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**Fundamental Properties of  
the Hamiltonian of a Dirac Particle  
Coupled to the Quantized Radiation Field**

**Asao Arai**

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# Fundamental Properties of the Hamiltonian of a Dirac Particle Coupled to the Quantized Radiation Field

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## Abstract

We study the Hamiltonian  $H(V, \mathbf{v})$  of a Dirac particle (a relativistic charged particle with spin 1/2) minimally coupled to the quantized radiation field, acting in the Hilbert space  $\mathcal{F} := [\oplus^4 L^2(\mathbb{R}^3)] \otimes \mathcal{F}_{\text{rad}}$ , where  $\mathcal{F}_{\text{rad}}$  is the Fock space of the quantized radiation field in the Coulomb gauge,  $V$  is an external field in which the Dirac particle moves, and  $\mathbf{v}$  is a momentum cutoff function for the interaction between the Dirac particle and the quantized radiation field. We first discuss the self-adjointness problem of  $H(V, \mathbf{v})$ . Then we investigate in detail properties of  $H := H(0, \mathbf{v})$ , the Hamiltonian in the case  $V = 0$ . In this case a unitary transform  $\tilde{H}$  of  $H$  has a direct integral decomposition  $\tilde{H} = \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{p}) d\mathbf{p}$ , where  $H(\mathbf{p})$  is an operator on  $\oplus^4 \mathcal{F}_{\text{rad}}$ , physically the polaron Hamiltonian of the Dirac particle with total momentum  $\mathbf{p} \in \mathbb{R}^3$ . We define a one parameter family  $\{H_\tau(\mathbf{p})\}_{\tau \in [0,1]}$  of deformations of  $H(\mathbf{p})$  such that  $H_1(\mathbf{p}) = H(\mathbf{p})$ . On the operator  $H_\tau(\mathbf{p})$ , we discuss the following aspects: (i) properties of the ground-state energy; (ii) existence of a ground state; (iii) the spectrum.

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# 1 Introduction

In this work we present a comprehensive mathematical analysis of the Hamiltonian of a model in the relativistic quantum electrodynamics (QED). The model we consider describes a quantum system of a Dirac particle (a relativistic charged particle with spin  $1/2$ , e.g., electron) minimally coupled to the quantized radiation field. This model is realistic in the sense that it explains atomic phenomena of one Dirac particle in which the effects of the quantized radiation field play essential roles (e.g., the Lamb shift and the Compton scattering), although the usual explanations are made by using informal physical perturbation theories [8, 15], to which no mathematically rigorous basis as yet has been given. We view the present work as a step towards a complete mathematical construction of the relativistic QED. Besides this point, we feel that mathematical analysis of the model is interesting also in its own right, because, as we shall see below, the Hamiltonian of the model belongs to a new class of symmetric operators on a Hilbert space of Fock type: it is given by a perturbation for the sum of a three-dimensional Dirac operator and a second quantized operator on a Fock space (an infinite dimensional Laplacian), containing both commutative and anticommutative operators. To our best knowledge, no mathematically rigorous analysis has been made on the model (cf., however, [18] where a *modified* one of the model is considered with relatively much mathematical rigor).

We now describe the model in more detail. For a Hilbert space  $\mathcal{H}$ , we denote its inner product and norm by  $(\cdot, \cdot)_{\mathcal{H}}$  (complex linear in the second variable) and  $\|\cdot\|_{\mathcal{H}}$  respectively, but, if there is no danger of confusion, then we often omit the subscript  $\mathcal{H}$  of them. For a linear operator  $T$  on  $\mathcal{H}$ , we denote its domain by  $D(T)$  and by  $\sigma(T)$  the spectrum of  $T$ . For two objects  $\mathbf{a} = (a_1, a_2, a_3)$  and  $\mathbf{b} = (b_1, b_2, b_3)$  such that products  $a_j b_j$ ,  $j = 1, 2, 3$ , and their sum can be defined,  $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$ .

The free Dirac particle of mass  $m_D \geq 0$  is described by the free Dirac operator

$$H_D := \boldsymbol{\alpha} \cdot (-i\nabla) + m_D \beta \quad (1.1)$$

acting in the Hilbert space

$$\mathcal{H}_D := \oplus^4 L^2(\mathbf{R}^3) \quad (1.2)$$

with domain  $D(H_D) := \oplus^4 H^1(\mathbf{R}^3)$  ( $H^1(\mathbf{R}^3)$  is the Sobolev space of order 1), where  $\alpha_j$ ,  $j = 1, 2, 3$  and  $\beta$  are  $4 \times 4$  Hermitian matrices satisfying

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \quad (1.3)$$

$$\{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \quad (1.4)$$

$\{A, B\} := AB + BA$ , and  $\nabla := (D_1, D_2, D_3)$ ,  $D_j$  being the generalized partial differential operator in the variable  $x_j$  [ $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ ]. It is well known that  $H_D$  is self-adjoint and essentially self-adjoint on  $\oplus^4 C_0^\infty(\mathbf{R}^3 \setminus \{0\})$  (e.g., [24, p.11, Theorem 1.1]). Moreover, the spectrum  $\sigma(H_D)$  of  $H_D$  is purely absolutely continuous and

$$\sigma(H_D) = (-\infty, -m_D] \cup [m_D, \infty). \quad (1.5)$$

As for the radiation field, we use the Coulomb gauge in quantizing it. In general, for a Hilbert space  $\mathcal{H}$ , we have the symmetric Fock space (or the Boson Fock space)

$$\mathcal{F}_s(\mathcal{H}) := \oplus_{n=0}^\infty (\otimes_s^n \mathcal{H}) \quad (1.6)$$

over  $\mathcal{H}$ , where  $\otimes_s^n \mathcal{H}$  denotes the  $n$ -fold symmetric tensor product Hilbert space of  $\mathcal{H}$  with convention  $\otimes_s^0 \mathcal{H} = \mathbf{C}$  (e.g., [19, p. 53,]). The Hilbert space of one photon states in momentum representation is given by

$$\mathcal{H}_{\text{ph}} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \quad (1.7)$$

where  $\mathbf{R}^3 := \{\mathbf{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R}, j = 1, 2, 3\}$  physically means the momentum space of photons. The symmetric Fock space

$$\mathcal{F}_{\text{rad}} := \mathcal{F}_s(\mathcal{H}_{\text{ph}}) \quad (1.8)$$

over  $\mathcal{H}_{\text{ph}}$  serves as a Hilbert space for the quantized radiation field in the Coulomb gauge.

We denote by  $a(F)$  ( $F \in \mathcal{H}_{\text{ph}}$ ) the annihilation operator with test vector  $F$  on  $\mathcal{F}_{\text{rad}}$  [20, §X.7]. By definition,  $a(F)$  is a densely defined closed linear operator and antilinear in  $F$ . The Segal field operator

$$\Phi_S(F) := \frac{a(F) + a(F)^*}{\sqrt{2}} \quad (1.9)$$

is self-adjoint [20, §X.7], where, for a closable operator  $T$ ,  $\bar{T}$  denotes its closure.

We take a nonnegative Borel measurable function  $\omega$  on  $\mathbf{R}^3$  to denote physically the one free photon energy. We assume that, for almost everywhere (a.e.)  $\mathbf{k} \in \mathbf{R}^3$  with respect to the Lebesgue measure on  $\mathbf{R}^3$ ,  $0 < \omega(\mathbf{k}) < \infty$ . Then the function  $\omega$  defines uniquely a multiplication operator on  $\mathcal{H}_{\text{ph}}$  which is nonnegative, self-adjoint and injective. We denote it by the same symbol  $\omega$  also. The free Hamiltonian of the quantized radiation field is defined by

$$H_{\text{rad}} := d\Gamma(\omega), \quad (1.10)$$

the second quantization of  $\omega$  [20, §X.7]. The operator  $H_{\text{rad}}$  is a nonnegative self-adjoint operator.

**Remark 1.1** Physically  $\omega$  is of the form  $\omega(\mathbf{k}) = |\mathbf{k}|$ , but, in this paper, for mathematical generality, we do not specify any concrete form of  $\omega$ .

For each  $j = 1, 2, 3$ , let  $v_j : \mathbf{x} \rightarrow v_j(\mathbf{x})$  be an  $\mathcal{H}_{\text{ph}}$ -valued strongly continuous function on  $\mathbf{R}^3$ . Then the quantized radiation field  $\mathbf{A}^v(\mathbf{x}) := (A_1^v(\mathbf{x}), A_2^v(\mathbf{x}), A_3^v(\mathbf{x}))$  with cutoff function  $\mathbf{v} := (v_1, v_2, v_3)$  is defined by

$$A_j^v(\mathbf{x}) := \Phi_S(v_j(\mathbf{x})), \quad j = 1, 2, 3. \quad (1.11)$$

**Remark 1.2** Usually the cutoff function  $v_j$  ( $j = 1, 2, 3$ ) is taken to be a function of the form  $\varrho_j$  defined as follows. For each  $\mathbf{k} \in \mathbf{R}^3$ , there exist vectors  $\mathbf{e}^{(r)}(\mathbf{k}) \in \mathbf{R}^3$ ,  $r = 1, 2$ , such that

$$\mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{e}^{(s)}(\mathbf{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)} \cdot \mathbf{k} = 0, \quad r, s = 1, 2. \quad (1.12)$$

These vectors are called the polarization vectors of one photon with momentum  $\mathbf{k}$  and can be taken to be Borel measurable. Let  $\varrho \in L^2(\mathbf{R}^3)$  such that  $\varrho/\sqrt{\omega} \in L^2(\mathbf{R}^3)$ . Then we can define an  $\mathcal{H}_{\text{ph}}$ -valued strongly continuous function  $\varrho_j$  by

$$\varrho_j(\mathbf{x})(\mathbf{k}) := \frac{\varrho(\mathbf{k})e^{-i\mathbf{k} \cdot \mathbf{x}}}{\sqrt{(2\pi)^3\omega(\mathbf{k})}} (e_j^{(1)}(\mathbf{k}), e_j^{(2)}(\mathbf{k})) \in \mathbf{C}^2, \quad \mathbf{x}, \mathbf{k} \in \mathbf{R}^3. \quad (1.13)$$

In this paper we do not restrict ourselves to this form of  $v_j$ .

We now move to the Hilbert space

$$\mathcal{F} := \mathcal{H}_D \otimes \mathcal{F}_{\text{rad}} \quad (1.14)$$

of state vectors for the coupled system of the Dirac particle and the quantized radiation field. This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4 \mathcal{F}_{\text{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\text{rad}} dx \quad (1.15)$$

the Hilbert space of  $\oplus^4 \mathcal{F}_{\text{rad}}$ -valued Lebesgue square integrable functions on  $\mathbf{R}^3$  (the constant fibre direct integral with base space  $(\mathbf{R}^3, dx)$  and fibre  $\oplus^4 \mathcal{F}_{\text{rad}}$ ), cf. [19, §II.4, Theorem II.10(b)] and [21, §XIII.6]. The identification is given in terms of the unitary operator determined by the correspondence

$$f \otimes \psi \rightarrow f(x)\psi = (f_1(x)\psi, f_2(x)\psi, f_3(x)\psi, f_4(x)\psi) \in \oplus^4 \mathcal{F}_{\text{rad}}$$

( $f = (f_j)_{j=1}^4 \in \mathcal{H}_D$ ,  $\psi \in \mathcal{F}_{\text{rad}}$ ). We freely use this identification.

Let  $F$  be an  $\mathcal{H}_{\text{ph}}$ -valued strongly continuous function on  $\mathbf{R}^3$ . Then, for all  $t \in \mathbf{R}$ , the operator valued function:  $\mathbf{x} \rightarrow \exp[it\Phi_S(F(\mathbf{x}))]$  is strongly continuous [20, §X.7, Theorem X.41(d)]. Applying a general convergence theorem [19, §VIII.7, Theorem VIII.21], we see that, for all  $z \in \mathbf{C} \setminus \mathbf{R}$ , the operator valued function:  $\mathbf{x} \rightarrow (\Phi_S(F(\mathbf{x})) - z)^{-1}$  on  $\mathbf{R}^3$  is strongly continuous. Therefore, one can define a self-adjoint, decomposable operator

$$[\Phi_S](F) := \int_{\mathbf{R}^3}^{\oplus} \Phi_S(F(\mathbf{x})) dx \quad (1.16)$$

acting on  $\mathcal{F} = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\text{rad}} dx$  [21, Theorem XIII.85(a)].

We set

$$A_j^\vee := [\Phi_S](v_j) = \int_{\mathbf{R}^3}^{\oplus} A_j^\vee(\mathbf{x}) dx \quad (1.17)$$

acting on  $\mathcal{F}$ .

We denote by  $q \in \mathbf{R} \setminus \{0\}$  the charge of the Dirac particle. We consider the situation where the Dirac particle is in an external field described by a  $4 \times 4$  Hermitian matrix-valued Borel measurable function  $V = (V_{ab})_{a,b=1,\dots,4}$  such that each  $V_{ab}$  is in  $L^2(\mathbf{R}^3)_{\text{loc}} := \{f : \mathbf{R}^3 \rightarrow \mathbf{C}; \text{Borel measurable} | \int_{|\mathbf{x}| \leq R} |f(\mathbf{x})|^2 dx < \infty \text{ for all } R > 0\}$ . Then the Hamiltonian of the Dirac particle is given by

$$H_D(V) := H_D + V \quad (1.18)$$

**Remark 1.3** If the Dirac particle is in an external magnetic field given by a vector potential (an  $\mathbf{R}^3$ -valued Borel measurable function on  $\mathbf{R}^3$ )  $\mathbf{A}^{\text{ex}} = (A_1^{\text{ex}}, A_2^{\text{ex}}, A_3^{\text{ex}})$  and in a scalar potential  $\phi^{\text{ex}}$ , then

$$V = -q\boldsymbol{\alpha} \cdot \mathbf{A}^{\text{ex}} + q\phi^{\text{ex}}.$$

The minimal interaction between the Dirac particle and the quantized radiation field with momentum cutoff  $\mathbf{v}$  is given by

$$H_I(\mathbf{v}) := -q\boldsymbol{\alpha} \cdot \mathbf{A}^{\mathbf{v}}. \quad (1.19)$$

Thus the total Hamiltonian of the coupled system is defined by

$$H(V, \mathbf{v}) := H_D(V) + H_{\text{rad}} + H_I(\mathbf{v}). \quad (1.20)$$

This is the main object of the mathematical analysis in the present paper.

In Section 2 we discuss self-adjointness of  $H(V, \mathbf{v})$ . We show that, for a class of the pair  $\langle V, \mathbf{v} \rangle$ ,  $H(V, \mathbf{v})$  has a self-adjoint extension which is natural in a sense. We also prove that, for a class of  $\langle V, \mathbf{v} \rangle$ ,  $H(V, \mathbf{v})$  is essentially self-adjoint. From Section 3 we focus our attention on the case where  $V = 0$  and  $\mathbf{v}$  is of the form  $v_j(\mathbf{x})(\mathbf{k}) = g_j(\mathbf{k})e^{-i\mathbf{k} \cdot \mathbf{x}}$ ,  $\mathbf{k}, \mathbf{x} \in \mathbb{R}^3$  with  $g_j \in \mathcal{H}_{\text{ph}}$ . In Section 3 we show that, under some conditions,  $H := H(0, \mathbf{v})$  is strongly translation invariant and that a unitary transform  $\widetilde{H}$  of  $H$  has a direct integral decomposition  $\widetilde{H} = \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{p}) d\mathbf{p}$  with  $H(\mathbf{p})$ , an operator on  $\oplus^4 \mathcal{F}_{\text{rad}}$ , physically the polaron Hamiltonian of the Dirac particle with total momentum  $\mathbf{p} \in \mathbb{R}^3$ . Thus the analysis of  $H$  is reduced to that of  $H(\mathbf{p})$ .

In Section 4 we first examine the boundedness from below of  $H(\mathbf{p})$ . It is shown that, in the physical case  $\omega(\mathbf{k}) = |\mathbf{k}|$ ,  $H(\mathbf{p})$  is *not* bounded from below. This is due to the term  $\boldsymbol{\alpha} \cdot \mathbf{P}^{\text{rad}}$  appearing in  $H(\mathbf{p})$ , where  $\mathbf{P}^{\text{rad}}$  is the momentum operator of the quantized radiation field. The unboundedness below of  $H(\mathbf{p})$  with  $\omega(\mathbf{k}) = |\mathbf{k}|$  suggests that an energy renormalization is needed to obtain a physically meaningful theory even in the case where a momentum cutoff in the interaction is made. In this sense the quantum theory described by  $H(\mathbf{p})$  with  $\omega(\mathbf{k}) = |\mathbf{k}|$  is somewhat singular. To avoid this difficulty, we introduce a one parameter family  $\{H_\tau(\mathbf{p})\}_{\tau \in [0,1]}$  of deformations of  $H(\mathbf{p})$  such that  $H_1(\mathbf{p}) = H(\mathbf{p})$ . Each  $H_\tau(\mathbf{p})$  is bounded from below (in the case  $\tau = 1$ , it is assumed that  $\omega(\mathbf{k}) \neq |\mathbf{k}|$ ). We investigate properties of the ground-state energy  $E_\tau(\mathbf{p})$  of  $H_\tau(\mathbf{p})$ . In Sections 5 and 6 we prove existence of a ground state of  $H_\tau(\mathbf{p})$ . In the former (resp. latter) section we treat the case where the photon is “massive” (resp. “massless”). The operator  $H_\tau(\mathbf{p})$  is *not* in the class of quantum field Hamiltonians introduced in [6] except for the case  $\tau = 0$ . Hence we must extend the methods used in [6] to the present model. The last section is devoted to locating the (essential) spectrum of  $H_\tau(\mathbf{p})$  and its direct integral  $\widetilde{H}_\tau := \int_{\mathbb{R}^3}^{\oplus} H_\tau(\mathbf{p}) d\mathbf{p}$ .

In the case  $\omega(\mathbf{k}) = |\mathbf{k}|$ , a renormalized theory may be constructed from defining the limit  $H_{\text{ren}}(\mathbf{p}) := \lim_{\tau \uparrow 1} (H_\tau(\mathbf{p}) - E_\tau(\mathbf{p}))$  in an appropriate sense. But, in this paper, we do not discuss this aspect.

## 2 Self-Adjointness of the Total Hamiltonian

In this section we discuss the self-adjointness problem of the total Hamiltonian  $H(V, \mathbf{v})$ .

## 2.1 Symmetricity and numerical range of $H(V, \mathbf{v})$

We denote by  $L^\infty(\mathbf{R}^3; \mathcal{H}_{\text{ph}})$  the set of all  $\mathcal{H}_{\text{ph}}$ -valued measurable functions  $f$  such that

$$\|f\|_\infty := \text{ess.sup}_{\mathbf{x} \in \mathbf{R}^3} \|f(\mathbf{x})\|_{\mathcal{H}_{\text{ph}}} < \infty, \quad (2.1)$$

where “ess.sup” means essential supremum. For a three-component function  $\mathbf{f} = (f_1, f_2, f_3)$  with  $f_j \in L^\infty(\mathbf{R}^3; \mathcal{H}_{\text{ph}})$ , we define  $\|\mathbf{f}\|_\infty$  by

$$\|\mathbf{f}\|_\infty := \sum_{j=1}^3 \|f_j\|_\infty. \quad (2.2)$$

We assume the following for the momentum cutoff function  $\mathbf{v}$ .

**Hypothesis (H.1)** Each  $j = 1, 2, 3$ ,  $v_j$  and  $v_j/\sqrt{\omega}$  are in  $L^\infty(\mathbf{R}^3; \mathcal{H}_{\text{ph}})$ .

**Lemma 2.1** Assume (H.1). Then  $D(H_{\text{rad}}^{1/2}) \subset D(H_I(\mathbf{v}))$  and, for all  $\Psi \in D(H_{\text{rad}}^{1/2})$  and  $\varepsilon > 0$ ,

$$\|H_I(\mathbf{v})\Psi\| \leq \sqrt{2(1+\varepsilon)}|q| \left\| \frac{\mathbf{v}}{\sqrt{\omega}} \right\|_\infty \|H_{\text{rad}}^{1/2}\Psi\| + \frac{|q|}{\sqrt{2}} \sqrt{1 + \frac{1}{\varepsilon}} \|\mathbf{v}\|_\infty \|\Psi\|. \quad (2.3)$$

*Proof.* It is well known that, for all  $F \in \mathcal{H}_{\text{ph}}$  and  $\psi \in D(H_{\text{rad}}^{1/2})$ ,  $\psi$  is in  $D(a(F)) \cap D(a(F)^*)$  and

$$\|a(F)\psi\| \leq \left\| \frac{F}{\sqrt{\omega}} \right\| \|H_{\text{rad}}^{1/2}\psi\|, \quad (2.4)$$

$$\|a(F)^*\psi\| \leq \left\| \frac{F}{\sqrt{\omega}} \right\| \|H_{\text{rad}}^{1/2}\psi\| + \|F\| \|\psi\|. \quad (2.5)$$

Hence,  $D(H_{\text{rad}}^{1/2}) \subset D(\Phi_S(F))$  and

$$\|\Phi_S(F)\psi\| \leq \sqrt{2} \left\| \frac{F}{\sqrt{\omega}} \right\| \|H_{\text{rad}}^{1/2}\psi\| + \frac{1}{\sqrt{2}} \|F\| \|\psi\|. \quad (2.6)$$

By this estimate and (1.11), we have for all  $\psi \in D(H_{\text{rad}}^{1/2})$

$$\|A_j^{\mathbf{v}}(\mathbf{x})\psi\|_{\mathcal{F}_{\text{rad}}} \leq \sqrt{2} \left\| \frac{v_j(\mathbf{x})}{\sqrt{\omega}} \right\| \|H_{\text{rad}}^{1/2}\psi\|_{\mathcal{F}_{\text{rad}}} + \frac{1}{\sqrt{2}} \|v_j(\mathbf{x})\| \|\psi\|_{\mathcal{F}_{\text{rad}}}. \quad (2.7)$$

Now, let  $\Psi \in D(H_{\text{rad}}^{1/2}) \subset \mathcal{F}$ . Then, by (2.7) and the elementary inequality

$$(a+b)^2 \leq (1+\varepsilon)a^2 + \left(1 + \frac{1}{\varepsilon}\right)b^2, \quad a, b \geq 0, \quad \varepsilon > 0, \quad (2.8)$$

we have

$$\begin{aligned}
& \|A_j^v(\mathbf{x})\Psi(\mathbf{x})\|_{\mathcal{F}_{\text{rad}}}^2 \\
& \leq 2(1+\varepsilon) \left\| \frac{v_j(\mathbf{x})}{\sqrt{\omega}} \right\|^2 \|H_{\text{rad}}^{1/2}\Psi(\mathbf{x})\|_{\mathcal{F}_{\text{rad}}}^2 + \frac{1}{2} \left(1 + \frac{1}{\varepsilon}\right) \|v_j(\mathbf{x})\|^2 \|\Psi(\mathbf{x})\|_{\mathcal{F}_{\text{rad}}}^2 \\
& \leq 2(1+\varepsilon) \left\| \frac{v_j}{\sqrt{\omega}} \right\|_{\infty}^2 \|H_{\text{rad}}^{1/2}\Psi(\mathbf{x})\|_{\mathcal{F}_{\text{rad}}}^2 + \frac{1}{2} \left(1 + \frac{1}{\varepsilon}\right) \|v_j\|_{\infty}^2 \|\Psi(\mathbf{x})\|_{\mathcal{F}_{\text{rad}}}^2.
\end{aligned}$$

This implies that  $\Psi \in D(A_j^v)$  and

$$\|A_j^v\Psi\|_{\mathcal{F}} \leq \sqrt{2}\sqrt{1+\varepsilon} \left\| \frac{v_j}{\sqrt{\omega}} \right\|_{\infty} \|H_{\text{rad}}^{1/2}\Psi\|_{\mathcal{F}} + \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\varepsilon}} \|v_j\|_{\infty} \|\Psi\|_{\mathcal{F}}. \quad (2.9)$$

Thus  $\Psi \in \cap_{j=1}^3 D(A_j^v) = D(H_I(\mathbf{v}))$ . By using the fact  $\|\alpha_j\| = 1$  and (2.9), we obtain (2.3).  $\blacksquare$

For a linear operator  $T$  on a Hilbert space  $\mathcal{X}$ , its numerical range is defined by

$$\Theta(T) := \{(u, Tu)_{\mathcal{X}} | u \in D(T), \|u\|_{\mathcal{X}} = 1\} \quad (2.10)$$

[17, Chapter V, §3-2].

**Proposition 2.2** *Assume (H.1). Then  $H(V, \mathbf{v})$  is a symmetric operator with*

$$D(H(V, \mathbf{v})) = D(H_D) \cap D(V) \cap D(H_{\text{rad}}). \quad (2.11)$$

Moreover

$$\Theta(H_D(V)) \subset \Theta(H(V, \mathbf{v})). \quad (2.12)$$

*Proof.* By Lemma 2.1, (2.11) holds. By the assumption that each component  $V_{ab}$  of  $V$  is in  $L^2(\mathbf{R}^3)_{\text{loc}}$ , we have

$$\oplus^4 C_0^\infty(\mathbf{R}^3) \subset D(V) \quad (2.13)$$

Hence

$$[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} D(H_{\text{rad}}) \subset D(H(V, \mathbf{v})), \quad (2.14)$$

where  $\otimes_{\text{alg}}$  denotes algebraic tensor product. This implies that  $D(H(V, \mathbf{v}))$  is dense. The Hermiticity of  $H(V, \mathbf{v})$  is obvious. Hence  $H(V, \mathbf{v})$  is symmetric.

