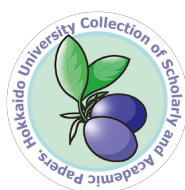




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**ASYMPTOTICS FOR THE PARTIAL
AUTOCORRELATION FUNCTION OF
A STATIONARY PROCESS**

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ASYMPTOTICS FOR THE PARTIAL AUTOCORRELATION FUNCTION OF A STATIONARY PROCESS

AKIHIKO INOUE

1. INTRODUCTION

The purpose of this paper is to study the long-time behaviour of the partial autocorrelation function of a stationary process.

Let $\{X_n\} = \{X_n : n \in \mathbb{Z}\}$ be a real, zero-mean, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$, which we shall simply call a *stationary process*. Throughout this paper, we assume that $\{X_n\}$ is *purely non-deterministic* (cf. §2). The *autocovariance function* $\gamma(\cdot)$ of $\{X_n\}$ is defined by

$$\gamma(n) := E[X_n X_0] \quad (n \in \mathbb{Z}).$$

We denote by H the closed real linear hull of $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then H is a real Hilbert space with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $n \geq 1$, we write $H_{[1, n]}$ for the subspace of H spanned by $\{X_1, X_2, \dots, X_n\}$, and $P_{[1, n]}$ for the orthogonal projection operator of H onto $H_{[1, n]}$.

The partial autocorrelation $\alpha(n)$ is the correlation coefficient of the two residuals obtained after regressing X_0 and X_n on the intermediate observations $X_1, \dots, X_{n-1}$. More precisely, the *partial autocorrelation function* $\alpha(\cdot)$ of $\{X_n\}$ is defined by

$$\alpha(n) := \frac{E[Z_n^+ Z_n^-]}{E[(Z_n^+)^2]^{1/2} \cdot E[(Z_n^-)^2]^{1/2}} \quad (n = 2, 3, \dots),$$

where

$$Z_n^+ := X_n - P_{[1, n-1]} X_n, \quad Z_n^- := X_0 - P_{[1, n-1]} X_0.$$

We also set $\alpha(1) := \gamma(1)/\gamma(0)$. We think of Z_n^+ as the part of X_n which cannot be explained by the intermediate observations $X_1, \dots, X_{n-1}$, and Z_n^- as the part of X_n which cannot be explained by the same ones. So the partial autocorrelation $\alpha(n)$ is a kind of ‘pure’ correlation coefficient between X_0 and X_n . See Brockwell-Davis [BD, §3.4 and §5.2] for background.

One of the important facts about the partial autocorrelation function $\alpha(\cdot)$ is that we can calculate the value of $\alpha(n)$ easily (at least numerically) from the values of $\gamma(0), \gamma(1), \dots, \gamma(n)$. To do that, one may just use the *Durbin-Levinson algorithm* (cf. [BD, Prop. 5.2.1]). Moreover, if we look at the algorithm carefully, then we see that the values of $\gamma(0), \alpha(1), \dots, \alpha(n)$ conversely determine the value of $\gamma(n)$. In this sense, the partial autocorrelation function $\alpha(\cdot)$ has the same information as the autocovariance function $\gamma(\cdot)$.

We are interested in the following seemingly simple problem: how does $\alpha(n)$ look like for n large? It turns out that this problem is more difficult than it looks at

first. The difficulty is related to the fact that the definition of partial autocorrelation function involves the prediction from a *finite* part of time. This setting makes the asymptotic analysis hard. For, since we wish to obtain the asymptotics, it would be inevitable to investigate $\alpha(\cdot)$ analytically but, generally speaking, it is not easy to do so for quantities related to such prediction.

We are especially interested in the case where $\{X_n\}$ is a *long-memory process*; roughly speaking, this means that the autocovariance $\gamma(n)$ tends to zero as $n \rightarrow \infty$ so slowly that $\gamma(\cdot)$ is not summable (cf. [BD, §13.2]). In our main theorem (Theorem 2.1), we will determine the desired asymptotics for the partial autocorrelation function, modulo absolute value, for a class of stationary processes which includes long-memory processes. Our result presents a surprising regularity in the asymptotics. More precisely, let $-\infty < d < \frac{1}{2}$ and ℓ be a slowly varying function at infinity (cf. §2). Then it will be shown that, under some conditions (on the $\text{MA}(\infty)$ and $\text{AR}(\infty)$ coefficients of $\{X_n\}$),

$$(1.1) \quad \gamma(n) \sim n^{2d-1} \ell(n) \quad (n \rightarrow \infty)$$

implies

$$(1.2) \quad |\alpha(n)| \sim \frac{\gamma(n)}{\sum_{k=-n}^n \gamma(k)} \quad (n \rightarrow \infty).$$

