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A two dimensional random crystalline algorithm for Gauss curvature flow

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ABSTRACT

In this paper we propose a two dimensional random crystalline algorithm for Gauss curvature flow.

1. Introduction.

Gauss curvature flow is a mathematical model of the wearing process of stones on beaches (see Firey [3] and also Chow [1] and Tso [10]).

Ishii [7] proposed a discrete time approximation scheme for Gauss curvature flow. Let us briefly introduce the idea. Take $h > 0$. Suppose that we are given a convex stone with a smooth surface at time $t = 0$. At time $t = h$, in every direction of outward normals, the stone is cut off the volume (which converges to zero as $h \rightarrow 0$). This process is continued until the extinction time of the stone. As $h \rightarrow 0$, the surface of a stone converges to a Gauss curvature flow in Hausdorff metric.

In this paper we propose a two dimensional random model. Roughly speaking, our model is as follows. Take $n \geq 5$ so that $2\pi n^{-1} < 2^{-1}\pi$. Suppose that we are given a convex polygon with n sides with outward normal $(\cos(2\pi i n^{-1}), \sin(2\pi i n^{-1}))$ ($i = 0, \dots, n-1$) at time $t = 0$. At a random time $\sigma > 0$, on a randomly chosen side, the set surrounded by the polygon is cut off the deterministic area (which converges to zero as $n \rightarrow \infty$). This process is continued until the extinction time. As $n \rightarrow \infty$, the polygon converges to a Gauss curvature flow in Hausdorff metric, in probability (see Corollary 2.2 in section 2). Such a model is more realistic in that the stone can not be cut off the portion in every direction at the same time.

In a two dimensional case, Gauss curvature flow is also a mean curvature flow. Our result can be also considered as a random crystalline approximation of a mean curvature flow. A crystalline approximation of a convex curve is useful in numerical analysis (see Girão [4] and also Girão and Kohn [5] and the references therein). In case an initial curve is not convex, Girão's result was generalized by K. Ishii and M. H. Soner (see Ishii and Soner [8] and the references therein for further information on this problem).

Let us introduce Girão's result in a special case since it plays a crucial role in this paper. Take a smooth simple closed convex curve Γ on $\mathbf{R}^2$ and denote by $\{\Gamma(t)\}_{0 \leq t < T^*}$ on $\mathbf{R}^2$ the mean curvature flow with $\Gamma(0) = \Gamma$. Here T^* denotes the extinction time of $\Gamma(t)$. More precisely speaking, the (inward) normal velocity at a point of $\Gamma(t)$ is equal to the curvature at the point. One can define, as below, polygons with n sides (n -polygon) $\{\Gamma_n(t)\}_{0 \leq t < T_n^*}$ which approximate $\{\Gamma(t)\}_{0 \leq t < T^*}$, where T_n^* denotes the extinction time of $\Gamma_n(t)$. For $n \geq 5$, $\Gamma_n(0)$ is a n -polygon of which the i th side is tangent to Γ at a point with outward normal $(\cos(2\pi i n^{-1}), \sin(2\pi i n^{-1}))$ ($i = 0, \dots, n-1$). For $t \in [0, T_n^*)$, the (inward) normal velocity $V_{n,i}(t)$ of the i th side of $\Gamma_n(t)$ is given by the following which is called a discrete curvature:

$$(1.1). \quad V_{n,i}(t) = 2(\tan(\pi n^{-1}))\ell_{n,i}(t)^{-1},$$

where $\ell_{n,i}(t)$ is a length of the i th side of $\Gamma_n(t)$. From (1.1), it can be shown that $\{\ell_{n,i}(t)\}_{i=0}^{n-1}$ satisfies the follows:

$$(1.2). \quad \begin{aligned} d[\ell_{n,i}(t)]/dt &= (2\ell_{n,i}(t)^{-1}\cos(2\pi n^{-1}) - \ell_{n,i+1}(t)^{-1} - \ell_{n,i-1}(t)^{-1})(\cos(\pi n^{-1}))^{-2}, \\ \ell_{n,n}(t) &= \ell_{n,0}(t). \end{aligned}$$

Let us recall the definition of Hausdorff metric: for compact sets A and $B \in \mathbf{R}^2$,

$$(1.3). \quad d_H(A, B) = \max(\max_{p \in A} \text{dist}(p, B), \max_{q \in B} \text{dist}(q, A)).$$

Denote by $\Omega(t)$ and $\Omega_{\ell,n}(t)$ closed sets surrounded by $\Gamma(t)$ and $\Gamma_n(t)$, respectively. The following was proved in Girão [4].

Theorem 1.1. For $t \in [0, T^*)$, $\liminf_{n \rightarrow \infty} T_n^* > t$ and $\sup_{0 \leq s \leq t} d_H(\Omega_{\ell, n}(s), \Omega(s))$ converges to 0 as $n \rightarrow \infty$.

Remark 1.1. Girão's result has not been generalized to a class of closed convex surfaces in $\mathbf{R}^d$ for $d \geq 3$. In Girão [4], they used a result on convergence of discrete curvatures of $\Gamma_n(t)$ to curvatures of $\Gamma(t)$. But Ishii and Soner [8] did not. This is a nice aspect of the viscosity solution approach.