Let  $\lambda \in \Theta(H_D(V))$ . Then  $\lambda = (f, H_D(V)f)$  for some  $f \in D(H_D(V))$  with  $\|f\| = 1$ . Let

$$\Omega_0 := \{1, 0, 0, 0, \dots\} \quad (2.15)$$

be the Fock vacuum of  $\mathcal{F}_{\text{rad}}$  and  $\Psi_f := f\Omega_0$ . Then,  $\Psi_f \in D(H(V, \mathbf{v}))$ ,  $\|\Psi_f\| = 1$  and

$$(\Psi_f, H(V, \mathbf{v})\Psi_f) = (f, H_D(V)f) = \lambda.$$

Hence  $\lambda \in \Theta(H(V, \mathbf{v}))$ . Thus (2.12) follows.  $\blacksquare$

**Remark 2.1** It is well known that, for a wide class of  $V$ ,  $H_D(V)$  is not semibounded (i.e., neither bounded from below nor above)(cf. [24, Chapter 4, §4.3]). Hence, by (2.12), for such a function  $V$ ,  $H(V, \mathbf{v})$  is not semibounded. In particular, in the case of the Coulomb potential

$$V(\mathbf{x}) = V_{\text{Cl}}(\mathbf{x}) := -\frac{Z}{|\mathbf{x}|} \quad (Z > 0 : \text{a constant}),$$

which is a physically important case, one can show that  $H(V_{\text{Cl}}, \mathbf{v})$  is not semibounded.

**Remark 2.2** The discussions above can be extended to a more general case where  $\mathcal{H}_{\text{ph}}$  and  $\omega$  are replaced by the abstract Hilbert space  $\mathcal{H}$  and a nonnegative self-adjoint operator  $A$  on  $\mathcal{H}$  with  $\ker A = \{0\}$ , respectively.

## 2.2 Self-adjoint extension

We define an operator  $P$  on  $\mathcal{H}_D$ , called the parity transformation (or the space-inversion), by

$$(Pf)(\mathbf{x}) := f(-\mathbf{x}), \quad f \in \mathcal{H}_D, \quad (2.16)$$

which is unitary.

By Pauli's lemma (e.g., cf. [24, p.14, p.74]), there exists a unitary operator  $U_C$  on  $\mathcal{H}_D$  such that

$$U_C^{-1} \alpha_j U_C = \bar{\alpha}_j, \quad j = 1, 2, 3, \quad (2.17)$$

$$U_C^{-1} \beta U_C = -\bar{\beta}, \quad (2.18)$$

where, for a matrix  $M$ ,  $\bar{M}$  denotes its complex conjugate. Let  $C_D$  be the complex conjugation on  $\mathcal{H}_D$  and

$$T_{\text{CP}} := U_C P C_D \quad (2.19)$$

on  $\mathcal{H}_D$ . Then  $T_{\text{CP}}$  is antiunitary. This operator is called the CP-transformation ("CP" means "charge conjugation and parity"), while  $U_C C_D$  is called the *charge conjugation* [24, p.14]. Using (2.17), one can easily prove the following lemma.

**Lemma 2.3**

$$\alpha \cdot (-i\nabla) T_{\text{CP}} \subset T_{\text{CP}} \alpha \cdot (-i\nabla). \quad (2.20)$$

Let  $C_{\text{ph}}$  be the complex conjugation on  $\mathcal{H}_{\text{ph}}$ :

$$C_{\text{ph}}(F_1 \oplus F_2) := \bar{F}_1 \oplus \bar{F}_2, \quad F_r \in L^2(\mathbf{R}^3), \quad r = 1, 2, \quad (2.21)$$

and  $j_C$  be the complex conjugation on  $\mathbf{C}$ . Then

$$J_{\text{rad}} := j_C \oplus (\oplus_{n=1}^{\infty} \otimes^n C_{\text{ph}}) \quad (2.22)$$

is a conjugation on  $\mathcal{F}_{\text{rad}}$  [ a mapping  $C$  on a Hilbert space is called a conjugation if it is antilinear and norm-preserving with  $C^2 = I(\text{identity})$  ].

**Lemma 2.4** For all  $F \in \mathcal{H}_{\text{ph}}$ ,

$$H_{\text{rad}} J_{\text{rad}} \subset J_{\text{rad}} H_{\text{rad}}, \quad (2.23)$$

$$\Phi_S(F) J_{\text{rad}} \subset J_{\text{rad}} \Phi_S(C_{\text{ph}} F). \quad (2.24)$$

*Proof.* Relation (2.23) follows from the reality of  $\omega$ . As for (2.24), we first show that  $\Phi_S(F) J_{\text{rad}} = J_{\text{rad}} \Phi_S(C_{\text{ph}} F)$  on the subspace of finite particle vectors

$$\mathcal{F}_{\text{rad},0} := \{\psi = \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_{\text{rad}} \mid \psi^{(n)} = 0 \text{ for all but finitely many } n\text{'s}\}. \quad (2.25)$$

Then, by a limiting argument using the fact that  $\mathcal{F}_{\text{rad},0}$  is a core of  $\Phi_S(F)$  [20, Theorem X.41], we obtain (2.24). ■

**Theorem 2.5** Assume (H.1). Suppose that

$$U_C^{-1} V(\mathbf{x}) U_C = \overline{V(-\mathbf{x})}, \quad (2.26)$$

$$v_j(\mathbf{x}) = \overline{v_j(-\mathbf{x})}, \quad j = 1, 2, 3, \quad (2.27)$$

for a.e.  $\mathbf{x}$ . Then  $H(V, \mathbf{v})$  has a self-adjoint extension.

*Proof.* The operator

$$J := T_{\text{CP}} \otimes J_{\text{rad}} \quad (2.28)$$

acting on  $\mathcal{F}$  is a conjugation. By Lemmas 2.3, 2.4 and the present assumption, we have

$$(H(V, \mathbf{v}) - m_D \beta) J \subset J (H(V, \mathbf{v}) - m_D \beta). \quad (2.29)$$

Hence, by von Neumann's theorem [20, Theorem X.3],  $H(V, \mathbf{v}) - m_D \beta$  has a self-adjoint extension  $H'(V, \mathbf{v})$ . Since  $m_D \beta$  is bounded,  $H'(V, \mathbf{v}) + m_D \beta$  is self-adjoint by the Kato-Rellich theorem [20, Theorem X.12], being a self-adjoint extension of  $H(V, \mathbf{v})$ . ■

**Remark 2.3** The Coulomb potential  $V = V_{\text{Cl}}$  (Remark 2.1) and the standard cutoff functions  $v_j = \varrho_j, j = 1, 2, 3$ , with  $\varrho$  real (Remark 1.2) satisfy (2.26) and (2.27) respectively.

**Remark 2.4** As is seen from the proof of Theorem 2.5, the self-adjoint extension of  $H(V, \mathbf{v})$  given in Theorem 2.5 is associated with the CP invariance (2.29) of  $H(V, \mathbf{v}) - m_D \beta$ . In this sense the self-adjoint extension of  $H(V, \mathbf{v})$  is physically natural.

## 2.3 Essential self-adjointness

We define

$$\Delta := \sum_{j=1}^3 D_j^2 \quad (2.30)$$

the Laplacian acting in  $\mathcal{H}_D$ .

For a subspace  $\mathcal{D}$  of  $\mathcal{H}_{\text{ph}}$ , we define  $\mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D}) \subset \mathcal{F}_{\text{rad}}$  to be the subspace algebraically spanned by  $\Omega_0$  and all the vectors of the form

$$a(F_1)^* \cdots a(F_n)^* \Omega_0, \quad n \geq 1, \quad F_j \in \mathcal{D}, \quad j = 1, \dots, n.$$

If  $\mathcal{D}$  is dense in  $\mathcal{H}_{\text{ph}}$ , then  $\mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$  is dense in  $\mathcal{F}_{\text{rad}}$ .

**Theorem 2.6** *Suppose in addition (H.1) that, for  $j = 1, 2, 3$ ,  $\omega v_j$  is strongly continuous with  $\omega v_j \in L^\infty(\mathbf{R}^3; \mathcal{H}_{\text{ph}})$  and  $v_j$  is strongly continuously differentiable with  $\partial_\ell v_j, \partial_\ell v_j / \sqrt{\omega} \in L^\infty(\mathbf{R}^3; \mathcal{H}_{\text{ph}})$ ,  $\ell = 1, 2, 3$  ( $\partial_\ell := \partial / \partial x_\ell$ ). Assume the following (V.1) and (V.2):*

(V.1)  $V$  is  $-\Delta$ -bounded.

(V.2) For each  $j = 1, 2, 3$  and  $a, b = 1, \dots, 4$ , the distribution  $D_j V_{ab}$  is in  $L^2(\mathbf{R}^3)_{\text{loc}}$  and there exists a constant  $c > 0$  such that, for all  $f \in \oplus^4 C_0^\infty(\mathbf{R}^3)$ ,

$$\|(D_j V)f\| \leq c \|(-\Delta + 1)^{1/2} f\|, \quad j = 1, 2, 3.$$

Let  $\mathcal{D} \subset \mathcal{H}_{\text{ph}}$  be a core of the self-adjoint operator  $\omega$ . Then  $H(V, \mathbf{v})$  is essentially self-adjoint on  $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$  and its closure is essentially self-adjoint on every core of  $-\Delta + H_{\text{rad}}$ .

*Proof.* The method of proof is to apply Nelson's commutator theorem [20, Theorem X.37]. We take as the comparison operator the self-adjoint operator

$$K_0 := -\Delta + H_{\text{rad}} + 1 \geq 1.$$

We recall a useful identity: Let  $T_j$ ,  $j = 1, 2, 3$ , be linear operators on a Hilbert space  $\mathcal{X}$  such that  $T_j T_\ell = T_\ell T_j$  on  $D(T_j T_\ell) \cap D(T_\ell T_j)$ ,  $j, \ell = 1, 2, 3$ , and set  $\mathbf{T} := (T_1, T_2, T_3)$ . Then

$$(\boldsymbol{\alpha} \cdot \mathbf{T})^2 = \mathbf{T}^2 \quad \text{on } \cap_{j,\ell=1}^3 [D(T_j T_\ell) \cap D(T_\ell T_j)], \quad (2.31)$$

where  $\alpha_j T_j$  ( $j = 1, 2, 3$ ) is considered as an operator on  $\oplus^4 \mathcal{X}$ . This follows from the anticommutation relations of  $\alpha_j$ 's (1.3).

Let  $f \in D(-\Delta)$ . Then, by (2.31),

$$\|\boldsymbol{\alpha} \cdot (-i\nabla)f\| = \|(-\Delta)^{1/2} f\| \leq \|f\|^{1/2} - \Delta f\|^{1/2}.$$

By the elementary inequality

$$xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2 \quad (2.32)$$

holding for all  $x, y \geq 0$  and  $\varepsilon > 0$ , we obtain

$$\|\alpha \cdot (-i\nabla)f\| \leq \varepsilon \|-\Delta f\| + \frac{1}{4\varepsilon} \|f\|.$$

By (V.1), there exists a constant  $c_1 > 0$  such that

$$\|Vf\| \leq c_1 \|(-\Delta + 1)f\|.$$

Hence

$$\|H_D(V)f\| \leq c_2 \|(-\Delta + 1)f\|, \quad (2.33)$$

where  $c_2 > 0$  is a constant.

By (2.32), we have for all  $\Psi \in D(H_{\text{rad}})$  and  $\varepsilon' > 0$

$$\begin{aligned} \sqrt{2(1+\varepsilon)}|q| \left\| \frac{\mathbf{v}}{\sqrt{\omega}} \right\|_{\infty} \|H_{\text{rad}}^{1/2}\Psi\| &\leq \sqrt{2(1+\varepsilon)}|q| \left\| \frac{\mathbf{v}}{\sqrt{\omega}} \right\|_{\infty} \|\Psi\|^{1/2} \|H_{\text{rad}}\Psi\|^{1/2} \\ &\leq \varepsilon' \|H_{\text{rad}}\Psi\| + \frac{(1+\varepsilon)q^2}{2\varepsilon'} \left\| \frac{\mathbf{v}}{\sqrt{\omega}} \right\|_{\infty}^2 \|\Psi\|. \end{aligned}$$

Hence, by Lemma 2.1, we obtain

$$\|H_I(\mathbf{v})\Psi\| \leq \varepsilon' \|H_{\text{rad}}\Psi\| + b_{\varepsilon'} \|\Psi\| \quad (2.34)$$

with

$$b_{\varepsilon'} := \frac{(1+\varepsilon)q^2}{2\varepsilon'} \left\| \frac{\mathbf{v}}{\sqrt{\omega}} \right\|_{\infty}^2 + \frac{|q|}{\sqrt{2}} \sqrt{1 + \frac{1}{\varepsilon}} \|\mathbf{v}\|_{\infty}. \quad (2.35)$$

It is easy to see that, for all  $\Phi \in D(K_0)$

$$\|(-\Delta + 1)\Phi\|^2 + \|H_{\text{rad}}\Phi\|^2 \leq \|K_0\Phi\|^2.$$

It follows from this estimate, (2.33) and (2.34) that

$$\|H(V, \mathbf{v})\Phi\| \leq c_3 \|K_0\Phi\|, \quad \Phi \in D(K_0), \quad (2.36)$$

where  $c_3 > 0$  is a constant.

Let  $\Psi \in [\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$ . Then

$$(V\Psi, K_0\Psi) - (K_0\Psi, V\Psi) = 2i \sum_{j=1}^3 \Im((D_j V)\Psi, D_j \Psi).$$

Hence

$$\begin{aligned}
|(V\Psi, K_0\Psi) - (K_0\Psi, V\Psi)| &\leq 2 \sum_{j=1}^3 \|(D_j V)\Psi\| \|D_j\Psi\| \\
&\leq 2 \left( \sum_{j=1}^3 \|(D_j V)\Psi\|^2 \right)^{1/2} \left( \sum_{j=1}^3 \|D_j\Psi\|^2 \right)^{1/2} \\
&\leq 2\sqrt{3}c \|(-\Delta + 1)^{1/2}\Psi\| \|(-\Delta)^{1/2}\Psi\|.
\end{aligned}$$

Since

$$\|(-\Delta + 1)^{1/2}\Phi\|^2 + \|H_{\text{rad}}^{1/2}\Phi\|^2 = \|K_0^{1/2}\Phi\|^2, \quad \Phi \in D(K_0^{1/2}), \quad (2.37)$$

we obtain

$$|(V\Psi, K_0\Psi) - (K_0\Psi, V\Psi)| \leq 2\sqrt{3}c \|K_0^{1/2}\Psi\|^2.$$

Since  $[H_D(0), K_0]\Psi = 0$ , it follows that

$$|(H_D(V)\Psi, K_0\Psi) - (K_0\Psi, H_D(V)\Psi)| \leq 2\sqrt{3}c \|K_0^{1/2}\Psi\|^2. \quad (2.38)$$

We have

$$\begin{aligned}
(H_I(\mathbf{v})\Psi, \Delta\Psi) - (\Delta\Psi, H_I(\mathbf{v})\Psi) &= \sum_{\ell=1}^3 \{(D_\ell\Psi, D_\ell H_I(\mathbf{v})\Psi) - (D_\ell H_I(\mathbf{v})\Psi, D_\ell\Psi)\} \\
&= -2iq \sum_{j,\ell=1}^3 \Im(D_\ell\Psi, \alpha_j[\Phi_S](\partial_\ell v_j)\Psi).
\end{aligned}$$

Hence

$$\begin{aligned}
&|(H_I(\mathbf{v})\Psi, -\Delta\Psi) - (-\Delta\Psi, H_I(\mathbf{v})\Psi)| \\
&\leq 2|q| \sum_{j,\ell=1}^3 \|D_j\Psi\| \|[\Phi_S](\partial_\ell v_j)\Psi\|
\end{aligned}$$

which, combined with Lemma 2.1 and (2.37), implies that there exists a constant  $c_4 > 0$  such that

$$|(H_I(\mathbf{v})\Psi, -\Delta\Psi) - (-\Delta\Psi, H_I(\mathbf{v})\Psi)| \leq c_4 \|K_0^{1/2}\Psi\|^2.$$

In general we have for all  $F \in D(\omega)$

$$[a(F), H_{\text{rad}}] = a(\omega F), \quad [a(F)^*, H_{\text{rad}}] = -a(\omega F)^* \quad (2.39)$$

on  $\mathcal{F}_{\text{rad}}^{\text{fin}}(D(\omega))$ , so that

$$[\Phi_S(F), H_{\text{rad}}] = i\Phi_S(i\omega F) \quad (2.40)$$

on  $\mathcal{F}_{\text{rad}}^{\text{fin}}(D(\omega))$ . Using this relation, we have

$$[H_I(\mathbf{v}), H_{\text{rad}}]\Psi = -iq \sum_{j=1}^3 \alpha_j[\Phi_S](i\omega v_j)\Psi.$$

Hence, Lemma 2.1 and (2.37), there exists a constant  $c_5 > 0$  such that

$$|(H_I(\mathbf{v})\Psi, H_{\text{rad}}\Psi) - (H_{\text{rad}}\Psi, H_I(\mathbf{v})\Psi)| \leq c_5 \|K_0^{1/2}\Psi\|^2.$$

Since  $[H_{\text{rad}}, K_0]\Psi = 0$ , it follows that

$$|([H_{\text{rad}} + H_I(\mathbf{v})]\Psi, K_0\Psi) - (K_0\Psi, [H_{\text{rad}} + H_I(\mathbf{v})]\Psi)| \leq (c_4 + c_5) \|K_0^{1/2}\Psi\|^2. \quad (2.41)$$

The subspace  $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$  is a core of  $K_0$ . Thus, by (2.36), (2.38) and (2.41), we can apply Nelson's commutator theorem [20, Theorem X.37] to obtain the desired result.  $\blacksquare$

As a corollary to Theorem 2.6, we have the following.

**Corollary 2.7** *Let  $V$  be bounded. Suppose that  $\mathbf{v}$  and  $\omega$  obey the same assumption as in Theorem 2.6. Then  $H(V, \mathbf{v})$  is essentially self-adjoint on  $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$ .*

*Proof.* We write  $H(V, \mathbf{v}) = H(0, \mathbf{v}) + V$ . By Theorem 2.6,  $H(0, \mathbf{v})$  is essentially self-adjoint on  $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$ . Since  $V$  is bounded, we can apply the Kato-Rellich theorem [20, Theorem X.12] to obtain the desired result.  $\blacksquare$

**Remark 2.5** Consider the case where  $V$  is of the form

$$V(\mathbf{x}) = V_s(\mathbf{x}) := \frac{C}{|\mathbf{x}|^s},$$

where  $C \in \mathbf{R} \setminus \{0\}$  and  $s > 0$  are constants. We have

$$D_j V_s(\mathbf{x}) = -\frac{sC x_j}{|\mathbf{x}|^{s+2}}, \quad j = 1, 2, 3, \quad \mathbf{x} \in \mathbf{R}^3 \setminus \{0\}.$$

Hence,  $D_j V_s \in L^2(\mathbf{R}^3)_{\text{loc}}$  if and only if  $s < 1/2$ . This means that Theorem 2.6 excludes the Coulomb case  $s = 1$ . In this sense Theorem 2.6 is not so strong.

Let  $0 < s < 1/2$ . Then it is easy to see that  $V_s, D_j V_s \in L^2(\mathbf{R}^3) + L^\infty(\mathbf{R}^3)$  and hence that  $V_s$  and  $D_j V_s$  are infinitesimally small with respect to  $-\Delta$  (cf. proof of [20, Theorem X.15]). In particular,  $V = V_s$  satisfies (V.1) and (V.2). Thus, under the assumption for  $\mathbf{v}$  and  $\omega$  stated in Theorem 2.6,  $H(V_s, \mathbf{v})$  with  $0 < s < 1/2$  is essentially self-adjoint on  $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(\mathcal{D})$ .

**Remark 2.6** It is not so difficult to show that the unperturbed Hamiltonian

$$T_0 := \boldsymbol{\alpha} \cdot (-i\nabla) + H_{\text{rad}},$$

which is essentially self-adjoint, is *not* closed on  $D(T_0) = D(\boldsymbol{\alpha} \cdot (-i\nabla)) \cap D(H_{\text{rad}})$ . Hence there are *no* constants  $c$  and  $d$  such that, for all  $\Psi \in D(T_0)$ ,  $\|\boldsymbol{\alpha} \cdot (-i\nabla)\Psi\| + \|H_{\text{rad}}\Psi\| \leq c\|T_0\Psi\| + d\|\Psi\|$ . This causes a difficulty in establishing essential self-adjointness by the method using relative boundedness with respect to  $\boldsymbol{\alpha} \cdot (-i\nabla)$ ,  $H_{\text{rad}}$  and  $T_0$ . For example, the Coulomb potential  $V_{\text{Cl}}$  (Remark 2.1) is  $\boldsymbol{\alpha} \cdot (-i\nabla)$ -bounded (cf. proof of [24, Theorem 4.2]), but it does not mean that  $V_{\text{Cl}}$  is  $T_0$ -bounded. Indeed, one can show that  $V_{\text{Cl}}$  is *not*  $T_0$ -bounded.

### 3 Translation Invariance and Direct Integral Decomposition

In this section we consider the total Hamiltonian without the external field  $V$

$$H(\mathbf{v}) := H(0, \mathbf{v}) = H_D + H_{\text{rad}} + H_I(\mathbf{v}). \quad (3.1)$$

This is a Hamiltonian of a *relativistic polaron with spin*. Gross [14] discusses a *spinless* relativistic polaron whose Hamiltonian is a different type from  $H(\mathbf{v})$ . For nonrelativistic polarons, see [2, 7, 9, 11, 12, 13, 22, 23] and references therein.

We first discuss the translation invariance and then a direct integral decomposition of  $H(\mathbf{v})$ . For this purpose, we introduce the total momentum operator of the quantum system under consideration. As is well known, the momentum operator  $\mathbf{P}^{\text{rad}} := (P_1^{\text{rad}}, P_2^{\text{rad}}, P_3^{\text{rad}})$  of the quantized radiation field is defined by

$$P_j^{\text{rad}} := d\Gamma(k_j), \quad (3.2)$$

the second quantization of the multiplication operator  $k_j$  on  $\mathcal{H}_{\text{ph}}$ , while the momentum operator of the Dirac particle is  $-i\nabla$ . Hence the total momentum operator  $\mathbf{P} := (P_1, P_2, P_3)$  is given by

$$P_j := -iD_j + P_j^{\text{rad}} \quad (3.3)$$

on  $\mathcal{F}$  ( $j = 1, 2, 3$ ).

The unitary group  $\{e^{itP_j}\}_{t \in \mathbf{R}}$  is interpreted as a unitary representation on  $\mathcal{F}$  of the translation group in the  $j$ -th direction in  $\mathbf{R}^3$ . Each  $P_j$  is self-adjoint and its spectrum is purely absolutely continuous with

$$\sigma(P_j) = \mathbf{R}. \quad (3.4)$$

The translation invariance of linear operators on  $\mathcal{F}$  is formulated in terms of the total momentum  $\mathbf{P}$  as follows.

**Definition 3.1** We say that a linear operator  $T$  on  $\mathcal{F}$  is *strongly translation invariant* if  $e^{itP_j}T \subset Te^{itP_j}$  for all  $t \in \mathbf{R}$  and  $j = 1, 2, 3$ .

**Remark 3.1** This notion of translation invariance is stronger than the conventional one:  $TP_j = P_jT$  on  $D(TP_j) \cap D(P_jT)$  for  $j = 1, 2, 3$ . The strong translation invariance implies the conventional translation invariance, but the converse is not true. If  $T$  is bounded, then the two notions of translation invariance are equivalent.

As we shall see below, it turns out that, for  $H(\mathbf{v})$  to be strongly translation invariant, the cutoff function  $\mathbf{v} = (v_1, v_2, v_3)$  must be of the form

$$v_j(\mathbf{x})(\mathbf{k}) = v_j^{\mathbf{g}}(\mathbf{x})(\mathbf{k}) := g_j(\mathbf{k})e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad j = 1, 2, 3, \quad \mathbf{k}, \mathbf{x} \in \mathbf{R}^3, \quad (3.5)$$

$$\mathbf{g} := (g_1, g_2, g_3) \quad (3.6)$$

with

$$g_j \in \mathcal{H}_{\text{ph}}, \quad \frac{g_j}{\sqrt{\omega}} \in \mathcal{H}_{\text{ph}}, \quad j = 1, 2, 3. \quad (3.7)$$

Henceforth we consider only this case and simply write  $H = H(\mathbf{v}^g)$ :

$$H := H_D + H_{\text{rad}} - q \sum_{j=1}^3 \alpha_j [\Phi_S](v_j^g). \quad (3.8)$$

We recall a notion of commutativity of self-adjoint operators. Two self-adjoint operators  $A$  and  $B$  on a Hilbert space is said to be *strongly commuting* if their spectral measures  $E_A$  and  $E_B$  commute. A set  $\{A_j\}_{j=1}^n$  of self-adjoint operators on a Hilbert space is said to be strongly commuting if, for all  $j, \ell$  ( $j \neq \ell$ ),  $A_j$  and  $A_\ell$  are strongly commuting. In this case we have an  $n$  variable functional calculus with respect to the  $n$ -dimensional spectral measure  $E_{A_1} \otimes \cdots \otimes E_{A_n}$  such that, for all Borel sets  $S_j$  ( $j = 1, \dots, n$ ) of  $\mathbf{R}$ ,  $(E_{A_1} \otimes \cdots \otimes E_{A_n})(S_1 \times \cdots \times S_n) = \prod_{j=1}^n E_{A_j}(S_j)$ . For criterion of strong commutativity, see [19, Theorem VIII.13].

We can prove the following proposition.

