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Takahiko Nakazi

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Riesz's Functions And Carleson Inequalities

by

Takahiko Nakazi*

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Abstract. Let μ be a finite positive Borel measure on the open unit disc D and H a set of all analytic functions on D . For each a in D , put

$$r(\mu, a) = \sup |f(a)|^2$$

where $f \in H$ and $\int_D |f|^2 d\mu \leq 1$. Unless the support set of μ is a finite set, $\int_D r(\mu, a) d\mu(a) = \infty$. However

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) < \infty$$

may happen where $D_t(z)$ denotes the Bergman disc in D . We study when this is possible. When ν is a discrete measure such that $d\nu = \sum_{a \in A} s(\mu, a) \delta_a$,

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) = \sup_{z \in D} \sum_{a \in A \cap D_t(z)} 1.$$

Under some condition on μ , we show that $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$ for a finite positive Borel measure ν on D if and only if (ν, μ) -Carleson inequality is valid.

§1. Introduction

Let D be the open unit disc in $\mathbb{C}$ and H a set of all analytic functions on D . When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu) = s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu ; f \in H \text{ and } f(a) = 1 \right\}$$

and

$$r(\mu) = r(\mu, a) = \sup \{ |f(a)|^2 ; f \in H \text{ and } \int_D |f|^2 d\mu \leq 1 \}.$$

In the previous paper [2], we noted the following : $r(\mu, a)s(\mu, a) = 1$ for $a \in D$, assuming $\infty \times 0 = 1$, $r(\mu)$ is lower semicontinuous and $s(\mu)$ is upper semicontinuous on D . r and s are called Riesz's functions. Corollary 2 in [2] shows that

$$\int_D r(\mu, a) d\mu(a) = \infty$$

if $\text{supp } \mu$ is not a finite set. By Theorem 8 and Lemma 2 in [2], when $(\text{supp } \mu) \cap D$ is a uniqueness set for H , $L_a^2(\mu) = H \cap L^2(\mu)$ is closed if and only if for all compact sets K in D

$$\int_K r(\mu, a) d\mu(a) < \infty.$$

For any z in D , let ϕ_z be the Möbius function on D and put

$$\beta(z, w) = \frac{1}{2} \log(1 + |\phi_z(w)|)(1 - |\phi_z(w)|)^{-1} \quad (z, w \in D).$$

For $0 < t < \infty$ and z in D , set

$$D_t(z) = \{w \in D ; \beta(z, w) < t\}$$

which is called the Bergman disc with "center" z and "radius" t . For all compact sets K in D $\int_K r(\mu, a) d\mu(a) < \infty$ if and only if for any z in D , $\int_{D_t(z)} r(\mu, a) d\mu(a) < \infty$. We are interested in when

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) < \infty.$$

In Section 2, we study a finite positive Borel measure μ such that

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) < \infty.$$

If $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) < \infty$ and ν is a finite positive Borel measure on D with $\nu \leq \mu$ then $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$. Even if $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) = \infty$, $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$ may happen when ν is enough small. In Section 3, we study a finite positive Borel measure ν for each μ such that $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$.

Throughout this paper, the measure m denotes the normalized Lebesgue area measure on D . We define an average of a finite positive Borel measure μ on $D_t(z)$ by $\hat{\mu}_t(z) = \mu(D_t(z))/m(D_t(z))$ ($z \in D$). We say that ν and μ satisfy the (ν, μ) -Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu$$

for all f in H . Under some condition on μ , the (ν, μ) -Carleson inequality is valid if and only if $\hat{\nu}_t \leq \gamma \hat{\mu}_t$ on D for some positive constant $\gamma > 0$ (see [1]). In Section 4, under the same condition above on μ , we show that the (ν, μ) -Carleson inequality is valid if and only if $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$. If $d\nu = \sum_{a \in A} s(\mu, a) \delta_a$ where A is a discrete set in D and δ_a denotes a point mass measure at a , then note that

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) = \sup_{z \in D} \sum_{a \in A \cap D_t(z)} 1.$$

Hence our result implies a generalization of a theorem of K.Zhu in [4, Theorem 1] when $\mu = m$.

For a finite positive Borel measure μ on D , put

$$\tilde{\mu}(a) = \int_D |k_a(z)|^2 d\mu(z) \quad (a \in D)$$

where $k_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2$. When $d\mu = w dm$, we denote the function by $\tilde{w}$ instead of $\tilde{\mu}$. We say that w is in $(A_2)_\partial$ if there exists a finite positive constant γ such that

$$\tilde{w}(a) \times (w^{-1})^\sim(a) \leq \gamma$$

for all a in D .