Let us introduce our random model. Take $T(n)$ such that

$$(1.4). \quad \lim_{n \rightarrow \infty} T(n)n^{-5} = \infty,$$

and put

$$(1.5). \quad \theta_n = 2\pi n^{-1},$$

$$(1.6). \quad h_n(x) = 2^{-1}(\tan \theta_n)\{-x + (x^2 + 4(\cot \theta_n)T(n)^{-1}\theta_n)^{1/2}\} \quad (x \in \mathbf{R}).$$

For $n \geq 5$, we consider a Markov process $\{(X_{n,i}(t), D_{n,i}(t))\}_{i=0}^{n-1}$ on $\mathbf{R}^{2n}$ such that

$$(1.7). \quad \{(X_{n,i}(0), D_{n,i}(0))\}_{i=0}^{n-1} = \{(\ell_{n,i}(0), 0)\}_{i=0}^{n-1}$$

and with the generator defined as follows: for a bounded Borel measurable function f from $\mathbf{R}^{2n}$ to $\mathbf{R}$,

$$(1.8). \quad Lf(x) = T(n)(\sin \theta_n)[\theta_n \cos^2(\theta_n/2)]^{-1} \sum_{i=0}^{n-1} I_{\{\min(x_{2(i-1)}, x_{2(i+1)}) \sin \theta_n > h_n(x_{2i})\}}(x) \\ \times [f(x + 2(\cot \theta_n)h_n(x_{2i})\mathbf{e}_{2i} - (\sin \theta_n)^{-1}h_n(x_{2i})(\mathbf{e}_{2(i-1)} + \mathbf{e}_{2(i+1)})) \\ + h_n(x_{2i})\mathbf{e}_{2i+1}) - f(x)]$$

(see Ethier and Kurtz [2], Chap. 4, section 2). Here we put $x = (x_i)_{i=0}^{2n-1}$, $\mathbf{e}_{2n+k} = \mathbf{e}_k$ and $x_{2n+k} = x_k$ ($k \in \mathbf{N}$), and $\{\mathbf{e}_k\}_{k=0}^{2n-1}$ denotes the standard normal base in $\mathbf{R}^{2n}$. (Notice

that $h_n(x) > 0$ for $x \in \mathbf{R}$.) Denote by $\Omega_{X,n}(t)$ a closed set surrounded by the n -polygon which has a length $X_{n,k}(t)$ on the k th side with outward normal $(\cos(k\theta_n), \sin(k\theta_n))$ ($k = 0, \dots, n-1$). We will show that $\Omega_{X,n}(t)$ converges to $\Omega(t)$ in Hausdorff metric, as $n \rightarrow \infty$, in probability (see Corollary 2.2 in section 2).

Let us discuss a heuristic meaning of our model defined above. If the i th side of $\Omega_{X,n}(0)$ is chosen randomly at a random time $\sigma > 0$, and if

$$(1.9). \quad \min(X_{n,i-1}(0), X_{n,i+1}(0)) \sin \theta_n > h_n(X_{n,i}(0)),$$

then $\Omega_{X,n}(0)(= \Omega_{\ell,n}(0))$ is cut off the isogonal trapezoid with the height $h_n(X_{n,i}(0))$ (and with an area $T(n)^{-1}\theta_n$) on the i th side. Thanks to (1.9), $\partial\Omega_{X,n}(\sigma)$ is again a n -polygon. After this process is repeated finite times $< (\text{the volume of } \Omega_{X,n}(0))/(T(n)^{-1}\theta_n)$, it is finished and $\Omega_{X,n}(t)$ remains to be a n -polygon for $t \geq 0$ since $2\pi n^{-1} < 2^{-1}\pi$. Notice that $h_n(x)$ is a positive solution to

$$(1.10). \quad (x + (\cot \theta_n)h_n(x))h_n(x) = T(n)^{-1}\theta_n.$$

For $x > 0$, the left hand side of (1.10) is an area of an isogonal trapezoid with a height $h_n(x)$ and with upper and lower sides of lengths x and $x + 2(\cot \theta_n)h_n(x)$, respectively.

In the proof of our result, we approximate $\Omega_{X,n}(t)$ by $\Omega_{\ell,n}(t)$ and make use of Theorem 1.1.

In section 2 we state our results which will be proved in section 4. Technical lemmas will be stated and proved in section 3.

2. Main results.

Before we show that a random crystalline model converges, in probability, to Gauss curvature flow, let us give some notation.

Put

$$(2.1). \quad d_{n,i}(t) = \int_0^t 2[\tan(\theta_n/2)]\ell_{n,i}(s)^{-1}ds \quad (i = 0, \dots, n-1)$$

(see (1.2) and (1.5) for notation).

Put the intersection point of the 0th and 1st side of $\Omega_{\ell,n}(0)$ at the origin. Then the coordinate of the intersection point of the i th and the $(i+1)$ th side of $\Omega_{X,n}(t)$ and $\Omega_{\ell,n}(t)$ can be written as follows, respectively:

$$(2.2). \quad \begin{aligned} Y_{n,0}(t) &= (-D_{n,0}(t), D_{n,0}(t) \cot \theta_n - D_{n,1}(t)(\sin \theta_n)^{-1}) \quad (i = 0), \\ Y_{n,i}(t) &= (-\sum_{k=1}^i X_{n,k}(t) \sin(k\theta_n) - D_{n,0}(t), \sum_{k=1}^i X_{n,k}(t) \cos(k\theta_n) \\ &\quad + D_{n,0}(t) \cot \theta_n - D_{n,1}(t)(\sin \theta_n)^{-1}) \quad (1 \leq i \leq n-1), \end{aligned}$$

$$(2.3). \quad \begin{aligned} y_{n,0}(t) &= (-d_{n,0}(t), d_{n,0}(t) \cot \theta_n - d_{n,1}(t)(\sin \theta_n)^{-1}) \quad (i = 0), \\ y_{n,i}(t) &= (-\sum_{k=1}^i \ell_{n,k}(t) \sin(k\theta_n) - d_{n,0}(t), \sum_{k=1}^i \ell_{n,k}(t) \cos(k\theta_n) \\ &\quad + d_{n,0}(t) \cot \theta_n - d_{n,1}(t)(\sin \theta_n)^{-1}) \quad (1 \leq i \leq n-1). \end{aligned}$$

Put for $\delta \in (0, T_n^*)$,

$$(2.4). \quad \begin{aligned} C_n(\delta) &= n \min\{\ell_{n,k}(s); 0 \leq k \leq n-1, 0 \leq s \leq T_n^* - \delta\}, \\ \tau_{n,\delta} &= \inf\{t > 0; C_n(\delta)/2 \geq n \min\{X_{n,k}(t); 0 \leq k \leq n-1\}\}. \end{aligned}$$

Then the following holds.