In particular, if $0 < d < \frac{1}{2}$, that is, $0 < 1 - 2d < 1$, then

$$(1.3) \quad |\alpha(n)| \sim \frac{d}{n} \quad (n \rightarrow \infty).$$

We explain the points of the result above. It should be noted that the assumption (1.1) simply says $\gamma(\cdot)$ is regularly varying with negative index (cf. Bingham *et al.* [BGT, §1.4.2]) since the index $2d - 1$ may take any non-negative values. It is perhaps surprising that there exists such a simple formula as (1.2). As one sees, the result (1.3) for the long-memory case $0 < d < 1$ is particularly simple; the index over n is one, whence independent of d , and the slowly varying function ℓ has even disappeared. We also notice that the quantity d , which is important to a long-memory process, appears so explicitly in (1.3).

We tackle the problem above via the asymptotic analysis of the relevant expected prediction error (Theorems 6.4, 6.6 and 6.7). The idea is to use the precise asymptotics for the sequence $\{c_n\}$ of $\text{MA}(\infty)$ coefficients and the sequence $\{a_n\}$ of $\text{AR}(\infty)$ coefficients. Here we note that the sequences $\{c_n\}$ and $\{a_n\}$ are defined for any purely non-deterministic stationary process (cf. §2). In order to deduce the desired asymptotic behaviour of the partial autocorrelation function from that of the prediction error, we will use a Tauberian argument. So naturally an adequate Tauberian condition will be needed. It turns out that the most elementary Tauberian condition, that is, monotonicity, is available here.

We will verify the desired monotonicity by an explicit representation of the prediction error in terms of $\{c_n\}$ and $\{a_n\}$ (Theorems 4.5 and 4.6). This representation, in turn, will be obtained by an argument on the geometry of the Hilbert space H (Theorem 4.1). Here is another idea, that is, to use a discrete-time analogue of the Seghler-Dym theorem in this context. Here the (original) Seghler-Dym theorem ([S], [Dy2]) concerns the intersection of past and future of a continuous-time stationary process. This theorem originates in the work of Levinson-McKean [LM]. We will prove an analogue of this theorem for discrete-time stationary processes (Theorem 3.1), and then apply it to our problem.

In the main theorem, we will assume some conditions which are given in terms of $\{c_n\}$ and $\{a_n\}$. As an example, we will consider the stationary processes whose autocovariance functions are completely monotone. This property for a stationary process is called *reflection positivity*. For example, if $-\infty < d < \frac{1}{2}$, then the stationary process with autocovariance function of the form $\gamma(n) = (1 + |n|)^{2d-1}$ has reflection positivity (cf. Example in §7). See Okabe [O] as well as [I2] and [OI] for earlier work. Since we wish to consider long-memory processes (as well as short-memory ones), our class of stationary processes with reflection positivity is different from those studied in them; the latter studied only short-memory processes. We will show that the stationary processes in our class satisfy the conditions of the main theorem (Theorem 7.3).

We state the main theorem in §2. In §3, we prove the Seghieri-Dym type theorem. In §4, we give some representation theorems in terms of $\{c_n\}$ and $\{a_n\}$. We obtain the necessary asymptotics for $\{c_n\}$ and $\{a_n\}$ in §5. In §6, we first show the necessary results on the asymptotics for the prediction error, and then prove the main theorem using them. In §7, we consider the class of stationary processes with reflection positivity, and show that they satisfy the conditions of the main theorem.

2. MAIN THEOREM

In this section, we shall state the main theorem. To do that, we need some notation.

Let $\{X_n\} = \{X_n : n \in \mathbb{Z}\}$ be a stationary process; as stated in §1, this means that $\{X_n\}$ is a real, zero-mean, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\gamma(\cdot)$ be the autocovariance function of $\{X_n\}$. As we also stated in §1, we write H for the real Hilbert space spanned by $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$, with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $I \subset \mathbb{Z}$, denote by H_I the closed real linear hull of $\{X_k : k \in I\}$ in H . In particular, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $m \leq n$, we write $H_{(-\infty, m]}$, $H_{[m, \infty)}$ and $H_{[m, n]}$ for H_I with $I = \{k \in \mathbb{Z} : -\infty < k \leq m\}$, $\{k \in \mathbb{Z} : m \leq k < \infty\}$ and $\{k \in \mathbb{Z} : m \leq k \leq n\}$, respectively. For $I \subset \mathbb{Z}$, we denote by P_I the orthogonal projection operator of H onto H_I . We write $P_I^\perp := I_H - P_I$, where I_H is the identity map of H . So $P_I^\perp$ is the orthogonal projection operator of H onto $H_I^\perp$.

