultraviolet cutoff) (e.g., [1, 4, 13, 14, 20, 21, 22, 25, 26, 27, 29, 30, 32, 35, 37, 38, 40]).

Mathematically rigorous analysis on the *infrared problem*, a number of problems related to the infrared catastrophe, was intiated by Fröhlich [20, 21]. He showed that the conventional picture mentioned above is true for dressed one electron states in a model of a scalar electron coupled to a massless scalar quantum field and discussed a C^* algebraic method (a GNS construction) to construct a dressed one electron state without infrared cutoff in a Hilbert space different from the original one where the model with infrared cutoff is defined [20]. For the absence of ground states of other models without infrared cutoff, see, e.g., [4] : the Pauli-Fierz Hamiltonian of nonrelativistic quantum electrodynamics with the dipole approximation and with a fixed electron momentum, [17, 19, 36, 38]: the standard spin-boson model, [11, §7.5]: the *massless* van Hove model.

We remark, however, that, for some classes of models with ultraviolet cutoff, each Hamiltonian may have a ground state even in the case without infrared cutoff, see, e.g., [2, 3, 5, 8, 9, 10, 39]. This suggests that there exists some subtlty on the absence of ground states in the case without infrared cutoff and that there are some classes of massless quantum field models with ultraviolet cutoff which behave differently in the case without infrared cutoff. It is interesting to classify massless quantum field models from this viewpoint. We will discuss this aspect in a separate paper.

In this paper we focus our attention on the absence of ground states or other eigenvectors of massless quantum field models without infrared cutoff (but with ultraviolet cutoff) and investigate *operator theoretical structures* underlying it. We take, as an important class of quantum field models, the generalized spin-boson(GSB) model introduced in [13] and analyze, by operator theoretical methods, *mathematical structures* of absence of ground states and the other eigenvectos of this model. As mentioned in [13], the GSB model is an abstract form of various models of quantum particles (or fields) interacting with a Bose field. We consider the case where the Bose field is massless and the interaction has no infrared cutoff (but with ultraviolet cutoff)(for the precise meaning of this, see the end of Section 2). We establish theorems on the absence of ground states and other eigenvectors of the GSB model with conditions in terms of correlation functions for some operators. We also discuss implications of these general results to concrete models including the standard spin-boson model.

Our method mainly consists of two steps: We first show that the aymptotic annihilation operators of the model exist and that every eigenvector of the Hamiltonian is in the kernels of them (Section 4). This can be done in a standard way (e.g., [28, 32]). Then we utilize these facts to analyze detailed properties of the eigenvectors of the Hamiltonian. In particular, we derive an explicit relation each ground state satisfies (Lemma 5.1). To our best knowledge, the second step of the method is new.

This paper is organized as follows: After briefly reviewing the GSB model in Section 2, we state in Section 3 the main results of this paper. Section 4 is devoted to proof of the existence of the asymptotic annihilation operators and basic properties of them, which constitutes part of the scattering theory for the model. In Section 5 (resp. Section 6) we prove the main theorems on the absence of ground states (resp. other eigenvectors), analyzing properties of ground states (resp. other eigenvectors) by the method mentioned above. In the last section, we discuss concrete models without infrared cutoff in view of the theorems established in the preceding sections.

2 A Brief Review of the GSB Model

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}_{\mathrm{b}}$ the Boson Fock space over $L^2(\mathbf{R}^\nu)$:

$$\mathcal{F}_{\mathrm{b}} := \bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbf{R}^\nu), \tag{2.1}$$

where $\otimes_s^n L^2(\mathbf{R}^\nu)$ denotes the n-fold symmetric tensor product of $L^2(\mathbf{R}^\nu)$, $\nu \geq 1$, with $\otimes_s^0 L^2(\mathbf{R}^\nu) := \mathbf{C}$. The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}.$$

Let $\omega : \mathbf{R}^\nu \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbf{R}^\nu$ with respect to the ν-dimensional Lebesgue measure and $\hat{\omega}$ be the multiplication operator by the function ω, acting in $L^2(\mathbf{R}^\nu)$. We denote by $d\Gamma(\hat{\omega})$ the second quantization of $\hat{\omega}$ [34, §X.7]. Let A be a self-adjoint operator on $\mathcal{H}$ bounded from below and $\mu > 0$ a constant. Then the unperturbed Hamiltonian of the model is defined by

$$H_0 := \frac{\mu}{2} A \otimes I + I \otimes d\Gamma(\hat{\omega}) \tag{2.2}$$

with domain $D(H_0) = D(A \otimes I) \cap D(I \otimes d\Gamma(\hat{\omega}))$, where I denotes identity operator and $D(T)$ the domain of an operator T. The operator H_0 is self-adjoint and bounded from below.

We denote by $a(f)$, $f \in L^2(\mathbf{R}^\nu)$, the smeared annihilation operators on $\mathcal{F}_{\mathrm{b}}$ [$a(f)$ is antilinear in f] [34, §X.7] and set

$$\phi(f) := \frac{1}{\sqrt{2}} \{a(f)^* + a(f)\}.$$

Let $\lambda_j \in L^2(\mathbf{R}^\nu)$, $j = 1, \cdots, J$ with $J \in \mathbf{N}$, B_j a symmetric operator on $\mathcal{H}$, $\alpha \in \mathbf{R} \setminus \{0\}$ a constant and

$$H_I := \alpha \sum_{j=1}^{J} B_j \otimes \phi(\lambda_j). \tag{2.3}$$

Then the total Hamiltonian H of the model is defined by

$$H := H_0 + H_I \tag{2.4}$$

acting in $\mathcal{F}$.

The inner product (resp. norm) of a Hilbert space $\mathcal{K}$ is denoted $(\cdot, \cdot)_{\mathcal{K}}$, complex linear in the second variable (resp. $\|\cdot\|_{\mathcal{K}}$). For each $s \in \mathbf{R}$, we define a Hilbert space

$$\mathcal{M}_s = \{f : \mathbf{R}^\nu \to \mathbf{C}, \text{Borel measurable} | \omega^{s/2} f \in L^2(\mathbf{R}^\nu)\}$$

with inner product $(f, g)_s := (\omega^{s/2} f, \omega^{s/2} g)_{L^2(\mathbf{R}^\nu)}$ and norm

$$\|f\|_s := \|\omega^{s/2} f\|_{L^2(\mathbf{R}^\nu)}, \quad f \in \mathcal{M}_s.$$

For a linear operator T, we denote its spectrum by $\sigma(T)$.

We shall assume the following (A.1) and (A.2):

(A.1) $\lambda_j \in \mathcal{M}_{-1} \cap \mathcal{M}_0, \quad j = 1, \cdots, J.$

(A.2) Let $\widetilde{A} := \mu[A - \inf \sigma(A)]/2$. Then $D(\widetilde{A}^{1/2}) \subset D(B_j)$, $j = 1, \cdots, J$, and there exist constants $a_j \geq 0$, $b_j \geq 0$, $j = 1, \cdots, J$, such that, for all $u \in D(\widetilde{A}^{1/2})$,

$$\|B_j u\|_0 \leq a_j \|\widetilde{A}^{1/2} u\|_0 + b_j \|u\|_0, \quad j = 1, \cdots, J,$$

$$|\alpha| \left(\sum_{j=1}^{J} a_j \|\lambda_j\|_{-1} \right) < 1.$$

Under these assumptions, H is self-adjoint on $D(H) = D(H_0)$ and bounded from below [13, Proposition 1.1].

We set

$$E_0(H) := \inf \sigma(H), \tag{2.5}$$

the ground state energy of H, and

$$\widehat{H} := H - E_0(H) \geq 0.$$

A non-zero vector in $\ker \widehat{H}$ (if it exists) is called a *ground state* of H.

Let

$$m_{\mathrm{b}} := \operatorname{ess.inf}_{k \in \mathbf{R}^\nu} \omega(k) \geq 0. \tag{2.6}$$

We say that the bosons of the quantum field under consideration are *massless* if $m_{\mathrm{b}} = 0$.

We denote by N the number operator on $\mathcal{F}_{\mathrm{b}}$: $N = d\Gamma(I)$ [34, §X.7]. As for the existence of a ground state of H and the spectrum of H, the following theorem has been established.