Theorem 2.1. *Suppose that (1.4) holds. Then for $\delta \in (0, T^*)$,*

$$(2.5). \quad \lim_{n \rightarrow \infty} n^2 E\left[\sup_{0 \leq t \leq \min(T_n^* - \delta, \tau_{n,\delta})} \sum_{i=0}^{n-1} |Y_{n,i}(t) - y_{n,i}(t)|^2\right] = 0.$$

As a corollary to Theorem 2.1, we obtain the following.

Corollary 2.2. Suppose that (1.4) holds. Then for any $t \in (0, T^*)$ and $\eta > 0$,

$$(2.6). \quad \lim_{n \rightarrow \infty} P\left(\sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega(s)) < \eta\right) = 1.$$

Let us state the outline of the proof of Theorem 2.1.

(Outline of the proof of Theorem 2.1). Take $n_0 \in \mathbb{N}$ sufficiently large so that $T_n^* > \delta$ for $n \geq n_0$, which is possible from Theorem 1.1.

The following will be proved in section 4: there exists $n_2 \geq n_0$ such that for $n \geq n_2$ and $t \in [0, T_n^* - \delta]$,

$$(2.7). \quad \begin{aligned} & E\left[\sup_{0 \leq s \leq \min(t, \tau_n, \delta)} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2\right] \\ & \leq 2T(n)^{-2} n^7 t |\theta_n ([C_n(\delta) 2^{-1}]^3 \sin \theta_n)^{-1}|^2 \\ & \quad + 8T(n)^{-1} n^3 t \theta_n (C_n(\delta)^2 \sin \theta_n \cos^2(\theta_n/2))^{-1} \\ & \quad + 32T(n)^{-1} n^2 \theta_n (C_n(\delta)^2 \sin \theta_n \cos^2(\theta_n/2))^{-1} \\ & \quad + 6T(n)^{-2} n^3 t |\theta_n (C_n(\delta) \sin \theta_n)^{-1}|^2 \\ & \quad + (4n^2 \sin^2 \theta_n ((\cos \theta_n) C_n(\delta)^2 \cos^2(\theta_n/2))^{-1} + 2 + 3) \\ & \quad \times \int_0^t E\left[\sup_{0 \leq u \leq \min(s, \tau_n, \delta)} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2\right] ds. \end{aligned}$$

By Gronwall's inequality, from (1.4), the proof is over, since $\{C_k(\delta)^{-1}\}_{k=n_3}^\infty$ is bounded (see (14) of Girão [4]).

Q. E. D.

Next we prove Corollary 2.2 from Theorem 2.1.

(Proof of Corollary 2.2). For any $t \in (0, T^*)$, take $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$(2.8). \quad \begin{aligned} & t < T_n^* - (T^* - t)/2, \\ & \sup_{0 \leq s \leq t} d_H(\Omega_{\ell,n}(s), \Omega(s)) < \eta/2, \end{aligned}$$

which is possible from Theorem 1.1. Put $\delta = (T^* - t)/2$. Then for $n \geq n_1$,

$$\begin{aligned}
(2.9). \quad & P\left(\sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega(s)) \geq \eta\right) \\
& \leq P\left(\sup_{0 \leq s \leq t} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2\right) \\
& \leq P\left(\sup_{0 \leq s \leq T_n^* - \delta} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2\right) \\
& \leq P(\tau_{n,\delta} < T_n^* - \delta) + P\left(\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2\right).
\end{aligned}$$

The first probability in (2.9) can be shown to converge to zero as $n \rightarrow \infty$ as follows: by Chebychev's inequality,

$$\begin{aligned}
(2.10). \quad & P(\tau_{n,\delta} < T_n^* - \delta) \\
& \leq P\left(\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |X_{n,k}(s) - \ell_{n,k}(s)| \geq C_n(\delta)(2n)^{-1}\right) \\
& \leq P\left(\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq C_n(\delta)(4n)^{-1}\right) \\
& \leq (4nC_n(\delta)^{-1})^2 E\left[\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2\right] \\
& \rightarrow 0, \text{ as } n \rightarrow \infty \text{ (from Theorem 2.1).}
\end{aligned}$$

Here we used the following:

$$(2.11). \quad |X_{n,k}(s) - \ell_{n,k}(s)| \leq |Y_{n,k}(s) - y_{n,k}(s)| + |Y_{n,k-1}(s) - y_{n,k-1}(s)|.$$

The second probability in (2.9) can be shown to converge to zero as $n \rightarrow \infty$ as follows: by Chebychev's inequality,

$$\begin{aligned}
(2.12). \quad & P\left(\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \geq \eta/2\right) \\
& \leq P\left(\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)| \geq \eta/2\right) \\
& \leq (\eta/2)^{-2} E\left[\sup_{0 \leq s \leq \min(T_n^* - \delta, \tau_{n,\delta})} \sum_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|^2\right] \\
& \rightarrow 0, \text{ as } n \rightarrow \infty \text{ (from Theorem 2.1).}
\end{aligned}$$

Here we used the following:

$$(2.13). \quad d_H(\Omega_{X,n}(s), \Omega_{\ell,n}(s)) \leq \max_{k=0}^{n-1} |Y_{n,k}(s) - y_{n,k}(s)|.$$

Q. E. D.

3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.