§2. $r(\mu, a) d\mu(a)$

It is easy to see that $\sup_{z \in D} \int_{D_t(z)} r(m, a) dm(a) < \infty$. In fact,

$$\begin{aligned} & \sup_{z \in D} \int_{D_t(z)} (1 - |a|^2)^{-2} dm(a) \\ & \leq \sup_{z \in D} \left\{ \sup_{a \in D_t(z)} (1 - |a|^2)^{-2} \right\} m(D_t(z)) \\ & = \sup_{z \in D} \left\{ \sup_{w \in D_t(0)} \left(1 - \left| \frac{z - w}{1 - \bar{z}w} \right|^2 \right)^{-2} \right\} \frac{(1 - |z|^2)^2 k^2}{(1 - |z|^2 k^2)^2} \\ & \leq \sup_{z \in D} (1 - |z|^2 k^2)^{-2} \times \frac{16}{k^2} \sup_{w \in D_t(0)} (1 - |w|^2)^{-2} < \infty \end{aligned}$$

because $k = \tanh t \in (0, 1)$. When $d\mu = \sum_{a \in A} s(\mu, a)\delta_a$ and A is a set of finitely many separated sequences in D , $\sup_{z \in D} \int_{D_t(z)} r(\mu, a)d\mu(a) < \infty$ because $r(\mu, a)d\mu(a) = \sum_{a \in A} \delta_a$. Put

$$\mathcal{R} = \{\mu ; \sup_{z \in D} \int_{D_t(z)} r(\mu, a)d\mu(a) < \infty\},$$

then m belongs to $\mathcal{R}$ and $d\mu = \sum_{a \in A} s(\mu, a)\delta_a$ belongs to A if and only if A is a set of finitely many separated sequences in D , that is, the supremum of the number of points in $A \cap D_t(z)$ is finite.

Lemma 1. *Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and fix $t > 0$. If $\beta(z, a) < t$ then there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \frac{\mu(D_t(a))}{\mu(D_t(z))} \leq C.$$

Proof. When $\beta(z, a) < t$, $D_t(a) \subset D_{2t}(z)$ and so it is enough to prove that

$$\sup_{z \in D} \frac{\mu(D_{2t}(z))}{\mu(D_t(z))} < \infty.$$

This is equivalent to that

$$\sup_{z \in D} \frac{\hat{\mu}_{2t}(z)}{\hat{\mu}_t(z)} < \infty.$$

$\hat{\mu}_t \leq \tilde{\mu}$ on D for any t by Lemma 4.3.3 in [5]. By hypothesis on w , there exists a finite positive constant γ such that $\tilde{\mu} \leq \gamma\hat{\mu}_t$ on D for any t (see [1, p157]). This implies that $\sup_{z \in D} \hat{\mu}_{2t}(z)/\hat{\mu}_t(z) < \infty$.

Lemma 2. *Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and fix $t > 0$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \frac{s(\mu, z)}{\mu(D_t(z))} \leq C \quad (z \in D).$$

Proof. Since $w \in (A_2)_\partial$, there exists a positive constant γ

$$\gamma^{-1}\tilde{w} \leq (\tilde{w}^{-1})^{-1} \leq \exp(\log \tilde{w})$$

on D . By Proposition 4 in [2]

$$\gamma^{-1}\tilde{w}(z) \leq \frac{s(\mu, z)}{(1 - |z|^2)^2} \leq \tilde{w}(z).$$

on D . By the proof of Lemma 1, $\tilde{w}(z)$ is equivalent to $\hat{\mu}_t(z) = (1 - |z|^2)^{-2} \mu(D_t(z))$ on D . This implies that $s(\mu, z)$ is equivalent to $\mu(D_t(z))$ on D .