Theorem 2.1 *Assume (A.1) and (A.2). Suppose that ω is continuous on $\mathbf{R}^\nu$ with*

$$\lim_{|k| \to \infty} \omega(k) = \infty$$

and there exist constants $\gamma > 0$ and $C > 0$ with

$$|\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma}(1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbf{R}^{\nu}.$$

Moreover suppose that each λ_j is continuous on $\mathbf{R}^{\nu}$,

$$\frac{\lambda_j}{\omega} \in L^2(\mathbf{R}^{\nu}), \quad j = 1, \cdots, J, \tag{2.7}$$

and A has compact resolvent. Then:

(i) *There exists a constant $\alpha_0 > 0$ such that, for all $|\alpha| < \alpha_0$, H has a ground state in $D(I \otimes N^{1/2})$.*

(ii) *If $m_{\mathrm{b}} > 0$, then, for all $|\alpha| < \alpha_0$, H has purely discrete spectrum in $[E_0(H), E_0(H) + m_{\mathrm{b}})$ and $\sigma_{\mathrm{ess}}(H) = [E_0(H) + m_{\mathrm{b}}, \infty)$, where $\sigma_{\mathrm{ess}}(H)$ denotes the essential spectrum of H.*

(iii) *If $m_{\mathrm{b}} = 0$, then, for all $|\alpha| < \alpha_0$,*

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [E_0(H), \infty). \tag{2.8}$$

Remark 2.1 Part (i) of this theorem is [13, Theorem 1.3]. Parts (ii) and (iii) are proven in [12]. If $m_{\mathrm{b}} > 0$, then (2.7) is automatically satisfied. Hence (2.7) becomes meaningful only in the case where the bosons are massless.

Remark 2.2 An explicit expression of $E_0(H)$ for the standard spin-boson model (see Section 7.1) is obtained in [24].

In this paper we consider the case where $\lambda_j/\omega \notin L^2(\mathbf{R}^{\nu})$ and $B_j \neq 0$ for some j, but (A.1) is satisfied. In this case we say that the interaction H_I has no infrared cutoff. Note that this case implies $m_{\mathrm{b}} = 0$, namely, we are in the case where the bosons are massless. The aim below is to discuss the absence of ground states and other eigenvectors of H in the case where H_I has no infrared cutoff.

Remark 2.3 Assume (A.1) and (A.2). Consider the case H_I has no infrared cutoff. In this case too, we can show that (2.8) continues to hold. Moreover, it can be proven that the asymptotic behavior of $E_0(H)$ as $|\alpha| \to \infty$ is given by the same formula as in the case where the assumption of Theorem 2.1 holds with some additional conditions, see Eq.(1.42) in [13].

3 Main Results

3.1 Absence of ground states

To formulate additional assumptions, we define, for each $n = 1, \cdots, \nu$, a subset $Y_n \subset \mathbf{R}^\nu$ by

$$Y_n := \{k = (k_1, \cdots, k_\nu) \in \mathbf{R}^\nu |\ k_n = 0\} \tag{3.1}$$

and set

$$Y := \cup_{n=1}^{\nu} Y_n. \tag{3.2}$$

We assume also the following (A.3) and (A.4):

(A.3) For $j = 1, \cdots, J$, $\lambda_j \in C^2(\mathbf{R}^\nu \setminus Y)$.

(A.4) $\omega \in C^3(\mathbf{R}^\nu \setminus Y)$ and $\partial\omega(k)/\partial k_n \neq 0$ on $\mathbf{R}^\nu \setminus Y$, $n = 1, \cdots, \nu$.

Example 3.1 The function $\omega(k) = |k|^p$ on $\mathbf{R}^\nu$ with a constant $p > 0$ satisfies (A.4).

For $\Psi \in D(B_j \otimes I)$, we define

$$m_j(\Psi) := (\Psi, (B_j \otimes I)\Psi)_{\mathcal{F}} \tag{3.3}$$

and, for $\Psi \in \cap_{j=1}^{J} D(B_j \otimes I)$,

$$m(\Psi) := \sum_{j=1}^{J} m_j(\Psi). \tag{3.4}$$

Theorem 3.1 *Assume (A.1)—(A.4). Let $\lambda_j \geq 0$, $j = 1, \cdots, J$, and suppose that, for some ℓ,*

$$\frac{\lambda_\ell}{\omega} \notin L^2(\mathbf{R}^\nu). \tag{3.5}$$

Then:

(i) *H has no ground states Ψ_0 in $D(I \otimes N^{1/2})$ such that $m_\ell(\Psi_0) > 0$ and $m_j(\Psi_0) \geq 0$ for $j = 1, \cdots, J$; $j \neq \ell$.*

(ii) *If each B_j $(j = 1, \cdots, J)$ is nonnegative and B_ℓ is strictly positive, then H has no ground states in $D(I \otimes N^{1/2})$.*

Remark 3.1 As is shown in Lemma 4.1, $D(\widehat{H}^{1/2}) \subset D(B_j \otimes I)$, $j = 1, \cdots, J$, so that, for all $\Psi \in D(\widehat{H}^{1/2})$, $m_j(\Psi)$ and $m(\Psi)$ are defined.

For a constant $K > 0$, we define

$$D_K := \{k \in \mathbf{R}^\nu | \omega(k) \leq K\}. \tag{3.6}$$

Theorem 3.2 *Assume (A.1)—(A.4). Suppose that there exists a function g on D_K with a constant $K > 0$ such that, for $j = 1, \cdots, J$, $\lambda_j = g$ on D_K and*

$$\int_{D_K} \frac{|g(k)|^2}{\omega(k)^2} dk = \infty.$$

Then H has no ground states Ψ_0 in $D(I \otimes N^{1/2})$ such that $m(\Psi_0) \neq 0$.

Remark 3.2 Physically the quantities $m_j(\Psi)$ and $m(\Psi)$ may be regarded as "order parameters" to characterize the absence of ground states or spontaneous symmetry breaking of H (cf. [38]).

We introduce

$$\widehat{B} := \sum_{j=1}^{J} B_j \otimes I \tag{3.7}$$

with

$$D(\widehat{B}) := \cap_{j=1}^{J} D(B_j \otimes I). \tag{3.8}$$

We denote by P_H the spectral measure of H and define

$$P_0 := P_H(\{E_0(H)\}), \tag{3.9}$$

which is the orthogonal projection onto $\ker \widehat{H}$. We define

$$M(\Psi) := \|P_0 \widehat{B} \Psi\|_{\mathcal{F}} \geq 0, \quad \Psi \in D(\widehat{B}). \tag{3.10}$$

This quantity also will play a role of an "order parameter" in our formulation on the absence of ground states of H.

Remark 3.3 (i) If H has no ground states (i.e., $\ker \widehat{H} = \{0\}$), then $M(\Psi) = 0$ for all $\Psi \in D(\widehat{B})$.

(ii) Suppose that H has a unique ground state Ψ_0 with $\|\Psi_0\|_{\mathcal{F}} = 1$. Then,

$$M(\Psi_0) = \left|(\Psi_0, \widehat{B}\Psi_0)_{\mathcal{F}}\right|,$$

the absolute value of the ground state expectation value of $\widehat{B}$.

Theorem 3.3 *Under the same assumption as in Theorem 3.2, H has no ground states Ψ_0 such that $\Psi_0 \in \left[\cap_{j=1}^{J} D\left((I \otimes N^{1/2})(B_j \otimes I)\right)\right] \cup D\left(I \otimes N^{1/2}\right)$ and $M(\Psi_0) > 0$.*

Remark 3.4 Note that $M(\Psi) = 0$ if and only if $\widehat{B}\Psi \in (\ker \widehat{H})^{\perp}$. Hence, under the assumption of Theorem 3.3, possible ground states Ψ_0 of H are such that

$$\Psi_0 \notin \left[\cap_{j=1}^{J} D\left((I \otimes N^{1/2})(B_j \otimes I)\right)\right] \cup D\left(I \otimes N^{1/2}\right)$$

or $\widehat{B}\Psi_0 \in (\ker \widehat{H})^{\perp}$. The former case physically corresponds to the situation where the expectation value of the boson number in the ground state Ψ_0 and in some $(B_j \otimes I)\Psi_0$ are infinite.

Remark 3.5 Physically the condition $M(\Psi) > 0$ or $m(\Psi) \neq 0$ may corresponds to the case where a possible spontaneous symmetry breaking of H is realized (cf. [38]). Hence the above theorems suggest that the ground states describing such cases, if exist, may be "out of" the Hilbert space $\mathcal{F}$. We will discuss this aspect in a future work.

Finally we consider a special case.

We say that a symmetric operator T on a Hilbert space $\mathcal{X}$ *strongly commutes* with a self-adjoint operator S on $\mathcal{X}$ if $e^{itS}T \subset Te^{itS}$ for all $t \in \mathbf{R}$.

Theorem 3.4 *Let the same assumption as in Theorem 3.2 be satisfied. Moreover, suppose that each $B_j \otimes I$ strongly commutes with H. Then H has no ground states Ψ_0 such that $\widehat{B}\Psi_0 \neq 0$.*

Remark 3.6 One of the simplest examples to illustrate Theorem 3.4 is the case where each B_j is a constant multiplication by a constant $\beta_j \in \mathbf{R}$. Then, putting $\lambda := \sum_{j=1}^{J} \beta_j \lambda_j$, we have $H = H_{\rm vH} := H_0 + \alpha I \otimes \phi(\lambda)$. This is a model of van Hove type (e.g., [18, p.17] and references cited there). In this model, one can show that, if $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$ with (A.1), then $H_{\rm vH}$ has no ground states (e.g., [11]). This is just a special case of Theorem 3.4.