Lemma 3.1. For $n \geq 5$, $i = 0, \dots, n-1$, and $s \in (0, T_n^*)$,

$$(3.1). \quad \begin{aligned} dy_{n,i}(s)/ds &= [\ell_{n,i+1}(s) \cos^2(\theta_n/2)]^{-1}(\sin(i\theta_n), -\cos(i\theta_n)) \\ &\quad - [\ell_{n,i}(s) \cos^2(\theta_n/2)]^{-1}(\sin((i+1)\theta_n), -\cos((i+1)\theta_n)). \end{aligned}$$

(proof). When $i = 0$, from (2.1) and (2.3),

$$(3.2). \quad \begin{aligned} dy_{n,0}(s)/ds &= (-2\ell_{n,0}(s)^{-1} \tan(\theta_n/2), 2\ell_{n,0}(s)^{-1} \tan(\theta_n/2) \cot \theta_n \\ &\quad - 2\ell_{n,1}(s)^{-1} \tan(\theta_n/2)(\sin \theta_n)^{-1}) \\ &= [\ell_{n,1}(s) \cos^2(\theta_n/2)]^{-1}(0, -1) - [\ell_{n,0}(s) \cos^2(\theta_n/2)]^{-1}(\sin \theta_n, -\cos \theta_n). \end{aligned}$$

Suppose that (3.1) is true for $i-1 (\geq 0)$. Then from (1.2) and (2.3),

$$(3.3). \quad \begin{aligned} dy_{n,i}(s)/ds &= dy_{n,i-1}(s)/ds + (2\ell_{n,i}(s)^{-1} \cos \theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1}) \\ &\quad \times (\cos^2(\theta_n/2))^{-1}(-\sin(i\theta_n), \cos(i\theta_n)) \\ &= [\cos^2(\theta_n/2)]^{-1}(\ell_{n,i}(s)^{-1} \sin((i-1)\theta_n) - \ell_{n,i-1}(s)^{-1} \sin(i\theta_n) \\ &\quad - (2\ell_{n,i}(s)^{-1} \cos \theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1}) \sin(i\theta_n), \end{aligned}$$

$$\begin{aligned}
& -\ell_{n,i}(s)^{-1} \cos((i-1)\theta_n) + \ell_{n,i-1}(s)^{-1} \cos(i\theta_n) \\
& + (2\ell_{n,i}(s)^{-1} \cos \theta_n - \ell_{n,i+1}(s)^{-1} - \ell_{n,i-1}(s)^{-1}) \cos(i\theta_n) \\
& = [\ell_{n,i+1}(s) \cos^2(\theta_n/2)]^{-1} (\sin(i\theta_n), -\cos(i\theta_n)) \\
& - [\ell_{n,i}(s) \cos^2(\theta_n/2)]^{-1} (\sin((i+1)\theta_n), -\cos((i+1)\theta_n)).
\end{aligned}$$

Here we used the following:

$$\begin{aligned}
(3.4). \quad & \sin((i-1)\theta_n) + \sin((i+1)\theta_n) = 2 \cos \theta_n \sin(i\theta_n), \\
& \cos((i-1)\theta_n) + \cos((i+1)\theta_n) = 2 \cos \theta_n \cos(i\theta_n).
\end{aligned}$$

Q. E. D.

Before we state and prove the following lemma, let us give some notation: for $y = (y_i)_{i=0}^{2n-1} \in \mathbf{R}^{2n}$, put

$$\begin{aligned}
(3.5). \quad \tilde{L}f(y) &= T(n)(\sin \theta_n)[\theta_n \cos^2(\theta_n/2)]^{-1} \\
&\times \sum_{i=0}^{n-1} \{f(y + h_n(\{|y_{2i} - y_{2(i-1)}|^2 + |y_{2i+1} - y_{2i-1}|^2)^{1/2}) \\
&\times (\sin \theta_n)^{-1}([\sin((i-1)\theta_n)]\mathbf{e}_{2(i-1)} - [\cos((i-1)\theta_n)]\mathbf{e}_{2i-1} \\
&\quad - [\sin((i+1)\theta_n)]\mathbf{e}_{2i} + [\cos((i+1)\theta_n)]\mathbf{e}_{2i+1})) - f(y)\}.
\end{aligned}$$

Put also $\mathbf{Y}_n(t) = (Y_{n,k}(t))_{k=0}^{n-1}$. Then the following can be proved by the Ito formula (see Ikeda and Watanabe [6] or Meyer [9]).

Lemma 3.2. *Suppose that (1.4) holds. Then for any $\delta \in (0, T^*)$, there exists $n_2 \in \mathbf{N}$ such that $\delta < T_n^*$ for $n \geq n_2$ and that for $n \geq n_2$ and any $f \in C_o^2(\mathbf{R}^{2n}; \mathbf{R})$*

$$(3.6). \quad f(\mathbf{Y}_n(\min(t, \tau_{n,\delta}))) = f(\mathbf{Y}_n(0)) + \int_0^{\min(t, \tau_{n,\delta})} \tilde{L}f(\mathbf{Y}_n(s)) ds + M(\min(t, \tau_{n,\delta}))$$

for $t \geq 0$, P-a.s., where $M(t)$ is a purely discontinuous martingale part.

(proof). Take $n_0 \in \mathbb{N}$ such that $\delta < T_n^*$ for $n \geq n_0$, which is possible from Theorem 1.1. First we show that there exists $n_2 \geq n_0$ such that for $n \geq n_2$ and $k = 0, \dots, n-1$

$$(3.7). \quad \min(X_{n,k-1}(t), X_{n,k+1}(t)) \sin \theta_n > h_n(X_{n,k}(t)) \quad \text{for } t \in [0, \tau_{n,\delta}) \text{ P-a.s..}$$

By (14) of Girão [4], $\{C_k(\delta)^{-1}\}_{k=n_0}^\infty$ defined in (2.4) is bounded. Therefore there exists $n_2 \geq n_0$ such that for $n \geq n_2$

$$C_n(\delta)(2n)^{-1} \sin \theta_n > (2n)C_n(\delta)^{-1} \theta_n T(n)^{-1}$$

by (1.4). From this, for $0 \leq k, i \leq n-1$ and $0 \leq t < \tau_{n,\delta}$

$$(3.8). \quad \begin{aligned} X_{n,k}(t) \sin \theta_n &> C_n(\delta)(2n)^{-1} \sin \theta_n > (2n)C_n(\delta)^{-1} \theta_n T(n)^{-1} \\ &> X_{n,i}(t)^{-1} \theta_n T(n)^{-1} > h_n(X_{n,i}(t)) \quad (\text{by (1.10)}) \text{ P-a.s.,} \end{aligned}$$

which implies (3.7).

