Singularities of Developable Surfaces

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Dedicated to the 60th birthday of Professor C.T.C. Wall.

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Abstract: In this survey article we explain the recent results on the singularities appearing in the tangent developables of space curves via the projective duality. First we examine, from our viewpoint, the classical local classification and Hartman-Nirenberg's theorem on developable surfaces. Then we show results on the local differentiable and topological classification of tangent developables of space curves. Lastly we present some related problems and questions, for instance, questions on the finite determinacy.

1 Introduction.

A smooth surface in Euclidean three space is called a developable surface if its Gaussian curvature vanishes everywhere on the surface. By Guass' fundamental theorem, a developable surface is locally isometric to a planer domain. This motivates the name "developable" (see [57]). Let $z = f(x, y)$ be a smooth surface. If we set $r = z_{xx}, s = z_{xy}, t = z_{yy}$, then the surface is a developable surface if and only if the Monge-Ampère equation $rt - s^2 = 0$ is satisfied on the surface [46][33].

A developable surface is a well-known object from the last century. For instance, analytic developable surfaces are locally classified into three types: *cones, cylinders* and *tangent developables.* In smooth case, we can rather freely change the three types through planer trapezia to make various (not necessarily analytic) non-singular portions of smooth developable surfaces.

A tangent developable is a part of the union of the one-parameter family of tangent lines to a fixed space curve. In the classical algebraic geometry, tangent developables play a significant role in the duality theory of space curves (cf. [10]). Tangent developables of space curves form a special class of ruled surfaces, namely, developable ruled surfaces, and have necessarily singularities at least along the original curves, while ruled surfaces generically have no non-isolated singularities.

Meanwhile, recently, developable surfaces are attracting our attention related to the computer sciences: They are widely used in industry. They are fundamental objects in computer-aided design [23]. Though the singularities should be avoided in the practical situations, the appearance of singularities in developable surfaces is essential in their nature.

It has become clear that it is useful, even in smooth category, to stand on the viewpoint of projection duality, or Legendre transformation [3], and use the modern singularity theory, for the full understanding of the structure of developable surfaces, locally and globally [53].

The tangent developable of a space curve γ has constant tangent space along each generating line: They are called osculating planes to γ, and form a space curve γ^*; the dual curve [53]. Then the tangent developable of γ is the projection to the original space of the projective conormal bundle over γ^* in the dual projective space. In other words, the tangent developable of γ is **the dual surface** of the dual curve γ^*.

In fact the tangent developable of a curve γ is regarded as the envelope of the one-parameter family of osculating planes if the dual curve γ^* is non-singular. If γ^* is singular, then the envelope consists of the tangent developable and the osculating plane to the base point [28].

Conversely, if a one-parameter family of planes is given, then its envelope contains a tangent developable as a component, provided the corresponding curve in the dual projective space is of finite type in the sense of §2. For instance, for the envelope of the family of normal planes to a space curve, we have the tangent developable of the focal curve [51].

Thus we can regard a tangent developable as a degenerate Legendre singularity or a wave front set of a possibly singular Legendre variety in the sense of Legendre singularity theory [2][3].

Then we are led to natural questions such as: *What kinds of singularities appear in tangent developables of curves? Can we classify the singularities at least locally?*

In a neighborhood of a non-singular point, the tangent developable has the parametrisation $(t, s) \mapsto \boldsymbol{x}(t) + s\dot{\boldsymbol{x}}(t)$. We shall give the parametrisation of the tangent developable near a possibly singular point (of finite type) of the curve, which covers also the non-singular case. See §4.

When we try to answer the questions above, several equivalence relations come in mind for space curve-germs and space surface-germs; the isometric equivalence, the affine equivalence, the differentiable equivalence, and the topological equivalence.

The classification under the affine equivalence seems to be the most natural one for tangent developables since the tangent lines to a space curve have invariant character under the affine equivalence. However the isometric and the affine equivalences are too strong to get a finite list after the classification, even for the simplest singularities, namely, cuspidal edges.

Therefore, in the following sections, we consider the local differentiable or topological classification. We pursue the classification of the geometric objects under diffeomorphisms or homeomorphisms by means of the geometric invariants. In fact we examine the condition that the leading terms of the parametrisation determine the equivalence class of the tangent developable: The determinacy is, also in our pursuits, the main guidepost for the root to the classification [42][59].