Theorem 1. *Let μ be a finite positive Borel measure on D .*

(1) *If μ_j ($j = 1, 2$) belong to $\mathcal{R}$, and λ_j ($j = 1, 2$) is a positive constant, then $\lambda_1\mu_1 + \lambda_2\mu_2$ belongs to $\mathcal{R}$.*

(2) *Suppose $d\mu = \sum_{a \in A} v(a)\delta_a$ and each point in A is isolated in D . Then μ belongs to $\mathcal{R}$ if and only if A is a set of finitely many separated sequences.*

(3) *Suppose $\sup_{z \in D} \int_{D_t(z)} (1 - |a|^2)^{-9/2} d\mu(a) < \infty$. If $d\mu = w dm$ and $\int_{K^c} w^{-1} dm < \infty$ for some compact set K in D , then μ belongs to $\mathcal{R}$.*

(4) *If $d\mu = w dm$ and w is in $(A_2)_\partial$, then μ belongs to $\mathcal{R}$.*

(5) *If $d\mu = w dm$ and $w = |f|^2$ for some f in H , then μ belongs to $\mathcal{R}$.*

Proof. (1) Note that for $j = 1, 2$, $\lambda_j\mu_j \leq \lambda_1\mu_1 + \lambda_2\mu_2$ and so $\lambda_j r(\mu_j, a) \geq r(\lambda_1\mu_1 + \lambda_2\mu_2, a)$ ($a \in D$). Then

$$\begin{aligned} & \sup_{z \in D} \int_{D_t(z)} r(\lambda_1\mu_1 + \lambda_2\mu_2, a) d(\lambda_1\mu_1 + \lambda_2\mu_2)(a) \\ & \leq \lambda_1^2 \sup_{z \in D} \int_{D_t(z)} r(\mu_1, a) d\mu_1(a) + \lambda_2^2 \sup_{z \in D} \int_{D_t(z)} r(\mu_2, a) d\mu_2(a) < \infty. \end{aligned}$$

(2) Since A is isolated in D , for each $a \in A$ there exists a function f in H such that $f(a) = 1$ and $f = 0$ on $A \setminus \{a\}$ (cf. [3, Theorem 15.11]). This implies that $s(\mu, a) = v(a)$ and so $r(\mu, a) = v(a)^{-1}$. Hence

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) = \sup_{z \in D} \sum_{a \in D_t(z) \cap A} 1.$$

This implies that μ belongs to $\mathcal{R}$ if and only if A is a set of finitely many separated sequences. (3) By Theorem 5 in [2], if $\int_{K^c} w^{-1} dm < \infty$ for some compact set K in D , then

$$s(\mu, a) \geq C(1 - |a|^2)^{9/2} \quad (a \in D)$$

for some positive constant C . Hence by hypothesis on μ ,

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) \leq C \sup_{z \in D} \int_{D_t(z)} (1 - |a|^2)^{-9/2} d\mu(a) < \infty.$$

Thus μ belongs to $\mathcal{R}$. (4) By Lemmas 1 and 2,

$$\begin{aligned} & \sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) \\ & \leq \sup_{z \in D} \left\{ \left(\sup_{a \in D_t(z)} r(\mu, a) \right) \times \mu(D_t(z)) \right\} \end{aligned}$$

$$\begin{aligned}
&= \sup_{z \in D} \left\{ \left(\sup_{a \in D_t(z)} \frac{1}{s(\mu, a)} \right) \times \mu(D_t(z)) \right\} \\
&\leq C_1 \sup_{z \in D} \left\{ \left(\sup_{a \in D_t(z)} \frac{1}{\mu(D_t(a))} \right) \times \mu(D_t(z)) \right\} \\
&\leq C_2 \sup_{z \in D} \left\{ \frac{1}{\mu(D_t(z))} \times \mu(D_t(z)) \right\} = C_2
\end{aligned}$$

where C_1 and C_2 are finite positive constant. (5) Since $w = |f|^2$ and $f \in H$, $\exp(\log w)^\sim(a) \geq w(a)$ ($a \in D$). For

$$(\log |f|^2)^\sim(a) = \int_D \log |f \circ \phi_a(z)|^2 dm(z) \geq \log |f \circ \phi_a(0)|^2.$$

Hence by (2) of Proposition 4 in [2]

$$\begin{aligned}
&\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) \\
&= \sup_{z \in D} \int_{D_t(z)} \frac{1}{s(\mu, a)} w(a) dm(a) \\
&\leq \sup_{z \in D} \int_{D_t(z)} \frac{w(a)}{(1 - |a|^2)^2 \exp(\log w)^\sim(a)} dm(a) \\
&\leq \sup_{z \in D} \int_{D_t(z)} (1 - |a|^2)^{-2} dm(a)
\end{aligned}$$

By the remark in the first line in this section, $m \in \mathcal{R}$ and so μ belongs to $\mathcal{R}$.