3.2 Absence of eigenvectors

For each $j = 1, \cdots, J$, we define an operator

$$T_j := \frac{\mu}{2}[A, B_j] \otimes I + \alpha \sum_{\ell=1}^{J} [B_\ell, B_j] \otimes \phi(\lambda_\ell), \tag{3.11}$$

where $[X, Y] := XY - YX$. Note that, for all $\Psi, \Phi \in D(H) \cap D(T_j)$,

$$(H\Psi, B_j \otimes I\Phi)_{\mathcal{F}} - (B_j \otimes I\Psi, H\Phi) = (\Psi, T_j\Phi)_{\mathcal{F}}. \tag{3.12}$$

The Heisenberg operator of T_j with respect to the Hamiltonian H is

$$T_j(t) := e^{itH} T_j e^{-itH}, \quad t \in \mathbf{R}.$$

Let

$$D_T := \cap_{j=1}^{J} D(T_j), \tag{3.13}$$

$$D_{T,\infty} := \cap_{t\geq 0} \cap_{j=1}^{J} D(T_j(t)). \tag{3.14}$$

We assume the following:

(A.5) The subspace D_T is dense in $\mathcal{F}$.

Under this assumption, one can show that, for all $\Phi \in D_{T,\infty}$ and $\Psi \in \mathcal{F}$, the function: $t \to (T_j(t)\Phi, \Psi)_{\mathcal{F}}$ on $\mathbf{R}$ is Lebesgue measurable.

Definition 3.5 We say that a vector $\Psi \in \mathcal{F}$ is in the set $\mathcal{T}$ if there exists a vector $\Phi \in D_{T,\infty}$ such that

$$\int_0^\infty |(T_j(t)\Phi, \Psi)_{\mathcal{F}}| dt < \infty. \tag{3.15}$$

For such a pair $\langle \Phi, \Psi \rangle$, we define functions $T^{(j)}_{\Phi,\Psi}$ on $\mathbf{R}$, $j = 1, \cdots, J$, by

$$T^{(j)}_{\Phi,\Psi}(E) := \int_0^\infty e^{itE}(T_j(t)\Phi, \Psi)_{\mathcal{F}} dt, \quad E \in \mathbf{R}. \tag{3.16}$$

Remark 3.7 Condition (3.15) is concerned with the decay property of the correlation functions $(T_j(t)\Phi, \Psi)_{\mathcal{F}}$ as $t \to \infty$. For some results on this aspect, see, e.g., [6, 23].

We remark the following.

Proposition 3.6 *Assume (A.1)–(A.5). If $\mathcal{T} = \emptyset$, then H has no eigenvectors in D_T.*

Proof. Suppose that H had an eigenvector Ψ in D_T. Then, it is easy to see that $\Psi \in D_{T,\infty}$ and, by (3.12), $(T_j(t)\Psi, \Psi)_{\mathcal{F}} = 0$ for all $t \in \mathbf{R}$. Hence $\Psi \in \mathcal{T}$, so that $\mathcal{T} \neq \emptyset$ This contradicts the assumption that $\mathcal{T} = \emptyset$. ■

In view of Proposition 3.6, as for the absence of eigenvectors of H in D_T, we need to consider only the case $\mathcal{T} \neq \emptyset$.

Remark 3.8 If each $B_j \otimes I$ commutes with H, then, by (3.12), $T_j = 0$, $j = 1, \cdots, J$. Hence, in this case, $\mathcal{T} = \mathcal{F}$ and, for all $\Psi \in \mathcal{F}$, $\mathcal{T}_\Psi = D_{T,\infty}$.

Definition 3.7 For $\Psi \in \mathcal{T}$, we define $\mathcal{T}_\Psi$ to be the set of all vectors $\Phi \in D_{T,\infty}$ with property (3.15).

For $\Psi \in \mathcal{T}$ and $\Phi \in \mathcal{T}_\Psi$, we define a function $S_{\Phi,\Psi}$ on $\mathbf{R}^\nu$ by

$$S_{\Phi,\Psi}(k) := \sum_{j=1}^J \left\{((B_j \otimes I)\Phi, \Psi)_{\mathcal{F}} - iT^{(j)}_{\Phi,\Psi}(\omega(k))\right\} \frac{\lambda_j(k)}{\omega(k)}, \quad k \in \mathbf{R}^\nu. \tag{3.17}$$

Definition 3.8 (i) We say that a vector $\Psi \in \mathcal{F}$ is in the set $\mathcal{E}_1$ if $\Psi \in \mathcal{T}$ and there exists a vector $\Phi \in \mathcal{T}_\Psi \cap D(I \otimes N^{1/2})$ such that $S_{\Phi,\Psi} \notin L^2(\mathbf{R}^\nu)$.

(ii) We say that a vector $\Psi \in \mathcal{F}$ is in the set $\mathcal{E}_2$ if $\Psi \in \mathcal{T} \cap D(I \otimes N^{1/2})$ and there exists a vector $\Phi \in \mathcal{T}_\Psi$ such that $S_{\Phi,\Psi} \notin L^2(\mathbf{R}^\nu)$.

Theorem 3.9 *Assume (A.1)—(A.5). Then H has no eigenvectors in $\mathcal{E}_1 \cup \mathcal{E}_2$.*

Theorem 3.10 *Assume (A.1)—(A.5). Let $\lambda_j \geq 0$, $j = 1, \cdots, J$, and suppose that, for some ℓ,*

$$\frac{\lambda_\ell}{\omega} \notin L^2(\mathbf{R}^\nu). \tag{3.18}$$

Then:

(i) *H has no eigenvectors Ψ in $D_T \cap D(I \otimes N^{1/2})$ such that $m_\ell(\Psi) > 0$ and $m_j(\Psi) \geq 0$ for $j = 1, \cdots, J$; $j \neq \ell$.*

(ii) *If each B_j $(j = 1, \cdots, J)$ is nonnegative and B_ℓ is strictly positive, then H has no eigenvectors in $D_T \cap D(I \otimes N^{1/2})$.*

Theorem 3.11 *Assume (A.1)—(A.5). Suppose that there exists a function g on D_K with a constant $K > 0$ such that, for $j = 1, \cdots, J$, $\lambda_j = g$ on D_K and*

$$\int_{D_K} \frac{|g(k)|^2}{\omega(k)^2} dk = \infty.$$

Then H has no eigenvectors Ψ in $D_T \cap D(I \otimes N^{1/2})$ such that $m(\Psi) \neq 0$.

For $\Psi \in \mathcal{T}$ and $\Phi \in \mathcal{T}_\Psi$, we define $M^*(\Phi, \Psi)$ by

$$M^*(\Phi, \Psi) := \sum_{j=1}^{J} \left\{ ((B_j \otimes I)\Phi, \Psi)_{\mathcal{F}} - iT^{(j)}_{\Phi,\Psi}(0) \right\} \tag{3.19}$$

We shall see that this quantity serves as an "order parameter" to characterize the absence of eigenvectors of H. Let

$$\begin{aligned} \mathcal{M}_1 &:= \{\Psi \in \mathcal{T} | \text{ there exists a } \Phi \in \mathcal{T}_\Psi \cap D(I \otimes N^{1/2}) & (3.20)\\ &\qquad \text{such that } M^*(\Phi, \Psi) \neq 0\}, & (3.21)\\ \mathcal{M}_2 &:= \{\Psi \in \mathcal{T} \cap D(I \otimes N^{1/2}) | \text{ there exists a } \Phi \in \mathcal{T}_\Psi & (3.22)\\ &\qquad \text{such that } M^*(\Phi, \Psi) \neq 0\}. & (3.23) \end{aligned}$$

Theorem 3.12 *Under the same assumption as in Theorem 3.11, H has no eigenvectors in $\mathcal{M}_1 \cup \mathcal{M}_2$.*

Theorem 3.13 *Let the same assumption as in Theorem 3.11 be satisfied. Moreover, suppose that $D_{T,\infty} \cap D(I \otimes N^{1/2})$ is dense in $\mathcal{F}$ and that each $B_j \otimes I$ strongly commutes with H. Then H has no eigenvectors Ψ such that $\widehat{B}\Psi \neq 0$.*

4 Asymptotic Annihilation Operators

As a preliminary to prove the main theorems stated in Section 3, we show that the strong limits of the operator

$$a_t(f) = e^{itH} I \otimes a\left(e^{-i\omega t} f\right) e^{-itH} \tag{4.1}$$

exist, as $t \to \pm\infty$, on $D(\widehat{H}^{1/2})$ and for all f in a dense subspace of $L^2(\mathbf{R}^\nu)$. The strong limits are called the *asymptotic annihilation operators.*

Lemma 4.1 *Assume (A.1) and (A.2). Then, for all $\Psi \in D(\widehat{H}^{1/2})$ and $t \in \mathbf{R}$, $D(\widehat{H}^{1/2}) \subset D(B_j \otimes I)$ and*

$$\|(B_j \otimes I)\Psi\| \le c_1 \|\widehat{H}^{1/2}\Psi\| + c_2\|\Psi\|, \quad j = 1, \cdots, J, \tag{4.2}$$

where c_1 and c_2 are constants.