After recalling the basic notions and the history of the study in §2, we explain the classical results about developable surfaces using the projective duality in §3. The parametrisations of tangent developables are given in §4. Then we give the differentiable classification in §5 and the main idea of the proof in §6. Further we present the topological classification in §7. A method to illustrate the topological equivalence classes of tangent developables is given in §8. The key ideas of the topological classification theorem are shown in §9. Lastly we present several open questions in §10.

We see in [36] [37] a general results on the bifurcation problem of curves and their tangent developables.

We find unexpected applications tangent developables, for example, in [58][50][22][24]. See also excellent expositions [9][51] on singularities of tangent developables and on the other related subjects.

In this article smoothness means infinite differentiability. For instance a smooth manifold is a C^∞ manifold, and a smooth mapping is a C^∞ mapping.

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2 Basic notions and the history.

To review the history of the classification, first we recall fundamental notions for space curves.

Let M be a one-dimensional smooth manifold (parameter space), $\gamma : M \to \mathbf{R}^3$ a germ of smoothly parametrised space curve and $t_0 \in M$. Represent γ as $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))$, using local smooth coordinates of M centred at t_0. Then we call γ *of finite type* at $t_0 \in M$ if the infinite vectors $\dot{\boldsymbol{x}}(t_0), \ddot{\boldsymbol{x}}(t_0), \cdots, \boldsymbol{x}^{(k)}(t_0), \cdots$, generate the three space. Then, after some smooth coordinate change of (M, t_0), and some affine coordinate change of $(\mathbf{R}^3, \gamma(t_0))$, if necessary, we write the curve γ under the form (standard form)

$$x_1(t) = t^m + o(t^m),\ x_2(t) = t^{m+s} + o(t^{m+s}),\ x_3(t) = t^{m+s+r} + o(t^{m+s+r}),$$

for some positive integers m, s, r. The triplet $(m, m+s, m+s+r)$ is independent of the choice of affine local coordinates, and it is called the *type* of the curve-germ γ: $\text{type}(\gamma) = (a_1, a_2, a_3) = (m, m+s, m+s+r)$. In fact, if γ is of finite type at t_0, then there exists a unique line (resp. a unique plane) having the most contact with γ at t_0. We call it **the tangent line** (resp. **the osculating plane**) to γ at t_0. Then a_1 (resp. a_2, a_3) is the order of contact of γ with the origin $\gamma(t_0)$ (resp. with the tangent line, with the osculating plane) at t_0. For the coordinates in the standard form, the tangent line (resp. the osculating plane) is given by x_1-axis (resp. (x_1, x_2)-plane). A curve-germ is of finite type if and only if the curve does not have infinite tangency with any affine plane.

We treat a curve which is of finite type at each point. Notice that only curve-germs of finite type appear in a generic multi-dimensional family of curves ([53] Theorem 2.2).

We call a point t_0 **an ordinary point** if $\text{type}(\gamma)$ at t_0 is $(1, 2, 3)$, in other words, if the Wronskian $|\dot{\boldsymbol{x}}(t), \ddot{\boldsymbol{x}}(t), \dddot{\boldsymbol{x}}(t)|$ does not vanish at t_0. Otherwise we call it **a special point.** Special points are isolated, since the Wronskian has a zero of finite order at each special point. Moreover we remark that the parametrised curve is nonsingular at t_0 if and only if $m = 1$.

Up to the present, several results are known on the local differentiable classification of developables.

At an ordinary point, it is classically known that the tangent developable of a space curve has the cuspidal edge along the curve itself. In fact the tangent developable has a parametrisation which is smoothly right-left equivalent to the map-germ $(x, t) \mapsto (x, t^2, t^3)$ at the origin. (See [9][51].)

For the simplest case of special points, namely in the case of type $(1, 2, 4)$, recently Cleave [11], Gaffney, du Plessis [17] and Scherbak [53] showed that the tangent developable has a parametrisation which is smoothly right-left equivalent to the map-germ $(x, t) \mapsto (x, t^2, xt^3)$, which is called the folded umbrella [1]. In this case the tangent developable has, besides the cuspidal edge, a self-intersection locus leaving from the special point.

Then Mond proceeded to give differentiable classification of tangent developable of curves of type $(1, 2, 2+r)$, $r \leq 5$, and of type $(1, 3, 4)$ in [43][44]. For instance, for the type $(1, 3, 4)$, the tangent developable has singularities along the original curve and along the tangent line to the origin, and has two branches of double point loci: It is called Mond surface.