By (2.2), we have the following:

$$(3.9). \quad \begin{aligned} \mathbf{Y}_n(s) = & -x_1 \sum_{k=0}^{n-1} \mathbf{e}_{2k} + (x_1 \cot \theta_n - x_3 (\sin \theta_n)^{-1}) \sum_{k=0}^{n-1} \mathbf{e}_{2k+1} \\ & + \sum_{i=1}^{n-1} x_{2i} [-(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1}], \end{aligned}$$

with $x_{2i} = X_{n,i}(s)$ and $x_{2i+1} = D_{n,i}(s)$ ($i = 0, \dots, n-1$). Therefore, from (1.8) and (3.7), by the Ito formula, the time-derivative of the continuous bounded variation part of $f(\mathbf{Y}_n(\min(t, \tau_{n,\delta})))$ can be written in the following manner:

$$(3.10). \quad T(n)(\sin \theta_n)[\theta_n \cos^2(\theta_n/2)]^{-1} \{(I) + (II) + (III) + (IV)\}.$$

Here we put as follows:

$$\begin{aligned}
(3.11). (I) = & f(\mathbf{Y}_n(s) - h_n(X_{n,0}(s))(\sin \theta_n)^{-1}[-(\sin \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1} \\
& - (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}] \\
& - h_n(X_{n,0}(s)) \sum_{k=0}^{n-1} \mathbf{e}_{2k} + h_n(X_{n,0}(s))(\cot \theta_n) \sum_{k=0}^{n-1} \mathbf{e}_{2k+1}) - f(\mathbf{Y}_n(s)),
\end{aligned}$$

$$\begin{aligned}
(3.12). (II) = & f(\mathbf{Y}_n(s) + 2(\cot \theta_n)h_n(X_{n,1}(s))[-(\sin \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,1}(s))(\sin \theta_n)^{-1}[-(\sin(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k} + (\cos(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,1}(s))(\sin \theta_n)^{-1} \sum_{k=0}^{n-1} \mathbf{e}_{2k+1}) - f(\mathbf{Y}_n(s)),
\end{aligned}$$

$$\begin{aligned}
(3.13). \quad (III) = & \sum_{i=2}^{n-2} \{f(\mathbf{Y}_n(s) + 2(\cot \theta_n)h_n(X_{n,i}(s)) \\
& \times [-(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,i}(s))(\sin \theta_n)^{-1}[-(\sin((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k+1} - (\sin((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k+1}]) - f(\mathbf{Y}_n(s))\},
\end{aligned}$$

$$\begin{aligned}
(3.14). \quad (IV) = & f(\mathbf{Y}_n(s) + 2(\cot \theta_n)h_n(X_{n,n-1}(s)) \\
& \times [-(\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}] \\
& - h_n(X_{n,n-1}(s))(\sin \theta_n)^{-1}[-(\sin((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k+1}]) - f(\mathbf{Y}_n(s)).
\end{aligned}$$

Noticing that $X_{n,k}(t) = |Y_{n,k}(t) - Y_{n,k-1}(t)|$, the proof is over by the following (3.15)-(3.18).

On (I), we have the following:

$$\begin{aligned}
(3.15). \quad & -h_n(X_{n,0}(s))(\sin \theta_n)^{-1}[-(\sin \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1} \\
& - (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} + (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1}] \\
& - h_n(X_{n,0}(s)) \sum_{k=0}^{n-1} \mathbf{e}_{2k} + h_n(X_{n,0}(s))(\cot \theta_n) \sum_{k=0}^{n-1} \mathbf{e}_{2k+1} \\
& = h_n(X_{n,0}(s))(\sin \theta_n)^{-1}[(\sin \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k} - (\cos \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1} \\
& + (\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} - (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1} \\
& - (\sin \theta_n) \sum_{k=0}^{n-1} \mathbf{e}_{2k} + (\cos \theta_n) \sum_{k=0}^{n-1} \mathbf{e}_{2k+1}] \\
& = h_n(X_{n,0}(s))(\sin \theta_n)^{-1}[(\sin((n-1)\theta_n))\mathbf{e}_{2(n-1)} \\
& - (\cos((n-1)\theta_n))\mathbf{e}_{2(n-1)+1} - (\sin \theta_n)\mathbf{e}_0 + (\cos \theta_n)\mathbf{e}_1].
\end{aligned}$$

On (II), we have the following:

$$\begin{aligned}
(3.16). \quad & 2(\cot \theta_n)h_n(X_{n,1}(s))[-(\sin \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k} + (\cos \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,1}(s))(\sin \theta_n)^{-1}[-(\sin(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k} + (\cos(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,1}(s))(\sin \theta_n)^{-1} \sum_{k=0}^{n-1} \mathbf{e}_{2k+1} \\
& = h_n(X_{n,1}(s))(\sin \theta_n)^{-1}[-(\sin(2\theta_n)) \sum_{k=1}^{n-1} \mathbf{e}_{2k} + 2(\cos^2 \theta_n) \sum_{k=1}^{n-1} \mathbf{e}_{2k+1}] \\
& + (\sin(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k} - (\cos(2\theta_n)) \sum_{k=2}^{n-1} \mathbf{e}_{2k+1} - \sum_{k=0}^{n-1} \mathbf{e}_{2k+1}] \\
& = h_n(X_{n,1}(s))(\sin \theta_n)^{-1}[-(\sin(2\theta_n))\mathbf{e}_2 + (\cos(2\theta_n))\mathbf{e}_3 - \mathbf{e}_1],
\end{aligned}$$

by (3.4).