Proof. As is shown in [13], H_I is relatively bounded with respect to

$$\widehat{H}_0 := \tilde{A} \otimes I + I \otimes d\Gamma(\omega) \ge 0$$

with relative bound smaller than one. Hence it follows from [34, Theorem X.18(a)] that H_I is relatively form-bounded with respect to $\widehat{H}_0$ with relative bound smaller than one. This implies that

$$\|\widehat{H}_0^{1/2}\Psi\| \le d_1\|\widehat{H}^{1/2}\Psi\| + d_2\|\Psi\|, \quad \Psi \in D(\widehat{H}^{1/2}) = D(\widehat{H}_0^{1/2}), \tag{4.3}$$

where d_1 and d_2 are constants. Assumption (A.2) implies that $B_j \otimes I$ is relatively form-bounded with respect to $\widehat{H}_0$ with relative bound a_j. Thus (4.2) follows. ∎

Lemma 4.2 *Assume (A.1) and (A.2). Then, for all $\Psi \in D(\widehat{H}^{1/2})$, $f \in \mathcal{M}_{-1} \cap \mathcal{M}_0$ and $t \in \mathbf{R}$,*

$$a_t(f)\Psi = I \otimes a(f)\Psi - \sum_{j=1}^{J} i\frac{\alpha}{\sqrt{2}} \int_0^t (f, e^{is\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} e^{isH}(B_j \otimes I)e^{-isH}\Psi ds. \tag{4.4}$$

Proof. Similar to the proof of [32, Theorem 4.2]. ∎

Lemma 4.3 *Assume (A.1)—(A.4). Then, for all $\Psi \in D(\widehat{H}^{1/2})$ and $f \in C_0^\infty(\mathbf{R}^\nu \setminus Y)$, the strong limits*

$$a_\pm(f)\Psi := \text{s-}\lim_{t\to\pm\infty} a_t(f)\Psi \tag{4.5}$$

exist and

$$a_\pm(f)\Psi = I \otimes a(f)\Psi - \sum_{j=1}^{J} i\frac{\alpha}{\sqrt{2}} \int_0^{\pm\infty} (f, e^{it\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} e^{itH}(B_j \otimes I)e^{-itH}\Psi dt, \tag{4.6}$$

where the integral is taken in the sense of strong topology of $\mathcal{F}$.

Proof. Let $\Psi \in D(\widehat{H}^{1/2})$ and $\partial_n := \partial/\partial k_n$. We have for all $s \in \mathbf{R} \setminus \{0\}$

$$\begin{aligned} e^{is\omega(k)} &= -\frac{1}{s^2}\frac{1}{\partial_n\omega(k)}\partial_n\frac{1}{\partial_m\omega(k)}\partial_m e^{is\omega(k)}, \quad k = (k_1, ..., k_\nu) \in \mathbf{R}^\nu \setminus Y, \\ & n, m = 1, ..., \nu. \end{aligned}$$

Hence, by integration by parts using (A.3) and (A.4), we have

$$(f, e^{is\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} = -\frac{1}{s^2}\int_{\mathbf{R}^\nu} g(k)e^{is\omega(k)}dk,$$

with

$$g(k) := \partial_m\frac{1}{\partial_m\omega(k)}\partial_n\frac{1}{\partial_n\omega(k)}\lambda_j(k)\overline{f(k)},$$

where $\bar{f}$ denotes the complex conjugate of f. Note that g is continuous on $\mathbf{R}^\nu \setminus Y$ and the support of g is included in $\mathbf{R}^\nu \setminus Y$. Hence g is integrable and we obtain

$$\left|(f, e^{is\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)}\right| \leq \frac{\int_{\mathbf{R}^\nu}|g(k)|dk}{s^2} \tag{4.7}$$

Hence, by (4.2), we have

$$\int_0^\infty \left\|\sum_{j=1}^J (f, e^{is\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} e^{isH}(B_j \otimes I)e^{-isH}\Psi\right\|_{\mathcal{F}} ds < \infty.$$

This fact and Lemma 4.2 imply the existence of the strong limits of $a_t(f)\Psi$ as $t \to \pm\infty$ and (4.6). ■

Lemma 4.4 *Let the same assumption as in Lemma 4.3 be satisfied. Let Ψ be an eigenvector of H. Then*

$$a_\pm(f)\Psi = 0 \tag{4.8}$$

for all $f \in C_0^\infty(\mathbf{R}^\nu \setminus Y)$.

Proof. Similar to [28, Theorem 2]. ■

5 Properties of Ground States and Proof of the Main Results (I)

We define

$$Q_0 := I - P_0, \tag{5.1}$$

which is the orthogonal projection onto the closed subspace

$$\mathcal{F}_+ := (\ker \widehat{H})^\perp. \tag{5.2}$$

For each $E \geq 0$, the operator $\widehat{H} + E$ can be regarded as an injective operator from $\mathcal{F}_+$ into itself. Hence we can define an operator $G(E)$ on $\mathcal{F}_+$ by

$$G(E) := \widehat{H}(\widehat{H} + E)^{-1}. \tag{5.3}$$

By using the spectral representation of $\widehat{H}$, we can show that $G(E)$ is densely defined, bounded and nonnegative. Hence $G(E)$ can be uniquely extended to a bounded linear operator on $\mathcal{F}_+$. We denote the extension by the same symbol $G(E)$. It follows

$$0 \leq (\Phi, G(E)\Phi)_{\mathcal{F}_+} \leq \|\Phi\|^2_{\mathcal{F}_+}, \quad \|G(E)\Phi\|_{\mathcal{F}_+} \leq \|\Phi\|_{\mathcal{F}_+}, \quad \Phi \in \mathcal{F}_+. \tag{5.4}$$

Lemma 5.1 *Assume (A.1)–(A.4). Suppose that H has a ground state Ψ_0. Then, for all $f \in C_0^\infty(\mathbf{R}^\nu \setminus Y)$,*

$$(I \otimes a(f))\Psi_0 = -\sum_{j=1}^J \frac{\alpha}{\sqrt{2}} \int_{\mathbf{R}^\nu} \frac{\overline{f(k)}\lambda_j(k)}{\omega(k)} \{(B_j \otimes I)\Psi_0 - G(\omega(k))Q_0(B_j \otimes I)\Psi_0\}\, dk \tag{5.5}$$

Proof. For a Lebesgue integrable function g on $\mathbf{R}^\nu$, we define

$$\langle g \rangle := \int_{L^2(\mathbf{R}^\nu)} g(k)dk. \tag{5.6}$$

By Lemmas 4.4, 4.3 and $e^{-itH}\Psi_0 = e^{-itE_0(H)}\Psi_0$, we have

$$I \otimes a(f)\Psi_0 = \sum_{j=1}^J i\frac{\alpha}{\sqrt{2}} \int_0^\infty (f, e^{it\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} e^{it\widehat{H}}(B_j \otimes I)\Psi_0 dt. \tag{5.7}$$

Let $\Psi \in D(H)$. Then

$$(\Psi, (I \otimes a(f))\Psi_0)_{\mathcal{F}} = \sum_{j=1}^J i\frac{\alpha}{\sqrt{2}} \int_0^\infty (f, e^{it\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} \left(e^{-it\widehat{H}}\Psi, (B_j \otimes I)\Psi_0\right)_{\mathcal{F}} dt.$$

In the same way as in the proof of (4.7), we can show that

$$\left|\left\langle \frac{e^{is\omega}\bar{f}\lambda_j}{\omega} \right\rangle\right| \leq \frac{D}{|s|^2}, \quad s \in \mathbf{R} \setminus \{0\},$$

with a positive constant D, which implies that

$$\lim_{s\to\infty} \left\langle \frac{e^{is\omega}\bar{f}\lambda_j}{\omega} \right\rangle = 0.$$