On the other hand, Arnold [1] gave the observation that the tangent developable of a singular curve of type $(2, 3, 4)$ is differentiably equivalent to the swallowtail from Legendre singularity theory [3]. See also [53].

In [55], Scherbak showed the picture of the tangent developable of a curve of type $(1, 3, 5)$ is, in the complex analytic category, holomorphically equivalent to the variety of irregular orbits of the finite reflection group H_3 in $\mathbf{C}^3$.

As for the topological classification, Mond [44] proved that the tangent developable of a curve of type $(1, 2, 2 + r)$, $r = 1, 2, \ldots$, is homeomorphic to the plane if r is odd and to the cone of the figure eight on the sphere if r is even. Nevertheless, no result was known succeeding Mond's result about the local topological classification of the tangent developables of space curves.

3 Projective duality.

Denote by $\mathbf{RP}^3$ the real projective space of dimension 3, and by $\mathbf{RP}^{3*}$ the dual projective space, which is identified with the space of projective planes in $\mathbf{RP}^3$. Consider the 5-dimensional manifold of incidence:

$$Q = \{(p,q) \in \mathbf{RP}^3 \times \mathbf{RP}^{3*} \mid p \in q^*\},$$

where $q^* \subset \mathbf{RP}^3$ is the plane defined by $q \in \mathbf{RP}^{3*}$. The double fibrations $\pi_1 : Q \to \mathbf{RP}^3$ and $\pi_2 : Q \to \mathbf{RP}^{3*}$ define a contact structure, that is, a completely non-integrable distribution of codimension one, $D = \mathrm{Ker}\pi_{1*} \oplus \mathrm{Ker}\pi_{2*}$ on Q. In fact the natural identifications of Q to the manifolds of contact elements, that is, the projective cotangent bundles $PT^*\mathbf{RP}^3$ and $PT^*\mathbf{RP}^{3*}$ are contact diffeomorphisms respectively [2][33].

Now we recall the fundamental duality theorem from [53].

Let $\gamma : (M, t_0) \to \mathbf{RP}^3$ be a smooth curve-germ. The osculating planes to γ, considered as projective planes, form a curve in the dual projective space: $\gamma^* : (M, t_0) \to \mathbf{RP}^{3*}$. We call γ^* the dual of γ. Then Arnold-Scherbak [53] gave an elegant proof to the following:

Lemma 3.1 *If γ is of type $(m, m+s, m+s+r)$ at t_0, then the dual γ^* is also of finite type and of type $(r, r+s, r+s+m)$ at t_0.*

To see this directly, first take homogeneous coordinates X_0, X_1, X_2, X_3 such that $x_i = X_i/X_0, i = 1,2,3$, give a standard form of γ. Second take homogeneous coordinates Y_0, Y_1, Y_2, Y_3 of $\mathbf{RP}^{3*}$ such that the equation

$$Y_0X_3 + Y_1X_2 + Y_2X_1 + Y_3X_0 = 0,$$

gives the incident relation. Then the dual curve

$$\gamma^*(t) = (y_1(t), y_2(t), y_3(t)), \quad y_i = Y_i/Y_0,$$

is obtained by solving the system of equations

$$\begin{aligned} x_3(t) + x_2(t)y_1 + x_1(t)y_2 + y_3 &= 0, \\ \dot{x}_3(t) + \dot{x}_2(t)y_1 + \dot{x}_1(t)y_2 &= 0, \\ \ddot{x}_3(t) + \ddot{x}_2(t)y_1 + \ddot{x}_1(t)y_2 &= 0, \end{aligned}$$

first for $t \neq 0$, and then extending the solutions to $t = 0$ smoothly.

Now we examine the classical results.

Any smooth surface S in $\mathbf{R}^3 \subset \mathbf{RP}^3$ lifts, with respect to π_1, to a Legendre surface, namely, an integral manifold of D; $\tilde{S} = \{(p, T_pS) \in Q \mid p \in S\}$. Here the tangent space T_pS is regarded as a projective plane in $\mathbf{RP}^3$. Then we project $\tilde{S}$ to another direction; $\pi_2 : \tilde{S} \to \mathbf{RP}^{3*}$. Then the essential point is that S is a developable surface if and only if the rank of the projection π_2 is at most one everywhere on $\tilde{S}$. Thus we see in [20][33]:

Lemma 3.2 *If S is a smooth developable surface in $\mathbf{R}^3$, then there exists a smooth one-dimensional foliation on S by straight lines such that S has a constant tangent space along each leaf.*

Using this fact we see the classical local classification is valid for the class of analytic developable surfaces.