On (III), we have the following: for $i = 2, \dots, n-2$,

$$\begin{aligned}
(3.17). \quad & 2(\cot \theta_n) h_n(X_{n,i}(s)) [-(\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} + (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1}] \\
& - h_n(X_{n,i}(s)) (\sin \theta_n)^{-1} [-(\sin((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k+1} - (\sin((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k+1}] \\
& = h_n(X_{n,i}(s)) (\sin \theta_n)^{-1} [-2 \cos \theta_n (\sin(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k} \\
& + 2 \cos \theta_n (\cos(i\theta_n)) \sum_{k=i}^{n-1} \mathbf{e}_{2k+1} \\
& + (\sin((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k} - (\cos((i-1)\theta_n)) \sum_{k=i-1}^{n-1} \mathbf{e}_{2k+1} \\
& + (\sin((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k} - (\cos((i+1)\theta_n)) \sum_{k=i+1}^{n-1} \mathbf{e}_{2k+1}] \\
& = h_n(X_{n,i}(s)) (\sin \theta_n)^{-1} [-(\sin((i+1)\theta_n)) \mathbf{e}_{2i} + (\cos((i+1)\theta_n)) \mathbf{e}_{2i+1} \\
& + (\sin((i-1)\theta_n)) \mathbf{e}_{2(i-1)} - (\cos((i-1)\theta_n)) \mathbf{e}_{2(i-1)+1}],
\end{aligned}$$

by (3.4).

On (IV), we have the following in the same way as in (3.17).

$$\begin{aligned}
(3.18). \quad & 2(\cot \theta_n) h_n(X_{n,n-1}(s)) [-(\sin((n-1)\theta_n)) \mathbf{e}_{2(n-1)} \\
& + (\cos((n-1)\theta_n)) \mathbf{e}_{2(n-1)+1}] \\
& - h_n(X_{n,n-1}(s)) (\sin \theta_n)^{-1} [-(\sin((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k} \\
& + (\cos((n-2)\theta_n)) \sum_{k=n-2}^{n-1} \mathbf{e}_{2k+1}]
\end{aligned}$$

$$\begin{aligned}
&= h_n(X_{n,n-1}(s))(\sin \theta_n)^{-1}[-(\sin(n\theta_n))\mathbf{e}_{2(n-1)} + (\cos(n\theta_n))\mathbf{e}_{2(n-1)+1} \\
&\quad + (\sin((n-2)\theta_n))\mathbf{e}_{2(n-2)} - (\cos((n-2)\theta_n))\mathbf{e}_{2(n-2)+1}].
\end{aligned}$$

Q. E. D.

4. Proof of Theorem 2.1.

In this section we prove (2.7) to complete the proof of Theorem 2.1. For $n \geq n_2$, by Lemmas 3.1 and 3.2, the following holds: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned}
(4.1). \quad &\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2 \\
&= 2 \sum_{k=0}^{n-1} \int_0^t \langle y_{n,k}(s) - Y_{n,k}(s), \ell_{n,k+1}(s)^{-1}(\sin(k\theta_n), -\cos(k\theta_n)) \\
&\quad + \ell_{n,k}(s)^{-1}(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) \rangle (\cos^2(\theta_n/2))^{-1} ds \\
&\quad - 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1} \sum_{k=0}^{n-1} \int_0^t [\langle y_{n,k-1}(t) - Y_{n,k-1}(s), \\
&\quad h_n(X_{n,k}(s))(\sin((k-1)\theta_n), -\cos((k-1)\theta_n)) + \langle y_{n,k}(t) - Y_{n,k}(s) \\
&\quad , h_n(X_{n,k}(s))(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) \rangle] ds \\
&\quad + 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1} (\sin \theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin \theta_n)^{-1}|^2 ds + M(t),
\end{aligned}$$

where $M(t)$ denotes a purely discontinuous martingale part. Hence

$$\begin{aligned}
(4.2). \quad &\sum_{k=0}^{n-1} |y_{n,k}(t) - Y_{n,k}(t)|^2 \\
&= 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t \langle y_{n,k}(s) - Y_{n,k}(s), \\
&\quad (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1})(\sin(k\theta_n), -\cos(k\theta_n)) \\
&\quad + (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) \rangle ds \\
&\quad + 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t \langle y_{n,k}(t) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1}
\end{aligned}$$

$$\begin{aligned}
& + X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1} \\
& - X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds \\
& + 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1}(\sin \theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin \theta_n)^{-1}|^2 ds \\
& + M(t).
\end{aligned}$$

To complete the proof we show the following: for $t \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned}
(4.3). \quad & 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), \\
& (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1})(\sin(k\theta_n), -\cos(k\theta_n)) \\
& + (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) > ds \\
& \leq 2 \sin^2 \theta_n ((\cos \theta_n) C_n(\delta) n^{-1} C_n(\delta) (2n)^{-1} \cos^2(\theta_n/2))^{-1} \\
& \times \int_0^t \sup_{0 \leq u \leq \min(s, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds,
\end{aligned}$$

$$\begin{aligned}
(4.4). \quad & 2(\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} \int_0^t < y_{n,k}(s) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} \\
& + X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1} \\
& - X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > ds \\
& \leq 2 \int_0^t \sup_{0 \leq u \leq \min(s, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2 ds \\
& + 2nt |\theta_n (T(n)[C_n(\delta)(2n)^{-1}]^3 \sin \theta_n)^{-1}|^2,
\end{aligned}$$

$$\begin{aligned}
(4.5). \quad & 2T(n)[\theta_n \cos^2(\theta_n/2)]^{-1}(\sin \theta_n) \sum_{k=0}^{n-1} \int_0^t |h_n(X_{n,k}(s))(\sin \theta_n)^{-1}|^2 ds \\
& \leq 8T(n)^{-1} n^3 t \theta_n [C_n(\delta)^2 \cos^2(\theta_n/2) \sin \theta_n]^{-1},
\end{aligned}$$