Hence, by integration by parts, we have

$$\begin{aligned}
(\Psi,(I\otimes a(f))\Psi_0)_{\mathcal{F}} &= \sum_{j=1}^{J}\frac{\alpha}{\sqrt{2}}\int_0^\infty \left\{\frac{d}{dt}\left\langle\frac{e^{it\omega}\bar{f}\lambda_j}{\omega}\right\rangle\right\}\left(e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}dt \\
&= -\sum_{j=1}^{J}\frac{\alpha}{\sqrt{2}}\left\langle\frac{\bar{f}\lambda_j}{\omega}\right\rangle(\Psi,(B_j\otimes I)\Psi_0)_{\mathcal{F}} \\
&\quad -\sum_{j=1}^{J}\frac{\alpha}{\sqrt{2}}\int_0^\infty\left\langle\frac{e^{it\omega}\bar{f}\lambda_j}{\omega}\right\rangle\left\{\frac{d}{dt}\left(e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}\right\}dt \\
&= -\sum_{j=1}^{J}\frac{\alpha}{\sqrt{2}}\left\{\left\langle\frac{\bar{f}\lambda_j}{\omega}\right\rangle(\Psi,(B_j\otimes I)\Psi_0)_{\mathcal{F}}+R_j(\Psi)\right\},
\end{aligned}$$

where

$$R_j(\Psi):=i\int_0^\infty\left\langle\frac{e^{it\omega}\bar{f}\lambda_j}{\omega}\right\rangle\left(\widehat{H}e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}dt.$$

To compute $R_j(\Psi)$, we note that

$$R_j(\Psi)=i\lim_{\varepsilon\downarrow 0}\int_0^\infty dt\,e^{-t\varepsilon}\int_{\mathbf{R}^\nu}dk\frac{e^{it\omega(k)}\overline{f(k)}\lambda_j(k)}{\omega(k)}\left(\widehat{H}e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}. \tag{5.8}$$

We have

$$\begin{aligned}
&\left|e^{-t\varepsilon}\frac{e^{it\omega(k)}\overline{f(k)}\lambda_j(k)}{\omega(k)}\left(\widehat{H}e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}\right| \\
&\le e^{-t\varepsilon}\frac{|f(k)|\,|\lambda_j(k)|}{\omega(k)}\|\widehat{H}\Psi\|_{\mathcal{F}}\|(B_j\otimes I)\Psi_0\|_{\mathcal{F}}.
\end{aligned}$$

The iterated integral $\int dt\int dk$ of the function on the right hand side is finite. Hence we can apply the Fubini theorem to interchange the integrals $\int dt$ and $\int dk$ in (5.8), and obtain

$$R_j(\Psi)=i\lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^\nu}dk\frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}\int_0^\infty dt\,e^{-t\varepsilon}e^{it\omega(k)}\left(\widehat{H}e^{-it\widehat{H}}\Psi,(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}.$$

It is easy to see that

$$\int_0^\infty dt\,e^{-t\varepsilon}e^{-it\omega(k)}e^{-it\widehat{H}}=(-i)(\widehat{H}+\omega(k)-i\varepsilon)^{-1}$$

in the topology of operator norm. Hence,

$$R_j(\Psi)=-\lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^\nu}dk\frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}\left(\Psi,(\widehat{H}+\omega(k)+i\varepsilon)^{-1}\widehat{H}Q_0(B_j\otimes I)\Psi_0\right)_{\mathcal{F}},$$

where we have used the fact that $\widehat{H}Q_0\Phi = \widehat{H}\Phi$ for all $\Phi \in D(\widehat{H})$. We have

$$\begin{aligned}
&\left|\frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}\left(\Psi, (\widehat{H}+\omega(k)+i\varepsilon)^{-1}\widehat{H}Q_0(B_j\otimes I)\Psi_0\right)_{\mathcal{F}}\right| \\
&\le \frac{|f(k)|\,|\lambda_j(k)|}{\omega(k)}\|\Psi\|_{\mathcal{F}}\left\|(\widehat{H}+\omega(k)+i\varepsilon)^{-1}\widehat{H}Q_0(B_j\otimes I)\Psi_0\right\|_{\mathcal{F}} \\
&\le \frac{|f(k)|\,|\lambda_j(k)|}{\omega(k)}\|\Psi\|_{\mathcal{F}}\,\|Q_0(B_j\otimes I)\Psi_0\|_{\mathcal{F}}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}\left(\Psi, (\widehat{H}+\omega(k)+i\varepsilon)^{-1}\widehat{H}Q_0(B_j\otimes I)\Psi_0\right)_{\mathcal{F}} \\
&= \frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}(\Psi, G(\omega(k))Q_0(B_j\otimes I)\Psi_0)_{\mathcal{F}}.
\end{aligned}$$

Hence, by the Lebesgue dominated convergence, we obtain

$$R_j(\Psi) = -\int_{\mathbf{R}^\nu} dk\frac{\overline{f(k)}\lambda_j(k)}{\omega(k)}(\Psi, G(\omega(k))Q_0(B_j\otimes I)\Psi_0)_{\mathcal{F}}.$$

Thus (5.5) follows. ■

For each $\Phi \in \mathcal{F}$ and $\Psi \in D(\widehat{B})$, we define a function $F_{\Phi,\Psi}$ on $\mathbf{R}^\nu$ by

$$F_{\Phi,\Psi}(k) := \sum_{j=1}^{J}\left\{(\Phi,(B_j\otimes I)\Psi)_{\mathcal{F}} - (\Phi, G(\omega(k))Q_0(B_j\otimes I)\Psi)_{\mathcal{F}}\right\}\frac{\lambda_j(k)}{\omega(k)}, \quad k\in\mathbf{R}^\nu. \tag{5.9}$$

Lemma 5.2 *Assume (A.1)—(A.4). Suppose that H has a ground state Ψ_0. Then the following hold.*

(i) *For all $\Psi \in D(I\otimes N^{1/2})$,*

$$F_{\Psi,\Psi_0} \in L^2(\mathbf{R}^\nu). \tag{5.10}$$

(ii) *If $\Psi_0 \in D(I\otimes N^{1/2})$, then, for all $\Psi\in\mathcal{F}$, (5.10) holds.*

Proof. It is well known (e.g., [34, §X.7]) that, for all $f \in L^2(\mathbf{R}^\nu)$,

$$\|a(f)\psi\|_{\mathcal{F}_b} \le \|f\|_{L^2(\mathbf{R}^\nu)}\|\|N^{1/2}\psi\|_{\mathcal{F}_b}, \tag{5.11}$$

$$\|a(f)^*\psi\|_{\mathcal{F}_b} \le \|f\|_{L^2(\mathbf{R}^\nu)}\|\|(N+I)^{1/2}\psi\|_{\mathcal{F}_b}, \quad \psi\in D(N^{1/2}). \tag{5.12}$$

(i) Let $f \in C_0^\infty(\mathbf{R}^\nu\setminus Y)$ and $\Psi\in D(I\otimes N^{1/2})$. Then

$$\begin{aligned}
|(\Psi,(I\otimes a(f))\Psi_0)_{\mathcal{F}}| &= |((I\otimes a(f)^*)\Psi,\Psi_0)_{\mathcal{F}}| \\
&\le \|I\otimes(N+I)^{1/2}\Psi\|_{\mathcal{F}}\|\Psi_0\|_{\mathcal{F}}\|f\|_{L^2(\mathbf{R}^\nu)}.
\end{aligned}$$

Since $C_0^\infty(\mathbf{R}^\nu \setminus Y)$ is dense in $L^2(\mathbf{R}^\nu)$, it follows from the Riesz lemma that there exists a vector $F \in L^2(\mathbf{R}^\nu)$ such that

$$(\Psi, (I \otimes a(g))\Psi_0)_{\mathcal{F}} = (g, F)_{L^2(\mathbf{R}^\nu)}, \quad g \in L^2(\mathbf{R}^\nu).$$

On the other hand, we have by Lemma 5.1

$$(\Psi, (I \otimes a(f))\Psi_0)_{\mathcal{F}} = -\frac{\alpha}{\sqrt{2}} \left\langle \bar{f} F_{\Psi,\Psi_0} \right\rangle. \tag{5.13}$$

Hence it follows that $F = -\alpha F_{\Psi,\Psi_0}/\sqrt{2}$. Thus (5.10) follows.

(ii) In this case we need only to note that, for all $\Psi \in \mathcal{F}$,

$$|(\Psi, (I \otimes a(f))\Psi_0)_{\mathcal{F}}| \le \|I \otimes N^{1/2}\Psi_0\|_{\mathcal{F}} \|\Psi\|_{\mathcal{F}} \|f\|_{L^2(\mathbf{R}^\nu)}.$$

Then the same argument as in part (i) gives (5.10). ∎

Remark 5.1 Assume (A.1)—(A.4) and let Ψ_0 be a ground state of H such that $\Psi_0 \in D(I \otimes N^{1/2})$. Then $(\Psi_0, G(\omega(k))Q_0 B_j \otimes I\Psi_0)_{\mathcal{F}} = 0$, since the range of $G(\omega(k))$ is included in $\mathcal{F}_+$. Hence, by Lemma 5.2(ii), we have

$$\sum_{j=1}^{J} (\Psi_0, (B_j \otimes I)\Psi_0)_{\mathcal{F}} \frac{\lambda_j}{\omega} \in L^2(\mathbf{R}^\nu). \tag{5.14}$$

On the other hand, this condition can be directly derived. Indeed, we have

$$(H\Psi_0, (I \otimes a(f/\omega))\Psi_0)_{\mathcal{F}} - ((I \otimes a(f/\omega)^*)\Psi_0, H\Psi_0)_{\mathcal{F}} = 0, \quad f \in \mathcal{M}_{-2}. \tag{5.15}$$

Computing the left hand side directly, we obtain

$$(\Psi_0, (I \otimes a(f))\Psi_0)_{\mathcal{F}} = -\sum_{j=1}^{J} \frac{\alpha}{\sqrt{2}} (\Psi_0, (B_j \otimes I)\Psi_0) \left\langle \frac{\bar{f}\lambda_j}{\omega} \right\rangle, \quad f \in \mathcal{M}_{-2}.$$

From this equation and the Riesz lemma, we obtain (5.14). Thus condition (5.10) may be regarded as a generalization of condition (5.14).