In fact, at each point in S, we take a germ of analytic transversal to the foliation passing through a point in S. Then, by lifting on $\tilde{S}$ and then by composing with π_2, we have an analytic curve β in the dual projective space, the projective conormal bundle of which coincides with the Legendre lift $\tilde{S}$ at least locally. If β collapses to a point q, then S is contained in the plane q^*, projective dual to q. If β is contained in a plane Π, then S is a cone made by lines passing through the point $\Pi^* \in \mathbf{RP}^3$. When Π^* lies in the plane at infinity, S becomes a cylinder. If β is not contained in any plane, then S is the tangent developable of the dual curve $\gamma = \beta^*$ in $\mathbf{RP}^3$.

We thus regard cones and cylinders as degenerations of tangent developables: A tangent developable of a space curve, which is not contained in any plane, is the dual surface of the dual curve to the original space curve. The dual curve, in this case, is not a plane curve. Then *a cone is the dual surface of a plane curve and a cylinder is the dual surface of a curve lying on the plane at infinity.*

If we start from a smooth space curve β in $\mathbf{RP}^{3*}$, which is not a plane curve, but has an infinite contact with a plane, then we find many examples of smooth developable surfaces which are not governed by the classical local classification.

Also the same argument as above is applied to showing the classical Hartman-Nirenberg's theorem [20][57][54]: *A properly embedded connected smooth developable surface S in $\mathbf{R}^3$ is necessarily a cylinder.*

In fact, near each point on S, the π_2-projection β is contained in a plane Π such that Π^* lies on the plane at infinity $\mathbf{RP}^3 - \mathbf{R}^3$. Moreover, near a point where the rank of the projection is equal to one, β is not contained in any line; if it were, then S would reduce to a line. Since S is connected, we see the global projection $\pi_2(\tilde{S})$ is contained in a plane. See for details and for the higher dimensional case in [33].

4 Parametrisations of tangent developables.

Let γ be a space curve of type $(m, m+s, m+s+r)$ at t_0. Then the dual γ^* is of type $(r, r+s, r+s+m)$ at t_0 by Lemma 3.1. We fix non-zero constants a, b, c, and represent γ^* as

$$\gamma^* : y_1 = at^r,\ y_2 = bt^{r+s} + o(t^{r+s}),\ y_3 = ct^{r+s+m} + o(t^{r+s+m}),$$

for a smooth coordinate t of (M, t_0), and for some affine coordinates y_1, y_2, y_3 of $(\mathbf{R}P^{3*}, \gamma^*(0))$.

We set

$$F(t, x) = y_3(t) + y_2(t)x_1 + y_1(t)x_2 + x_3.$$

Then we have a parametrisation of the tangent developable of γ (resp. the curve γ itself), solving the equation $F = \partial F/\partial t = 0$ (resp. $F = \partial F/\partial t = \partial^2 F/\partial t^2 = 0$), first for $t \neq 0$, and then extending the solution to $t = 0$.

Setting $x_1 = x$, we have $x_2 = -\dot{y}_3/\dot{y}_1 - x\dot{y}_2/\dot{y}_1$, from $\partial F/\partial t = 0$. Further from $F = 0$, we see that $x_3 = x_3(t, x)$ satisfies $\partial x_3/\partial t = -y_1 \partial x_2/\partial t$.

If we set $a = -(r+s)(r+s+m), b = -r(r+s+m), c = -r(r+s)$, just for simplifying the expression, we get the parametrisation $f = \mathrm{dev}(\gamma) : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ defined by

$$\begin{aligned} x_1 &= x, \\ x_2 &= t^{s+m} + t^{s+m+1}\varphi(t) + x(t^s + t^{s+1}\psi(t)), \\ x_3 &= (r+s)(r+s+m)\int_0^t u^r \frac{\partial x_2(u, x)}{\partial u} du \\ &= (r+s)(s+m)t^{r+s+m} + \cdots + x\{s(r+s+m)t^{r+s} + \cdots\}, \end{aligned}$$

for some smooth functions-germs $\varphi(t)$ and $\psi(t)$. See [32].