As we stated in §1, we assume throughout this paper that the stationary process $\{X_n\}$ is *purely non-deterministic*, that is,

$$\bigcap_{n=-\infty}^{\infty} H_{(-\infty, n]} = \{0\}$$

or equivalently there exists a positive even and integrable function $\Delta(\cdot)$ on $(-\pi, \pi)$ such that

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\theta} \Delta(\theta) d\theta \quad (n \in \mathbb{Z}), \quad \int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$$

(see [BD, §5.7] and Rozanov [Ro, Ch. II]; in the latter, the term *linearly regular* is used instead of purely non-deterministic). We call $\Delta(\cdot)$ the *spectral density* of $\{X_n\}$. It should be pointed out that there exists an a.e. ambiguity for $\Delta(\cdot)$. We define the *outer function* $h(\cdot)$ of $\{X_n\}$ by

$$(2.1) \quad h(z) := \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \Delta(\theta) d\theta \right\} \quad (z \in \mathbb{C}, |z| < 1).$$

The function $h(\cdot)$ is actually an outer function which is in the Hardy space H^{2+} of class 2 over the unit disk $|z| < 1$ (cf. Rudin [Ru, Def. 17.14]).

Let c_n be the power series coefficients of $(2\pi)^{1/2}h(z)$:

$$(2\pi)^{1/2}h(z) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| < 1).$$

The coefficients c_n are real and satisfy $\sum_{n=0}^{\infty} (c_n)^2 < \infty$ (cf. [Ru, Thm. 17.12]). We call c_n the n -th MA(∞) coefficient of $\{X_n\}$ (see (4.7) below for background). The sequence $\{c_n\}$ is often called the *canonical representation kernel* of $\{X_n\}$, too. Now the outer function $h(z)$ has no zeros in $|z| < 1$, whence we have another holomorphic function $h(z)^{-1}$ in $|z| < 1$. Let a_n be the power series coefficients of the function $-1/\{(2\pi)^{1/2}h(z)\}$:

$$-\frac{1}{(2\pi)^{1/2}h(z)} = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1).$$

Then a_n are also real. We call a_n the n -th AR(∞) coefficient of $\{X_n\}$ (see (4.9) below for background). Since

$$(2.2) \quad \left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} c_n z^n \right) = -1 \quad (|z| < 1),$$

we have the following relation between $\{c_n\}$ and $\{a_n\}$:

$$(2.3) \quad \sum_{j=0}^n a_j c_{n-j} = -\delta_{n0} \quad (n \geq 0).$$

Our main theorem will be stated under the following conditions on the sequences $\{c_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$ and $\{a_n - a_{n+1}\}_{n=0}^{\infty}$:

- (C1) $c_n \geq 0$ for all $n \geq 0$;
- (C2) $\{c_n\}$ is eventually decreasing to zero;
- (A1) $\{a_n\}$ is eventually decreasing to zero;
- (A2) $\{a_n - a_{n+1}\}$ is eventually decreasing to zero.

We write $\mathcal{R}_0$ for the class of slowly varying functions at infinity: the class of positive, measurable ℓ , defined on some neighbourhood $[A, \infty)$ of infinity, such that

$$\forall \lambda > 0, \quad \lim_{x \rightarrow \infty} \ell(\lambda x) / \ell(x) = 1$$

(see [BGT, Ch. 1] for background). Let $\ell \in \mathcal{R}_0$, and choose B so large that $\ell(\cdot)$ is locally bounded on $[B, \infty)$ (cf. [BGT, Cor. 1.4.2]). When we say $\int_B^{\infty} \ell(s) ds / s = \infty$, it means that $\int_B^{\infty} \ell(s) ds / s = \infty$. If so, then we define another slowly varying function $\tilde{\ell}$ by

$$(2.4) \quad \tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \quad (x \geq B)$$

(see [BGT, §1.5.6]). The asymptotic behaviour of $\tilde{\ell}(x)$ as $x \rightarrow \infty$ does not depend on the choice of B by the assumption $\int_B^{\infty} \ell(s) ds / s = \infty$.