Proof of Theorem 3.1

(i) Suppose that H had a ground state Ψ_0 in $D(I \otimes N^{1/2})$ such that $m_\ell(\Psi_0) > 0$ with $m_j(\Psi_0) \ge 0$ for $j = 1, \cdots, J; j \ne \ell$. Then we have $(\Psi_0, G(\omega(\cdot))Q_0(B_j \otimes I)\Psi_0) = 0$ (Remark 5.1). Hence

$$F_{\Psi_0,\Psi_0} = \sum_{j=1}^{J} m_j(\Psi_0) \frac{\lambda_j}{\omega} \ge m_\ell(\Psi_0) \frac{\lambda_\ell}{\omega}.$$

By the assumption, $m_\ell(\Psi_0)\lambda_\ell/\omega \notin L^2(\mathbf{R}^\nu)$. Hence $F_{\Psi_0,\Psi_0} \notin L^2(\mathbf{R}^\nu)$. But this contradicts Lemma 5.2(i).

(ii) Suppose that H had a ground state Ψ_0 in $D(I \otimes N^{1/2})$. By the assumption on B_j, $m_j(\Psi_0) \ge 0$, $j = 1, \cdots, J$ and there exists a constant $\delta > 0$ such that $m_\ell(\Psi_0) \ge \delta \|\Psi_0\|^2 > 0$. But this contradicts part (i). ∎

Proof of Theorem 3.2

Suppose that H had a ground state Ψ_0 in $D(I \otimes N^{1/2})$ such that $m(\Psi_0) \neq 0$. For all $k \in D_K$, we have $F_{\Psi_0,\Psi_0} = m(\Psi_0)g/\omega$. Hence $F_{\Psi_0,\Psi_0} \notin L^2(\mathbf{R}^\nu)$, which contradicts Lemma 5.2(i). ■

Remark 5.2 As is seen, Theorems 3.1 and 3.2 can be derived directly from the fact stated in Remark 5.1.

Proof of Theorem 3.3

Suppose that H had a ground state Ψ_0 in $\left[\cap_{j=1}^J D((I \otimes N^{1/2})(I \otimes B_j))\right] \cup D(I \otimes N^{1/2})$ with $M(\Psi_0) > 0$. Put

$$\Phi_B := \sum_{j=1}^J (B_j \otimes I)\Psi_0.$$

Then, applying Lemma 5.2 with $\Psi = \Phi_B$, we have $F_{\Phi_B,\Psi_0} \in L^2(\mathbf{R}^\nu)$. For all $k \in D_K$, we have

$$F_{\Phi_B,\Psi_0}(k) = [(\Phi_B, \Phi_B)_{\mathcal{F}} - (Q_0\Phi_B, G(\omega(k))Q_0\Phi_B)_{\mathcal{F}}]\frac{g}{\omega}$$

We note that

$$\begin{aligned}(\Phi_B, \Phi_B)_{\mathcal{F}} - (Q_0\Phi_B, G(\omega(k))Q_0\Phi_B)_{\mathcal{F}} &\geq (\Phi_B, \Phi_B)_{\mathcal{F}} - (\Phi_B, Q_0\Phi_B)_{\mathcal{F}} \\ &= (\Phi_B, P_0\Phi_B)_{\mathcal{F}} = M(\Psi_0)^2.\end{aligned}$$

Hence

$$|F_{\Phi_B,\Psi_0}(k)| \geq M(\Psi_0)^2 \frac{|g(k)|}{\omega(k)}, \quad k \in D_K,$$

which implies that $F_{\Phi_B,\Psi_0} \notin L^2(\mathbf{R}^\nu)$. This is a contradiction. ■

Proof of Theorem 3.4

Suppose that H had a ground state Ψ_0 such that $\Phi_0 := \widehat{B}\Psi_0 \neq 0$. The strong commutativity of B_j with H implies that $(B_j \otimes I)\Psi_0 \in \ker \widehat{H}$. Hence $Q_0(B_j \otimes I)\Psi_0 = 0$. Hence, for all $\Psi \in \mathcal{F}$, $F_{\Psi,\Psi_0}(k) = (\Psi, \Phi_0)_{\mathcal{F}} g(k)/\omega(k)$, $k \in D_K$. Since $\Phi_0 \neq 0$ and $D(I \otimes N^{1/2})$ is dense in $\mathcal{F}$, there exists a vector $\Xi \in D(I \otimes N^{1/2})$ such that $(\Xi, \Phi_0)_{\mathcal{F}} \neq 0$. Then we have $F_{\Xi,\Psi_0} \notin L^2(\mathbf{R}^\nu)$. But this contradicts Lemma 5.2(i). ■

6 Proof of the Main Results (II)

In this section we prove Theorems 3.9–3.13.

Lemma 6.1 *Assume (A.1)-(A.5). Suppose that H has an eigenvector Ψ in $\mathcal{T}$. Then, for all $\Phi \in \mathcal{T}_\Psi$ and $f \in C_0^\infty(\mathbf{R}^\nu \setminus Y)$,*

$$(\Phi, (I \otimes a(f))\Psi)_{\mathcal{F}} = -\frac{\alpha}{\sqrt{2}} \int_{\mathbf{R}^\nu} \overline{f(k)} S_{\Phi,\Psi}(k) dk. \tag{6.1}$$

Proof. Let $\Phi \in \mathcal{T}_\Psi$ and $f \in C_0^\infty(\mathbf{R}^\nu \setminus Y)$. Let $H\Psi = E\Psi$ with $E \geq E_0(H)$ and $H_E := H - E$. In the same way as in the proof of Lemma 5.1, we have

$$I \otimes a(f)\Psi = \sum_{j=1}^J i \frac{\alpha}{\sqrt{2}} \int_0^\infty (f, e^{it\omega}\lambda_j)_{L^2(\mathbf{R}^\nu)} e^{itH_E}(B_j \otimes I)\Psi dt \tag{6.2}$$

and we can show that

$$(\Phi, (I \otimes a(f))\Psi)_{\mathcal{F}} = -\sum_{j=1}^J \frac{\alpha}{\sqrt{2}} \left\{ \left\langle \frac{\bar{f}\lambda_j}{\omega} \right\rangle (\Phi, (B_j \otimes I)\Psi)_{\mathcal{F}} + L_j \right\},$$

where

$$L_j := i \int_0^\infty \left\langle \frac{e^{it\omega}\bar{f}\lambda_j}{\omega} \right\rangle \left(H_E e^{-itH_E}\Phi, (B_j \otimes I)\Psi \right)_{\mathcal{F}} dt.$$

Using that $H_E\Psi = 0$, we have

$$\left(H_E e^{-itH_E}\Phi, (B_j \otimes I)\Psi \right)_{\mathcal{F}} = -\left(T_j(t)\Phi, \Psi \right)_{\mathcal{F}}.$$

Hence

$$L_j = -i \int_0^\infty \left\langle \frac{e^{it\omega}\bar{f}\lambda_j}{\omega} \right\rangle (T_j(t)\Phi, \Psi)_{\mathcal{F}}.$$

Since

$$\left| \frac{e^{it\omega(k)}\overline{f(k)}\lambda_j(k)}{\omega(k)} (T_j(t)\Phi, \Psi)_{\mathcal{F}} \right| \leq \frac{|f(k)||\lambda_j(k)|}{\omega(k)} \left| (T_j(t)\Phi, \Psi)_{\mathcal{F}} \right|$$

and the iterated integral of the right hand side with $\int dt \int dk$ is finite, we can apply the Fubini theorem to obtain

$$L_j = -i \int_{\mathbf{R}^\nu} dk \frac{e^{it\omega(k)}\overline{f(k)}\lambda_j(k)}{\omega(k)} T^{(j)}_{\Phi,\Psi}(\omega(k)).$$

Thus (6.1) follows. ∎

Lemma 6.2 *Assume (A.1)—(A.5). Suppose that H has an eigenvector Ψ in $\mathcal{T}$. Then the following hold.*