From the concrete parametrisation, we see the tangent developable of γ is singular along the original curve γ, and further along the tangent line at a special point if $s \geq 2$, besides self-intersection loci. Remark that the parametrisation $\mathrm{dev}(\gamma)$ has very degenerate singularities even in the sense of Legendre singularity theory, because the singularities along $\{t = 0\}$ are more degenerate than the ordinary cuspidal edge if $s \geq 3$ or $r \geq 2$.

To classify the singularities arising from the tangent developables, thus, we are naturally led to the problem on the determinacy of $\mathrm{dev}(\gamma)$ by the leading terms.

5 Differentiable classification.

Two smooth map-germs $f, f' : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ are called **differentiably (right-left) equivalent** if there exist diffeomorphism-germs $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and $\tau : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ such that $\tau \circ f = f' \circ \sigma$.

Generalising all of know results on the local differentiable classification of tangent developables, we have the following determinacy result in [28][30]:

Theorem 5.1 *The local differentiable equivalence class of the tangent developable is determined by the type* $\mathbf{A}$ *of a space curve, namely,*

$$\#\left(\{\mathrm{dev}(\gamma) \mid \mathrm{type}(\gamma) = \mathbf{A}\}/\underset{\mathrm{diff}}{\sim}\right) = 1$$

if and only if $\mathbf{A}$ *is one of the following:*

$$(1, 2, 2+r); (r : \text{positive integer}), (2, 3, 4), (1, 3, 4), (3, 4, 5), (1, 3, 5).$$

For example, if $\mathbf{A} = (1, 3, 5)$, then the normal form of the tangent developable under the differentiable equivalence is given by

$$(t, x) \mapsto (x,\ t^3 + xt^2,\ 12t^5 + 10xt^4),$$

which we call Scherbak surface.

Also the higher dimensional generalisation of Theorem 5.1 is given in [28][30]. The complex analytic analogue to Theorem 5.1 holds in the same way [29].

6 Technical digression.

Two aspects are useful to have the differentiable normal forms of the singularities of tangent developables: One is the Legendre singularity theory [3] as in [1][53][55], and the other is analysing directly the parametrisation as in [9][51][43][44]. In fact we need both aspects to complete the differentiable classification in [28][30].

Recall the parametrisation $\mathrm{dev}(\gamma) = (x_1, x_2, x_3)$ in §4 satisfies

$$x_1 = x, \quad \partial x_3/\partial t = (r+s)(r+s+m)t^r \partial x_2/\partial t.$$

Then we see the exterior derivative of the third component satisfies

$$x_3 = \{\partial x_3/\partial x - (r+s)(r+s+m)t^r \partial x_2/\partial x\}dx_1 + (r+s)(r+s+m)t^r dx_2.$$

In this sense the tangent developables are governed by some algebraic structure.

When we trivialise a family of tangent developables of curves of constant type using the infinitesimal method [42], we try to solve the corresponding variational equations in terms of vector fields. Then we encounter a certain module describing the ramification of a finite mapping [25][45]. This object also appears in the classification of isotropic mappings [26], and in the characterisation of certain stabilities [31] in the symplectic geometry. We also apply it to classifying differential equations [21].

Let $g = (g_1, \dots, g_p) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a finite smooth map-germ $(n \le p)$. We define **the ramification module** R_g of g by the set of smooth function-germs $e : (\mathbf{R}^n, 0) \to \mathbf{R}$ such that the exterior differential de of e is a functional linear combination of $dg_1, \dots, dg_p$: $de = a_1 dg_1 + \cdots + a_n dg_n$, for some smooth function-germs $a_i : (\mathbf{R}^n, 0) \to \mathbf{R}, 1 \le i \le n$. This is a variant of the ring of functions constant along components of fibres of a map-germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ when $n > p$ [41]. We see R_g is a module over the algebra of all smooth function-germs $(\mathbf{R}^p, 0) \to \mathbf{R}$ via the pull-back by g.

For the parametrisation of a tangent developable, setting $g = (x_1, x_2)$, we see that x_3 belongs to the ramification module R_g.

We study on the structure of the ramification module in [25] [27] [26] [29] [30]. The key point is that R_g is a differentiable algebra in the sense of Malgrange [40] so that we can apply Malgrange's preparation theorem for differentiable algebras to solve the variational equation.

We expect this object has other applications to the analysis of various geometrical objects.

7 Topological classification.

The result on the differentiable classification turns our concern to the topological classification of tangent developables.

Two smooth map-germs $f, f' : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ are called **topologically (right-left) equivalent** if there exist homeomorphism-germs $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and $\tau : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ such that $\tau \circ f = f' \circ \sigma$.