**Proposition 3.2** *The Hamiltonian  $H$  is strongly translation invariant.*

*Proof.* The self-adjoint operators  $-iD_j, P_\ell^{\text{rad}}$  ( $j, \ell = 1, 2, 3$ ) and  $H_{\text{rad}}$  are mutually strongly commuting. In particular, we have for all  $t \in \mathbf{R}$

$$e^{itP_j} = e^{it(-iD_j)} e^{itP_j^{\text{rad}}} = e^{itP_j^{\text{rad}}} e^{it(-iD_j)} \quad (3.9)$$

and

$$e^{itP_j}(H_D + H_{\text{rad}}) \subset (H_D + H_{\text{rad}})e^{itP_j}. \quad (3.10)$$

For all  $\Psi \in D(H) = D(H_D) \cap D(H_{\text{rad}})$  and  $\ell = 1, 2, 3$ , we have

$$\begin{aligned} (e^{itP_j}[\Phi_S](v_\ell^g)\Psi)(\mathbf{x}) &= e^{it(-iD_j)} e^{itP_j^{\text{rad}}} \Phi_S(v_\ell^g(\mathbf{x}))\Psi(\mathbf{x}) \\ &= e^{it(-iD_j)} \Phi_S(e^{itk_j} v_\ell^g(\mathbf{x})) e^{itP_j^{\text{rad}}} \Psi(\mathbf{x}) \\ &= \Phi_S(e^{itk_j} v_\ell^g(\mathbf{x} + te_j))(e^{itP_j}\Psi)(\mathbf{x}), \end{aligned}$$

where

$$\mathbf{e}_1 := (1, 0, 0), \quad \mathbf{e}_2 := (0, 1, 0), \quad \mathbf{e}_3 := (0, 0, 1).$$

Note that

$$e^{itk_j} v_\ell^g(\mathbf{x} + te_j)(\mathbf{k}) = v_j^g(\mathbf{x})(\mathbf{k}).$$

Hence

$$e^{itP_j}[\Phi_S](v_\ell^g)\Psi = [\Phi_S](v_\ell^g)e^{itP_j}\Psi. \quad (3.11)$$

It follows from (3.10) and (3.11) that

$$e^{itP_j} H \subset H e^{itP_j}.$$

Thus the desired result is proven. ■

Proposition 3.2 suggests that  $H$  may have a direct integral decomposition in a spectral representation of the total momentum  $\mathbf{P}$ . To see this aspect, we first define a unitary transformation which brings  $\mathbf{P}$  to a multiplication operator. For all  $\mathbf{x} \in \mathbf{R}^3$ , the operator

$$Q(\mathbf{x}) := \sum_{j=1}^3 x_j P_j^{\text{rad}} \quad (3.12)$$

acting in  $\mathcal{F}$  is self-adjoint. Since the correspondence:  $\mathbf{x} \rightarrow e^{iQ(\mathbf{x})}$  is strongly continuous, we can define a decomposable operator

$$W := \int_{\mathbf{R}^3}^{\oplus} e^{iQ(\mathbf{x})} d\mathbf{x} \quad (3.13)$$

on  $\mathcal{F} = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\text{rad}} d\mathbf{x}$ . It follows that  $W$  is unitary.

The Fourier transform on  $\mathcal{H}_D = \oplus^4 L^2(\mathbf{R}^3)$  can be naturally extended to a unitary operator on  $\mathcal{F}$  by

$$(U_F \Psi)(\mathbf{p}) := \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbf{R}^3} e^{-i\mathbf{p} \cdot \mathbf{x}} \Psi(\mathbf{x}) d\mathbf{x}, \quad \text{a.e. } \mathbf{p} \in \mathbf{R}^3, \Psi \in \mathcal{F}. \quad (3.14)$$

(For a.e.  $\mathbf{x}$ ,  $\Psi(\mathbf{x}) \in \oplus^4 \mathcal{F}_{\text{rad}}$ .)

We define a unitary operator on  $\mathcal{F}$  by

$$U := U_F W. \quad (3.15)$$

Then we have a direct integral decomposition

$$U\mathcal{F} = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\text{rad}} d\mathbf{p}. \quad (3.16)$$

In quite the same way as in [9, §5], we can show that, for  $j = 1, 2, 3$ ,

$$U P_j^{\text{rad}} U^{-1} = P_j^{\text{rad}}, \quad (3.17)$$

$$U(-iD_j)U^{-1} = \overline{p_j - P_j^{\text{rad}}}. \quad (3.18)$$

Hence

$$U P_j U^{-1} = \int_{\mathbf{R}^3}^{\oplus} p_j d\mathbf{p}. \quad (3.19)$$

Thus the Hilbert space  $U\mathcal{F}$  carries a spectral representation of  $\mathbf{P}$  and the index parameter  $\mathbf{p}$  in the decomposition (3.16) physically means an observed value of the total momentum  $\mathbf{P}$ .

It is easy to see that

$$U H_{\text{rad}} U^{-1} = H_{\text{rad}}. \quad (3.20)$$

Moreover we have

$$U[\Phi_S](v_j^g)U^{-1} = \Phi_S(g_j), \quad j = 1, 2, 3. \quad (3.21)$$

Thus, introducing operators

$$H_{I,0}(g) := -q \sum_{j=1}^3 \alpha_j \Phi_S(g_j), \quad (3.22)$$

and

$$H(p) := \alpha \cdot (p - P^{\text{rad}}) + m_D \beta + H_{\text{rad}} + H_{I,0}(g), \quad p \in \mathbb{R}^3, \quad (3.23)$$

we have

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(p) dp. \quad (3.24)$$

We introduce a subspace of  $\mathcal{F}_{\text{rad}}$ :

$$\mathcal{F}_{\text{rad},0}^{\infty} := \mathcal{F}_{\text{rad}}^{\text{fin}} (C_0^{\infty}(\mathbb{R}^3) \oplus C_0^{\infty}(\mathbb{R}^3)). \quad (3.25)$$

**Theorem 3.3** *Assume in addition to (3.7) that  $|k|g_j, |k|g_j/\sqrt{\omega}, \omega g_j \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, 2, 3$ . Suppose that  $\omega \in L^2(\mathbb{R}^3)_{\text{loc}}$ . Then, for all  $p \in \mathbb{R}^3$ ,  $H(p)$  is essentially self-adjoint on  $\oplus^4 \mathcal{F}_{\text{rad},0}^{\infty}$ .*

*Proof.* Let  $p \in \mathbb{R}^3$  be fixed and  $L := \sum_{j=1}^3 (p_j - P_j^{\text{rad}})^2 + H_{\text{rad}}$ . Then  $L$  is self-adjoint, nonnegative and reduced by each closed subspace  $\oplus^4 (\otimes_s^n \mathcal{H}_{\text{ph}})$  with its reduced part  $L_n$  being the multiplication self-adjoint operator by the function  $(p - \sum_{j=1}^n k_j)^2 + \sum_{j=1}^n \omega(k_j)$ . By the present assumption, this function is in  $L^2(\mathbb{R}^{3n})_{\text{loc}}$ . Hence  $L_n$  is essentially self-adjoint on  $\oplus^4 S_n (\otimes_{\text{alg}}^n [C_0^{\infty}(\mathbb{R}^3) \oplus C_0^{\infty}(\mathbb{R}^3)])$ , where  $S_n$  denotes the symmetrizer on  $\otimes^n \mathcal{H}_{\text{ph}}$ . It follows that  $L$  is essentially self-adjoint on  $\oplus^4 \mathcal{F}_{\text{rad},0}^{\infty}$ . Let  $\psi \in \oplus^4 \mathcal{F}_{\text{rad},0}^{\infty}$ . Then

$$\begin{aligned} \|\alpha \cdot (p - P^{\text{rad}}) \psi\|^2 &= \left( \psi, (p - P^{\text{rad}})^2 \psi \right) \leq \|\psi\| \left\| (p - P^{\text{rad}})^2 \psi \right\| \\ &\leq \varepsilon \left\| (p - P^{\text{rad}})^2 \psi \right\|^2 + \frac{1}{4\varepsilon} \|\psi\|^2 \\ &\leq \varepsilon \|L\psi\|^2 + \frac{1}{4\varepsilon} \|\psi\|^2, \end{aligned}$$

where  $\varepsilon > 0$  is an arbitrary constant. We have

$$\|H_{I,0}(g)\psi\| \leq \sqrt{2}|q| \left\| \frac{g}{\sqrt{\omega}} \right\| \|H_{\text{rad}}^{1/2} \psi\| + \frac{|q|}{\sqrt{2}} \|g\| \|\psi\|.$$

By these estimates, we obtain

$$\|H(p)\psi\| \leq c_1 \|(L + 1)\psi\|,$$

where  $c_1 > 0$  is a constant. In the same way as in the proof of Theorem 2.6, we can show that

$$|(H(\mathbf{p})\psi, L\psi) - (L\psi, H(\mathbf{p})\psi)| \leq c_2 \|(L+1)^{1/2}\psi\|^2,$$

where  $c_2 > 0$  is a constant. Thus we can apply Nelson's commutator theorem [20, Theorem X.37] to obtain the desired result. ■

For a linear operator  $T$  on a Hilbert space, we denote its resolvent set by  $\varrho(T)$ . For each  $z \in \varrho(\overline{H(\mathbf{p})})$ , we define

$$R_z(\mathbf{p}) := (\overline{H(\mathbf{p})} - z)^{-1}. \quad (3.26)$$

**Lemma 3.4** *Let  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then, under the same assumption as in Theorem 3.3, the operator-valued function:  $\mathbf{p} \rightarrow R_z(\mathbf{p})$  is strongly continuous.*

*Proof.* Let  $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^3$ . Then, for all  $\psi \in \oplus^4 \mathcal{F}_{\text{rad},0}^\infty$ , we have by (2.31)

$$\begin{aligned} \|H(\mathbf{p}')\psi - H(\mathbf{p})\psi\| &= \|\alpha \cdot (\mathbf{p}' - \mathbf{p})\psi\| \\ &= |\mathbf{p}' - \mathbf{p}| \|\psi\| \rightarrow 0 \quad (\mathbf{p}' \rightarrow \mathbf{p}). \end{aligned}$$

Since  $\oplus^4 \mathcal{F}_{\text{rad},0}^\infty$  is a common core for the family  $\{\overline{H(\mathbf{p})}\}_{\mathbf{p} \in \mathbb{R}^3}$  of self-adjoint operators (Theorem 3.3), it follows from a convergence theorem [19, Theorem VIII.25(a)] that  $R_z(\mathbf{p}') \rightarrow R_z(\mathbf{p})$  strongly as  $\mathbf{p}' \rightarrow \mathbf{p}$ . ■

**Theorem 3.5** *Under the same assumption as in Theorem 3.3,  $H$  is essentially self-adjoint and*

$$U\overline{H}U^{-1} = \int_{\mathbb{R}^3}^{\oplus} \overline{H(\mathbf{p})} d\mathbf{p}. \quad (3.27)$$

*Proof.* The essential self-adjointness of  $H$  follows from the present assumption and Corollary 2.7. By Theorem 3.3, Lemma 3.4 and [21, Theorem XIII.85(a)],  $H' := \int_{\mathbb{R}^3}^{\oplus} \overline{H(\mathbf{p})} d\mathbf{p}$  is self-adjoint. By (3.24), we have  $U\overline{H}U^{-1} \subset H'$ . The essential self-adjointness of  $H$  implies that of  $U\overline{H}U^{-1}$ . Thus (3.27) follows. ■

## 4 Properties of the Ground-State Energy of the Polaron Hamiltonian with a Fixed Total Momentum

For each  $\mathbf{p} \in \mathbb{R}^3$ , the operator  $\overline{H(\mathbf{p})}$  acting in the Hilbert space  $\oplus^4 \mathcal{F}_{\text{rad}}$  describes the polaron Hamiltonian with total momentum  $\mathbf{p}$ . From now on we focus our attention on analyzing this operator. In this section we discuss fundamental properties of the ground-state energy of  $\overline{H(\mathbf{p})}$  defined by

$$E(\mathbf{p}) := \inf \sigma(\overline{H(\mathbf{p})}), \quad (4.1)$$

provided that  $H(\mathbf{p})$  is essentially self-adjoint (cf. Theorem 3.3). At this stage, however,  $\overline{H(\mathbf{p})}$  is not necessarily bounded below. Namely it may happen that  $E(\mathbf{p}) = -\infty$ . We first consider this aspect.

#### 4.1 Unboundedness below of $H(\mathbf{p})$ in the physical case

In this subsection we consider the physical case  $\omega(\mathbf{k}) = |\mathbf{k}|$ ,  $\mathbf{k} \in \mathbf{R}^3$  and prove the following proposition. We write

$$g_j = \langle g_j^{(1)}, g_j^{(2)} \rangle \in \mathcal{H}_{\text{ph}}, \quad j = 1, 2, 3, \quad (4.2)$$

and define

$$\mathbf{g}^{(r)} := (g_1^{(r)}, g_2^{(r)}, g_3^{(r)}), \quad r = 1, 2. \quad (4.3)$$

**Proposition 4.1** *Consider the case  $\omega(\mathbf{k}) = |\mathbf{k}|$ ,  $\mathbf{k} \in \mathbf{R}^3$ . Suppose that there exist positive constants  $R_1$  and  $R_2$  such that at least one of the functions  $|g_j^{(r)}|$ ,  $j = 1, 2, 3; r = 1, 2$ , is positive on  $D_{R_1, R_2} := \{\mathbf{k} \in \mathbf{R}^3 | R_1 \leq |\mathbf{k}| \leq R_2\}$ . Then  $H(\mathbf{p})$  is not bounded below.*

*Proof.* For  $z = (z_1, z_2, z_3, z_4) \in \mathbf{C}^4$  with  $\|z\| = 1$  and  $\psi \in D(H_{\text{rad}}) \subset \mathcal{F}_{\text{rad}}$  with  $\|\psi\| = 1$ , we define

$$\Psi_z := (z_1\psi, z_2\psi, z_3\psi, z_4\psi) \in \oplus^4 \mathcal{F}_{\text{rad}}, \quad (4.4)$$

$$u_j(z) := (z, \alpha_j z)_{\mathbf{C}^4}, \quad j = 1, 2, 3, \quad (4.5)$$

$$\mathbf{u}(z) := (u_1(z), u_2(z), u_3(z)) \in \mathbf{R}^3, \quad (4.6)$$

$$\xi_z(\mathbf{k}) := |\mathbf{k}| - \mathbf{u}(z) \cdot \mathbf{k}. \quad (4.7)$$

Then  $\|\Psi_z\| = 1$  and we have

$$(\Psi_z, H(\mathbf{p})\Psi_z) = \mathbf{u}(z) \cdot \mathbf{p} + m_D(z, \beta z)_{\mathbf{C}^4} + (\psi, H_z \psi)_{\mathcal{F}_{\text{rad}}}, \quad (4.8)$$

where

$$H_z := d\Gamma(\xi_z) - q \sum_{j=1}^3 u_j(z) \Phi_S(g_j). \quad (4.9)$$

Hence

$$\inf_{\Psi \in D(H(\mathbf{p})), \|\Psi\|=1} (\Psi, H(\mathbf{p})\Psi) \leq \mathbf{u}(z) \cdot \mathbf{p} + m_D(z, \beta z)_{\mathbf{C}^4} + (\psi, H_z \psi)_{\mathcal{F}_{\text{rad}}}. \quad (4.10)$$

We have

$$\mathbf{u}(z) \cdot \mathbf{k} = (z, \alpha \cdot \mathbf{k} z). \quad (4.11)$$

By (2.31), we have

$$(\alpha \cdot \mathbf{k})^2 = \mathbf{k}^2, \quad \mathbf{k} \in \mathbf{R}^3, \quad (4.12)$$

implying

$$\|\alpha \cdot k\| = |k|. \quad (4.13)$$

Hence we have

$$|u(z) \cdot k| \leq |k|. \quad (4.14)$$

Therefore

$$\xi_z(k) \geq 0, \quad k \in \mathbb{R}^3. \quad (4.15)$$

For  $\varepsilon > 0$ , we define

$$H_z^{(\varepsilon)} := d\Gamma(\xi_z + \varepsilon) - q \sum_{j=1}^3 u_j(z) \Phi_S(g_j). \quad (4.16)$$

This is a Hamiltonian of van Hove type model (e.g., [10, pp.17-28]). Hence it follows that  $H_z^{(\varepsilon)}$  is self-adjoint on  $D(d\Gamma(\xi_z + \varepsilon))$  and that the ground-state energy  $E_0(H_z^{(\varepsilon)}) := \inf \sigma(H_z^{(\varepsilon)})$  is given by

$$E_0(H_z^{(\varepsilon)}) = - \sum_{r=1}^2 \frac{q^2}{2} \int_{\mathbb{R}^3} dk \frac{|u(z) \cdot g^{(r)}(k)|^2}{\xi_z(k) + \varepsilon}.$$

Consider the case where  $|g_1^{(1)}| \geq \delta > 0$  on  $D_{R_1, R_2}$ , and let  $z_+ \in \mathbb{C}^4$  be a normalized eigenvector of  $\alpha_1$ :  $\alpha_1 z_+ = z_+$ . Then the anticommutation relations (1.3) and (1.4) imply that

$$u_j(z_+) = \delta_{1j}, \quad j = 1, 2, 3, \quad (z_+, \beta z_+)_{\mathbb{C}^4} = 0. \quad (4.17)$$

Hence

$$\begin{aligned} E_0(H_{z_+}^{(\varepsilon)}) &\leq -\frac{1}{2} \int_{D_{R_1, R_2}} dk \frac{|g_1^{(1)}(k)|^2}{|k| - u(z_+) \cdot k + \varepsilon} \\ &\leq -\frac{\delta^2}{2} \int_{D_{R_1, R_2}} dk \frac{1}{|k| - u(z_+) \cdot k + \varepsilon} \\ &= -\pi \delta^2 \int_{R_1}^{R_2} dk k \log \frac{2k + \varepsilon}{\varepsilon}. \end{aligned}$$

This implies that

$$E_0(H_{z_+}^{(\varepsilon)}) \rightarrow -\infty$$

as  $\varepsilon \rightarrow 0$ . By (4.10) and (4.17), we have

$$\inf_{\Psi \in D(H(\mathbf{p})), \|\Psi\|=1} (\Psi, H(\mathbf{p})\Psi) \leq p_1 + E_0(H_{z_+}^{(\varepsilon)}). \quad (4.18)$$

Thus  $\inf_{\Psi \in D(H(\mathbf{p})), \|\Psi\|=1} (\Psi, H(\mathbf{p})\Psi) = -\infty$ . Similarly we can prove this fact in the case  $|g_j^{(r)}| \geq \delta > 0$  on  $D_{R_1, R_2}$  with  $j, r \neq 1$ . ■

**Remark 4.1** If  $\omega(k) = |k|^a$  with  $0 < a < 1$ , one can easily show that  $H(p)$  is not bounded below.

Proposition 4.1 shows that, to have a physically meaningful polaron theory in the physical case  $\omega(k) = |k|$ , we need a regularization so that the Hamiltonian is deformed to a self-adjoint operator bounded from below. In what follows, we present a regularization method and investigate a regularized theory.

## 4.2 A one-parameter family of deformed polaron Hamiltonians

By (4.13),

$$\sigma(\alpha \cdot k) = \sigma_p(\alpha \cdot k) = \{\pm|k|\}, \quad k \in \mathbb{R}^3, \quad (4.19)$$

where, for a linear operator  $T$ ,  $\sigma_p(T)$  denotes the point spectrum of  $T$  (the set of eigenvalues of  $T$ ). The multiplicity of the eigenvalues  $|k|$  and  $-|k|$  of  $\alpha \cdot k$  are respectively two.

Let  $0 \leq \tau \leq 1$  and

$$\nu_{\pm, \tau}(k) := \omega(k) \pm \tau \alpha \cdot k, \quad (4.20)$$

$$\mu_{\pm, \tau}(k) := \omega(k) \pm \tau |k|. \quad (4.21)$$

Then, by the above mentioned fact on the eigenvalues of  $\alpha \cdot k$ , we have

$$\sigma(\nu_{+, \tau}(k)) = \sigma_p(\nu_{+, \tau}(k)) = \sigma(\nu_{-, \tau}(k)) = \sigma_p(\nu_{-, \tau}(k)) = \{\mu_{+, \tau}(k), \mu_{-, \tau}(k)\}, \quad (4.22)$$

where the multiplicity of the eigenvalues  $\mu_{+, \tau}(k)$  and  $\mu_{-, \tau}(k)$  are respectively two. In particular,

$$\nu_{\pm, \tau}(k) \geq \mu_{-, \tau}(k). \quad (4.23)$$

In what follows, we fix  $\tau \in [0, 1]$  and assume the following:

### Hypothesis (H.2)

(i)  $\mu_{-, \tau} > 0$  a.e.

(ii)  $g_j, g_j/\sqrt{\mu_{-, \tau}} \in \mathcal{H}_{\text{ph}}, j = 1, 2, 3$ .

Condition (H.2)-(i) may be regarded as a spectral condition for the photon energy-momentum operator  $(\omega(k), k)$ , implying that, for a.e.  $k \in \mathbb{R}^3$ ,  $\mu_{-, \tau}(k)^{-1}$  exists. Hence, for a.e.  $k \in \mathbb{R}^3$ , the Hermitian matrices  $\nu_{\pm, \tau}(k)$  are invertible with

$$\nu_{\pm, \tau}(k)^{-1} = \omega(k)^{-1} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\tau^n (\alpha \cdot k)^n}{\omega(k)^n}. \quad (4.24)$$

**Remark 4.2** In the physical case  $\omega(\mathbf{k}) = |\mathbf{k}|$ , (H.2) holds only if  $0 \leq \tau < 1$ , and the parameter  $\tau$  may be regarded as a regularization parameter. In this context, the limit  $\tau \rightarrow 1$  corresponds to the removal of the regularization.

It is easy to see that  $H_{\text{rad}}$  and  $\boldsymbol{\alpha} \cdot \mathbf{P}^{\text{rad}}$  are strongly commuting. Hence  $H_{\text{rad}} - \tau \boldsymbol{\alpha} \cdot \mathbf{P}^{\text{rad}}$  is essentially self-adjoint. Thus we can define a self-adjoint operator

$$L(\tau) := \overline{H_{\text{rad}} - \tau \boldsymbol{\alpha} \cdot \mathbf{P}^{\text{rad}}}. \quad (4.25)$$

Using the fact that

$$\nu_{-, \tau} \geq 0, \quad (4.26)$$

one can show that  $L(\tau)$  is nonnegative.

Let

$$h_D(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta, \quad \mathbf{p} \in \mathbf{R}^3, \quad (4.27)$$

and define

$$H_\tau(\mathbf{p}) := h_D(\mathbf{p}) + L(\tau) + H_{I,0}(\mathbf{g}) \quad (4.28)$$

This is a deformation of  $H(\mathbf{p})$  with

$$H_1(\mathbf{p}) = H(\mathbf{p}). \quad (4.29)$$

**Theorem 4.2** Assume (H.2). Then  $H_\tau(\mathbf{p})$  is self-adjoint on  $D(L(\tau))$  and essentially self-adjoint on every core of  $L(\tau)$ . Moreover,  $H_\tau(\mathbf{p})$  is bounded from below.

*Proof.* For notational simplicity, we denote an element of  $\mathbf{R}^{3n}$  ( $n \geq 1$ ) as

$$\mathbf{k}^{(n)} = (k_1, \dots, k_n) \in \mathbf{R}^{3n} \quad (4.30)$$

and, for a permutation  $\pi \in S_n$  (the symmetric group of order  $n$ ),

$$\pi(\mathbf{k}^{(n)}) := (k_{\pi(1)}, \dots, k_{\pi(n)}) \in \mathbf{R}^{3n}. \quad (4.31)$$

The symmetric tensor product  $\otimes_s^n L^2(\mathbf{R}^3)$  can be naturally identified with the Hilbert space  $L^2_{\text{sym}}(\mathbf{R}^{3n})$  consisting of all elements  $\psi$  in  $L^2(\mathbf{R}^{3n})$  such that, for a.e.  $\mathbf{k}^{(n)} \in \mathbf{R}^{3n}$  and all  $\pi \in S_n$ ,

$$\psi(\mathbf{k}^{(n)}) = \psi(\pi(\mathbf{k}^{(n)})) \quad (4.32)$$

(cf. [19, Theorem II.10(a)]). Hence we have the natural identification

$$\mathcal{F}_s(L^2(\mathbf{R}^3)) = \oplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbf{R}^{3n}) \quad (4.33)$$

with  $L^2_{\text{sym}}(\mathbf{R}^{3n})|_{n=0} := \mathbf{C}$ .

For  $m, n \in \mathbf{N}$  we denote by  $L^2_{\text{sym}}(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$  the Hilbert space consisting of all elements  $\psi$  in  $L^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$  such that, for all permutations  $\pi \in S_m$  and  $\pi' \in S_n$ ,

$$\psi(\mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) = \psi(\pi(\mathbf{k}^{(m)}); \pi'(\mathbf{k}'^{(n)})), \quad \text{a.e. } \mathbf{k}^{(m)}, \mathbf{k}'^{(n)}, \quad (4.34)$$

with  $L_{\text{sym}}^2(\mathbf{R}^{3n})|_{n=0} := \mathbb{C}$ .

For  $m, n \in \mathbb{N}$  we denote by  $L_{\text{sym}}^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$  the Hilbert space consisting of all elements  $\psi$  in  $L^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$  such that, for all permutations  $\pi \in S_m$  and  $\pi' \in S_n$ ,

$$\psi(\mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) = \psi(\pi(\mathbf{k}^{(m)}); \pi'(\mathbf{k}'^{(n)})), \quad \text{a.e. } \mathbf{k}^{(m)}, \mathbf{k}'^{(n)}, \quad (4.34)$$

where we set

$$L_{\text{sym}}^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})|_{m=0} := L_{\text{sym}}^2(\mathbf{R}^{3n}), \quad L_{\text{sym}}^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})|_{n=0} := L_{\text{sym}}^2(\mathbf{R}^{3m}).$$

Then we have the natural identification

$$\mathcal{F}_{\text{rad}} = \oplus_{m,n=0}^{\infty} L_{\text{sym}}^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n}). \quad (4.35)$$

Henceforth we use this identification freely.