(i) *For all $\Phi \in \mathcal{T}_\Psi \cap D(I \otimes N^{1/2})$, $S_{\Phi,\Psi} \in L^2(\mathbf{R}^\nu)$.*

(ii) *If $\Psi \in D(I \otimes N^{1/2})$, then, for all $\Phi \in \mathcal{T}_\Psi$, $S_{\Phi,\Psi} \in L^2(\mathbf{R}^\nu)$.*

Proof. Similar to the proof of Lemma 5.2 (use Lemma 6.1). ∎

Proof of Theorem 3.9

Suppose that H had an eigenvector Ψ in $\mathcal{E}_1 \cup \mathcal{E}_2$. Let $\Psi \in \mathcal{E}_1$. Then $\Psi \in \mathcal{T}$ and there exists a vector $\Phi \in \mathcal{T}_\Psi \cap D(I \otimes N^{1/2})$ such that $S_{\Phi,\Psi} \notin L^2(\mathbf{R}^\nu)$. But this contradicts Lemma 6.2(i). Similarly, if $\Psi \in \mathcal{E}_2$, then this leads to a contradiction with Lemma 6.2(ii). Thus the desired assertion follows. ■

Proof of Theorem 3.10

(i) Suppose that H had an eigenvector Ψ in $D_T \cap D(I \otimes N^{1/2})$ such that $m_\ell(\Psi) > 0$ with $m_j(\Psi) \geq 0$ for $j = 1, \cdots, J$; $j \neq \ell$. Then $\Psi \in D_T$ and $(T_j(t)\Psi, \Psi) = 0$. Hence $\Psi \in \mathcal{T}$ and $\Psi \in \mathcal{T}_\Psi$ with

$$S_{\Psi,\Psi}(k) = \sum_{j=1}^{J} m_j(\Psi)\frac{\lambda_j}{\omega} \geq m_\ell(\Psi)\frac{\lambda_\ell}{\omega}.$$

By the assumption, $m_\ell(\Psi)\lambda_\ell/\omega \notin L^2(\mathbf{R}^\nu)$. Hence $S_{\Psi,\Psi} \notin L^2(\mathbf{R}^\nu)$. Hence $\Psi \in \mathcal{E}_1$. This contradicts Theorem 3.9. Thus the desired assertion follows.

(ii) Suppose that H had an eigenvector Ψ in $D_T \cap D(I \otimes N^{1/2})$. By the assumption on B_j, $m_j(\Psi) \geq 0$, $j = 1, \cdots, J$, and there exists a constant $\delta > 0$ such that $m_\ell(\Psi) \geq \delta \|\Psi\|_{\mathcal{F}}^2 > 0$. But this contradicts part (i). ■

Proof of Theorem 3.11

Suppose that H had an eigenvector Ψ in $D_T \cap D(I \otimes N^{1/2})$ with $m(\Psi) \neq 0$. In the same way as in the proof of Theorem 3.10, we have $S_{\Psi,\Psi} = m(\Psi)g/\omega$ on D_K. Hence $S_{\Psi,\Psi} \notin L^2(\mathbf{R}^\nu)$. Hence $\Psi \in \mathcal{E}_1$. But this contradicts Theorem 3.9. ■

Proof of Theorem 3.12

Suppose that H had an eigenvector Ψ in $\mathcal{M}_1 \cup \mathcal{M}_2$. Consider the case $\Psi \in \mathcal{M}_1$ first. Then there exists a vector $\Phi \in \mathcal{T}_\Psi \cap D(I \otimes N^{1/2})$ such that $M^*(\Phi, \Psi) \neq 0$. Let $k \in D_K$. Then

$$|S_{\Phi,\Psi}(k)| = |L(\omega(k))| \left|\frac{g(k)}{\omega(k)}\right|,$$

where

$$L(E) := \sum_{j=1}^{J} \left\{(\Phi, (B_j \otimes I)\Psi)_{\mathcal{F}} - T_{\Phi,\Psi}^{(j)}(E)\right\}, \quad E \in \mathbf{R}.$$

It is easy to see that $T_{\Phi,\Psi}^{(j)}$ is continuous on $\mathbf{R}$. Hence, for every $\varepsilon > 0$, there exists a constant K_ε such that, for all $k \in D_{K_\varepsilon}$,

$$\left|T_{\Phi,\Psi}^{(j)}(\omega(k)) - T_{\Phi,\Psi}^{(j)}(0)\right| < \varepsilon, \; j = 1, \cdots, J.$$

We have $L(0) = M^*(\Phi, \Psi) \neq 0$. Take $\varepsilon > 0$ as $|L(0)|/(M+1) > \varepsilon$. Then, by the triangle inequality, we have for all $k \in D_{K_\varepsilon}$

$$\begin{aligned} |L(\omega(k))| &\geq |L(0)| - |L(\omega(k)) - L(0)| \\ &\geq |L(0)| - M\varepsilon \\ &> \varepsilon. \end{aligned}$$

Hence

$$|S_{\Phi,\Psi}(k)| \geq \varepsilon \left| \frac{g(k)}{\omega(k)} \right|, \quad k \in D_{K_\varepsilon},$$

which implies $S_{\Phi,\Psi} \notin L^2(\mathbf{R}^\nu)$. But this contradicts Lemma 6.2(i). Thus H has no eigenvectors in $\mathcal{M}_1$. Similarly we can show that H has no eigenvectors in $\mathcal{M}_2$. ■

Proof of Theorem 3.13

Suppose that H had an eigenvector Ψ in $\mathcal{T}$ with $H\Psi = E\Psi$ $(E \geq E_0(H))$ such that $\widehat{B}\Psi \neq 0$. By the strong commutativity of B_j with H, we have $(B_j \otimes I)\Psi \in \ker H_E$. It follows that, for all $\Xi \in D_{T,\infty}$, $(T_j(t)\Xi, \Psi)_{\mathcal{F}} = 0$. Hence $S_{\Xi,\Psi} = \sum_{j=1}^J (\Xi, (B_j \otimes I)\Psi)\lambda_j/\omega$. In particular, $S_{\Xi,\Psi} = (\Xi, \widehat{B}\Psi)g/\omega$ on D_K. Since $\widehat{B}\Psi \neq 0$ and $D_{T,\infty} \cap D(I \otimes N^{1/2})$ is dense, there exists a vector Ξ_0 in $D_{T,\infty} \cap D(I \otimes N^{1/2})$ such that $(\Xi_0, \widehat{B}\Psi)_{\mathcal{F}} \neq 0$. Hence $S_{\Xi_0,\Psi} \notin L^2(\mathbf{R}^\nu)$. But this contardicts Lemma 6.2(i). ■

7 Examples

In this section, we discuss what the preceding general results imply about concrete models.

7.1 The spin-boson model

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Pauli matrices. The Hamiltonian H_{SB} of the spin-boson model is defined by putting $\mathcal{H} = \mathbf{C}^2$, $A = \sigma_1$, $J = 1$, $B_1 = \sigma_3$ and $\lambda_1 = \lambda$ in H:

$$\begin{aligned} H_{\mathrm{SB}} &:= \frac{\mu}{2}\sigma_1 \otimes I + I \otimes d\Gamma(\hat{\omega}) + \alpha\sigma_3 \otimes \phi(\lambda), \\ \mathcal{F}_{\mathrm{SB}} &:= \mathbf{C}^2 \otimes \mathcal{F}_{\mathrm{b}}. \end{aligned}$$

In this model Assumption (A.2) is trivially satisfied with $\alpha \in \mathbf{R} \setminus \{0\}$ arbitrary.

In what follows, we simply write as $S \otimes I = S$ (resp. $I \otimes S' = S'$) with an operator S (resp. S') on $\mathbf{C}^2$ (resp. $\mathcal{F}_{\mathrm{b}}$) and $(\,\cdot\,,\,\cdot\,)_{\mathcal{F}_{\mathrm{SB}}} = (\,\cdot\,,\,\cdot\,)$.

We assume the following.

(SB) Assumptions (A.1), (A.3) with $J = 1$ and $\lambda_1 = \lambda$, and (A.4) hold.

Theorem 3.2 immediately gives the following.

Theorem 7.1 *Suppose that $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$. Then $H_{\rm SB}$ has no ground states Ψ_0 in $D(N^{1/2})$ such that $(\Psi_0, \sigma_3\Psi_0) \neq 0$.*

Remark 7.1 This result is already obtained by Spohn by a different method [37, 38] (cf. also [40]).

We denote by $P_0^{(\rm SB)}$ the orthogonal projection onto $\ker \widehat{H}_{\rm SB}$. Theorem 3.3 yields the following.