Let $\alpha(\cdot)$ be the partial autocorrelation function of $\{X_n\}$. Here is the main theorem.

Theorem 2.1. Let $-\infty < d < \frac{1}{2}$ and $\ell \in \mathcal{R}_0$. We assume (C1), (C2), (A1), and (A2). Suppose

$$(2.5) \quad \gamma(n) \sim n^{2d-1} \ell(n) \quad (n \rightarrow \infty).$$

Then

$$(2.6) \quad |\alpha(n)| \sim \frac{\gamma(n)}{\sum_{k=-n}^n \gamma(k)} \quad (n \rightarrow \infty).$$

In other words,

(1) if $0 < d < \frac{1}{2}$, then

$$(2.7) \quad |\alpha(n)| \sim \frac{d}{n} \quad (n \rightarrow \infty);$$

(2) if $d = 0$ and $\int^\infty \ell(s) ds/s = \infty$, then

$$(2.8) \quad |\alpha(n)| \sim n^{-1} \frac{\ell(n)}{2\bar{\ell}(n)} \quad (n \rightarrow \infty);$$

(3) if $-\infty < d \leq 0$, and if further $\int^\infty \ell(s) ds/s < \infty$ for $d = 0$, then

$$(2.9) \quad |\alpha(n)| \sim \frac{n^{2d-1} \ell(n)}{\sum_{-\infty}^\infty \gamma(k)} \quad (n \rightarrow \infty).$$

We must point out that what we actually do below is to prove (2.7)–(2.9) separately rather than to prove (2.6) directly. It is easy to show that the asymptotics (2.7)–(2.9) are unified in (2.6) but it is still mysterious why they are so.

3. INTERSECTION OF PAST AND FUTURE

In this section, we shall prove a discrete analogue of the Seghler-Dym theorem ([S], [Dy2]; see also Levinson-McKean [LM, §6c], Dym-McKean [DM, §4.3] and Dym [Dy1, Thm. 2.1]). It will play a crucial role in this paper. We notice that, as stated in §2, the stationary process $\{X_n\}$ is assumed to be purely non-deterministic.

Theorem 3.1. If the spectral density $\Delta(\cdot)$ of $\{X_n\}$ satisfies $\int_{-\pi}^\pi \Delta(\theta)^{-1} d\theta < \infty$, then for any $n \geq 0$

$$(3.1) \quad H_{(-\infty, 0]} \cap H_{[-n, \infty)} = H_{[-n, 0]}.$$

Proof. Step 1. We denote by $H^\mathbb{C}$ the closed complex linear hull of $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then $H^\mathbb{C}$ is a complex Hilbert space with inner product $(Y_1, Y_2) := E[Y_1 \overline{Y_2}]$. We define its closed subspaces $H_{(-\infty, 0]}^\mathbb{C}$, $H_{[-n, \infty)}^\mathbb{C}$ and $H_{[-n, 0]}^\mathbb{C}$ as we defined $H_{(-\infty, 0]}$, $H_{[-n, \infty)}$ and $H_{[-n, 0]}$ above, but replacing $\mathbb{R}$ by $\mathbb{C}$. We shall prove

$$(3.2) \quad H_{(-\infty, 0]}^\mathbb{C} \cap H_{[-n, \infty)}^\mathbb{C} = H_{[-n, 0]}^\mathbb{C} \quad (\forall n \geq 0).$$

The result (3.1) for the real case follows from this.

We write L for the complex Hilbert space $L^2((-\pi, \pi), \Delta(\theta) d\theta)$ with the inner product

$$(f, g)_L := \int_{-\pi}^\pi f(\theta) \overline{g(\theta)} \Delta(\theta) d\theta.$$

For $I \subset \mathbb{Z}$, we denote by L_I the closed complex linear hull of $\{e^{ik\theta} : k \in I\}$ in L . In particular, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $m \leq n$, we write $L_{(-\infty, m]}$, $L_{[m, \infty)}$

and $L_{[m,n]}$ for L_I with $I = \{k \in \mathbb{Z} : -\infty < k \leq m\}$, $\{k \in \mathbb{Z} : m \leq k < \infty\}$ and $\{k \in \mathbb{Z} : m \leq k \leq n\}$, respectively.