§3. $r(\mu, a)d\nu(a)$

If v is a Borel function such that $0 \leq v \leq s(\mu)$ on D , and $d\nu = vdm$, then

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) \leq \int_D r(\mu, a) v(a) dm(a) \leq 1.$$

Put

$$\mathcal{R}^\mu = \{ \nu ; \sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty, \nu \text{ is a finite positive Borel measure} \}$$

for each finite positive Borel measure μ . Then the above measure vdm belongs to $\mathcal{R}^\mu$. In this section, we study the set $\mathcal{R}^\mu$.

Theorem 2. μ , ν and σ denote finite positive Borel measures on D .

(1) If μ is in $\mathcal{R}$ and $\nu \leq \gamma\mu$ for some positive constant γ then ν belongs to $\mathcal{R}^\mu$.

(2) Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$. If ν is in $\mathcal{R}^\mu$ and $\hat{\sigma}_t \leq \gamma\hat{\nu}_t$ on D for some $\gamma > 0$ and some $t > 0$ then σ belongs to $\mathcal{R}^\mu$.

(3) When λ is a (not necessarily finite) positive Borel measure on D and $d\nu = s(\mu, a)d\lambda(a)$, ν belongs to $\mathcal{R}^\mu$ if and only if $\sup_{z \in D} \lambda(D_t(z)) < \infty$ for some $t > 0$. In particular, when $d\lambda = \sum_{a \in A} \delta_a$ and A is a discrete set in D , ν belongs to $\mathcal{R}^\mu$ if and only if A is a set of finitely many separated sequences.

Proof. (1) is clear. (2) Suppose $w \in (A_2)_\partial$. By Lemmas 1 and 2, there exists a positive finite constant C such that

$$\frac{1}{C} \sup_{z \in D} \frac{\alpha(D_t(z))}{\mu(D_t(z))} \leq \sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\alpha(a) \leq C \sup_{z \in D} \frac{\alpha(D_t(z))}{\mu(D_t(z))}$$

because $r(\mu, a) s(\mu, a) \equiv 1$. These inequalities imply (2) by applying it for $\alpha = \nu$ and $\alpha = \sigma$. In fact, as $\alpha = \nu$ in the above inequalities

$$\sup_{z \in D} \frac{\nu(D_t(z))}{\mu(D_t(z))} < \infty$$

because $\nu \in \mathcal{R}^\mu$. As $\alpha = \sigma$,

$$\sup_{z \in D} \frac{\sigma(D_t(z))}{\mu(D_t(z))} \leq \sup_{z \in D} \frac{\nu(D_t(z))}{\mu(D_t(z))} \sup_{z \in D} \frac{\sigma(D_t(z))}{\nu(D_t(z))} < \infty$$

because $\hat{\sigma}_t \leq \gamma\hat{\nu}_t$. (3) If $d\nu = s(\mu, a)d\lambda(a)$, then

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) = \sup_{z \in D} \lambda(D_t(z)).$$

This implies (3).

§4. (ν, μ) -Carleson inequality

If $\int_D r(\mu, a) d\nu(a) < \infty$ then (ν, μ) -Carleson inequality is valid (see [2, Theorem 7]). The following question is natural. Is (ν, μ) -Carleson inequality valid for ν in $\mathcal{R}^\mu$? Theorems 2 and 3 answer for it positively when $d\mu = wdm$ and w is in $(A_2)_\partial$. Corollary 1 is a generalization of a result of K.Zhu [4, Theorem 1].

Theorem 3 *Let μ be a finite positive Borel measure and λ a (not necessarily finite) positive Borel measure on D . Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and $d\nu = s(\mu, a)d\lambda(a)$. $\sup_{z \in D} \lambda(D_t(z)) < \infty$ if and only if (ν, μ) -Carleson inequality is valid.*

Proof. By Theorem 3 in [1], in order to prove this theorem, it is sufficient to show that $\sup_{z \in D} \lambda(D_t(z)) < \infty$ if and only if

$$\sup_{z \in D} \frac{\hat{\nu}_t(z)}{\hat{\mu}_t(z)} < \infty.$$

The proof of Theorem 2 shows this.

Corollary 1. *Suppose $d\mu = wdm$ is a finite positive Borel measure with w in $(A_2)_\partial$ and $d\nu = \sum_{a \in A} s(\mu, a)\delta_a$ with a discrete set A in D . A is a set of finitely many separated sequences in D if and only if (ν, μ) -Carleson inequality is valid.*

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Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

nakazi @ math.sci.hokudai.ac.jp