Proceeding to the pioneering work of Mond [44], we have in [32]:

Theorem 7.1 *If s, r are not both even, then*

$$\#\left(\{\mathrm{dev}(\gamma) \mid \mathrm{type}(\gamma) = (m, m+s, m+s+r)\}/\underset{\mathrm{top}}{\sim}\right) = 1.$$

If m is odd, s, r are both even, then

$$\#\left(\{\mathrm{dev}(\gamma) \mid \mathrm{type}(\gamma) = (m, m+s, m+s+r)\}/\underset{\mathrm{top}}{\sim}\right) = 1,$$

when $m = 1$, and

$$\#\left(\{\mathrm{dev}(\gamma) \mid \mathrm{type}(\gamma) = (m, m+s, m+s+r)\}/\underset{\mathrm{top}}{\sim}\right) = 2,$$

when $m \geq 3$.

If m, s, r are all even, then

$$\#\left(\{\mathrm{dev}(\gamma) \mid \mathrm{type}(\gamma) = (m, m+s, m+s+r)\}/\underset{\mathrm{top}}{\sim}\right)$$

is uncountably infinite.

Moreover the topological types are determined by just the parities of types except for the case m, s, r are all even, or s, r are both even, m is odd and ≥ 3.

Furthermore the homeomorphisms can be taken to preserve the original curves and the tangent lines to the base points.

In particular we see that two non-singular space curves ($m = 1$) of the same type have the homeomorphic tangent developables to each other.

Here we observe two kinds of determinacies: Topological determinacy with respect to higher order perturbations of the original curve, and topological determinacy by the parity of the type, namely, the orders of leading terms of the curve.

We observe the similarity of Theorem 7.1 to the following well-known fact: Non-flat smooth function-germs $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ are topologically equivalent to the function-germ $y = x$ or $y = x^2$ depending on the parity of the order. Furthermore the number of topological types of flat functions is uncountably infinite. Here we call $f : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ flat if its infinite jet $j^\infty f(0) = 0$.

For simplicity of the explanation, we divide all types into eight classes by the parities of (m, s, r) and show with the names of the topological equivalence classes as follows:

	$s \setminus r$	odd	even
m: odd	odd	(i) cuspidal edge	(ii) folded umbrella
m: odd	even	(iii) Mond surface	(iv) (pseudo) Scherbak surface
m: even	odd	(v) swallowtail	(vi) folded pleat
m: even	even	(vii) butterfly	(viii) not determined

For the concrete pictures of tangent developables in each case, see [32].

8 Section and Projection.

For the illustration of the topological types of tangent developables, it is useful the method of "section and projection", which is originated by Cayley [10], and later clarified by Piene [48] and Scherbak [53]. This is also used to show the indeterminacy result in Theorem 7.1 [32].

Let $f = \text{dev}(\gamma) : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ be the parametrisation of the tangent developable of γ in its normal form as in §4. Since f is a finite map-germ, the germ at 0 of the image $f(\mathbf{R}^2)$ is well-defined, which is denoted $\text{DEV}(\gamma)$.

Set $\Pi_c = \{x_1 = c\}$, for sufficiently small $c \neq 0$. Remark that the plane Π_c intersects transversely to γ. Take a section $\text{DEV}_c(\gamma) = \Pi_c \cap \text{DEV}(\gamma)$. Consider the line ℓ_t obtained from osculating planes O_t to γ at t by cutting by the plane Π_c. Then the section $\text{DEV}_c(\gamma)$ is the envelope of the family of lines ℓ_t. We see the dual ℓ_t^* of ℓ_t is the line connecting two points O_t^* and Π_c^* in the dual projective space and, the dual of $\text{DEV}_c(\gamma)$ is the set $\{\ell_t^* \mid t \in (\mathbf{R}, 0)\}$ in the projective plane consisting of lines passing through the point Π_c^*.

Thus we see the section is the planer projective dual [8][60] to the projection of γ^* from the corresponding point in $\mathbf{RP}^{3*}$.

If (y_1, y_2, y_3) represents γ^*, then the projection is given by the plane curve $(y_1, cy_2 + y_3)$, where $\mathrm{ord}(y_1, y_2, y_3) = (r, r+s, r+s+m)$.

Example 8.1 : First let type$(\gamma) = (1,2,3)$. Then type$(\gamma^*) = (1,2,3)$. Thus we have a projection $(t + \cdots, c(t^2 + \cdots) + t^3 + \cdots)$. This plane curve has exactly one inflection point. Therefore the dual has one cuspidal point, which is a section of DEV(γ). Thus we see the tangent developable is a cuspidal edge.