The stationary process $\{X_n\}$ permits a spectral representation of the form

$$(3.3) \quad X(k) = \int_{-\pi}^{\pi} e^{ik\theta} Z(d\theta) \quad (n \in \mathbb{Z}),$$

where Z is the spectral measure such that $E[Z(A)\overline{Z(B)}] = \int_{A \cap B} \Delta(\theta) d\theta$ (see [BD, §4.8]). The mapping $f \mapsto \int_{-\pi}^{\pi} f(\theta) Z(d\theta)$ gives a Hilbert space isomorphism of L onto $H^{\mathbb{C}}$. For $I \subset \mathbb{Z}$, the subspace L_I is mapped to $H_I^{\mathbb{C}}$. So in order to prove (3.2), it is enough to prove

$$L_{(-\infty, 0]} \cap L_{[-n, \infty)} = L_{[-n, 0]} \quad (\forall n \geq 0).$$

However the implication $\supset$ is trivial, hence we shall prove only the opposite one $\subset$.

We write H^{2+} for the Hardy space H^{2+} of class 2 over the unit disk $|z| < 1$, and H^{2-} for that over the region $|z| > 1$ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. As usual, we identify each function $f(z)$ in H^{2+} or H^{2-} with its boundary function $f(e^{i\theta})$ and regard both H^{2+} and H^{2-} as subspaces of $L^2((-\pi, \pi), d\theta)$.

We define an outer function h in H^{2+} by (2.1), and $h^* \in H^{2-}$ by $h^*(z) := \overline{h(1/\bar{z})}$ ($|z| > 1$). We define $h_n \in H^{2+}$ by $h_n(z) := z^n h(z)$. Then since $L_{[-n, \infty)} = e^{-in\theta} L_{[0, \infty)}$, it follows that

$$L_{[-n, \infty)} = \frac{1}{h_n} H^{2+}, \quad L_{(-\infty, 0]} = \frac{1}{h^*} H^{2-}$$

(see Ibragimov and Rozanov [IR, II.2, Theorem 1]; Beurling's theorem is essential here). So, for any $f(e^{i\theta}) \in L_{(-\infty, 0]} \cap L_{[-n, \infty)}$, there exist $g_+ \in H^{2+}$ and $g_- \in H^{2-}$ such that

$$f(e^{i\theta}) = \frac{g_+(e^{i\theta})}{h_n(e^{i\theta})} = \frac{g_-(e^{i\theta})}{h^*(e^{i\theta})} \quad \text{a.e. on } (-\pi, \pi).$$

For these g_+ and g_- , we put $f(z) := g_+(z)/h_n(z)$ for $|z| < 1$, and $f(z) := g_-(z)/h^*(z)$ for $|z| > 1$. Then f is meromorphic in $|z| < 1$ and possibly has a unique pole at zero, and the order of pole is at most n . On the other hand, f is holomorphic in the region $|z| > 1$ of the Riemann sphere. We claim that the function f can be continued analytically from $|z| < 1$ to $|z| > 1$ across the unit circle $|z| = 1$.

This claim implies that the function f so obtained is meromorphic over the Riemann sphere, whence it is a rational function. By the above description of singularity, f must be of the form $f(z) = \sum_{k=0}^n a_k z^{-k}$ with some $a_k \in \mathbb{C}$, whence $f(e^{i\theta}) = \sum_{k=0}^n a_k e^{-ik\theta}$. Therefore $f(e^{i\theta}) \in L_{[-n, 0]}$, and this gives the desired implication $L_{(-\infty, 0]} \cap L_{[-n, \infty)} \subset L_{[-n, 0]}$.

Step 2. We complete the proof by proving the claim above. It should be pointed out that the argument below is parallel to that of [LM, §6c].

Put $A_k := \{\theta \in (-\pi, \pi) : \sup_{1-(1/k) < r < 1} |f(re^{i\theta})| \geq k\}$. Then $A_k \supset A_{k+1}$. Moreover, since

$$\int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta \leq \left\{ \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \Delta(\theta) d\theta \right\}^{1/2} \left\{ \int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta \right\}^{1/2} < \infty,$$

Egoroff's theorem implies that the Lebesgue measure of A_k tends to zero as $k \rightarrow \infty$.