and for $t \in [0, T_n^* - \delta]$,

$$\begin{aligned}
(4.6). \quad & \{E[\sup_{0 \leq s \leq \min(t, \tau_{n,\delta})} |M(s)|^2]\}^{1/2} \\
& \leq 32T(n)^{-1}n^2\theta_n(C_n(\delta)^2 \sin \theta_n \cos^2(\theta_n/2))^{-1} \\
& \quad + 6T(n)^{-2}n^3t|\theta_n(C_n(\delta) \sin \theta_n)^{-1}|^2 \\
& \quad + 3 \int_0^t E[\sup_{0 \leq u \leq \min(s, \tau_{n,\delta})} \sum_{k=0}^{n-1} |y_{n,k}(u) - Y_{n,k}(u)|^2] ds.
\end{aligned}$$

(4.3) is true, since for $s \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned}
& \sum_{k=0}^{n-1} \langle y_{n,k}(s) - Y_{n,k}(s), (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1})(\sin(k\theta_n), -\cos(k\theta_n)) \\
& \quad + (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1})(-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) \rangle > \\
& = \sum_{k=0}^{n-1} \langle y_{n,k}(s) - Y_{n,k}(s), (\sin(k\theta_n), -\cos(k\theta_n)) \rangle (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1}) \\
& \quad + \sum_{k=0}^{n-1} \langle y_{n,k-1}(s) - Y_{n,k-1}(s) + (\ell_{n,k}(s) - X_{n,k}(s))(-\sin(k\theta_n), \cos(k\theta_n)), \\
& \quad (-\sin((k+1)\theta_n), \cos((k+1)\theta_n)) \rangle (\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1}) \\
& = \sum_{k=0}^{n-1} \langle y_{n,k}(s) - Y_{n,k}(s), (\sin(k\theta_n) - \sin((k+2)\theta_n), -\cos(k\theta_n) + \cos((k+2)\theta_n)) \rangle \\
& \quad \times (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1}) \\
& \quad + \sum_{k=0}^{n-1} (\sin(k\theta_n) \sin((k+1)\theta_n) + \cos(k\theta_n) \cos((k+1)\theta_n)) \\
& \quad \times (\ell_{n,k}(s) - X_{n,k}(s))(\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1}) \\
& = \sum_{k=0}^{n-1} \langle y_{n,k}(s) - Y_{n,k}(s), -2(\sin \theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) \rangle \\
& \quad \times (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1}) \\
& \quad + \sum_{k=0}^{n-1} (\cos \theta_n)(\ell_{n,k}(s) - X_{n,k}(s))(\ell_{n,k}(s)^{-1} - X_{n,k}(s)^{-1}) \\
& \leq \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 \sin^2 \theta_n ((\cos \theta_n) \ell_{n,k+1}(s) X_{n,k+1}(s))^{-1} \\
& \leq \sin^2 \theta_n ((\cos \theta_n) C_n(\delta) n^{-1} C_n(\delta) (2n)^{-1})^{-1} \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2
\end{aligned}$$

(see (2.4)). Here we used the following:

$$\begin{aligned}
& < y_{n,k}(s) - Y_{n,k}(s), -2(\sin \theta_n)(\cos((k+1)\theta_n), \sin((k+1)\theta_n)) > \\
& \quad \times (\ell_{n,k+1}(s)^{-1} - X_{n,k+1}(s)^{-1}) \\
& \leq 2|y_{n,k}(s) - Y_{n,k}(s)|(\sin \theta_n)[(\cos \theta_n)\ell_{n,k+1}(s)X_{n,k+1}(s)]^{-1/2} \\
& \quad \times [(\cos \theta_n)(\ell_{n,k+1}(s)X_{n,k+1}(s))^{-1}]^{1/2}|\ell_{n,k+1}(s) - X_{n,k+1}(s)| \\
& \leq |y_{n,k}(s) - Y_{n,k}(s)|^2(\sin^2 \theta_n)[(\cos \theta_n)\ell_{n,k+1}(s)X_{n,k+1}(s)]^{-1} \\
& \quad + (\cos \theta_n)(\ell_{n,k+1}(s)X_{n,k+1}(s))^{-1}(\ell_{n,k+1}(s) - X_{n,k+1}(s))^2.
\end{aligned}$$

(4.4) is true, since for $s \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned}
& (\cos^2(\theta_n/2))^{-1} \sum_{k=0}^{n-1} < y_{n,k}(s) - Y_{n,k}(s), [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} \\
& \quad + X_{n,k+1}(s)^{-1}](\sin(k\theta_n), -\cos(k\theta_n)) + [T(n)h_n(X_{n,k}(s))\theta_n^{-1} \\
& \quad - X_{n,k}(s)^{-1}](\sin((k+1)\theta_n), -\cos((k+1)\theta_n)) > \\
& \leq \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 \\
& \quad + 2^{-1} \sum_{k=0}^{n-1} [-T(n)h_n(X_{n,k+1}(s))\theta_n^{-1} + X_{n,k+1}(s)^{-1}]^2 (\cos^2(\theta_n/2))^{-2} \\
& \quad + 2^{-1} \sum_{k=0}^{n-1} [T(n)h_n(X_{n,k}(s))\theta_n^{-1} - X_{n,k}(s)^{-1}]^2 (\cos^2(\theta_n/2))^{-2} \\
& \leq \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 + \sum_{k=0}^{n-1} |\theta_n(T(n)X_{n,k}(s)^3 \sin \theta_n)^{-1}|^2 \\
& \leq \sum_{k=0}^{n-1} |y_{n,k}(s) - Y_{n,k}(s)|^2 + n|\theta_n(T(n)[C_n(\delta)(2n)^{-1}]^3 \sin \theta_n)^{-1}|^2
\end{aligned}$$