It is not so difficult to show that  $D(L(\tau))$  is the set of vectors  $\Psi = \{\Psi^{(m,n)}\}_{m,n=0}^{\infty} \in \oplus^4 \mathcal{F}_{\text{rad}}$  [ $\Psi^{(m,n)} \in \oplus^4 L_{\text{sym}}^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$ ] such that

$$\sum_{m,n=0}^{\infty} \int_{\mathbf{R}^{3m} \times \mathbf{R}^{3n}} \left\| \left[ \sum_{j=1}^m \nu_{-, \tau}(\mathbf{k}_j) + \sum_{\ell=1}^n \nu_{-, \tau}(\mathbf{k}'_{\ell}) \right] \Psi^{(m,n)}(\mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) \right\|_{\mathbb{C}^4}^2 d\mathbf{k}^{(m)} d\mathbf{k}'^{(n)} < \infty$$

and

$$(L(\tau)\Psi)^{(m,n)}(\mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) = \left[ \sum_{j=1}^m \nu_{-, \tau}(\mathbf{k}_j) + \sum_{\ell=1}^n \nu_{-, \tau}(\mathbf{k}'_{\ell}) \right] \Psi^{(m,n)}(\mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) \quad (4.36)$$

for all  $\Psi \in D(L(\tau))$ ,  $m, n \geq 0$ . Using these facts, we can show in the same way as in the case of (2.4) and (2.5) that, for all  $\Psi \in D(L(\tau)^{1/2})$  and  $F = \langle F^{(1)}, F^{(2)} \rangle \in \mathcal{H}_{\text{ph}}$ ,  $\Psi$  is in  $D(a(F)^{\#})$  and

$$\|a(F)\Psi\| \leq \|F\nu_{-, \tau}^{-1/2}\| \|L(\tau)^{1/2}\Psi\|, \quad (4.37)$$

$$\|a(F)^*\Psi\| \leq \|F\nu_{-, \tau}^{-1/2}\| \|L(\tau)^{1/2}\Psi\| + \|F\| \|\Psi\|, \quad (4.38)$$

with

$$\|F\nu_{-, \tau}^{-1/2}\|^2 := \sum_{r=1}^2 \int_{\mathbf{R}^3} \|F^{(r)}(\mathbf{k})\nu_{-, \tau}(\mathbf{k})^{-1/2}\|_{M_4(\mathbb{C})}^2 d\mathbf{k},$$

where  $M_4(\mathbb{C})$  is the set of  $4 \times 4$  complex matrices and, for  $T \in M_4(\mathbb{C})$ ,  $\|T\|_{M_4(\mathbb{C})} := \sup_{z \in \mathbb{C}^4, \|z\|_{\mathbb{C}^4}=1} \|Tz\|_{\mathbb{C}^4}$ . By (4.22) we have

$$\|\nu_{-, \tau}(\mathbf{k})^{-1/2}\|_{M_4(\mathbb{C})} = \mu_{-, \tau}(\mathbf{k})^{-1/2}, \quad j = 1, 2. \quad (4.39)$$

Hence

$$D(L(\tau)) \subset D(L(\tau)^{1/2}) \subset D(H_{I,0}(\mathbf{g})) \quad (4.40)$$

where

$$b_\epsilon(\mathbf{g}) := \frac{q^2}{2\epsilon} \left\| \frac{\mathbf{g}}{\sqrt{\mu_{-, \tau}}} \right\|^2 + \frac{|q| \|\mathbf{g}\|}{\sqrt{2}}. \quad (4.43)$$

Hence  $H_{I,0}(\mathbf{g})$  is infinitesimally small with respect to  $L(\tau)$ . Hence, by the Kato-Rellich theorem (e.g., [20, p.162, Theorem X.12]), we obtain the desired result. ■

Let

$$v_{\tau,j}^{\mathbf{g}}(\mathbf{x}) := v_j^{\mathbf{g}}(\tau\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^3, \quad j = 1, 2, 3, \quad (4.44)$$

and define

$$H_\tau := H_D + H_{\text{rad}} - q \sum_{j=1}^3 \alpha_j [\Phi_S](v_{\tau,j}^{\mathbf{g}}) \quad (4.45)$$

acting in  $\mathcal{F}$ .

A deformation of the unitary operator  $W$  defined by (3.13) is given by

$$W_\tau := \int_{\mathbf{R}^3} e^{i\tau Q(\mathbf{x})} d\mathbf{x}, \quad (4.46)$$

which is also unitary. Let

$$U_\tau := U_F W_\tau. \quad (4.47)$$

Then, in the same way as in Theorem 3.5, we can prove the following theorem.

**Theorem 4.3** *Under the same assumption as in Theorem 3.3,  $H_\tau$  and  $H_\tau(\mathbf{p})$  are essentially self-adjoint and*

$$U_\tau \overline{H_\tau} U_\tau^{-1} = \int_{\mathbf{R}^3}^{\oplus} \overline{H_\tau(\mathbf{p})} d\mathbf{p}. \quad (4.48)$$

**Remark 4.3** We have  $H_1 = H$ . On the other hand, the case  $\tau = 0$  physically corresponds to the case of the dipole approximation, i.e., the case where the Dirac particle interacts with the quantized radiation field only near the origin  $\mathbf{x} = 0$ . Thus the parameter  $\tau \in [0, 1]$  may be regarded as an interpolation parameter between the full interaction and the dipole-approximated one.

### 4.3 Bounds of the ground-state energy of $H_\tau(\mathbf{p})$

Assume (H.2). Then, by Theorem 4.2, the ground-state energy of  $H_\tau(\mathbf{p})$

$$E_\tau(\mathbf{p}) := \inf \sigma(H_\tau(\mathbf{p})) \quad (4.49)$$

is finite.

In general, for every self-adjoint operator  $A$  on a Hilbert space  $\mathcal{K}$  which is bounded below,

$$\inf \sigma(A) = \inf_{\psi \in D(A); \|\psi\|_{\mathcal{K}}=1} (\psi, A\psi)_{\mathcal{K}} \quad (4.50)$$

(the variational principle).

Applying this fact, we have

$$E_\tau(\mathbf{p}) = \inf_{\Psi \in D(H_\tau(\mathbf{p})); \|\Psi\|=1} (\Psi, H_\tau(\mathbf{p})\Psi). \quad (4.51)$$

We shall use this formula to estimate  $E_\tau(\mathbf{p})$ .

We first estimate  $E_\tau(\mathbf{p})$  from below. We introduce a  $4 \times 4$  Hermitian matrix:

$$R_\tau(\mathbf{g}) := \sum_{r=1}^2 \frac{1}{2} \int_{\mathbf{R}^3} d\mathbf{k} \, \alpha \cdot \overline{\mathbf{g}^{(r)}(\mathbf{k})} \nu_{-, \tau}(\mathbf{k})^{-1} \alpha \cdot \mathbf{g}^{(r)}(\mathbf{k}). \quad (4.52)$$

We have

$$\|\nu_{-, \tau}(\mathbf{k})^{-1}\|_{M_4(\mathbf{C})} = \frac{1}{\mu_{-, \tau}(\mathbf{k})}. \quad (4.53)$$

By (4.13), we have

$$\alpha \cdot \mathbf{g}^{(r)}(\mathbf{k})\|_{M_4(\mathbf{C})} = |\mathbf{g}^{(r)}(\mathbf{k})|. \quad (4.54)$$

Hence

$$\|\alpha \cdot \overline{\mathbf{g}^{(r)}(\mathbf{k})} \nu_{-, \tau}(\mathbf{k})^{-1} \alpha \cdot \mathbf{g}^{(r)}(\mathbf{k})\|_{M_4(\mathbf{C})} \leq \frac{|\mathbf{g}^{(r)}(\mathbf{k})|^2}{\mu_{-, \tau}(\mathbf{k})}. \quad (4.55)$$

Hence, by Hypothesis (H.2),  $R_\tau(\mathbf{g})$  is well defined. Moreover it is positive semi-definite:

$$R_\tau(\mathbf{g}) \geq 0. \quad (4.56)$$

Let

$$\omega_D(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m_D^2}, \quad (4.57)$$

the relativistic energy of the free Dirac particle with momentum  $\mathbf{p}$ .

**Proposition 4.4** *Assume (H.2). Suppose that  $\omega$  is in  $L^2(\mathbf{R}^3)_{\text{loc}}$ . Then, for all  $\mathbf{p} \in \mathbf{R}^3$ ,*

$$H_\tau(\mathbf{p}) \geq h_D(\mathbf{p}) - q^2 R_\tau(\mathbf{g}) \quad (4.58)$$

In particular,

$$E_\tau(\mathbf{p}) \geq -\omega_D(\mathbf{p}) - q^2 \|R_\tau(\mathbf{g})\|_{M_4(\mathbf{C})}. \quad (4.59)$$

Before going into the proof of Proposition 4.4, we define some objects. For  $r = 1, 2$  and  $f \in L^2(\mathbf{R}^3)$  we define an operator  $a^{(r)}(f)$  by

$$a^{(1)}(f) := a(\langle f, 0 \rangle), \quad a^{(2)}(f) := a(\langle 0, f \rangle). \quad (4.60)$$

Then we have for all  $\psi \in D(a^{(1)}(f))$  and  $\phi \in D(a^{(2)}(f))$

$$(a^{(1)}(f)\psi)^{(m,n)}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}) = \sqrt{m+1} \int_{\mathbf{R}^3} \overline{f(\mathbf{k})} \psi^{(m+1,n)}(\mathbf{k}, \mathbf{k}^{(m)}; \mathbf{k}'^{(n)}) d\mathbf{k}, \quad (4.61)$$

$$(a^{(2)}(f)\phi)^{(m,n)}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}) = \sqrt{n+1} \int_{\mathbf{R}^3} \overline{f(\mathbf{k})} \phi^{(m,n+1)}(\mathbf{k}^{(m)}; \mathbf{k}, \mathbf{k}'^{(n)}) d\mathbf{k},$$

$$\text{a.e.}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}), \quad m, n \geq 0. \quad (4.62)$$

We have for all  $F = \langle F^{(1)}, F^{(2)} \rangle \in \mathcal{H}_{\text{ph}}$

$$a(F) = \sum_{r=1}^2 a^{(r)}(F^{(r)}) \quad (4.63)$$

on  $D(a^{(1)}(F^{(1)})) \cap D(a^{(2)}(F^{(2)}))$ .

We denote by  $C(\mathbf{R}^{3m} \times \mathbf{R}^{3n})$  the space of continuous functions on  $\mathbf{R}^{3m} \times \mathbf{R}^{3n}$  and set

$$L^2 C_{\text{sym}}(\mathbf{R}^{3m} \times \mathbf{R}^{3n}) := C(\mathbf{R}^{3m} \times \mathbf{R}^{3n}) \cap L^2_{\text{sym}}(\mathbf{R}^{3m} \times \mathbf{R}^{3n}). \quad (4.64)$$

Then

$$\mathcal{F}_{\text{rad},0}^C := \left\{ \psi = \{\psi^{(m,n)}\}_{m,n=0}^\infty \in \mathcal{F}_{\text{rad},0} \mid \psi^{(m,n)} \in L^2 C_{\text{sym}}(\mathbf{R}^{3m} \times \mathbf{R}^{3n}), \right. \quad (4.65)$$

$$\left. m, n \geq 0 \right\} \quad (4.66)$$

is a dense subspace in  $\mathcal{F}_{\text{rad}}$ .

For each  $\mathbf{k} \in \mathbf{R}^3$ , we can define an operator  $a^{(r)}(\mathbf{k})$  with  $D(a^{(r)}(\mathbf{k})) := \mathcal{F}_{\text{rad},0}^C$  ( $r = 1, 2$ ) by

$$(a^{(1)}(\mathbf{k})\psi)^{(m,n)}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}) = \sqrt{m+1} \psi^{(m+1,n)}(\mathbf{k}, \mathbf{k}^{(m)}; \mathbf{k}'^{(n)}), \quad (4.67)$$

$$(a^{(2)}(\mathbf{k})\psi)^{(m,n)}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}) = \sqrt{n+1} \psi^{(m,n+1)}(\mathbf{k}^{(m)}; \mathbf{k}, \mathbf{k}'^{(n)}),$$

$$\text{a.e.}(\mathbf{k}^{(m)}, \mathbf{k}'^{(n)}), \psi \in \mathcal{F}_{\text{rad},0}^C, m, n \geq 0. \quad (4.68)$$

Note that  $D(a^{(r)}(\mathbf{k})^*) = \{0\}$  and hence  $a^{(r)}(\mathbf{k})^*$  is a trivial operator (cf. [20, §X.7, pp. 218–220]). We have for all  $\psi \in \mathcal{F}_{\text{rad},0}^C$  and  $f \in L^2(\mathbf{R}^3)$

$$a^{(r)}(f)\psi = \int_{\mathbf{R}^3} d\mathbf{k} \overline{f(\mathbf{k})} a^{(r)}(\mathbf{k})\psi, \quad r = 1, 2, \quad (4.69)$$

where the integral is taken as strong Bochner integral.

#### Proof of Proposition 4.4

Let  $\mathcal{F}_{\text{rad},0}^\infty$  be as in (3.25). Then we have

$$\mathcal{F}_{\text{rad},0}^\infty \subset \mathcal{F}_{\text{rad},0}^C. \quad (4.70)$$

Let

$$\eta^{(r)}(\mathbf{k}) = \frac{1}{\sqrt{2}} \nu_{-,r}(\mathbf{k})^{-1} \boldsymbol{\alpha} \cdot \mathbf{g}^{(r)}(\mathbf{k}), \quad r = 1, 2. \quad (4.71)$$

By (4.53) and (4.54),  $\|\eta^{(r)}(\cdot)\|_{M_4(\mathbf{C})} \in L^2(\mathbf{R}^3)$ . For a.e.  $\mathbf{k} \in \mathbf{R}^3$  and  $r = 1, 2$ , we define an operator  $b^{(r)}(\mathbf{k})$  acting in  $\oplus^4 \mathcal{F}_{\text{rad}}$  with  $D(b^{(r)}(\mathbf{k})) = \oplus^4 \mathcal{F}_{\text{rad},0}^\infty$  by

$$b^{(r)}(\mathbf{k}) = a^{(r)}(\mathbf{k}) - q\eta^{(r)}(\mathbf{k}). \quad (4.72)$$

Then, by direct computation, we can show that, for all  $\Psi \in \oplus^4 \mathcal{F}_{\text{rad},0}^\infty$ ,

$$\begin{aligned} & \sum_{r=1}^2 \int_{\mathbf{R}^3} (b^{(r)}(\mathbf{k})\Psi, \nu_{-,r}(\mathbf{k})b^{(r)}(\mathbf{k})\Psi) d\mathbf{k} \\ &= (\Psi, H_\tau(\mathbf{p})\Psi) - (\Psi, [h_D(\mathbf{p}) - q^2 R_\tau(\mathbf{g})]\Psi). \end{aligned} \quad (4.74)$$

Since the left hand side of (4.74) is nonnegative, it follows that

$$(\Psi, H_\tau(\mathbf{p})\Psi) \geq (\Psi, [h_D(\mathbf{p}) - q^2 R_\tau(\mathbf{g})]\Psi). \quad (4.75)$$

By the assumption that  $\omega \in L^2(\mathbf{R}^3)_{\text{loc}}$ , the  $M_4(\mathbf{C})$ -valued function  $\sum_{j=1}^m \nu_{-,r}(\mathbf{k}_j) + \sum_{l=1}^n \nu_{-,r}(\mathbf{k}'_l)$  on  $\mathbf{R}^{3m} \times \mathbf{R}^{3n}$  is in  $L^2(\mathbf{R}^{3m} \times \mathbf{R}^{3n})_{\text{loc}}$ . Hence, it follows that  $\oplus^4 \mathcal{F}_{\text{rad},0}^\infty$  is a core of  $L(\tau)$ . Hence, by Theorem 4.2, it is a core of  $H_\tau(\mathbf{p})$ . Thus (4.75) implies (4.58).

By (4.58), we have

$$H_\tau(\mathbf{p}) \geq -\|h_D(\mathbf{p})\|_{M_4(\mathbf{C})} - q^2 \|R_\tau(\mathbf{g})\|_{M_4(\mathbf{C})}.$$

It is well known that

$$\sigma(h_D(\mathbf{p})) = \sigma_{\mathbf{p}}(h_D(\mathbf{p})) = \{\pm\omega_D(\mathbf{p})\}, \quad (4.76)$$

the multiplicity of each eigenvalue being two, so that

$$\|h_D(\mathbf{p})\|_{M_4(\mathbf{C})} = \omega_D(\mathbf{p}). \quad (4.77)$$

Hence, by the variational principle, we obtain (4.59). ■

We next estimate  $E_\tau(\mathbf{p})$  from above. For  $z \in \mathbf{C}^4$  with  $\|z\| = 1$ , we define

$$\xi_{z,\tau}(\mathbf{k}) := \omega(\mathbf{k}) - \tau \mathbf{u}(z) \cdot \mathbf{k}. \quad (4.78)$$

By (4.14),

$$\xi_{z,\tau}(\mathbf{k}) \geq \mu_{-,r}(\mathbf{k}). \quad (4.79)$$

By this fact, we can define

$$C_\tau(z) := \sum_{r=1}^2 \frac{1}{2} \|\xi_{z,\tau}^{-1/2} \mathbf{u}(z) \cdot \mathbf{g}^{(r)}\|_{L^2(\mathbf{R}^3)}^2. \quad (4.80)$$

We set

$$\beta(z) := (z, \beta z)_{\mathbf{C}^4}, \quad z \in \mathbf{C}^4. \quad (4.81)$$

**Proposition 4.5** *Assume (H.2). Then*

$$E_\tau(\mathbf{p}) \leq \inf_{z \in \mathbf{C}^4, \|z\|=1} \{ \mathbf{u}(z) \cdot \mathbf{p} + m_D \beta(z) - q^2 C_\tau(z) \} \quad (4.82)$$

*Proof.* Let  $z \in \mathbb{C}^4$  and  $\psi \in D(H_{\text{rad}}) \subset \mathcal{F}_{\text{rad}}$  with  $\|z\| = 1$  and  $\|\psi\| = 1$ . Then, by the variational principle, we have

$$E_\tau(\mathbf{p}) \leq (z \otimes \psi, H_\tau(\mathbf{p})z \otimes \psi). \quad (4.83)$$

Let

$$H_{z,\tau} := d\Gamma(\xi_{z,\tau}) - q \sum_{j=1}^3 u_j(z) \Phi_S(g_j). \quad (4.84)$$

Then we have

$$(z \otimes \psi, H_\tau(\mathbf{p})z \otimes \psi) = \mathbf{u}(z) \cdot \mathbf{p} + m_D \beta(z) + (\psi, H_{z,\tau} \psi). \quad (4.85)$$

Hence

$$E_\tau(\mathbf{p}) \leq \mathbf{u}(z) \cdot \mathbf{p} + m_D \beta(z) + (\psi, H_{z,\tau} \psi). \quad (4.86)$$

As in Theorem 4.2, we can show that  $H_{z,\tau}$  is self-adjoint on  $D(H_{z,\tau}) = D(d\Gamma(\xi_{z,\tau}))$  and bounded below. Let  $\varepsilon > 0$  and

$$U_z^{(\varepsilon)} := \exp \left[ -iq\Phi_S \left( i(\xi_{z,\tau} + \varepsilon)^{-1} \mathbf{u}(z) \cdot \mathbf{g} \right) \right] \quad (4.87)$$

Then we have

$$U_z^{(\varepsilon)} d\Gamma(\xi_{z,\tau} + \varepsilon) U_z^{-1} = H_{z,\tau} + q^2 C_\tau^{(\varepsilon)}(z) + \varepsilon N, \quad (4.88)$$

where

$$C_\tau^{(\varepsilon)}(z) := \sum_{r=1}^2 \frac{1}{2} \|(\xi_{z,\tau} + \varepsilon)^{-1/2} \mathbf{u}(z) \cdot \mathbf{g}^{(r)}\|_{L^2(\mathbb{R}^3)}^2$$

and

$$N := d\Gamma(I) \quad (4.89)$$

is the number operator on  $\mathcal{F}_{\text{rad}}$ . Thus, putting  $\psi_\varepsilon := U_z^{(\varepsilon)} \Omega_0$ , we have  $\|\psi_\varepsilon\| = 1$  and

$$\begin{aligned} (\psi_\varepsilon, H_{z,\tau} \psi_\varepsilon) &= -q^2 C_\tau^{(\varepsilon)}(z) - \varepsilon (\psi_\varepsilon, N \psi_\varepsilon) \\ &\leq -q^2 C_\tau^{(\varepsilon)}(z). \end{aligned}$$

Hence

$$E_\tau(\mathbf{p}) \leq \mathbf{u}(z) \cdot \mathbf{p} + m_D \beta(z) - q^2 C_\tau^{(\varepsilon)}(z).$$

It is easy to see that

$$\lim_{\varepsilon \rightarrow 0} C_\tau^{(\varepsilon)}(z) = C_\tau(z).$$

Thus (4.82) follows. ■

To get a more explicit upper bound of  $E_\tau(\mathbf{p})$ , we estimate the right hand side of (4.82). As is well known (e.g., [24, §1.4]), the  $4 \times 4$  matrix

$$w(\mathbf{p}) := \frac{m_D + \omega_D(\mathbf{p}) + \beta \boldsymbol{\alpha} \cdot \mathbf{p}}{\sqrt{2\omega_D(\mathbf{p})(m_D + \omega_D(\mathbf{p}))}} \quad (4.90)$$

is a unitary matrix diagonalizing  $h_D(\mathbf{p})$ :

$$w(\mathbf{p})h_D(\mathbf{p})w(\mathbf{p})^{-1} = \beta\omega_D(\mathbf{p}). \quad (4.91)$$

Let  $z_0 \in \mathbb{C}^4$  be a normalized eigenvector of  $\beta$  with eigenvalue  $-1$ :  $\beta z_0 = -z_0$  and

$$z_0(\mathbf{p}) := w(\mathbf{p})^{-1}z_0. \quad (4.92)$$

Then we have  $\|z_0(\mathbf{p})\| = 1$  and

$$h_D(\mathbf{p})z_0(\mathbf{p}) = -\omega_D(\mathbf{p})z_0(\mathbf{p}), \quad (4.93)$$

so that

$$(z_0(\mathbf{p}), h_D(\mathbf{p})z_0(\mathbf{p})) = -\omega_D(\mathbf{p}). \quad (4.94)$$

By the anticommutation relations in (1.4), one can show that

$$u_j(z_0) = 0, \quad j = 1, 2, 3. \quad (4.95)$$

Moreover (1.3) and (1.4) imply that

$$[\alpha_j, w(\mathbf{p})] = -\frac{2p_j}{\sqrt{2\omega_D(\mathbf{p})(m_D + \omega_D(\mathbf{p}))}}\beta. \quad (4.96)$$

Using these relations, we see that

$$u(z_0(\mathbf{p})) = -\frac{\mathbf{p}}{\omega_D(\mathbf{p})}. \quad (4.97)$$

Hence we obtain

$$F_\tau(\mathbf{p}) := C_\tau(z_0(\mathbf{p})) = \sum_{r=1}^2 \frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{\omega(\mathbf{k}) + \tau \frac{\mathbf{p} \cdot \mathbf{k}}{\omega_D(\mathbf{p})}} \left| \frac{\mathbf{p} \cdot \mathbf{g}^{(r)}(\mathbf{k})}{\omega_D(\mathbf{p})} \right|^2 d\mathbf{k} \geq 0. \quad (4.98)$$

Thus, by (4.82), we obtain the following result.

**Proposition 4.6** *Assume (H.2). Then, for all  $\mathbf{p} \in \mathbb{R}^3$ ,*

$$E_\tau(\mathbf{p}) \leq -\omega_D(\mathbf{p}) - q^2 F_\tau(\mathbf{p}). \quad (4.99)$$

**Remark 4.4** If the Lebesgue measure of the set  $\{\mathbf{k} \in \mathbb{R}^3 | \mathbf{p} \cdot \mathbf{g}^{(r)}(\mathbf{k}) \neq 0\}$  is positive, then  $F_\tau(\mathbf{p}) > 0$  and hence  $E_\tau(\mathbf{p}) < -\omega_D(\mathbf{p})$ , i.e., the interaction makes the ground-state energy strictly lower [note that  $E_\tau(\mathbf{p})|_{q=0} = -\omega_D(\mathbf{p})$ ].

**Proposition 4.7** *Assume (H.2). Suppose that  $\omega \in L^2(\mathbb{R}^3)_{\text{loc}}$ . Then:*

(i)

$$\lim_{q \rightarrow 0} E_\tau(\mathbf{p}) = -\omega_D(\mathbf{p}). \quad (4.100)$$

(ii)

$$\lim_{|\mathbf{p}| \rightarrow \infty} \frac{E_\tau(\mathbf{p})}{\omega_D(\mathbf{p})} = -1. \quad (4.101)$$

*Proof.* (i) We have from Propositions 4.4 and 4.6

$$-\omega_D(\mathbf{p}) - q^2 \|R_\tau(\mathbf{g})\|_{M_4(\mathbf{C})} \leq E_\tau(\mathbf{p}) \leq -\omega_D(\mathbf{p}) - q^2 F_\tau(\mathbf{p}). \quad (4.102)$$

Hence, taking  $q \rightarrow 0$  in this inequality, we obtain (4.100).