Theorem 7.2 *Suppose that $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$. $H_{\rm SB}$ has no ground states Ψ_0 in $D(N^{1/2})$ such that $(\Psi_0, \sigma_3 P_0^{(\rm SB)}\sigma_3\Psi_0) \neq 0$.*

We next consider implications of Theorems in Subsection 3.2. Using the relations

$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2,$$

we have

$$T_1 = -i\mu\sigma_2$$

with $D(T_1) = D(\phi(\lambda))$. Hence (A.5) is satisfied. Theorem 3.11 implies the following.

Theorem 7.3 *Suppose that $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$. $H_{\rm SB}$ has no eigenvectors Ψ in $D(N^{1/2}) \cap D(\phi(\lambda))$ such that $(\Psi, \sigma_3\Psi) \neq 0$.*

Let

$$\sigma_j(t) := e^{itH_{\rm SB}}\sigma_j e^{-itH_{\rm SB}}, \quad j = 1, 2, 3,$$

and

$$R_j(t;\Psi) := (\Psi, \sigma_j\sigma_2(t)\Psi). \tag{7.1}$$

We introduce subsets of $\mathcal{F}_{\rm SB}$: We say that $\Psi \in \mathcal{F}_{\rm SB}$ is in the set $\mathcal{S}_{\rm SB}^{(j)}$ if

$$\int_0^\infty |R_j(t;\Psi)|\, dt < \infty.$$

For $\Psi \in \mathcal{S}_{\rm SB}^{(j)}$, we define

$$\chi_j(\Psi) := (\Psi, \sigma_j\sigma_3\Psi) + \mu\int_0^\infty R_j(t;\Psi)dt. \tag{7.2}$$

Explicitly we have

$$\chi_1(\Psi) := -i(\Psi, \sigma_2\Psi) + \mu\int_0^\infty R_1(t;\Psi)dt, \tag{7.3}$$

$$\chi_2(\Psi) := i(\Psi, \sigma_1\Psi) + \mu\int_0^\infty R_2(t;\Psi)dt, \tag{7.4}$$

$$\chi_3(\Psi) := \|\Psi\|^2 + \mu\int_0^\infty R_3(t;\Psi)dt. \tag{7.5}$$

Theorem 7.4 *Suppose that $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$. Then, for each $j = 1,2,3$, H_{SB} has no eigenvectors Ψ in $\mathcal{S}_{\mathrm{SB}}^{(j)} \cap D(N^{1/2})$ such that $\chi_j(\Psi) \neq 0$.*

Proof. Suppose that H_{SB} had an eigenvector Ψ in $\mathcal{S}_{\mathrm{SB}}^{(j)} \cap D(N^{1/2})$ such that $\chi_j(\Psi) \neq 0$. Then $\Psi \in \mathcal{T} \cap D(N^{1/2})$ and $\sigma_j \Psi \in \mathcal{T}_\Psi$. Note that, in the present case, $M^*(\sigma_j \Phi, \Psi) = \chi_j(\Psi)$. Hence $\Psi \in \mathcal{M}_2$. This contradicts Theorem 3.12. ■

Corollary 7.5 *Suppose that $\omega(k) \sim c|k|^p$ as $|k| \to 0$ with some constants $c, p > 0$. Let λ be of the form*

$$\lambda = \frac{\rho}{\omega^r} \in \mathcal{M}_{-1} \cap \mathcal{M}_0$$

with a constant $r > 0$ such that $2p(r+1) > \nu - 1$, where ρ is a continuous function on $\mathbf{R}^\nu$ such that $\rho \neq 0$ on D_K for some $K > 0$. Then, for each $j = 1,2,3$, H has no eigenvectors Ψ in $\mathcal{S}_{\mathrm{SB}}^{(j)} \cap D(I \otimes N^{1/2})$ such that $\chi_j(\Psi) \neq 0$.

Proof. It is easy to see that, under the assumption for λ, $\lambda/\omega \notin L^2(\mathbf{R}^\nu)$. Hence Theorem 7.4 gives the desired assertion. ■

7.2 A model of quantum particles interacting with a massless quantum scalar field

Consider the case where $\mathcal{H} = L^2(\mathbf{R}^{\nu N})$ $(N \geq 1)$. We denote a point x in $\mathbf{R}^{\nu N}$ as $x = (x_1, \cdots, x_N)$ with $x_n = (x_{n1}, \cdots, x_{n\nu}) \in \mathbf{R}^\nu$, $n = 1, \cdots, N$. Let Δ_{x_n} be the generalized Laplacian on $L^2(\mathbf{R}^{\nu N})$ in the variable x_n and take

$$\frac{\mu}{2} A = H_{\mathrm{P}} := -\sum_{n=1}^{N} \frac{\Delta_{x_n}}{2M_n} + V,$$

where $M_n > 0$ is a constant and V is a real-valued measurable function on $\mathbf{R}^{\nu N}$ such that H_{p} is self-adjoint on $D(H_{\mathrm{p}}) := [\cap_{n=1}^N D(\Delta_{x_n})] \cap D(V)$ and bounded from below. Let $\lambda_{nj} \in \mathcal{M}_0 \cap \mathcal{M}_{-1}$, $j = 1, \cdots, J$, $n = 1, \cdots, N$, and

$$H_{\mathrm{PB}} := H_{\mathrm{P}} \otimes I + I \otimes d\Gamma(\hat{\omega}) + \alpha \sum_{n=1}^{N} \sum_{j=1}^{J} x_{nj} \otimes \phi(\lambda_{nj}).$$

Assume that there exist constants $b \geq 0, c > 0$ such that

$$V(x) + b \geq c|x|^2, \quad x \in \mathbf{R}^{\nu N}.$$

Then it is easy to see that (A.2) is satisfied with B_j and λ_j replaced by x_{nj} and λ_{nj} respectively. Hence, for all $|\alpha| < c_0$ with some constant $c_0 > 0$, H_{PB} is self-adjoint and bounbded from below [13, Proposition 1.1]. Note that, in this model, the operator T_j defined by (3.11) takes the form

$$T_{nj} := -\sum_{n=1}^{N} \frac{1}{2M_n}[\Delta_{x_n}, x_{nj}] = -i\frac{p_{nj}}{M_n},$$

where $p_{nj} := -iD_{x_{nj}}$, the generalized partial differential operator in the variable x_{nj}. Thus, in this model, conditions on the absence of ground states or eigenvectors are given in terms of expectation values of the position operators x_{nj} and the momentum operators $p_{nj}(t) = e^{itH_{\rm PB}}p_{nj}e^{-itH_{\rm PB}}$, $t \in \mathbf{R}$, $j = 1, \cdots, J$, $n = 1, \cdots, N$. As in the spin-boson model, we can write down theorems on the absence of ground states or eigenvectors of $H_{\rm PB}$ by translating the general main theorems stated in Section 3 into the present context. But we omit doing it.

Remark 7.2 As mentioned in Section 1, for some classes of potentials V, $H_{\rm PB}$ without infrared cutoff may have a ground state Ψ_0. For example, models discussed in [2, 8, 10]. In these models, however, $(\Psi_0, x_{nj}\Psi_0) = 0$ and $\Psi_0 \in D(I \otimes N^{1/2})$, which are compatible with the general theorems in Section 3.

7.3 The Pauli-Fierz model with the dipole approximation and without A^2 term

We carry over the notation in the preceding example. We consider a model whose Hamiltonian is given by

$$H_{\rm PF} := H_{\rm P} \otimes I + I \otimes d\Gamma(\hat{\omega}) + \alpha \sum_{n=1}^{N}\sum_{j=1}^{J} p_{nj} \otimes \phi(\lambda_{nj}).$$

This is a simplified version of the Pauli-Fierz model in nonrelativistic quantum electrodynamics [33](cf. also [7, 14, 15]). It is easy to see that (A.2) is satisfied with B_j and λ_j replaced by p_{nj} and λ_{nj} respectively. Hence, for all $|\alpha| < d_0$ with some constant $d_0 > 0$, $H_{\rm PF}$ is self-adjoint and bounbded from below [13, Proposition 1.1]. Assume that V is a distribution such that $V_{nj} := D_{x_{nj}}V \in L^2_{\rm loc}(\mathbf{R}^{\nu N})$. Then the operator T_j defined by (3.11) takes the form

$$T_{nj} := [V, p_{nj}] = iV_{nj}$$

Thus, in this model, conditions on the absence of ground states or eigenvectors are given in terms of expectation values of the momentum operators p_{nj} and the operators $e^{itH_{\rm PF}}V_{nj}e^{-itH_{\rm PF}} = V_{nj}(x(t))$, $t \in \mathbf{R}$, $j = 1, \cdots, J$, $n = 1, \cdots, N$, where $x(t) := (x_1(t), \cdots, x_N(t))$, $x_{nj}(t) := e^{itH_{\rm PF}}x_{nj}e^{-itH_{\rm PF}}$. We omit to write down theorems on the absence of ground states or eigenvectors of $H_{\rm PF}$.

Remark 7.3 A remark similar to Remark 7.2 can be made on the present model too (cf. [10]).

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