(see (2.4)). Here we used the following: for $x > 0$,

$$\begin{aligned}
& x^{-1} - T(n)h_n(x)\theta_n^{-1} \\
& = \theta_n(T(n)x^3 \sin \theta_n)^{-1} \{2(1 + (1 + 4x^{-2}T(n)^{-1}\theta_n \cot \theta_n)^{1/2})^{-1}\}^2 \cos \theta_n,
\end{aligned}$$

and

$$\cos \theta_n = \cos^2(\theta_n/2) - \sin^2(\theta_n/2) < \cos^2(\theta_n/2).$$

(4.5) is true, since for $s \in [0, \min(T_n^* - \delta, \tau_{n,\delta})]$,

$$\begin{aligned} \sum_{k=0}^{n-1} |h_n(X_{n,k}(s))(\sin \theta_n)^{-1}|^2 &\leq \sum_{k=0}^{n-1} |\theta_n(T(n)X_{n,k}(s) \sin \theta_n)^{-1}|^2 \\ &\leq n|\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 \end{aligned}$$

(see (2.4)). Here we used that $h_n(x) < \theta_n(T(n)x)^{-1}$ from (1.10).

(4.6) is true, since

$$\begin{aligned} M(t) &= \sum_{0 \leq s \leq t} \sum_{k=0}^{n-1} \{|y_{n,k}(s) - Y_{n,k}(s)|^2 - |y_{n,k}(s) - Y_{n,k}(s-)|^2\} \\ &\quad - T(n)(\sin \theta_n)[\theta_n \cos^2(\theta_n/2)]^{-1} \sum_{i=0}^{n-1} \int_0^t 2[-h_n(X_{n,i}(s))(\sin \theta_n)^{-1} \\ &\quad \times \langle y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) \rangle \\ &\quad + \langle y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) \rangle \\ &\quad + |h_n(X_{n,i}(s))(\sin \theta_n)^{-1}|^2] ds, \end{aligned}$$

and henceforth for $t \in [0, T_n^* - \delta]$,

$$\begin{aligned}
& E\left[\sup_{0 \leq s \leq \min(t, \tau_{n,\delta})} |M(s)|^2\right] \\
& \leq 4E[|M(\min(t, \tau_{n,\delta}))|^2] \quad (\text{by Doob's inequality}) \\
& = 4T(n)(\sin \theta_n)(\theta_n \cos^2(\theta_n/2))^{-1} \sum_{i=0}^{n-1} E\left[\int_0^{\min(t, \tau_{n,\delta})} (-2h_n(X_{n,i}(s))(\sin \theta_n)^{-1} \right. \\
& \quad \left[\langle y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) \rangle \right. \\
& \quad \left. + \langle y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) \rangle \right] \\
& \quad \left. + 2|h_n(X_{n,i}(s))(\sin \theta_n)^{-1}|^2 ds\right] \\
& \leq 16T(n)(\sin \theta_n)(\theta_n \cos^2(\theta_n/2))^{-1} |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 \\
& \quad \times \sum_{i=0}^{n-1} E\left[\int_0^{\min(t, \tau_{n,\delta})} (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \right. \\
& \quad \left. + |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|)^2 ds\right] \\
& \leq \{(16T(n)(\sin \theta_n)(\theta_n \cos^2(\theta_n/2))^{-1} |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 \\
& \quad + \sum_{i=0}^{n-1} E\left[\int_0^{\min(t, \tau_{n,\delta})} (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \right. \\
& \quad \left. + |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|)^2 ds\right])/2\}^2 \\
& \leq \{(16T(n)(\sin \theta_n)(\theta_n \cos^2(\theta_n/2))^{-1} |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 \\
& \quad + 3nt|\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 \\
& \quad + 6 \sum_{i=0}^{n-1} E\left[\int_0^t |y_{n,i}(\min(s, \tau_{n,\delta})) - Y_{n,i}(\min(s, \tau_{n,\delta}))|^2 ds\right])/2\}^2.
\end{aligned}$$

Here we considered as the following. For $s \in [0, \min(t, \tau_{n,\delta}))$,

$$\begin{aligned}
& \sum_{i=0}^{n-1} (-2h_n(X_{n,i}(s))(\sin \theta_n)^{-1} \\
& \quad [\langle y_{n,i-1}(s) - Y_{n,i-1}(s), (\sin((i-1)\theta_n), -\cos((i-1)\theta_n)) \rangle \\
& \quad + \langle y_{n,i}(s) - Y_{n,i}(s), (-\sin((i+1)\theta_n), \cos((i+1)\theta_n)) \rangle] \\
& \quad + 2|h_n(X_{n,i}(s))(\sin \theta_n)^{-1}|^2)^2 \\
& \leq \sum_{i=0}^{n-1} 4|h_n(X_{n,i}(s))(\sin \theta_n)^{-1}|^2 (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
& \quad + |h_n(X_{n,i}(s))(\sin \theta_n)^{-1}|)^2 \\
& \leq \sum_{i=0}^{n-1} 4|\theta_n(T(n)X_{n,i}(s) \sin \theta_n)^{-1}|^2 (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
& \quad + |\theta_n(T(n)X_{n,i}(s) \sin \theta_n)^{-1}|)^2 \quad (\text{from (1.10)}) \\
& \leq \sum_{i=0}^{n-1} 4|\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|^2 (|y_{n,i-1}(s) - Y_{n,i-1}(s)| + |y_{n,i}(s) - Y_{n,i}(s)| \\
& \quad + |\theta_n(T(n)C_n(\delta)(2n)^{-1} \sin \theta_n)^{-1}|)^2 \quad (\text{from (2.4)}).
\end{aligned}$$

We also used the following: for $x, y, z \in \mathbf{R}$,

$$xy \leq \{(x+y)/2\}^2, \quad (x+y+z)^2 \leq 3(x^2 + y^2 + z^2).$$

Q. E. D.

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