(ii) By (4.102), we have

$$-1 - \frac{q^2 \|R_\tau(\mathbf{g})\|_{M_4(\mathbf{C})}}{\omega_D(\mathbf{p})} \leq \frac{E_\tau(\mathbf{p})}{\omega_D(\mathbf{p})} \leq -1.$$

It is obvious that

$$\lim_{|\mathbf{p}| \rightarrow \infty} \frac{1}{\omega_D(\mathbf{p})} = 0. \quad (4.103)$$

Hence (4.101) follows. ■

#### 4.4 Effective mass of the polaron

The *effective mass (physical mass)* of the polaron is defined by

$$m_\tau^* := -E_\tau(0) \quad (4.104)$$

Assume (H.2) and suppose that  $\omega \in L_{\text{loc}}^2(\mathbf{R}^3)$ . Then it follows from Propositions 4.4 and 4.6 that

$$\sup_{z \in \mathbf{C}^4; \|z\|=1} \{q^2 C_\tau(z) - m_D \beta(z)\} \leq m_\tau^* \leq m_D + q^2 \|R_\tau(\mathbf{g})\|_{M_4(\mathbf{C})}. \quad (4.105)$$

In particular,

$$\lim_{q \rightarrow 0} m_\tau^* = m_D, \quad (4.106)$$

$$\lim_{m_D \rightarrow \infty} \frac{m_\tau^*}{m_D} = 1. \quad (4.107)$$

#### 4.5 Analytical properties of $E_\tau(\mathbf{p})$ as a function of $\mathbf{p}$

**Proposition 4.8** *Assume (H.2). Then, for all  $\mathbf{p}, \mathbf{p}' \in \mathbf{R}^3$ ,*

$$|E_\tau(\mathbf{p}) - E_\tau(\mathbf{p}')| \leq |\mathbf{p} - \mathbf{p}'|. \quad (4.108)$$

*Proof.* Let  $\Psi \in D(L(\tau))$  with  $\|\Psi\| = 1$ . Then we have

$$\begin{aligned} |(\Psi, H_\tau(\mathbf{p})\Psi) - (\Psi, H_\tau(\mathbf{p}')\Psi)| &= |(\Psi, \boldsymbol{\alpha} \cdot (\mathbf{p} - \mathbf{p}')\Psi)| \\ &\leq |\mathbf{p} - \mathbf{p}'|. \end{aligned}$$

Hence

$$(\Psi, H_\tau(\mathbf{p})\Psi) \leq (\Psi, H_\tau(\mathbf{p}')\Psi) + |\mathbf{p} - \mathbf{p}'|.$$

By the variational principle, the left hand side is more than or equal to  $E_\tau(\mathbf{p})$ . Hence  $E_\tau(\mathbf{p}) \leq (\Psi, H_\tau(\mathbf{p}')\Psi) + |\mathbf{p} - \mathbf{p}'|$ , which, together with the variational principle again, implies that  $E_\tau(\mathbf{p}) \leq E_\tau(\mathbf{p}') + |\mathbf{p} - \mathbf{p}'|$ . This result and symmetry with  $\mathbf{p}$  and  $\mathbf{p}'$  give (4.108). ■

Proposition 4.7 shows that the function  $E_\tau : \mathbf{p} \rightarrow E_\tau(\mathbf{p})$  on  $\mathbf{R}^3$  is Lipschitz continuous.

We denote by  $SO(3)$  the three dimensional rotation group. For  $T \in SO(3)$  and a function  $f$  on  $\mathbf{R}^3$ , we define  $\hat{T}f$  by

$$(\hat{T}f)(\mathbf{k}) := f(T^{-1}\mathbf{k}), \quad \mathbf{k} \in \mathbf{R}^3. \quad (4.109)$$

It is easy to see that  $\hat{T}$  is unitary on  $L^2(\mathbf{R}^3)$  and naturally extended to a unitary operator on  $\mathcal{H}_{\text{ph}}$ .

A  $\mathbf{C}$ -valued function  $f$  on  $\mathbf{R}^3$  is said to be rotation invariant if  $\hat{T}f = f$  for all  $T \in SO(3)$ . A  $\mathbf{C}^3$ -valued function  $\mathbf{f} = (f_1, f_2, f_3)$  on  $\mathbf{R}^3$  is said to be rotation invariant if  $(\hat{T}f_1, \hat{T}f_2, \hat{T}f_3) = T^{-1}\mathbf{f}$  for all  $T \in SO(3)$ .

**Proposition 4.9** *Assume (H.2). Let  $\mathbf{g}^{(r)}$  be rotation invariant ( $r = 1, 2$ ). Then the function  $\mathbf{p} \rightarrow E_\tau(\mathbf{p})$  is rotation invariant.*

*Proof.* Let  $T \in SO(3)$ . Then there exists a unitary matrix  $u_T$  such that

$$u_T \alpha_j u_T^{-1} = \sum_{k=1}^3 T_{jk} \alpha_k, \quad u_T \beta u_T^{-1} = \beta.$$

Hence

$$H_\tau(T^{-1}\mathbf{p}) = u_T H_\tau^{(T)}(\mathbf{p}) u_T^{-1},$$

where

$$H_\tau^{(T)}(\mathbf{p}) := h_D(\mathbf{p}) + H_{\text{rad}} - \tau \boldsymbol{\alpha} \cdot T \mathbf{p}^{\text{rad}} - q \sum_{j,\ell=1}^3 \alpha_j T_{j\ell} \Phi_S(g_\ell).$$

In general, for every unitary operator  $S$  on  $\mathcal{H}_{\text{ph}}$ , the operator

$$\Gamma(S) := \bigoplus_{n=0}^{\infty} (\otimes^n S) \quad (4.110)$$

acting in  $\mathcal{F}_{\text{rad}}$  is unitary ( $\otimes^0 S := 1$ ) and, for all self-adjoint operator  $A$  on  $\mathcal{H}_{\text{ph}}$ ,

$$\Gamma(S) d\Gamma(A) \Gamma(S)^{-1} = d\Gamma(SAS^{-1}), \quad (4.111)$$

$$\Gamma(S) a(F) \Gamma(S)^{-1} = a(SF), \quad F \in \mathcal{H}_{\text{ph}}, \quad (4.112)$$

see, e.g., [20, §X.7]. The operator  $\Gamma(S)$  can be naturally extended to  $\oplus^4 \mathcal{F}_{\text{rad}}$ . In particular, we have

$$\Gamma(\hat{T}^{-1})H_{\text{rad}}\Gamma(\hat{T}) = H_{\text{rad}}, \quad \Gamma(\hat{T}^{-1})\mathbf{P}^{\text{rad}}\Gamma(\hat{T}) = T\mathbf{P}^{\text{rad}}.$$

By the rotation invariance of  $\mathbf{g}^{(r)}$ ,

$$\Gamma(\hat{T}^{-1})\Phi_S(g_j)\Gamma(\hat{T}) = \sum_{\ell=1}^3 T_{j\ell}\Phi_S(g_\ell).$$

Hence we have

$$H_\tau(T^{-1}\mathbf{p}) = u_T\Gamma(\hat{T}^{-1})H_\tau(\mathbf{p})\Gamma(\hat{T})u_T^{-1}.$$

Since  $u_T\Gamma(\hat{T}^{-1})$  is unitary with  $[u_T\Gamma(\hat{T}^{-1})]^{-1} = \Gamma(\hat{T})u_T^{-1}$ , it follows that  $H_\tau(T^{-1}\mathbf{p})$  is unitarily equivalent to  $H_\tau(\mathbf{p})$ . Hence  $E_\tau(T^{-1}\mathbf{p}) = E_\tau(\mathbf{p})$ . ■

**Remark 4.5** Consider the case where

$$g_j(\mathbf{k}) = \frac{\varrho(\mathbf{k})}{\sqrt{(2\pi)^3\omega(\mathbf{k})}} (e_j^{(1)}(\mathbf{k}), e_j^{(2)}(\mathbf{k}))$$

(see Remark 1.2). Take  $\mathbf{e}^{(r)}(\mathbf{k})$  such that, for a.e.  $\mathbf{k}$  and all  $T \in \text{SO}(3)$ ,  $\mathbf{e}^{(r)}(T\mathbf{k}) = T\mathbf{e}^{(r)}(\mathbf{k})$ ,  $r = 1, 2$ . Then  $\mathbf{g}$  is rotation invariant if and only if  $\varrho/\sqrt{\omega}$  is rotation invariant.

**Proposition 4.10** Assume (H.2).

(i) (concavity) For all  $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^3$  and  $\lambda \in [0, 1]$ ,

$$\lambda E_\tau(\mathbf{p}) + (1 - \lambda)E_\tau(\mathbf{p}') \leq E_\tau(\lambda\mathbf{p} + (1 - \lambda)\mathbf{p}'). \quad (4.113)$$

(ii) For all  $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^3$  and  $\varepsilon, \lambda \in [0, 1]$ ,

$$E_\tau(\lambda\mathbf{p} + (1 - \lambda)\mathbf{p}') \leq \varepsilon E_\tau(\mathbf{p}) + (1 - \varepsilon)E_\tau(\mathbf{p}') + (\varepsilon + \lambda - 2\varepsilon\lambda)|\mathbf{p} - \mathbf{p}'|. \quad (4.114)$$

In particular,

$$E_\tau(\lambda\mathbf{p} + (1 - \lambda)\mathbf{p}') \leq \lambda E_\tau(\mathbf{p}) + (1 - \lambda)E_\tau(\mathbf{p}') + 2(1 - \lambda)\lambda|\mathbf{p} - \mathbf{p}'|. \quad (4.115)$$

*Proof.* (i) It follows from the definition of  $H_\tau(\mathbf{p})$  and Theorem 4.2 that

$$\begin{aligned} H_\tau(\lambda\mathbf{p} + (1 - \lambda)\mathbf{p}') &= \lambda H_\tau(\mathbf{p}) + (1 - \lambda)H_\tau(\mathbf{p}') \\ &\geq \lambda E_\tau(\mathbf{p}) + (1 - \lambda)E_\tau(\mathbf{p}'). \end{aligned}$$

Hence, by the variational principle, we obtain (4.113).

(ii) By the variational principle, we have for all  $\Psi \in D(H_\tau(\mathbf{p})) = D(L(\tau))$  with  $\|\Psi\| = 1$

$$\begin{aligned}
E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') &\leq (\Psi, H_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') \Psi) \\
&= \lambda (\Psi, H_\tau(\mathbf{p}) \Psi) + (1 - \lambda) (\Psi, H_\tau(\mathbf{p}') \Psi) \\
&= (\Psi, H_\tau(\mathbf{p}) \Psi) + (1 - \lambda) (\Psi, [H_\tau(\mathbf{p}') - H_\tau(\mathbf{p})] \Psi) \\
&= (\Psi, H_\tau(\mathbf{p}) \Psi) + (1 - \lambda) (\Psi, \boldsymbol{\alpha} \cdot (\mathbf{p}' - \mathbf{p}) \Psi) \\
&\leq (\Psi, H_\tau(\mathbf{p}) \Psi) + (1 - \lambda) |\mathbf{p} - \mathbf{p}'|.
\end{aligned}$$

Hence, by the variational principle again, we obtain

$$E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') \leq E_\tau(\mathbf{p}) + (1 - \lambda) |\mathbf{p} - \mathbf{p}'|.$$

Similarly we have

$$E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') \leq E_\tau(\mathbf{p}') + \lambda |\mathbf{p} - \mathbf{p}'|.$$

Combining these inequalities with the identity

$$E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') = \varepsilon E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}') + (1 - \varepsilon) E_\tau(\lambda \mathbf{p} + (1 - \lambda) \mathbf{p}'),$$

we obtain (4.114). ■

## 5 Existence of a Ground State of $H_\tau(\mathbf{p})$ (I)—The Massive Case

In this section we prove the existence of a ground state of  $H_\tau(\mathbf{p})$  under certain conditions for  $\omega$ . We assume the following two conditions (H.3) and (H.4).

**Hypothesis (H.3)** There exists a constant  $M > 0$  such that, for all  $\mathbf{k} \in \mathbf{R}^3$ ,

$$\mu_{-, \tau}(\mathbf{k}) \geq M, \tag{5.1}$$

where  $M$  may depend on  $\tau$ .

**Hypothesis (H.4)**

(i)  $g_j \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, 2, 3$ .

(ii) The function  $\omega$  is uniformly continuous on  $\mathbf{R}^3$ .

Condition (5.1) and (H.4)-(i) implies that, for all  $s > 0$

$$\mu_{-, \tau}^{-s} g_j \in \mathcal{H}_{\text{ph}}, \quad j = 1, 2, 3.$$

Hence (H.3) and (H.4)-(i) imply (H.2).

We have from (5.1)

$$\omega(\mathbf{k}) \geq M. \quad (5.2)$$

Hence (H.3) implies that the photon is “massive”.

We introduce

$$\Delta_\tau(\mathbf{p}) := \inf_{n \geq 1} \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbb{R}^3} \left\{ E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) \right\} - E_\tau(\mathbf{p}). \quad (5.3)$$

By Proposition 4.8,

$$\begin{aligned} E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) - E_\tau(\mathbf{p}) &\geq \sum_{j=1}^n \omega(\mathbf{k}_j) - \left| \sum_{j=1}^n \tau \mathbf{k}_j \right| \\ &\geq \sum_{j=1}^n \mu_{-, \tau}(\mathbf{k}_j) \end{aligned} \quad (5.4)$$

$$\geq nM. \quad (5.5)$$

Hence

$$\Delta_\tau(\mathbf{p}) \geq M. \quad (5.6)$$

The main result of this section is stated in the following theorem.

**Theorem 5.1** *Assume (H.3) and (H.4). Suppose that*

$$\lim_{|\mathbf{k}| \rightarrow \infty} \mu_{-, \tau}(\mathbf{k}) = \infty. \quad (5.7)$$

*Then, for all  $\mathbf{p} \in \mathbb{R}^3$ ,  $H_\tau(\mathbf{p})$  has purely discrete spectrum in  $[E_\tau(\mathbf{p}), E_\tau(\mathbf{p}) + \Delta_\tau(\mathbf{p})]$ . In particular,  $H_\tau(\mathbf{p})$  has a ground state.*

**Example 5.1** Let  $m_{\text{ph}} > 0$  be a constant parameter and  $\omega(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m_{\text{ph}}^2}$ ,  $\mathbf{k} \in \mathbb{R}^3$  (the usual form of one free massive photon with mass  $m_{\text{ph}}$  and momentum  $\mathbf{k}$ ). Then  $\lim_{m_{\text{ph}} \rightarrow 0} \omega(\mathbf{k}) = |\mathbf{k}|$  and

$$\mu_{-, \tau}(\mathbf{k}) \geq \sqrt{1 - \tau^2} m_{\text{ph}}$$

for all  $\mathbf{k} \in \mathbb{R}^3$ . Hence, if  $0 \leq \tau < 1$ , then (5.1) and (5.7) hold with  $M = \sqrt{1 - \tau^2} m_{\text{ph}}$ . But, in the case  $\tau = 1$ , neither (5.1) nor (5.7) holds. Thus, in this model of  $\omega$ , the original case  $\tau = 1$  is somewhat singular in view of conditions (5.1) and (5.7).

**Example 5.2** An artificial example is given by

$$\omega_{a,M}(\mathbf{k}) := (1 + M^a)|\mathbf{k}| + M, \quad \mathbf{k} \in \mathbf{R}^3,$$

with  $a > 0$  and  $M > 0$  constant parameters. In this case,  $\mu_{-, \tau}(\mathbf{k}) = (1 + M^a - \tau)|\mathbf{k}| + M$ . Hence  $\omega = \omega_{a,M}$  satisfies (5.1) and (5.7) for all  $\tau \in [0, 1]$ . We have  $\lim_{M \rightarrow 0} \omega_{a,M}(\mathbf{k}) = |\mathbf{k}|$  for all  $\mathbf{k} \in \mathbf{R}^3$ . Hence  $\omega_{a,M}$  can serve as a function approximating the energy function  $|\mathbf{k}|$  of a free massless photon.

To prove Theorem 5.1, we use the method similar to that developed in [6].

## 5.1 Finite volume approximations

Following the method given in [6], we first discuss the case where the quantized radiation field is over a finite box  $[-V, V]^3 \subset \mathbf{R}^3$  with a size parameter  $V > 0$ . Let

$$\Gamma_V = \frac{2\pi\mathbf{Z}^3}{V} = \left\{ \mathbf{k} = \left( \frac{2\pi n_1}{V}, \frac{2\pi n_2}{V}, \frac{2\pi n_3}{V} \right) \mid n_j \in \mathbf{Z}, j = 1, 2, 3 \right\} \quad (5.8)$$

and  $F_{1V}$  be the closed subspace of  $L^2(\mathbf{R}^3)$  consisting of all elements which are a.e. constant on each cube

$$C(\mathbf{k}, V) := \left[ k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right] \times \left[ k_2 - \frac{\pi}{V}, k_2 + \frac{\pi}{V} \right] \times \left[ k_3 - \frac{\pi}{V}, k_3 + \frac{\pi}{V} \right] \quad (5.9)$$

indexed by  $\mathbf{k} \in \Gamma_V$ . Note that  $F_{1V}$  is naturally identified with  $\ell^2(\Gamma_V)$ , the Hilbert space of square summable sequences on  $\Gamma_V$ . Then the symmetric Fock space

$$\mathcal{F}_{\text{rad}, V} := \mathcal{F}_s(F_{1V} \oplus F_{1V}) \quad (5.10)$$

over  $F_{1V} \oplus F_{1V}$  serves as a Hilbert space for the quantized radiation field on the finite box.

For each  $\mathbf{k} \in \Gamma_V$  we define

$$\chi_{\mathbf{k}, V} := \chi_{C(\mathbf{k}, V)} \quad (5.11)$$

the characteristic function of  $C(\mathbf{k}, V)$  and introduce

$$a_V^{(r)}(\mathbf{k}) := \left( \frac{V}{2\pi} \right)^{3/2} a^{(r)}(\chi_{\mathbf{k}, V}), \quad \mathbf{k} \in \Gamma_V, r = 1, 2. \quad (5.12)$$

It is easy to see that, for all  $s, r = 1, 2$  and  $\mathbf{k}, \mathbf{k}' \in \Gamma_V$ ,

$$[a_V^{(r)}(\mathbf{k}), a_V^{(s)}(\mathbf{k}')^*] = \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}, \quad (5.13)$$

$$[a_V^{(r)}(\mathbf{k}), a_V^{(s)}(\mathbf{k}')] = 0, \quad [a_V^{(r)}(\mathbf{k})^*, a_V^{(s)}(\mathbf{k}')^*] = 0, \quad (5.14)$$

on  $\mathcal{F}_{\text{rad}, 0}$ .

For each  $\mathbf{k} \in \mathbf{R}^3$ , we denote by  $\mathbf{k}_V = (k_{V1}, k_{V2}, k_{V3})$  a point in  $\Gamma_V$  closest to  $\mathbf{k}$ :

$$\mathbf{k}_V \in \Gamma_V, \quad |k_j - k_{Vj}| \leq \frac{\pi}{V}, \quad j = 1, 2, 3.$$

Hence

$$|\mathbf{k} - \mathbf{k}_V| \leq \frac{\sqrt{3}\pi}{V}. \quad (5.15)$$

In what follows we assume (H.3) and (H.4).

The continuity of  $\omega$  allows one to define

$$\omega_V(\mathbf{k}) := \omega(\mathbf{k}_V), \quad \mathbf{k} \in \mathbf{R}^3. \quad (5.16)$$

The free Hamiltonian and the momentum operator with volume cutoff  $V$  are defined by

$$H_{\text{rad},V} := d\Gamma(\omega_V), \quad (5.17)$$

$$\mathbf{P}_V^{\text{rad}} := (P_{V1}^{\text{rad}}, P_{V2}^{\text{rad}}, P_{V3}^{\text{rad}}), \quad P_{Vj}^{\text{rad}} := d\Gamma(k_{Vj}), \quad j = 1, 2, 3, \quad (5.18)$$

respectively. The correspondent to the operator  $L(\tau)$  is given by

$$L_V(\tau) := \overline{H_{\text{rad},V} - \tau \boldsymbol{\alpha} \cdot \mathbf{P}_V^{\text{rad}}}, \quad (5.19)$$

which is self-adjoint and nonnegative.

Let  $K > 0$  and

$$H_{I,K,V}(\mathbf{g}) := - \sum_{j=1}^3 \sum_{r=1}^2 \sum_{\mathbf{k} \in \Gamma_V; |\mathbf{k}_\ell| \leq K, \ell=1,2,3} \frac{q}{\sqrt{2}} \left( \frac{2\pi}{V} \right)^{3/2} \alpha_j \quad (5.20)$$

$$\times \left[ a_V^{(r)}(\mathbf{k})^* g_j^{(r)}(\mathbf{k}) + a_V^{(r)}(\mathbf{k}) \overline{g_j^{(r)}(\mathbf{k})} \right]. \quad (5.21)$$

We also need a momentum cutoff interaction:

$$H_{I,K}(\mathbf{g}) := -q \sum_{j=1}^3 \alpha_j \Phi_S(g_{j,K}), \quad (5.22)$$

with

$$g_{j,K}(\mathbf{k}) := \chi_K(k_1) \chi_K(k_2) \chi_K(k_3) g_j(\mathbf{k}), \quad \mathbf{k} \in \mathbf{R}^3, \quad (5.23)$$

where  $\chi_K$  is the characteristic function of  $\{\lambda \in \mathbf{R} \mid |\lambda| \leq K\}$ . Hamiltonians with cutoffs are defined by

$$H_\tau(\mathbf{p}; K, V) := h_D(\mathbf{p}) + L_V(\tau) + H_{I,K,V}(\mathbf{g}), \quad (5.24)$$

$$H_\tau(\mathbf{p}; K) := h_D(\mathbf{p}) + L(\tau) + H_{I,K}(\mathbf{g}). \quad (5.25)$$

**Lemma 5.2** *Let  $s > 0$ . Then:*

(i)  $D(L(\tau)^s) \subset D(N^s)$  and, for all  $\Psi \in D(L(\tau)^s)$ ,

$$\|N^s \Psi\| \leq M^{-s} \|L(\tau)^s \Psi\|, \quad (5.26)$$

where  $N$  is the number operator on  $\mathcal{F}_{\text{rad}}$  given by (4.89).

(ii)  $D(L_V(\tau)^s) \subset D(N^s)$  and, for all  $\Psi \in D(L_V(\tau)^s)$ ,

$$\|N^s \Psi\| \leq M^{-s} \|L_V(\tau)^s \Psi\|. \quad (5.27)$$

*Proof.* (i) For all  $\mathbf{k}_j \in \mathbf{R}^3, j = 1, \dots, n$ , we have the following operator inequality:

$$\sum_{j=1}^n \{\omega(\mathbf{k}_j) - \tau \boldsymbol{\alpha} \cdot \mathbf{k}_j\} \geq \sum_{j=1}^n \mu_{-, \tau}(\mathbf{k}_j) \geq nM,$$

where we have used (H.3). From this estimate and the spectral theorem of self-adjoint operator it follows that  $D(L(\tau)^s) \subset D(N^s)$  and, for all  $\Psi \in D(L(\tau)^s)$ ,

$$\|L(\tau)^s \Psi\|^2 \geq M^{2s} \|N^s \Psi\|^2. \quad (5.28)$$

Hence we obtain (5.26).

(ii) Similar to the proof of part (i). ■

**Remark 5.1** It is easy to see that, for all  $t \in \mathbf{R}$ ,  $e^{itN} L(\tau) \subset L(\tau) e^{itN}$ , which implies that  $N$  and  $L(\tau)$  strongly commute.

Hypothesis (H.4)-(ii) implies that, for every  $\varepsilon > 0$ , there exists a constant  $\delta(\varepsilon) > 0$  such that, for all  $\mathbf{k}, \mathbf{k}' \in \mathbf{R}^3$  with  $|\mathbf{k} - \mathbf{k}'| < \delta(\varepsilon)$

$$|\omega(\mathbf{k}) - \omega(\mathbf{k}')| < \varepsilon. \quad (5.29)$$

Hence, putting

$$V_\varepsilon := \frac{\sqrt{3}\pi}{\delta(\varepsilon)}, \quad (5.30)$$

we have for all  $V > V_\varepsilon$ ,

$$|\omega(\mathbf{k}) - \omega_V(\mathbf{k})| < \varepsilon, \quad \mathbf{k} \in \mathbf{R}^3. \quad (5.31)$$

**Lemma 5.3** Let  $V > V_\varepsilon$  and

$$C_V(\varepsilon) := \varepsilon + \frac{\sqrt{3}\pi}{V}. \quad (5.32)$$

Then  $D(L_V(\tau)) = D(L(\tau))$  and, for all  $\Psi \in D(L(\tau))$ ,

$$\|L_V(\tau)\Psi - L(\tau)\Psi\| \leq C_V(\varepsilon) \|N\Psi\|. \quad (5.33)$$

*Proof.* For all  $\mathbf{k}_j \in \mathbf{R}^3, j = 1, \dots, n$ , we have

$$\sum_{j=1}^n (\omega(\mathbf{k}_j) - \tau \boldsymbol{\alpha} \cdot \mathbf{k}_j) = G_V(\mathbf{k}_1, \dots, \mathbf{k}_n) + \sum_{j=1}^n (\omega_V(\mathbf{k}_j) - \tau \boldsymbol{\alpha} \cdot \mathbf{k}_{jV}) \quad (5.34)$$

where

$$G_V(\mathbf{k}_1, \dots, \mathbf{k}_n) := \sum_{j=1}^n \{[\omega(\mathbf{k}_j) - \omega_V(\mathbf{k}_j)] - \tau \boldsymbol{\alpha} \cdot (\mathbf{k}_j - \mathbf{k}_{jV})\}.$$

By (5.31) and (5.15), we have

$$\|G_V(\mathbf{k}_1, \dots, \mathbf{k}_n)\|_{M_4(\mathbf{C})} \leq C_V(\varepsilon)n. \quad (5.35)$$

It follows that  $D(N) \cap D(L_V(\tau)) \subset D(L(\tau))$  and, for all  $\Psi \in D(N) \cap D(L_V(\tau))$ ,

$$\|L(\tau)\Psi\| \leq \sqrt{2}(C_V(\varepsilon)\|N\Psi\| + \|L_V(\tau)\Psi\|).$$

But, by Lemma 5.2,  $D(N) \cap D(L_V(\tau)) = D(L_V(\tau))$ . Hence  $D(L_V(\tau)) \subset D(L(\tau))$ . Replacing the role of  $L(\tau)$  and  $L_V(\tau)$  in this argument, we obtain  $D(L_V(\tau)) \supset D(L(\tau))$ . Thus the first assertion follows.

Estimate (5.33) follows from (5.35). ■

For the time being, we assume the following (H.5) in addition to (H.3) and (H.4):

**Hypothesis (H.5)** Each  $g_j^{(r)}$  ( $j = 1, 2, 3; r = 1, 2$ ) is continuous on  $\mathbf{R}^3$ .

Under Hypothesis (H.5), we can define

$$g_{j,K,V}^{(r)}(\mathbf{k}) := \sum_{\mathbf{k}' \in \Gamma_V; |\mathbf{k}'| \leq K, \ell=1,2,3} g_j^{(r)}(\mathbf{k}') \chi_{\mathbf{k}',V}(\mathbf{k}). \quad (5.36)$$

Let

$$g_{j,K,V} := (g_{j,K,V}^{(1)}, g_{j,K,V}^{(2)}) \in \mathcal{H}_{\text{ph}}. \quad (5.37)$$

Then we have

$$H_{I,K,V}(\mathbf{g}) = -q \sum_{j=1}^3 \alpha_j \Phi_S(g_{j,K,V}). \quad (5.38)$$

**Lemma 5.4** Assume (H.5). Then

$$\lim_{V \rightarrow \infty} \|g_{j,K,V}^{(r)} - g_{j,K}^{(r)}\|_{L^2(\mathbf{R}^3)} = 0. \quad (5.39)$$

*Proof.* Similar to the proof of [6, Lemma 3.2]. ■

**Lemma 5.5** Assume (H.2). Let  $b_\varepsilon(\mathbf{g})$  be as in (4.43). Then, for all  $\varepsilon \in (0, 1)$  and  $\Psi \in D(L(\tau))$ ,

$$\|L(\tau)\Psi\| \leq \frac{1}{1-\varepsilon}\|H_\tau(\mathbf{p})\Psi\| + \frac{b_\varepsilon(\mathbf{g}) + \omega_D(\mathbf{p})}{1-\varepsilon}\|\Psi\|. \quad (5.40)$$

In particular, for all  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $L(\tau)(H_\tau(\mathbf{p}) - z)^{-1}$  is bounded.