Next let type$(\gamma) = (1,2,4)$, type$(\gamma^*) = (2,3,4)$. Then, according to the sign of c, the projection is a beak-like cusp with no double points and no double tangents, or, with one double point and one double tangent, which is self-dual in each case. Then we have a folded umbrella as the tangent developable.

Consider more complicated case type$(\gamma) = (2,4,5)$, type$(\gamma^*) = (1,3,5)$. Then for a sign of c, the projection has three inflection points and three double tangents, therefore its dual has three cuspidal edges and three self-intersections. For another sign of c, the projection has only one inflection point and no double tangents. Thus we have a butterfly [35] as the tangent developable.

Similarly, we get the topological pictures also for other several cases. However, to get the accurate classification, we need to analyse directly the parametrisations as in the next section.

9 Topological triviality.

Consider the typical parametrisation $f_0 : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ defined by

$$f_0(t,x) = (x,\ t^{s+m} + xt^s,\ (r+s)(s+m)t^{r+s+m} + s(r+s+m)xt^{r+s}),$$

setting $\varphi(t) = 0, \psi(t) = 0$ for f.

To describe the double point loci, set $f_0(t_1, x) = f_0(t_2, x)$. Eliminating x we have

$$d_0(t_1,t_2) = \begin{vmatrix} t_2^s - t_1^s, & t_2^{s+m} - t_1^{s+m} \\ s(r+s+m)(t_2^{r+s} - t_1^{r+s}), & (r+s)(s+m)(t_2^{r+s+m} - t_1^{r+s+m}) \end{vmatrix}$$

must vanish for such t_1 and t_2. Then we set

$$g(z) \;=\; (1/rm)d_0(1,z)$$

$$\begin{aligned} = \quad & z^{r+2s+m} - \frac{(r+s)(s+m)}{rm} z^{r+s+m} + \frac{s(r+s+m)}{rm} z^{r+s} \\ & + \frac{s(r+s+m)}{rm} z^{s+m} - \frac{(r+s)(s+m)}{rm} z^{s} + 1. \end{aligned}$$

We call $g(z)$ **the characteristic polynomial** of $\mathbf{A} = (m, m+s, m+s+r)$. For this very concrete polynomial, we easily see the following:

(0) $z = \xi$ is a real root of $g(z) = 0$ if and only if $t_2 = \xi t_1$ is a real branch of $\{d_0(t_1, t_2) = 0\}$.

(1) $z = 1$ is a multiple root of $g(z) = 0$ of multiplicity 2.

(2) If ξ is a root, then also $1/\xi$ is a root of $g(z) = 0$.

(3) If there exist a triple point of f_0 then there exist non-trivial ($\neq 1$) roots ξ, η of $g(z) = 0$ such that the product $\xi\eta$ is also a non-trivial root of $g(z) = 0$.

Then we recall the theorem of Descartes: *The number of real positive roots of real algebraic equation $g = 0$, counted with multiplicities, is at most the number of sign-changes of terms of g.*

Our $g(z)$ has coefficients which signs vary as $+ - + + - +$. Therefore we see $g(z) = 0$ has at most 2 solutions except for the multiple root $z = 1$. Since $g''(1) > 0$, $\lim_{z\to\infty} g(z) = \infty$, $g(0) = 1$, we see $g(z) = 0$ has no real positive root other than $z = 1$.

For negative roots of $g(z) = 0$, we apply the theorem of Descartes to $g(-z)$, and check up all seven classes. For instance, for the case (vii), the number of sign-changes is equal to 3 if $r > m$, and 5 if $r < m$. Further $g(-1) = 0$, $(dg/dz)(-1) = -2(s+m)s/r < 0$ and $\lim_{z\to-\infty} g(z) = -\infty$. Therefore we see there exist just 3 negative roots. The arguments for remaining cases are similar.

Thus we observe the following: The real roots other than 1 of the equation $g(z) = 0$ are all negative and simple in all seven cases from (i) to (vii). In each case, the number of them is equal to (i): 0, (ii): 1, (iii): 2, (iv): 3, (v): 1, (vi): 0, (vii): 3.