*Proof.* By (4.42), we have

$$\begin{aligned} \|L(\tau)\Psi\| &= \|[H_\tau(\mathbf{p}) - H_{I,0}(\mathbf{g}) - h_D(\mathbf{p})]\Psi\| \\ &\leq \|H_\tau(\mathbf{p})\Psi\| + \|H_{I,0}(\mathbf{g})\Psi\| + \omega_D(\mathbf{p})\|\Psi\| \\ &\leq \|H_\tau(\mathbf{p})\Psi\| + \varepsilon\|L(\tau)\Psi\| + b_\varepsilon(\mathbf{g})\|\Psi\| + \omega_D(\mathbf{p})\|\Psi\|. \end{aligned}$$

Hence (5.40) follows. ■

Let

$$E_\tau(\mathbf{p}; K, V) := \inf \sigma(H_\tau(\mathbf{p}; K, V)), \quad E_\tau(\mathbf{p}; K) := \inf \sigma(H_\tau(\mathbf{p}; K)), \quad (5.41)$$

and  $R_{\tau,V}(\mathbf{g}_{K,V})$  be the constant  $R_\tau(\mathbf{g})$  with  $\nu_{-, \tau}(\mathbf{k})$  and  $g_j$  replaced by  $\nu_{-, \tau}(\mathbf{k}_V)$  and  $g_{j,K,V}$  respectively. Then, in the same way as in Proposition 4.4, we can show that

$$E_\tau(\mathbf{p}; K, V) \geq -\omega_D(\mathbf{p}) - q^2\|R_{\tau,V}(\mathbf{g}_{K,V})\|, \quad (5.42)$$

$$E_\tau(\mathbf{p}; K) \geq -\omega_D(\mathbf{p}) - q^2\|R_\tau(\mathbf{g}_K)\|. \quad (5.43)$$

Let

$$Q_\tau(\mathbf{p}; \mathbf{g}) := \omega_D(\mathbf{p}) + q^2\|R_\tau(\mathbf{g})\|. \quad (5.44)$$

By Lemma 5.4, we have

$$\lim_{V \rightarrow \infty} \|R_{\tau,V}(\mathbf{g}_{K,V})\| = \|R_\tau(\mathbf{g}_K)\|. \quad (5.45)$$

Hence, for every  $\delta > 0$ , there exists a constant  $R_\delta > 0$  such that, for all  $V > R_\delta$ ,

$$q^2 \left| \|R_{\tau,V}(\mathbf{g}_{K,V})\| - \|R_\tau(\mathbf{g}_K)\| \right| < \delta, \quad (5.46)$$

$$(-\infty, -Q_\tau(\mathbf{p}; \mathbf{g}_K) - \delta] \subset \varrho(H_\tau(\mathbf{p}; K, V)) \cap \varrho(H_\tau(\mathbf{p}; K)). \quad (5.47)$$

**Lemma 5.6** Assume (H.3)–(H.5). Let  $\delta > 0$ . Then, for all  $z \in \mathbb{C} \setminus \mathbb{R}$  or all  $z < -\delta - Q_\tau(\mathbf{p}; \mathbf{g}_K)$ ,

$$\lim_{V \rightarrow \infty} \|(H_\tau(\mathbf{p}; K, V) - z)^{-1} - (H_\tau(\mathbf{p}; K) - z)^{-1}\| = 0. \quad (5.48)$$

*Proof.* Let  $V > \max\{V_\varepsilon, R_\delta\}$ . Then, by Lemma 5.3, we can write

$$(H_\tau(\mathbf{p}; K, V) - z)^{-1} - (H_\tau(\mathbf{p}; K) - z)^{-1} = I_V + II_V,$$

where

$$\begin{aligned} I_V &:= (H_\tau(\mathbf{p}; K, V) - z)^{-1}[L(\tau) - L_V(\tau)](H_\tau(\mathbf{p}; K) - z)^{-1}, \\ II_V &:= (H_\tau(\mathbf{p}; K, V) - z)^{-1}[H_{I,K}(\mathbf{g}) - H_{I,K,V}(\mathbf{g})](H_\tau(\mathbf{p}; K) - z)^{-1}. \end{aligned}$$

By Lemma 5.5,  $L(\tau)(H_\tau(\mathbf{p}; K) - z)^{-1}$  is bounded (the case where  $g_j^{(r)}$  is replaced by  $g_{j,K}^{(r)}$ ).

Let  $z \in \mathbb{C} \setminus \mathbb{R}$  first. Then, by (5.33) and (5.26), we have

$$\|I_V\| \leq M^{-1}|\Im z|^{-1}C_V(\varepsilon)\|L(\tau)(H_\tau(\mathbf{p}; K) - z)^{-1}\|,$$

which implies that  $\limsup_{V \rightarrow \infty} \|I_V\| \leq \varepsilon M^{-1}|\Im z|^{-1}\|L(\tau)(H_\tau(\mathbf{p}; K) - z)^{-1}\|$ . Taking  $\varepsilon \rightarrow 0$  gives  $\lim_{V \rightarrow \infty} \|I_V\| = 0$ .

As for  $II_V$ , we have from (5.38) and (4.42)

$$\|[H_{I,K}(\mathbf{g}) - H_{I,K,V}(\mathbf{g})]\Psi\| \leq \varepsilon\|L(\tau)\Psi\| + b_\varepsilon(\mathbf{g}_K - \mathbf{g}_{K,V})\|\Psi\|, \quad \Psi \in D(L(\tau)), \quad (5.49)$$

where

$$\mathbf{g}_K := (g_{1,K}, g_{2,K}, g_{3,K}), \quad \mathbf{g}_{K,V} := (g_{1,K,V}, g_{2,K,V}, g_{3,K,V}).$$

Hence

$$\|II_V\| \leq |\Im z|^{-1}(\varepsilon\|L(\tau)(H_\tau(\mathbf{p}; K) - z)^{-1}\| + b_\varepsilon(\mathbf{g}_K - \mathbf{g}_{K,V})|\Im z|^{-1}).$$

By Lemma 5.4 and the estimate

$$\left\| \frac{\mathbf{g}}{\sqrt{\mu_{-, \tau}}} \right\| \leq \frac{1}{\sqrt{M}}\|\mathbf{g}\|,$$

we have

$$\lim_{V \rightarrow \infty} b_\varepsilon(\mathbf{g}_K - \mathbf{g}_{K,V}) = 0.$$

Hence

$$\limsup_{V \rightarrow \infty} \|II_V\| \leq |\Im z|^{-1}\varepsilon\|L(\tau)(H_\tau(\mathbf{p}; K) - z)^{-1}\|.$$

Taking  $\varepsilon \rightarrow 0$  gives  $\lim_{V \rightarrow \infty} \|II_V\| = 0$ . Thus we obtain (5.48). Similarly we can prove (5.48) for all  $z < -\delta - Q_\tau(\mathbf{p}; \mathbf{g}_K)$ .  $\blacksquare$

**Lemma 5.7** Assume (H.3) and (H.4). Then, for all  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$\lim_{V \rightarrow \infty} \|(H_\tau(\mathbf{p}; K) - z)^{-1} - (H_\tau(\mathbf{p}) - z)^{-1}\| = 0. \quad (5.50)$$

*Proof.* We have

$$\begin{aligned} & \| (H_\tau(\mathbf{p}; K) - z)^{-1} - (H_\tau(\mathbf{p}) - z)^{-1} \| \\ &= (H_\tau(\mathbf{p}; K) - z)^{-1} [H_I(\mathbf{g}) - H_{I,K}(\mathbf{g})] (H_\tau(\mathbf{p}) - z)^{-1}. \end{aligned}$$

Using the Lebesgue dominated convergence theorem, we can show that

$$\lim_{K \rightarrow \infty} \| \mathbf{g} - \mathbf{g}_K \| = 0. \quad (5.51)$$

Hence, in the same way as in the proof of Lemma 5.6, we obtain (5.50).  $\blacksquare$

**Remark 5.2** In Lemma 5.7, we do not need Hypothesis (H.5), since (5.51) holds for all  $g_j \in \mathcal{H}_{\text{ph}}, j = 1, 2, 3$ .

**Lemma 5.8** Assume (H.3) and (H.4). Let  $\delta > 0$  and take  $R_\delta$  such that (5.46) and (5.47) hold. Let  $t \in \mathbf{R}$  such that  $t_K := t - q^2 \|R_\tau(\mathbf{g}_K)\| - \delta > 0$ . Then, for all  $V > R_\delta$ ,

$$-t - \omega_D(\mathbf{p}) \in \varrho(H_\tau(\mathbf{p}; K, V)) \cap \varrho(H_\tau(\mathbf{p}; K)) \quad (5.52)$$

and

$$\lim_{V \rightarrow \infty} \| (H_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} - (H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1} \| = 0 \quad (5.53)$$

uniformly in  $\mathbf{p} \in \mathbf{R}^3$ . Moreover,

$$\lim_{V \rightarrow \infty} E_\tau(\mathbf{p}; K, V) = E_\tau(\mathbf{p}; K) \quad (5.54)$$

uniformly in  $\mathbf{p} \in \mathbf{R}^3$ .

*Proof.* Let  $\varepsilon > 0$  and  $V > \max\{V_\varepsilon, R_\delta\}$ . Then (5.52) follows from (5.47). In the same way as in the proof of Lemma 5.6 we can write

$$(H_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} - (H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1} = A_V + B_V,$$

where

$$\begin{aligned} A_V &:= (H_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} [L(\tau) - L_V(\tau)] (H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}, \\ B_V &:= (H_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} [H_{I,K}(\mathbf{g}) - H_{I,K,V}(\mathbf{g})] \\ &\quad \times (H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}. \end{aligned}$$

We have

$$\| (H_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} \| = \frac{1}{|E_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t|}. \quad (5.55)$$

By (5.42) and (5.46), we have

$$\begin{aligned} E_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t &\geq t - q^2 \|R_\tau(\mathbf{g}_{K,V})\| \\ &\geq t_K > 0. \end{aligned} \quad (5.56)$$

By this estimate and Lemma 5.3, we have

$$\begin{aligned} \|A_V\| &\leq t_K^{-1} \|[L(\tau) - L_V(\tau)](H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\| \\ &\leq t_K^{-1} C_V(\varepsilon) Y(\mathbf{p}), \end{aligned}$$

where

$$Y(\mathbf{p}) := \|N(H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\|.$$

By Hypothesis (H.3), we have

$$L(\tau) - MN \geq 0. \quad (5.57)$$

Note that  $N$  strongly commutes with the nonnegative self-adjoint operator  $L(\tau) - MN + h_D(\mathbf{p}) + \omega_D(\mathbf{p})$ . Hence, for all  $\Psi \in D(L(\tau))$ ,

$$\begin{aligned} M\|N\Psi\| &\leq \|[MN + [L(\tau) - MN + h_D(\mathbf{p}) + \omega_D(\mathbf{p})]]\Psi\| \\ &= \|[L(\tau) + h_D(\mathbf{p}) + \omega_D(\mathbf{p})]\Psi\| \\ &= \|[H_\tau(\mathbf{p}) + \omega_D(\mathbf{p}) - H_{I,0}(\mathbf{g})]\Psi\| \\ &\leq \|[H_\tau(\mathbf{p}) + \omega_D(\mathbf{p})]\Psi\| + \|H_{I,0}(\mathbf{g})\Psi\|. \end{aligned}$$

By the well known estimates

$$\begin{aligned} \|a(F)\psi\| &\leq \|F\| \|N^{1/2}\psi\|, \quad \|a(F)^*\psi\| \leq \|F\| \|(N+1)^{1/2}\psi\|, \\ \psi &\in D(N^{1/2}), \quad F \in \mathcal{H}_{\text{ph}}, \end{aligned} \quad (5.58)$$

we have

$$\begin{aligned} \|H_{I,0}(\mathbf{g})\Psi\| &\leq \sqrt{2}|q|\|\mathbf{g}\|\|(N+1)^{1/2}\Psi\| \\ &\leq \sqrt{2}|q|\|\mathbf{g}\|(\|\Psi\|^{1/2}\|N\Psi\|^{1/2} + \|\Psi\|) \\ &\leq a\|N\Psi\| + c_a(\mathbf{g})\|\Psi\|, \end{aligned} \quad (5.59)$$

where

$$c_a(\mathbf{g}) := \sqrt{2}|q|\|\mathbf{g}\| + \frac{q^2}{2a}\|\mathbf{g}\|^2$$

with  $a > 0$  arbitrary. Hence, taking  $0 < a < M$ , we obtain

$$(M - a)\|N\Psi\| \leq \|[H_\tau(\mathbf{p}) + \omega_D(\mathbf{p})]\Psi\| + c_a(\mathbf{g})\|\Psi\|. \quad (5.60)$$

Hence

$$\begin{aligned} (M - a)Y(\mathbf{p}) &\leq \|[H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p})](H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\| \\ &\quad + c_a(\mathbf{g}_K)\|[H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t]^{-1}\| \\ &\leq 1 + t\|[H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t]^{-1}\| \\ &\quad + c_a(\mathbf{g}_K)\|[H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t]^{-1}\|. \end{aligned}$$

We can show that

$$\|(H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\| = (E_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1} \quad (5.61)$$

$$\leq (t_K + \delta)^{-1}. \quad (5.62)$$

Hence  $\sup_{\mathbf{p} \in \mathbf{R}^3} Y(\mathbf{p}) < \infty$ . Thus it follows that  $\lim_{V \rightarrow \infty} \|A_V\| = 0$  uniformly in  $\mathbf{p}$ .

As for  $B_V$ , we have by (5.49) and (5.59)

$$\begin{aligned} \|B_V\| &\leq t_K^{-1} \|[H_{I,K}(\mathbf{g}) - H_{I,K,V}(\mathbf{g})](H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\| \\ &\leq t_K^{-1} (a \|N(H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\| \\ &\quad + c_a(\mathbf{g}_{K,V} - \mathbf{g}_K) \|(H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}\|). \end{aligned}$$

By Lemma 5.4,

$$c_a(\mathbf{g}_{K,V} - \mathbf{g}_K) \rightarrow 0 \quad (V \rightarrow \infty). \quad (5.63)$$

By this fact and the preceding discussions, one can easily see that  $\lim_{V \rightarrow \infty} \|B_V\| = 0$  uniformly in  $\mathbf{p}$ . Thus (5.53) holds uniformly in  $\mathbf{p} \in \mathbf{R}^3$ .

(5.53) implies that

$$\lim_{V \rightarrow \infty} (E_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) + t)^{-1} = (E_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1}$$

uniformly in  $\mathbf{p} \in \mathbf{R}^3$ . By (5.56), (5.62) and Proposition 4.6,

$$\begin{aligned} t_K &\leq \sup_{V \gg 1, \mathbf{p} \in \mathbf{R}^3} E_\tau(\mathbf{p}; K, V) + \omega_D(\mathbf{p}) < \infty, \\ t_K + \delta &\leq \sup_{\mathbf{p} \in \mathbf{R}^3} E_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) < \infty. \end{aligned}$$

Thus (5.54) holds uniformly in  $\mathbf{p} \in \mathbf{R}^3$ . ■

**Lemma 5.9** *Assume (H.3) and (H.4). Let  $t > q^2 \|R_\tau(\mathbf{g})\|$ . Then, for all sufficiently large  $K > 0$ ,  $-t - \omega_D(\mathbf{p}) \in \varrho(H_\tau(\mathbf{p}; K)) \cap \varrho(H_\tau(\mathbf{p}))$  and*

$$\lim_{K \rightarrow \infty} \|(H_\tau(\mathbf{p}; K) + \omega_D(\mathbf{p}) + t)^{-1} - (H_\tau(\mathbf{p}) + \omega_D(\mathbf{p}) + t)^{-1}\| = 0 \quad (5.64)$$

uniformly in  $\mathbf{p} \in \mathbf{R}^3$ . Moreover,

$$\lim_{V \rightarrow \infty} E_\tau(\mathbf{p}; K) = E_\tau(\mathbf{p}) \quad (5.65)$$

uniformly in  $\mathbf{p} \in \mathbf{R}^3$ .

*Proof.* Similar to the proof of Lemma 5.8. ■

## 5.2 Reduction of $H_\tau(\mathbf{p}; K, V)$

We have the orthogonal decomposition

$$\mathcal{F}_{\text{rad}} = \mathcal{F}_{\text{rad},V} \oplus \mathcal{F}_{\text{rad},V}^\perp. \quad (5.66)$$

Hence

$$\oplus^4 \mathcal{F}_{\text{rad}} = (\oplus^4 \mathcal{F}_{\text{rad},V}) \oplus (\oplus^4 \mathcal{F}_{\text{rad},V}^\perp). \quad (5.67)$$

We denote by  $P_V$  the orthogonal projection from  $\oplus^4 \mathcal{F}_{\text{rad}}$  onto  $\oplus^4 \mathcal{F}_{\text{rad},V}$ .

**Lemma 5.10** *Assume (H.3)–(H.5). Then, all  $\mathbf{p} \in \mathbf{R}^3$ ,  $V > 0$  and  $K > 0$ ,*

$$P_V H_\tau(\mathbf{p}; K, V) \subset H_\tau(\mathbf{p}; K, V) P_V \quad (5.68)$$

*i.e.,  $H_\tau(\mathbf{p}; K, V)$  is reduced by  $\oplus^4 \mathcal{F}_{\text{rad},V}$ .*

*Proof.* We can write

$$L^2(\mathbf{R}^3) = F_{1V} \oplus F_{1V}^\perp. \quad (5.69)$$

Let  $p_V$  be the orthogonal projection from  $L^2(\mathbf{R}^3)$  onto  $F_{1V}$  (i.e., for all  $f \in C(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$ ,  $(p_V f)(\mathbf{k}) = f(\mathbf{k}')$  if  $\mathbf{k} \in C(\mathbf{k}', V)$ ). We use the natural identification

$$\oplus^4 \mathcal{F}_{\text{rad}} = \mathbf{C}^4 \otimes \mathcal{F}_{\text{rad}}.$$

Let  $f_j, g_\ell \in L^2(\mathbf{R}^3)$ ,  $j = 1, \dots, m$ ,  $\ell = 1, \dots, n$ ,  $z \in \mathbf{C}^4$  and  $\Phi \in \mathbf{C}^4 \otimes \mathcal{F}_{\text{rad}}$  be of the form

$$\Phi = z \otimes a^{(1)}(f_1)^* \dots a^{(1)}(f_m)^* a^{(2)}(g_1)^* \dots a^{(2)}(g_n)^* \Omega_0.$$

Then we have

$$P_V \Phi = z \otimes a^{(1)}(p_V f_1)^* \dots a^{(1)}(p_V f_m)^* a^{(2)}(p_V g_1)^* \dots a^{(2)}(p_V g_n)^* \Omega_0.$$

Using this expression and the easily proven fact that

$$p_V \omega_V \subset \omega_V p_V, \quad (5.70)$$

we see that  $H_{\text{rad},V} P_V = P_V H_{\text{rad},V}$  on  $\mathbf{C}^4 \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(D(\omega_V))$ . Since  $\mathbf{C}^4 \otimes_{\text{alg}} \mathcal{F}_{\text{rad}}^{\text{fin}}(D(\omega_V))$  is a core of  $H_{\text{rad},V}$ , it follows that

$$P_V H_{\text{rad},V} \subset H_{\text{rad},V} P_V. \quad (5.71)$$

Similarly we have

$$P_V P_{Vj}^{\text{rad}} \subset P_{Vj}^{\text{rad}} P_V, \quad j = 1, 2, 3. \quad (5.72)$$

Hence we obtain

$$P_V L_V(\tau) \subset L_V(\tau) P_V. \quad (5.73)$$

Note that  $g_{j,K,V}^{(r)}$  is in  $F_{1V}$ . Hence it follows that  $H_{I,K,V}(\mathbf{g}) P_V = P_V H_{I,K,V}(\mathbf{g})$  on  $D(L_V(\tau))$ . It is obvious that  $P_V h_D(\mathbf{p}) \subset h_D(\mathbf{p}) P_V$ . Thus we obtain (5.68).  $\blacksquare$

Relation (5.71) implies that  $H_{\text{rad},V}$  is reduced by  $\oplus^4 \mathcal{F}_{\text{rad},V}$ . We can show that the reduced part  $H_{\text{rad},V}|_{\oplus^4 \mathcal{F}_{\text{rad},V}}$  is of the form

$$H_{\text{rad},V}|_{\oplus^4 \mathcal{F}_{\text{rad},V}} = d\Gamma_V(\omega_V), \quad (5.74)$$

the second quantization of  $\omega_V$  in  $\mathcal{F}_{\text{rad},V}$ . An informal expression of  $H_{\text{rad},V}|_{\oplus^4 \mathcal{F}_{\text{rad},V}}$  is given by

$$H_{\text{rad},V}|_{\oplus^4 \mathcal{F}_{\text{rad},V}} = \sum_{r=1}^2 \sum_{\mathbf{k} \in \Gamma_V} \omega(\mathbf{k}) a_V^{(r)}(\mathbf{k})^* a^{(r)}(\mathbf{k})|_{\oplus^4 \mathcal{F}_{\text{rad},V}}.$$

Similarly it follows from (5.72) that each  $P_{Vj}^{\text{rad}}$  is by  $\oplus^4 \mathcal{F}_{\text{rad},V}$  and

$$P_{Vj}^{\text{rad}}|_{\oplus^4 \mathcal{F}_{\text{rad},V}} = d\Gamma_V(k_{Vj}). \quad (5.75)$$

**Lemma 5.11** *Let  $A$  and  $B$  be strongly commuting self-adjoint operators on a Hilbert space  $\mathcal{X}$  such that  $D(B) \subset D(A)$  and, for all  $\psi \in D(B)$ ,*

$$0 \leq (\psi, A\psi) \leq (\psi, B\psi). \quad (5.76)$$

*Let  $s, c > 0$ . Then:*

(i) *For all  $\psi \in \mathcal{X}$ ,*

$$\|(B+c)^{-s}\psi\| \leq \|(A+c)^{-s}\psi\|. \quad (5.77)$$

(ii) *If  $(A+c)^{-s}$  is compact, then  $(B+c)^{-s}$  is compact.*

*Proof.* (i) We denote by  $E_A$  the spectral measure of the self-adjoint operator  $A$ . By the strong commutativity of  $A$  and  $B$ , there exists a two-dimensional spectral measure  $E_{A,B}$  such that, for all Borel sets  $S_1, S_2 \subset \mathbf{R}$ ,  $E_{A,B}(S_1 \times S_2) = E_A(S_1)E_B(S_2)$  and  $A = \int_{\mathbf{R}^2} \lambda dE_{A,B}(\lambda, \mu)$ ,  $B = \int_{\mathbf{R}^2} \mu dE_{A,B}(\lambda, \mu)$ . Then (5.76) implies that the support of  $E_{A,B}$  is a subset of  $S_{A,B} := \{(\lambda, \mu) \in \mathbf{R}^2 | 0 \leq \lambda \leq \mu\}$ . Hence

$$\begin{aligned} & \| (A+c)^{-s}\psi \|^2 - \| (B+c)^{-s}\psi \|^2 \\ &= \int_{S_{A,B}} \left\{ \frac{1}{(\lambda+c)^{2s}} - \frac{1}{(\mu+c)^{2s}} \right\} d\|E_{A,B}(\lambda, \mu)\psi\|^2 \\ &= \int_{S_{A,B}} \frac{(\mu+c)^{2s} - (\lambda+c)^{2s}}{(\lambda+c)^{2s}(\mu+c)^{2s}} d\|E_{A,B}(\lambda, \mu)\psi\|^2 \\ &\geq 0. \end{aligned}$$

Thus (5.77) follows.

(ii) Let  $(A+c)^{-s}$  be compact and  $\{\psi_n\}_n$  be a bounded sequence of  $\mathcal{X}$ . Then there is a subsequence  $\{\phi_n\}_n$  of  $\{\psi_n\}_n$  such that  $(A+c)^{-s}\phi_n$  converges to a vector in  $\mathcal{X}$  as  $n \rightarrow \infty$ . By (5.77),  $\{(B+c)^{-s}\phi_n\}_n$  is a Cauchy sequence. Hence it converges as  $n \rightarrow \infty$ . Thus  $(B+c)^{-s}$  is compact. ■

By (5.73), we can define

$$L_V(\tau)_{\text{red}} := L_V(\tau)|_{\oplus^4 \mathcal{F}_{\text{rad},V}} \quad (5.78)$$

the reduced part of  $L_V(\tau)$  to  $\oplus^4 \mathcal{F}_{\text{rad},V}$  of  $L_V(\tau)$ .