In particular, we see that -1 is a root of $g(z) = 0$ just in the cases (ii), (iv), (v) and (vii). However if s and r are both even and m is odd, then there do not exist points (x, t) and $(x, -t)$ near the origin having same image by f_0. Nevertheless, it is possible that one extra component of double point loci emerges after a perturbation of f_0 from the root -1 of the characteristic polynomial in this case, that is, the case (iv). If $m = 1$, then it is impossible, but if $m \geq 3$, then it is in fact the case. Thus the determinacy fails for (iv), $m \geq 3$. However the position of the extra component of double point loci is restricted, and we conclude that exactly two topological classes appear: Scherbak surface and pseudo Scherbak surface.

In the case (viii) : m, s and r are all even, the polynomial $g(z)$ has -1 as a multiple root. This implies essentially the indeterminacy of the topological type of the parametrisation $\mathrm{dev}(\gamma)$ in this case.

Now we apply Kuiper-Kuo theorem [38][39]to $d(t_1, t_2)$ divided by $(t_1-t_2)^2$, provided it has isolated singularities, that is, when the type $\mathbf{A}$ does not belong to the class (iv), $m \geq 3$ or (viii). Then we see no bifurcation occurs by a type-preserving deformation of γ. Thus we construct a stratification of the corresponding unfolding of $\mathrm{dev}(\gamma)$. In this sense, the study on isolated singularities is applied to that on non-isolated singularities.

Then, according to the standard prescription, we further show that the stratification is Thom regular and apply Thom's isotopy lemma [18]. Thus we see the topological triviality of the family of tangent developables of space curves of constant type.

In the process of the proof, we find out general local properties for the tangent developables: Except for the case (viii), the tangent developable has, near the base point, no double points on the curve itself and on the tangent line at the base point, no points of self-tangency, no triple points, and a topologically cone structure.

10 Open problems and questions.

We summarise the situation we are facing into the correspondence

$$\{\text{Space Curves}\} \longrightarrow \{\text{Space Singular Surfaces}\},$$

by taking tangent developables. Then we concern with, not just singular surfaces themselves, but the interrelation between these two objects, for instance, the determinacy of the "shape" of tangent developables by leading terms of curves.

We conclude the present survey by several promising open problems and questions. The first one is:

(1) Give the local differentiable classification, or holomorphic classification over the complex, of tangent developables of curves of type other than

those in Theorem 5.1, for instance, of type $(1,3,6)$.

Extending the usual notion of determinacy [59], we recognise the notion of determinacy remains significant also in our classification problem.

A space curve-germ $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$ is called **finitely differentiably** (resp. **topologically) determined relatively to the developable** if there exists a positive integer k such that any curve-germ $\gamma' : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$ with $j^k\gamma'(0) = j^k\gamma(0)$ has the tangent developable which is differentiably (resp. topologically) equivalent to that of γ.

We see γ is finitely differentiably determined relatively to the developable if the type of γ at 0 is one of types in Theorem 5.1. Also we see, from Theorem 7.1, γ is finitely topologically determined relatively to the developable if, for the type $(m, m+s, m+s+r)$ of γ at 0, s and r are not both even, or, s and r are both even and $m = 1$. Then we ask the following:

(2) For any γ of type $(m, m+s, m+s+r)$, and for any positive integer k, does there exist a finitely differentiably (or at least topologically) determined space curve-germ γ' relatively to the developable with $j^k\gamma'(0) = j^k\gamma(0)$?

The general methods to show the topological triviality of a family of complex varieties are (a) finding an algebraic torsion to construct controlled vector fields, and (b) showing the constancy of some invariants, which implies regularities of stratifications. See for instance, [12][13][15][16]. Then the question is:

(3) Can the general framework above be applied to the local topological classification of tangent developables in the complex analytic case?

Recently there appear several global results on the topology of tangent developables of generic space curves. For a generic space curve, the tangent developable has, as topological singular points, (folded) umbrellas and generic triple points, as well as transversal double points, and therefore it is homeomorphic to the image of a stable mapping locally at each point [4]. See also [5][52][6][33][56]. Remark that Piene [49] gave the global enumerative geometry of tangent developables of algebraic space curves over the complex (or over an algebraically closed field).

Also there are several results on the topology of dual surfaces of generic space curves, that is, the tangent developables of the dual curves of generic space curves [14][47][7].

From the local classification given in the previous sections, it is natural to ask the global topology of tangent developables of more degenerate curves.

In other words,

(4) In the global context, is the tangent developable of a generic (in the sense of Tougeron) curves has locally topologically cone structure?

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