**Lemma 5.12** *Assume (5.7). Then  $L_V(\tau)_{\text{red}}$  has compact resolvent.*

*Proof.* Let

$$\mu_{-, \tau, V}(\mathbf{k}) := \mu_{-, \tau}(\mathbf{k}_V), \quad \mathbf{k} \in \mathbb{R}^3,$$

and  $d\Gamma_V(\mu_{-, \tau, V})$  be the second quantization of  $\mu_{-, \tau, V}$  in  $\mathcal{F}_{\text{rad},V}$ . Then we have

$$\begin{aligned} \sigma(d\Gamma_V(\mu_{-, \tau, V})) &= \sigma_{\text{p}}(d\Gamma_V(\mu_{-, \tau, V})) \\ &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \mu_{-, \tau}(\mathbf{k}_j) \mid \mathbf{k}_j \in \Gamma_V, j = 1, \dots, n \right\}}. \end{aligned}$$

By the assumption on  $\mu_{-, \tau}$ , it turns out that  $d\Gamma_V(\mu_{-, \tau, V})$  has purely discrete spectrum and the eigenvalues of  $d\Gamma_V(\mu_{-, \tau, V})$  are labeled as  $\{\lambda_n\}_{n=1}^{\infty}$ , where  $\lambda_1 < \lambda_2 < \dots$  with the multiplicity of each eigenvalue  $\lambda_j$  being finite and  $\lim_{n \rightarrow \infty} \lambda_n \rightarrow \infty$ . Hence, by a general criterion [21, Theorem XIII.64(v)],  $d\Gamma_V(\mu_{-, \tau, V})$  has compact resolvent.

It is easy to see that, for all  $t \in \mathbb{R}$ ,

$$e^{itd\Gamma_V(\mu_{-, \tau, V})} L_V(\tau)_{\text{red}} \subset L_V(\tau)_{\text{red}} e^{itd\Gamma_V(\mu_{-, \tau, V})},$$

which implies that  $L_V(\tau)_{\text{red}}$  and  $d\Gamma_V(\mu_{-, \tau, V})$  strongly commute. By (4.23), we have

$$0 \leq d\Gamma_V(\mu_{-, \tau, V}) \leq L_V(\tau)_{\text{red}}. \quad (5.79)$$

Hence, applying Lemma 5.9, we conclude that  $L_V(\tau)_{\text{red}}$  has compact resolvent. ■

We denote by  $H_{\tau}(\mathbf{p}; K, V)_{\text{red}}$  the reduced part of  $H_{\tau}(\mathbf{p}; K, V)$  to  $\oplus^4 \mathcal{F}_{\text{rad},V}$ :

$$H_{\tau}(\mathbf{p}; K, V)_{\text{red}} := H_{\tau}(\mathbf{p}; K, V)|_{\oplus^4 \mathcal{F}_{\text{rad},V}}. \quad (5.80)$$

**Lemma 5.13** *Assume (H.3)–(H.5) and (5.7). Then  $H_{\tau}(\mathbf{p}; K, V)_{\text{red}}$  has compact resolvent.*

*Proof.* The operator  $H_{\tau}(\mathbf{p}; K, V)_{\text{red}}$  is self-adjoint on

$$D(L_V(\tau)_{\text{red}}) = D(L_V(\tau)_{\text{red}}) \cap D(h_D(\mathbf{p}) + H_{I,K,V}(\mathbf{g})_{\text{red}})$$

and bounded below, where  $H_{I,K,V}(\mathbf{g})_{\text{red}}$  is the restriction of  $H_{I,K,V}(\mathbf{g})$  to  $\oplus^4 \mathcal{F}_{\text{rad},V}$ . By Lemma 5.10,  $L_V(\tau)_{\text{red}}$  has compact resolvent. Hence, applying [6, Lemma 3.8], we obtain the desired result. ■

**Remark 5.3** Since  $h_D(\mathbf{p}) + H_{I,K,V}(\mathbf{g})$  is infinitesimally small with respect to  $L_V(\tau)$ , Lemma 5.13 can be proven also by using the min-max principle (cf. [16, Proposition 4.11], [21, Theorems XIII.1, XIII.64]).

### 5.3 Lower bound of $H_\tau(\mathbf{p}; K, V)$ on $(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp$

Let

$$\Delta_\tau(\mathbf{p}; K, V) := \inf_{n \geq 1} \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbb{R}^3} \left\{ E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j; K, V \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) \right\} - E_\tau(\mathbf{p}; K, V). \quad (5.81)$$

**Lemma 5.14** *Assume (H.3)–(H.5). Then*

$$H_\tau(\mathbf{p}; K, V)|(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp \geq E_\tau(\mathbf{p}; K) + \Delta_\tau(\mathbf{p}; K, V). \quad (5.82)$$

*Proof.* Let  $\mathcal{H}_V := F_{1V} \oplus F_{1V} \subset \mathcal{H}_{\text{ph}}$ . Then we have  $\mathcal{H}_{\text{ph}} = \mathcal{H}_V \oplus \mathcal{H}_V^\perp$ . For a Hilbert space  $\mathcal{X}$ , we denote by  $\mathcal{F}^{(n)}(\mathcal{X}) (= \otimes_s^n \mathcal{X})$  (resp.  $a_{\mathcal{X}}(\cdot)$ ,  $\Omega_{\mathcal{X}}$ ) the  $n$ -particle space of  $\mathcal{F}_s(\mathcal{X})$  (resp. the annihilation operator on  $\mathcal{F}_s(\mathcal{X})$ , the Fock vacuum of  $\mathcal{F}_s(\mathcal{X})$ ). Let

$$\mathcal{H}^{(n)} := \oplus_{j=0}^n \mathcal{F}^{(j)}(\mathcal{H}_V) \otimes \mathcal{F}^{(n-j)}(\mathcal{H}_V^\perp), \quad n \geq 0.$$

Then  $\mathcal{F}_{\text{rad}}$  has the following natural identification:

$$\mathcal{F}_{\text{rad}} = \oplus_{n=0}^\infty \mathcal{H}^{(n)},$$

where the unitary correspondence for this identification is given by

$$a(f) \rightarrow a_{\mathcal{H}_V}(f_V) \otimes I + I \otimes a_{\mathcal{H}_V^\perp}(f_V^\perp), \quad (5.83)$$

$$\Omega_0 \rightarrow \Omega_{\mathcal{H}_V} \otimes \Omega_{\mathcal{H}_V^\perp}, \quad (5.84)$$

where  $f = f_V + f_V^\perp$ ,  $f_V \in \mathcal{H}_V$ ,  $f_V^\perp \in \mathcal{H}_V^\perp$ . It follows that

$$\mathcal{F}_{\text{rad}, V} = \oplus_{n=0}^\infty [\mathcal{F}^{(n)}(\mathcal{H}_V) \otimes \mathcal{F}^{(0)}(\mathcal{H}_V^\perp)], \quad (5.85)$$

$$\begin{aligned} \mathcal{F}_{\text{rad}, V}^\perp &= \oplus_{n=1}^\infty [\oplus_{r=0}^{n-1} \mathcal{F}^{(r)}(\mathcal{H}_V) \otimes \mathcal{F}^{(n-r)}(\mathcal{H}_V^\perp)] \\ &= \oplus_{n=1}^\infty \mathcal{F}_s(\mathcal{H}_V) \otimes \mathcal{F}^{(n)}(\mathcal{H}_V^\perp). \end{aligned} \quad (5.86)$$

Moreover we use the identification

$$(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp = \mathbb{C}^4 \otimes \mathcal{F}_{\text{rad}, V}^\perp = \mathcal{F}_{\text{rad}, V}^\perp \otimes \mathbb{C}^4.$$

Since  $\omega_V$  and  $k_{Vj}$  are reduced by  $\mathcal{H}_V$  (see (5.70)), it follows that  $L_V(\tau)|(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp$  is of the form

$$L_V(\tau)_{\text{red}} \otimes I_{\mathcal{F}_s(\mathcal{H}_V^\perp)} + I_{\mathcal{F}_{\text{rad}, V}} \otimes L_V(\tau),$$

where  $I_{\mathcal{X}}$  denotes the identity operator on the Hilbert space  $\mathcal{X}$ . Since  $g_{j,K,V}$  is in  $\mathcal{H}_V$ , it follows that  $[h_D(\mathbf{p}) + H_{I,K,V}(\mathbf{g})]|(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp$  is of the form

$$[h_D(\mathbf{p}) + H_{I,K,V}(\mathbf{g})] \otimes I_{\mathcal{F}_s(\mathcal{H}_V^\perp)}.$$

Hence  $H' := H_\tau(\mathbf{p}; K, V) |(\oplus^4 \mathcal{F}_{\text{rad}, V})^\perp$  is of the form

$$H' := H_\tau(\mathbf{p}; K, V) \otimes I_{\mathcal{F}_s(\mathcal{H}_V^\perp)} + I_{\mathcal{F}_{\text{rad}, V}} \otimes L_V(\tau).$$

Let  $\Psi_n = \psi \otimes \phi^{(n)}$ ,  $\psi \in \mathcal{F}_s(\mathcal{H}_V) \otimes \mathbb{C}^4$ ,  $\phi^{(n)} \in \mathcal{F}^{(n)}(\mathcal{H}_V^\perp)$  with  $n \geq 1$ . Then, by the fact just mentioned, we see that

$$\begin{aligned} (\Psi_n, H' \Psi_n) &= \int_{\mathbb{R}^{3n}} (\psi, [H_\tau(\mathbf{p} - \tau \sum_{j=1}^n \mathbf{k}_{jV}; K, V) + \sum_{j=1}^n \omega(\mathbf{k}_{jV})] \psi) |\phi^{(n)}(\mathbf{k}^{(n)})|^2 d\mathbf{k}^{(n)} \\ &\geq \int_{\mathbb{R}^{3n}} \left[ E_\tau \left( \mathbf{p} - \tau \sum_{j=1}^n \mathbf{k}_{jV}; K, V \right) + \sum_{j=1}^n \omega(\mathbf{k}_{jV}) \right] \|\psi\|^2 \\ &\quad \times |\phi^{(n)}(\mathbf{k}^{(n)})|^2 d\mathbf{k}^{(n)} \\ &\geq \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbb{R}^3} \left\{ E_\tau \left( \mathbf{p} - \tau \sum_{j=1}^n \mathbf{k}_j; K, V \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) \right\} \|\Psi_n\|^2 \\ &\geq [E_\tau(\mathbf{p}; K, V) + \Delta_\tau(\mathbf{p}; K, V)] \|\Psi_n\|^2. \end{aligned}$$

Since  $H'$  is reduced by each  $\mathcal{F}_s(\mathcal{H}_V) \otimes \mathbb{C}^4 \otimes \mathcal{F}^{(n)}(\mathcal{H}_V^\perp)$ , (5.82) follows.  $\blacksquare$

## 5.4 Proof of Theorem 6.1

We first consider the case where (H.5) is satisfied in addition to (H.3) and (H.4). We set

$$\Delta_\tau(\mathbf{p}; K) := \inf_{n \geq 1} \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbb{R}^3} \left\{ E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j; K \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) \right\} - E_\tau(\mathbf{p}; K). \quad (5.87)$$

By Lemmas 5.13 and 5.14,  $H_\tau(\mathbf{p}; K, V)$  has purely discrete spectrum in the interval  $[E_\tau(\mathbf{p}; K, V), E_\tau(\mathbf{p}; K, V) + \Delta_{\tau, \mathbf{p}}(K, V)]$ . By Lemma 5.8,

$$\lim_{V \rightarrow \infty} E_\tau(\mathbf{p}; K, V) = E_\tau(\mathbf{p}; K), \quad \lim_{V \rightarrow \infty} \Delta_\tau(\mathbf{p}; K, V) = \Delta_\tau(\mathbf{p}; K).$$

By these facts and Lemma 5.8, we can apply [16, Lemma 4.6] to conclude that  $H_\tau(\mathbf{p}; K)$  has purely discrete spectrum in  $[E_\tau(\mathbf{p}; K), E_\tau(\mathbf{p}; K) + \Delta_\tau(\mathbf{p}; K)]$ . Since we have Lemmas 5.8 and 5.9, we can apply [16, Lemma 4.6] again to conclude that  $H_\tau(\mathbf{p})$  has purely discrete spectrum in  $[E_\tau(\mathbf{p}), E_\tau(\mathbf{p}) + \Delta_\tau(\mathbf{p})]$ . Hence Theorem 5.1 is proved.

We next consider the case where each  $g_j$  is a general element of  $\mathcal{H}_{\text{ph}}$ . To make explicit the  $\mathbf{g}$ -dependence of  $H_\tau(\mathbf{p})$ , we write  $H_\tau(\mathbf{p}) = H_{\tau, \mathbf{g}}(\mathbf{p})$  and put  $E_{\tau, \mathbf{g}}(\mathbf{p}) := \inf \sigma(H_{\tau, \mathbf{g}}(\mathbf{p}))$ . For each  $j = 1, 2, 3$ , there exists a sequence  $\{g_{jn}\}_n \subset \mathcal{H}_{\text{ph}}$  such that  $g_{jn}^{(\tau)}$  is continuous on  $\mathbb{R}^3$  and  $\|g_{jn} - g_j\|_{\mathcal{H}_{\text{ph}}} \rightarrow 0$  ( $n \rightarrow \infty$ ). Let  $\mathbf{g}_n = (g_{1n}, g_{2n}, g_{3n})$ . Then, by the preceding result,  $H_{\tau, \mathbf{g}_n}(\mathbf{p})$  has purely discrete spectrum

in  $[E_{\tau, g_n}(\mathbf{p}), E_{\tau, g_n}(\mathbf{p}) + \Delta_{\tau, g_n}(\mathbf{p})]$ , where  $\Delta_{\tau, g_n}(\mathbf{p})$  is  $\Delta_{\tau}(\mathbf{p})$  with  $g$  replaced by  $g_n$ . Then, in the same way as in Lemma 5.9, we can show that  $H_{\tau, g_n}(\mathbf{p})$  converges to  $H_{\tau, g}(\mathbf{p})$  in the strong resolvent sense as  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} E_{\tau, g_n}(\mathbf{p}) = E_{\tau, g}(\mathbf{p})$  uniformly in  $\mathbf{p} \in \mathbf{R}^3$ . Hence we can apply [16, Lemma 4.6] again to conclude that  $H_{\tau, g}(\mathbf{p})$  has purely discrete spectrum in  $[E_{\tau, g}(\mathbf{p}), E_{\tau, g}(\mathbf{p}) + \Delta_{\tau}(\mathbf{p})]$ . Thus Theorem 5.1 is proved.

## 6 Existence of a Ground State of $H_{\tau}(\mathbf{p})$ (II)— The Massless Case

In this section we consider the case where Hypothesis (H.3) does not necessarily hold. We prove the following theorem.

**Theorem 6.1** *Assume (H.2), (H.4)-(ii) and (5.7). Suppose that  $g_j/\mu_{-, \tau} \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, 2, 3$  with*

$$|q| \left\| \frac{\mathbf{g}}{\mu_{-, \tau}} \right\| < \sqrt{2}. \quad (6.1)$$

*Then, for all  $\mathbf{p} \in \mathbf{R}^3$ ,  $H_{\tau}(\mathbf{p})$  has a ground state  $\Psi_{\tau}(\mathbf{p})$  with  $\|\Psi_{\tau}(\mathbf{p})\| = 1$ . Moreover,  $\Psi_{\tau}(\mathbf{p}) \in D(N^{1/2})$  and*

$$\|N^{1/2}\Psi_{\tau}(\mathbf{p})\| \leq \frac{|q|}{\sqrt{2}} \left\| \frac{\mathbf{g}}{\mu_{-, \tau}} \right\|. \quad (6.2)$$

**Remark 6.1** Theorem 7.1 does not cover the original physical case where  $\omega(\mathbf{k}) = |\mathbf{k}|$  and  $\tau = 1$ . But, for  $0 \leq \tau < 1$ , Theorem 7.1 can be applied to the case  $\omega(\mathbf{k}) = |\mathbf{k}|$ .

**Lemma 6.2** *Assume (H.3) and (H.5). Suppose that  $g_j \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, 2, 3$  and  $\omega$  is continuous on  $\mathbf{R}^3$ . Let  $\Psi_0$  be any ground state of  $H_{\tau}(\mathbf{p})$  with  $\|\Psi_0\| = 1$ . Then  $\Psi_0 \in D(N^{1/2})$  and*

$$\|N^{1/2}\Psi_0\| \leq \frac{|q|}{\sqrt{2}} \left\| \frac{\mathbf{g}}{\mu_{-, \tau}} \right\|. \quad (6.3)$$

*Proof.* For notational simplicity, we set

$$\begin{aligned} H &:= H_{\tau}(\mathbf{p}) \\ \hat{H} &:= H - E_{\tau}(\mathbf{p}) \geq 0. \end{aligned}$$

For each  $\mathbf{k} \in \mathbf{R}^3$  and all  $\Psi \in \oplus^4 \mathcal{F}_{\text{rad}, 0}^S$ , we have

$$[H, a^{(r)}(\mathbf{k})]\Psi = -\nu_{-, \tau}(\mathbf{k})a^{(r)}(\mathbf{k})\Psi + \sum_{j=1}^3 \frac{q}{\sqrt{2}} \alpha_j g_j^{(r)}(\mathbf{k})\Psi, \quad r = 1, 2. \quad (6.4)$$

Hence we have

$$\begin{aligned} (a^{(r)}(\mathbf{k})\Psi, \hat{H}a^{(r)}(\mathbf{k})\Psi) &= (a^{(r)}(\mathbf{k})\Psi, a^{(r)}(\mathbf{k})\hat{H}\Psi) - (a^{(r)}(\mathbf{k})\Psi, \nu_{-, \tau}(\mathbf{k})a^{(r)}(\mathbf{k})\Psi) \\ &\quad + \sum_{j=1}^3 \frac{q}{\sqrt{2}} (a^{(r)}(\mathbf{k})\Psi, \alpha_j g_j^{(r)}(\mathbf{k})\Psi). \end{aligned}$$

Note that the left hand side is nonnegative. Hence

$$\begin{aligned} &(a^{(r)}(\mathbf{k})\Psi, \nu_{-, \tau}(\mathbf{k})a^{(r)}(\mathbf{k})\Psi) \\ &\leq \Re(a^{(r)}(\mathbf{k})\Psi, a^{(r)}(\mathbf{k})\hat{H}\Psi) + \sum_{j=1}^3 \frac{q}{\sqrt{2}} \Re(a^{(r)}(\mathbf{k})\Psi, \alpha_j g_j^{(r)}(\mathbf{k})\Psi). \end{aligned}$$

Multiplying  $\mu_{-, \tau}^{-1}$  to the both sides and integrating with respect to  $d\mathbf{k}$  with summation in  $r = 1, 2$ , we obtain

$$\begin{aligned} &\sum_{r=1}^2 \int_{\mathbb{R}^3} (a^{(r)}(\mathbf{k})\Psi, \mu_{-, \tau}^{-1} \nu_{-, \tau}(\mathbf{k})a^{(r)}(\mathbf{k})\Psi) d\mathbf{k} \\ &\leq \Re(d\Gamma(\mu_{-, \tau}^{-1})\Psi, \hat{H}\Psi) + \sum_{j=1}^3 \frac{q}{\sqrt{2}} \Re(a(\mu_{-, \tau}^{-1} g_j)\Psi, \alpha_j \Psi). \end{aligned}$$

By the operator inequality

$$\mu_{-, \tau}^{-1} \nu_{-, \tau}(\mathbf{k}) \geq 1,$$

we have

$$\sum_{r=1}^2 \int_{\mathbb{R}^3} (a^{(r)}(\mathbf{k})\Psi, \mu_{-, \tau}^{-1} \nu_{-, \tau}(\mathbf{k})a^{(r)}(\mathbf{k})\Psi) d\mathbf{k} \geq (\Psi, N\Psi).$$

Hence

$$(\Psi, N\Psi) \leq \Re(d\Gamma(\mu_{-, \tau}^{-1})\Psi, \hat{H}\Psi) + \sum_{j=1}^3 \frac{q}{\sqrt{2}} \Re(a(\mu_{-, \tau}^{-1} g_j)\Psi, \alpha_j \Psi). \quad (6.5)$$

Since  $\oplus^4 \mathcal{F}_{\text{rad}, 0}^S$  is a core of  $L(\tau)$  (and hence of  $\hat{H}$ ), there exists a sequence  $\{\Psi_n\}_n \subset D(L(\tau))$  such that  $\Psi_n \rightarrow \Psi_0$  and  $\hat{H}\Psi_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\mu_{-, \tau}^{-1} \leq 1/M$ , it follows that

$$\begin{aligned} \|d\Gamma(\mu_{-, \tau}^{-1})\Psi\| &\leq \frac{1}{M} \|N\Psi\| \leq \frac{1}{M^2} \|L(\tau)\Psi\| \\ &\leq C \|(\hat{H} + 1)\Psi\|, \end{aligned}$$

where  $C$  is a constant. Hence it follows that  $d\Gamma(\mu_{-, \tau}^{-1})\Psi_n \rightarrow d\Gamma(\mu_{-, \tau}^{-1})\Psi_0$  as  $n \rightarrow \infty$ . Similarly we can show that

$$a(\mu_{-, \tau}^{-1} g_j)\Psi_n \rightarrow a(\mu_{-, \tau}^{-1} g_j)\Psi_0 \quad (n \rightarrow \infty).$$

Hence (6.5) implies that  $\{N^{1/2}\Psi_n\}_n$  is a Cauchy sequence. Hence  $\Psi_0 \in D(N^{1/2})$  and  $N^{1/2}\Psi_n \rightarrow N^{1/2}\Psi_0$  ( $n \rightarrow \infty$ ). Taking  $n \rightarrow \infty$  in (6.5) with  $\Psi = \Psi_n$ , we obtain

$$\begin{aligned} \|N^{1/2}\Psi_0\|^2 &\leq \sum_{j=1}^3 \frac{q}{\sqrt{2}} \Re(a(\mu_{-, \tau}^{-1} g_j) \Psi_0, \alpha_j \Psi_0) \\ &\leq \sum_{j=1}^3 \frac{|q|}{\sqrt{2}} \|a(\mu_{-, \tau}^{-1} g_j) \Psi_0\| \\ &\leq \sum_{j=1}^3 \frac{|q|}{\sqrt{2}} \|\mu_{-, \tau}^{-1} g_j\| \|N^{1/2}\Psi_0\| \end{aligned}$$

Thus we obtain (6.3). ■

*In the rest of this section, we suppose that the assumption of Theorem 6.1 holds.*  
For each  $M > 0$ , we define

$$\omega_M(\mathbf{k}) := \omega(\mathbf{k}) + M, \quad \mathbf{k} \in \mathbf{R}^3,$$

and

$$\mu_{-, \tau}^{(M)}(\mathbf{k}) := \omega_M(\mathbf{k}) - \tau|\mathbf{k}|, \quad \mathbf{k} \in \mathbf{R}^3.$$

We fix  $\tau$  and  $\mathbf{p}$  and denote by  $H_M$  the operator  $H = H_\tau(\mathbf{p})$  with  $\omega$  replaced by  $\omega_M$ . Let

$$E_M := \inf \sigma(H_M).$$

Note that the assumption of Theorem 5.1 is satisfied with  $\omega$  replaced by  $\omega_M$ . Hence, by Theorem 5.1, there exists a ground state  $\Psi_M$  of  $H_M$  with  $\|\Psi_M\| = 1$ :  $H_M \Psi_M = E_M \Psi_M$ . There is a sequence  $\{M_j\}_{j=1}^\infty$  such that  $M_1 > M_2 > \dots > 0$ ,  $M_j \rightarrow 0$  ( $j \rightarrow \infty$ ) and the weak limit

$$\Psi_0 := \text{w-}\lim_{j \rightarrow \infty} \Psi_{M_j} \tag{6.6}$$

exists. We shall show that this vector  $\Psi_0$  is a ground state of  $H$ .

**Lemma 6.3** (i) *For all  $\Psi \in \oplus^4 \mathcal{F}_{\text{rad}, 0}^S$ ,*

$$\lim_{M \rightarrow 0} H_M \Psi = H \Psi.$$

(ii) *For all  $z \in \mathbf{C} \setminus \mathbf{R}$  and  $\Psi \in \oplus^4 \mathcal{F}_{\text{rad}}$*

$$\lim_{M \rightarrow 0} (H_M - z)^{-1} \Psi = (H - z)^{-1} \Psi.$$

*Proof.* (i) This follows from direct computation using the fact that

$$\lim_{M \rightarrow 0} \omega_M(\mathbf{k}) = \omega(\mathbf{k}), \quad \mathbf{k} \in \mathbf{R}^3.$$

(ii) Note that  $\oplus^4 \mathcal{F}_{\text{rad},0}^S$  is a common core of  $H_M$  and  $H$ . By this fact and part (i), we can apply a general convergence theorem [19, Theorem VIII.25(a)] to obtain the desired result.  $\blacksquare$

We set

$$E_0 := E_\tau(p). \quad (6.7)$$

**Lemma 6.4**

$$\lim_{M \rightarrow 0} E_M = E_0. \quad (6.8)$$

*Proof.* For all  $\Psi \in \oplus^4 \mathcal{F}_{\text{rad},0}^S$ , we have

$$(\Psi, H_M \Psi) = (\Psi, H \Psi) + M(\Psi, N \Psi) \geq (\Psi, H \Psi).$$

Hence

$$(\Psi, H_M \Psi) \geq E_0 \|\Psi\|^2.$$

By this inequality and the fact that  $\oplus^4 \mathcal{F}_{\text{rad},0}^S$  is a core of  $H_M$ , we obtain

$$E_M \geq E_0. \quad (6.9)$$

Hence

$$\liminf_{M \rightarrow 0} E_M \geq E_0.$$

By (6.9),  $E_0 - 1$  is in the resolvent set of  $H_M$ . Hence, by Lemma 6.3(ii), we have

$$\|(H - E_0 + 1)^{-1}\| \leq \liminf_{M \rightarrow \infty} \|(H_M - E_0 + 1)^{-1}\|.$$

Since  $\|(H - E_0 + 1)^{-1}\| = 1$  and  $\|(H_M - E_0 + 1)^{-1}\| = (E_M - E_0 + 1)^{-1}$ , it follows that

$$\limsup_{M \rightarrow 0} E_M \leq E_0.$$

Thus we obtain (6.8).  $\blacksquare$

## Proof of Theorem 7.1

Let  $P_0$  be the orthogonal projection from  $\mathcal{F}_{\text{rad}}$  onto the Fock vacuum sector  $\{\alpha \Omega_0 | \alpha \in \mathbb{C}\}$ . Then  $P_0 \geq I - N$ . Hence, by Lemma 6.2,

$$\begin{aligned} (\Psi_{M_j}, I \otimes P_0 \Psi_{M_j}) &\geq 1 - (\Psi_{M_j}, I \otimes N \Psi_{M_j}) \\ &\geq 1 - \frac{|q|}{\sqrt{2}} \left\| \frac{\mathbf{g}}{\mu_{-, \tau}^{(M_j)}} \right\|. \end{aligned}$$

It is easy to see that

$$\lim_{M \rightarrow 0} \left\| \frac{\mathbf{g}}{\mu_{-, \tau}^{(M)}} \right\| = \left\| \frac{\mathbf{g}}{\mu_{-, \tau}} \right\|.$$

Moreover the dimension of the range of  $I \otimes P_0$  is four. Hence (6.6) implies that  $\lim_{j \rightarrow \infty} (\Psi_{M_j}, I \otimes P_0 \Psi_{M_j}) = (\Psi_0, I \otimes P_0 \Psi_0)$ . Therefore it follows that

$$(\Psi_0, I \otimes P_0 \Psi_0) \geq 1 - \frac{|q|}{\sqrt{2}} \left\| \frac{\mathbf{g}}{\mu_{-, \tau}} \right\| > 0.$$

Hence  $\Psi_0 \neq 0$ . Thus we can apply [6, Lemma 4.9] to conclude that  $\Psi_0$  is a ground state of  $H$ . ■

## 7 Spectrum

In this section we locate the (essential) spectrum of  $H_\tau(\mathbf{p})$  and that of  $H_\tau$  defined by (4.45).

### 7.1 Essential spectrum of $H_\tau(\mathbf{p})$

We define

$$S_\tau := \{\mathbf{k} \in \mathbf{R}^3 \mid \mu_{-, \tau}(\mathbf{k}) > 0\}. \quad (7.1)$$

The key fact to the analysis of the essential spectrum of  $H_\tau(\mathbf{p})$  is the following lemma.

**Lemma 7.1** *Assume (H.2). Suppose that  $\omega \in C(\mathbf{R}^3)$ . Let  $\lambda \in S_\tau$ . Then, there exists a sequence  $\{\Psi_n\}_{n=1}^\infty \subset D(L(\tau))$  such that  $\|\Psi_n\| = 1$ ,  $n \geq 1$ ,*

$$\lim_{n \rightarrow \infty} \|[H_\tau(\mathbf{p}) - E_\tau(\mathbf{p}) - \nu_{-, \tau}(\lambda)]\Psi_n\| = 0, \quad (7.2)$$

and

$$\text{w-} \lim_{n \rightarrow \infty} \Psi_n = 0. \quad (7.3)$$

*Proof.* Throughout the proof, we set  $H = H_\tau(\mathbf{p})$ ,  $\hat{H} = H_\tau(\mathbf{p}) - E_\tau(\mathbf{p})$  and  $E_0 = E_\tau(\mathbf{p})$ . Let  $P$  be the spectral measure of  $\hat{H}$  and  $0 < \varepsilon \leq 1$ . Since  $0 \in \sigma(\hat{H})$ , it follows that  $P([0, \varepsilon)) \neq 0$ . Hence there exists a vector  $\Phi_\varepsilon \in D(\hat{H}) = D(L(\tau))$  such that  $\|\Phi_\varepsilon\| = 1$  and  $\Phi_\varepsilon \in \text{Ran}(P([0, \varepsilon)))$ , where  $\text{Ran}(T)$  denotes the range of an operator  $T$ . We denote by  $T_j$  the multiplication operator by the function  $k_j$  ( $\mathbf{k} \in \mathbf{R}^3$ ). Let  $\Psi \in \mathcal{F}_{\text{rad}, 0}^\infty$  and  $f \in \mathcal{H}_{\text{ph}}$  be such that

$$\omega f, k_j f, \omega f / \sqrt{\mu_{-, \tau}}, k_j f / \sqrt{\mu_{-, \tau}} \in \mathcal{H}_{\text{ph}}.$$

Then we have

$$(\hat{H}\Psi, a(f)^*\Phi_\varepsilon) = ([a(f), \hat{H}]\Psi, \Phi_\varepsilon) + (a(f)\Psi, \hat{H}\Phi_\varepsilon).$$

It is easy to see that

$$[a(f), H]\Psi = a(\omega f)\Psi - \tau \sum_{j=1}^3 \alpha_j a(T_j f)\Psi - \sum_{j=1}^3 \frac{q}{\sqrt{2}} \alpha_j (f, g_j)\Psi.$$

By the functional calculus, we see that  $\hat{H}\Phi_\varepsilon \in D(\hat{H})$  and

$$\|\hat{H}\Phi_\varepsilon\| \leq \varepsilon. \quad (7.4)$$

Recall also that  $D(H) = D(L(\tau)) \subset D(a(F)) \cap D(a(F)^*)$  for all  $F \in \mathcal{H}_{\text{ph}}$  such that  $F/\sqrt{\mu_{-, \tau}} \in \mathcal{H}_{\text{ph}}$ . By these facts, we obtain

$$\begin{aligned} (\hat{H}\Psi, a(f)^*\Phi_\varepsilon) &= \left( \Psi, \left[ a(\omega f)^* - \tau \sum_{j=1}^3 \alpha_j a(T_j f)^* - \sum_{j=1}^3 \frac{q}{\sqrt{2}} \alpha_j (g_j, f) \right] \Phi_\varepsilon \right) \\ &\quad + (\Psi, a(f)^*\hat{H}\Phi_\varepsilon), \end{aligned}$$

Since  $\mathcal{F}_{\text{rad}, 0}^\infty$  is a core of  $\hat{H}$ , it follows that  $a(f)^*\Phi_\varepsilon \in D(\hat{H})$  and

$$\begin{aligned} \hat{H}a(f)^*\Phi_\varepsilon &= \left[ a(\omega f)^* - \tau \sum_{j=1}^3 \alpha_j a(T_j f)^* \right] \Phi_\varepsilon - \sum_{j=1}^3 \frac{q}{\sqrt{2}} (f, g_j) \alpha_j \Phi_\varepsilon \\ &\quad + a(f)^*\hat{H}\Phi_\varepsilon. \end{aligned}$$

Hence, for all  $\lambda \in \mathbf{R}^3$ ,

$$\begin{aligned} (\hat{H} - \nu_{-, \tau}(\lambda))a(f)^*\Phi_\varepsilon &= \left[ a((\omega - \omega(\lambda))f)^* + \tau \sum_{j=1}^3 \alpha_j a((\lambda_j - T_j)f)^* \right] \Phi_\varepsilon \\ &\quad - \sum_{j=1}^3 \frac{q}{\sqrt{2}} (f, g_j) \alpha_j \Phi_\varepsilon + a(f)^*\hat{H}\Phi_\varepsilon. \end{aligned} \quad (7.5)$$

Let  $h \in C_0^\infty(\mathbf{R}^3)$  be such that  $\int_{\mathbf{R}^3} h(\mathbf{k}) d\mathbf{k} = 1$ ,  $h(\mathbf{k}) = 0$  for  $|\mathbf{k}| \geq 1$ , and define  $h_n(\mathbf{k}) = n^3 h(n(\mathbf{k} - \lambda))$ ,  $n \geq 1$ . Then, for all continuous functions  $\phi$  on  $\mathbf{R}^3$ ,

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^3} h_n(\mathbf{k}) \phi(\mathbf{k}) d\mathbf{k} = \phi(\lambda). \quad (7.6)$$

Let  $f_n := \langle \sqrt{h_n}, \sqrt{h_n} \rangle / \sqrt{2}$ . Then  $f_n \in \mathcal{H}_{\text{ph}}$  with  $\|f_n\| = 1$  and  $\omega f_n, k_j f_n \in \mathcal{H}_{\text{ph}}$ . It follows from the condition  $\mu_{-, \tau}(\lambda) > 0$  and the continuity of  $\omega$  that

$$\inf_{|\mathbf{k}| \leq 1} \mu_{-, \tau} \left( \lambda + \frac{\mathbf{k}}{n} \right) \geq c > 0 \quad (7.7)$$

for all  $n \geq n_0$  with some  $n_0 > 0$ , where  $c$  is a constant independent of  $n$ . Using this fact, one can show that  $\omega f_n / \sqrt{\mu_{-, \tau}}$ ,  $k_j f_n / \sqrt{\mu_{-, \tau}} \in \mathcal{H}_{\text{ph}}$ . In general, by the canonical commutation relations, for all  $F \in \mathcal{H}_{\text{ph}}$  and  $\Psi \in D(a(F)) (\subset D(a(F)^*))$ ,

$$\|a(F)^* \Psi\|^2 = \|a(F) \Psi\|^2 + \|F\|^2 \|\Psi\|^2 \geq \|F\|^2 \|\Psi\|^2.$$

Hence

$$\|a(f_n)^* \Phi_\varepsilon\| \geq 1. \quad (7.8)$$

This allows one to define a unit vector

$$\Phi_{n, \varepsilon} := \frac{a(f_n)^* \Phi_\varepsilon}{\|a(f_n)^* \Phi_\varepsilon\|}.$$

By (7.5) and (7.8),

$$\begin{aligned} & \|(\hat{H} - \nu_{-, \tau}(\lambda)) \Phi_{n, \varepsilon}\| \\ & \leq \|a((\omega - \omega(\lambda)) f_n)^* \Phi_\varepsilon\| + \tau \sum_{j=1}^3 \|a((\lambda_j - T_j) f_n)^* \Phi_\varepsilon\| \\ & \quad + \sum_{j=1}^3 \frac{|q|}{\sqrt{2}} |(f_n, g_j)| + \|a(f_n)^* \hat{H} \Phi_\varepsilon\| \\ & \leq \|\mu_{-, \tau}^{-1/2} (\omega - \omega(\lambda)) f_n\| \|L(\tau)^{1/2} \Phi_\varepsilon\| + \|\mu_{-, \tau}^{-1/2} (\omega - \omega(\lambda)) f_n\| \\ & \quad + \tau \sum_{j=1}^3 \|\mu_{-, \tau}^{-1/2} (\lambda_j - T_j) f_n\| \|L(\tau)^{1/2} \Phi_\varepsilon\| + \tau \sum_{j=1}^3 \|\mu_{-, \tau}^{-1/2} (\lambda_j - T_j) f_n\| \\ & \quad + \sum_{j=1}^3 \frac{|q|}{\sqrt{2}} |(f_n, g_j)| + \|\mu_{-, \tau}^{-1/2} f_n\| \|L(\tau)^{1/2} \hat{H} \Phi_\varepsilon\| + \|\hat{H} \Phi_\varepsilon\|. \end{aligned}$$

By (7.6), we have

$$\lim_{n \rightarrow \infty} \|(\omega - \omega(\lambda)) f_n\| = 0, \quad \lim_{n \rightarrow \infty} \|(\lambda_j - T_j) f_n\| = 0.$$

By (7.7), one can show that

$$\lim_{n \rightarrow \infty} \|\mu_{-, \tau}^{-1/2} (\omega - \omega(\lambda)) f_n\| = 0, \quad \lim_{n \rightarrow \infty} \|\mu_{-, \tau}^{-1/2} (\lambda_j - T_j) f_n\| = 0$$

and

$$\|\mu_{-, \tau}^{-1/2} f_n\| \leq \frac{1}{\sqrt{c}}.$$

Let  $f \in C(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$ . Then we have by change of variables

$$\left( \sqrt{h_n}, f \right)_{L^2(\mathbf{R}^3)} = n^{-3/2} \int_{|k| \leq 1} \sqrt{h(k)} f \left( \lambda + \frac{k}{n} \right) dk.$$

Since  $f$  is continuous,

$$\sup_{n \geq 1} \sup_{|k| \leq 1} \left| f \left( \lambda + \frac{k}{n} \right) \right| < \infty, \quad \lim_{n \rightarrow \infty} f \left( \lambda + \frac{k}{n} \right) = f(\lambda).$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \left( \sqrt{h_n}, f \right)_{L^2(\mathbf{R}^3)} = 0. \quad (7.9)$$

Since  $\|\sqrt{h_n}\|_{L^2(\mathbf{R}^3)} = 1$  and  $C(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$  is dense in  $L^2(\mathbf{R}^3)$ , (7.9) extends to all  $f \in L^2(\mathbf{R}^3)$ . Hence we obtain

$$\text{w-} \lim_{n \rightarrow \infty} f_n = 0. \quad (7.10)$$

In particular,

$$\lim_{n \rightarrow \infty} (f_n, g_j) = 0.$$

Thus we obtain

$$\limsup_{n \rightarrow \infty} \|(\hat{H} - \nu_{-, \tau}(\lambda))\Phi_{n, \varepsilon}\| \leq c^{-1/2} \|L(\tau)^{1/2} \hat{H} \Phi_\varepsilon\| + \|\hat{H} \Phi_\varepsilon\|. \quad (7.11)$$

By Lemma 5.5 and [20, p.168, Theorem X.18(a)], there exists a constant  $c_1 > 0$  independent of  $\varepsilon$  such that

$$\|L(\tau)^{1/2} \hat{H} \Phi_\varepsilon\| \leq c_1 \|(\hat{H} + 1)^{1/2} \hat{H} \Phi_\varepsilon\|.$$

By the functional calculus, we have

$$\|(\hat{H} + 1)^{1/2} \hat{H} \Phi_\varepsilon\| \leq (\varepsilon + 1)^{1/2} \varepsilon \leq \sqrt{2} \varepsilon.$$

By this fact, (7.4) and (7.11), we obtain

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \|(\hat{H} - \nu_{-, \tau}(\lambda))\Phi_{n, \varepsilon}\| = 0.$$

This implies that there exist sequences  $\{n_k\}_{k=1}^\infty$  and  $\{\varepsilon_k\}_{k=1}^\infty$  with  $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$ ,  $n_k \rightarrow \infty$  ( $k \rightarrow \infty$ ) and  $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_k > \varepsilon_{k+1} > \dots$ ,  $\varepsilon_k \rightarrow 0$  ( $k \rightarrow \infty$ ) such that  $\lim_{k \rightarrow \infty} \|(\hat{H} - \nu_{-, \tau}(\lambda))\Phi_{n_k, \varepsilon_k}\| = 0$ . Thus, putting  $\Psi_k := \Phi_{n_k, \varepsilon_k}$ , we obtain (7.2).

To prove (7.3), it is sufficient to show that, for all  $F_j \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, \dots, m$ ,  $m \geq 0$ , and  $z \in \mathbf{C}^4$ ,

$$\lim_{k \rightarrow \infty} (z \otimes a(F_1)^* \dots a(F_m)^* \Omega_0, \Psi_k) = 0. \quad (7.12)$$

We have

$$\begin{aligned} C_k &:= (z \otimes a(F_1)^* \dots a(F_m)^* \Omega_0, \Psi_k) \\ &= \sum_{j=1}^m (F_j, f_{n_k}) (z \otimes a(F_1) \dots a(F_{j-1})^* a(F_{j+1})^* \dots a(F_m)^* \Omega_0, \Phi_{\varepsilon_k}) \\ &\quad \times \|a(f_{n_k})^* \Phi_{\varepsilon_k}\|^{-1}. \end{aligned}$$

Hence

$$|C_k| \leq \sum_{j=1}^m |(F_j, f_{n_k})| \|(z \otimes a(F_1) \cdots a(F_{j-1})^* a(F_{j+1})^* \cdots a(F_m)^* \Omega_0)\|.$$

By this estimate and (7.10), we have  $C_k \rightarrow 0$  ( $n \rightarrow \infty$ ). Thus (7.3) follows.  $\blacksquare$

For a self-adjoint  $A$ , we denote by  $\sigma_{\text{ess}}(A)$  the essential spectrum of  $A$ .

**Theorem 7.2** *Assume (H.2). Suppose that  $\omega \in C(\mathbb{R}^3)$ . Then, for all  $\mathbf{p} \in \mathbb{R}^3$ ,*

$$\overline{\{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k}) | \mathbf{k} \in \mathbb{R}^3\}} \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p})). \quad (7.13)$$

*Proof.* Let  $\boldsymbol{\lambda} \in S_\tau$ . Then, Lemma 7.1 and a general criterion on essential spectrum (e.g., [1, Lemma 5.19]) imply that

$$\omega(\boldsymbol{\lambda}) + E_\tau(\mathbf{p}) \in \sigma_{\text{ess}}(H_\tau(\mathbf{p}) + \tau \boldsymbol{\alpha} \cdot \boldsymbol{\lambda}).$$

Note that

$$H_\tau(\mathbf{p}) + \tau \boldsymbol{\alpha} \cdot \boldsymbol{\lambda} = H_\tau(\mathbf{p} + \tau \boldsymbol{\lambda}).$$

Hence

$$\omega(\boldsymbol{\lambda}) + E_\tau(\mathbf{p}) \in \sigma_{\text{ess}}(H_\tau(\mathbf{p} + \tau \boldsymbol{\lambda})). \quad (7.14)$$

This holds for all  $\mathbf{p} \in \mathbb{R}^3$  and all  $\boldsymbol{\lambda} \in S_\tau$ . For each  $\mathbf{p} \in \mathbb{R}^3$  and  $\mathbf{k} \in S_\tau$ , we set  $\mathbf{p}' := \mathbf{p} - \tau \mathbf{k}$ . Then, applying (7.14) with  $\boldsymbol{\lambda}$  and  $\mathbf{p}$  replaced by  $\mathbf{k}$  and  $\mathbf{p}'$  respectively, we have

$$\overline{\{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k}) | \mathbf{k} \in S_\tau\}} \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p})), \quad (7.15)$$

where we have used that  $\sigma_{\text{ess}}(H_\tau(\mathbf{p}))$  is closed. Since the function  $\mu_{-, \tau}$  is continuous on  $\mathbb{R}^3$ , it follows that

$$\overline{\{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k}) | \mathbf{k} \in S_\tau\}} = \overline{\{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k}) | \mathbf{k} \in \mathbb{R}^3\}}.$$

Thus (7.13) is derived.  $\blacksquare$

We define

$$\delta_\tau(\mathbf{p}) := \inf_{\mathbf{k} \in \mathbb{R}^3} \{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k})\} - E_\tau(\mathbf{p}). \quad (7.16)$$

**Corollary 7.3** *Let the same assumption as in Theorem 7.2 be satisfied. Assume (5.7). Then:*

(i) For all  $\mathbf{p} \in \mathbb{R}^3$ ,

$$[E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty) \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p})). \quad (7.17)$$

(ii) If  $\omega(0) = 0$ , then

$$\sigma(H_\tau(\mathbf{p})) = \sigma_{\text{ess}}(H_\tau(\mathbf{p})) = [E_\tau(\mathbf{p}), \infty). \quad (7.18)$$

**Remark 7.1** Under Hypothesis (H.2)-(i),

$$\delta_\tau(\mathbf{p}) \geq \inf_{\mathbf{k} \in \mathbf{R}^3} \mu_{-, \tau}(\mathbf{k}) \geq 0, \quad (7.19)$$

see (5.4).

*Proof.* (i) Let  $\mathbf{p} \in \mathbf{R}^3$  be fixed. Then the function  $F$  on  $\mathbf{R}^3$  defined by  $F(\mathbf{k}) := E_\tau(\mathbf{p} - \tau\mathbf{k}) + \omega(\mathbf{k})$ ,  $\mathbf{k} \in \mathbf{R}^3$ , is continuous. By (5.4),  $F(\mathbf{k}) \geq E_\tau(\mathbf{p}) + \mu_{-, \tau}(\mathbf{k})$ . Hence  $F(\mathbf{k}) \rightarrow \infty$  as  $|\mathbf{k}| \rightarrow \infty$ . From this fact and that

$$\delta_\tau(\mathbf{p}) + E_\tau(\mathbf{p}) = \inf_{\mathbf{k} \in \mathbf{R}^3} F(\mathbf{k}),$$

we have

$$[E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty) = \overline{\{F(\mathbf{k}) | \mathbf{k} \in \mathbf{R}^3\}},$$

which, together with Theorem 7.2, implies (7.17).

(ii) Let  $\omega(0) = 0$ . Then  $\delta_\tau(\mathbf{p}) \leq F(0) - E_\tau(\mathbf{p}) = 0$ . Hence  $\delta_\tau(\mathbf{p}) = 0$ . By this result and part (i), we have  $[E_\tau(\mathbf{p}), \infty) \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p}))$ . On the other hand, it is obvious that  $\sigma_{\text{ess}}(H_\tau(\mathbf{p})) \subset \sigma(H_\tau(\mathbf{p})) \subset [E_\tau(\mathbf{p}), \infty)$ . Thus (7.18) follows. ■

**Remark 7.2** General theorems on locating essential spectrum of a class of self-adjoint operators on an abstract Hilbert space of Fock type are established in [4, 5] ([5] is a refinement of [4]). But these theorems cannot be applied to the operator  $H_\tau(\mathbf{p})$  *directly*. This is due to the noncommuting coefficients  $\alpha_j$  coupled to  $d\Gamma(k_j)$ . In the proof of Theorem 7.2, we have applied only some basic ideas in [4, 5].

Corollary 7.3(ii) shows that the (essential) spectrum of  $H_\tau(\mathbf{p})$  in the *massless case* is completely located under a weaker condition than in Theorem 6.1. If we impose stronger conditions than in Theorem 7.2, then we can completely locate the essential spectrum of  $H_\tau(\mathbf{p})$  in the *massive case* too:

**Theorem 7.4** *Let the same assumption as in Theorem 5.1 be satisfied. Suppose that, for all  $\mathbf{k}, \mathbf{k}' \in \mathbf{R}^3$ ,*

$$\omega(\mathbf{k} + \mathbf{k}') \leq \omega(\mathbf{k}) + \omega(\mathbf{k}'). \quad (7.20)$$

*Then*

$$\sigma_{\text{ess}}(H_\tau(\mathbf{p})) = [E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty). \quad (7.21)$$

*Proof.* Condition (7.20) implies that

$$\sum_{j=1}^n \omega(\mathbf{k}_j) \geq \omega\left(\sum_{j=1}^n \mathbf{k}_j\right), \quad \mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbf{R}^3.$$

Hence

$$\begin{aligned}
& \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbf{R}^3} \left\{ E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j \right) + \sum_{j=1}^n \omega(\mathbf{k}_j) \right\} - E_\tau(\mathbf{p}) \\
& \geq \inf_{\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbf{R}^3} \left\{ E_\tau \left( \mathbf{p} - \sum_{j=1}^n \tau \mathbf{k}_j \right) + \omega \left( \sum_{j=1}^n \mathbf{k}_j \right) \right\} - E_\tau(\mathbf{p}) \\
& = \delta_\tau(\mathbf{p}).
\end{aligned}$$

Hence  $\Delta_\tau(\mathbf{p}) = \delta_\tau(\mathbf{p})$ . By this fact and Theorem 6.1, we have  $\sigma_{\text{ess}}(H_\tau(\mathbf{p})) \subset [E_\tau(\mathbf{p}) + \delta_\tau(\mathbf{p}), \infty)$ . Under the present assumption, the assumption of Corollary 7.3 is satisfied. Hence Corollary 8.3(i) holds. Thus (7.21) follows. ■

## 7.2 Essential spectrum of $H_\tau$

Let  $H_\tau$  be as in (4.45).

**Theorem 7.5** *Assume (H.2). Suppose that  $|\mathbf{k}|g_j$ ,  $|\mathbf{k}|g_j/\sqrt{\omega}$ ,  $\omega g_j \in \mathcal{H}_{\text{ph}}$ ,  $j = 1, 2, 3$  with  $\omega \in C(\mathbf{R}^3)$  and that  $\mu_{-\tau}(\mathbf{k}) \rightarrow \infty$  as  $|\mathbf{k}| \rightarrow \infty$ . Then, for all  $\tau \in [0, 1]$ ,*

$$\sigma(H_\tau) = \mathbf{R}. \quad (7.22)$$

*Proof.* Let  $H' := \int_{\mathbf{R}^3}^\oplus H_\tau(\mathbf{p}) d\mathbf{p}$  and, for  $\lambda \in \mathbf{R}$  and  $\varepsilon > 0$ ,

$$K_\varepsilon(\lambda) := \{\mathbf{p} \in \mathbf{R}^3 \mid \sigma(H_\tau(\mathbf{p})) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}$$

Then, by Theorem 4.2 and a general theorem [21, p.284, Theorem XIII.85],  $H'$  is self-adjoint and

$$\sigma(H') = \{\lambda \in \mathbf{R} \mid |K_\varepsilon(\lambda)| > 0 \text{ for all } \varepsilon > 0\},$$

where  $|K_\varepsilon(\lambda)|$  denotes the three-dimensional Lebesgue measure of  $K_\varepsilon(\lambda)$ . Let  $a := \inf_{\mathbf{k} \in \mathbf{R}^3} \{E_\tau(\mathbf{p} - \tau \mathbf{k}) + \omega(\mathbf{k})\}$ . Then, by Theorem 7.2,  $[a, \infty) \subset \sigma_{\text{ess}}(H_\tau(\mathbf{p}))$ . Putting  $\omega_0 := \omega(0)$ , we have

$$a \leq E_\tau(\mathbf{p}) + \omega_0.$$

Hence, for all  $\varepsilon > 0$ ,

$$\{\mathbf{p} \in \mathbf{R}^3 \mid E_\tau(\mathbf{p}) + \omega_0 \leq \lambda\} \subset K_\varepsilon(\lambda).$$

By Proposition 4.6, we have  $E_\tau(\mathbf{p}) \leq -\omega_D(\mathbf{p})$ . Hence

$$\{\mathbf{p} \in \mathbf{R}^3 \mid -\omega_D(\mathbf{p}) + \omega_0 \leq \lambda\} \subset K_\varepsilon(\lambda).$$

It is easy to see that, for all  $\lambda \in \mathbf{R}$ ,  $|\{\mathbf{p} \in \mathbf{R}^3 \mid -\omega_D(\mathbf{p}) + \omega_0 \leq \lambda\}| = \infty$ . Hence, for all  $\lambda \in \mathbf{R}$  and  $\varepsilon > 0$ ,  $|K_\varepsilon(\lambda)| = \infty$ . Thus  $\sigma(H') = \mathbf{R}$ . By this result and Theorem 4.3, we obtain (7.22). ■

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