Now

$$\begin{aligned} \int_{A_k} |f(re^{i\theta})| d\theta &\leq r^{-n} \left\{ \int_{A_k} |g_+(re^{i\theta})|^2 d\theta \right\}^{1/2} \left\{ \int_{A_k} |h(re^{i\theta})|^{-2} d\theta \right\}^{1/2} \\ &\leq r^{-n} \|g_+\|_{2+} \left\{ \int_{A_k} |h(re^{i\theta})|^{-2} d\theta \right\}^{1/2}, \end{aligned}$$

where $\|g_+\|_{2+}$ is the H^{2+} -norm of g_+ . Let

$$(3.4) \quad P_r(\theta) := \frac{1-r^2}{1-2r\cos\theta+r^2}$$

be the Poisson kernel. By Jensen's inequality,

$$\begin{aligned} |h(re^{i\theta})|^{-2} &= \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) \log \Delta(t)^{-1} dt \right\} \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) \Delta(t)^{-1} dt, \end{aligned}$$

whence, for $k \geq 2$,

$$\begin{aligned} &\sup_{1-(1/k) < r < 1} \int_{A_k} |f(re^{i\theta})| d\theta \\ &\leq 2^n \|g_+\|_{2+} \left[\int_{-\pi}^{\pi} \left\{ \sup_{1-(1/k) < r < 1} \frac{1}{2\pi} \int_{A_k} P_r(t-\theta) d\theta \right\} \Delta(t)^{-1} dt \right]^{1/2}. \end{aligned}$$

For fixed $m \in \mathbb{N}$, we have for almost all $t \in (-\pi, \pi)$

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow \infty} \sup_{1-(1/k) < r < 1} \frac{1}{2\pi} \int_{A_k} P_r(t-\theta) d\theta \\ &\leq \lim_{k \rightarrow \infty} \sup_{1-(1/k) < r < 1} \frac{1}{2\pi} \int_{A_m} P_r(t-\theta) d\theta = I_{A_m}(t). \end{aligned}$$

Let $m \rightarrow \infty$. Then we obtain

$$\lim_{k \rightarrow \infty} \sup_{1-(1/k) < r < 1} \frac{1}{2\pi} \int_{A_k} P_r(t-\theta) d\theta = 0 \quad \text{a.e. on } (-\pi, \pi).$$

Consequently,

$$\lim_{k \rightarrow \infty} \sup_{1-(1/k) < r < 1} \int_{A_k} |f(re^{i\theta})| d\theta = 0,$$

so that

$$\lim_{r \uparrow 1} \int_{-\pi}^{\pi} |f(e^{i\theta}) - f(re^{i\theta})| d\theta = 0.$$

The analogous result for $r \downarrow 1$ follows similarly from the fact $g_- \in H^{2-}$.

Choose $\alpha \in (-\pi, \pi)$ so that $f(re^{i\alpha})$ tends boundedly to $f(e^{i\alpha})$ as $r \rightarrow 1$. For $z = re^{i\theta}$ in the region $D := \{re^{i\theta} : \frac{1}{2} < r < 2, \alpha < \theta < \alpha + 2\pi\}$, define $F(z)$ by $F(z) = \int_{\Gamma} f(w) dw$, where the path $\Gamma = \gamma_1 + \gamma_2$ from $e^{i\alpha}$ to z is defined by $\gamma_1(t) := te^{i\alpha}$ with t from 1 to r and then $\gamma_2(t) := re^{it}$ with t from α to θ . The function F is holomorphic in $D_0 := D \setminus \{z \in D : |z| = 1\}$ and continuous in D , whence F is holomorphic in D by the reflection principle. Since $f = F'$ in D_0 , this implies that f can be continued analytically in D across $|z| = 1$. Since we can

choose a different α , we conclude that f can be continued analytically across the whole unit circle $|z| = 1$, as claimed. This completes the proof. $\square$

4. REPRESENTATIONS

In this section, we shall prove for a purely non-deterministic stationary process $\{X_n\}$ some representation theorems which are given in terms of the MA(∞) coefficients c_n and AR(∞) coefficients a_n . They will enable us to carry out in §6 the asymptotic analysis via $\{c_n\}$ and $\{a_n\}$.

For $Y \in H$ and $I \subset \mathbb{Z}$, we think of $P_I Y$ as the best prediction of Y on the observations $\{X_k : k \in I\}$, whence $P_I^\perp Y = Y - P_I Y$ as its prediction error. By the orthogonal decompositions

$$P_{[-n,0]}^\perp = P_{(-\infty,0]}^\perp + P_{[-n,0]}^\perp P_{(-\infty,0]} = P_{[-n,\infty)}^\perp + P_{[-n,0]}^\perp P_{[-n,\infty)},$$

we have

$$\begin{aligned} \|P_{[-n,0]}^\perp Y\|^2 &= \|P_{(-\infty,0]}^\perp Y\|^2 + \|P_{[-n,0]}^\perp P_{(-\infty,0]} Y\|^2 \\ (4.1) \quad &= \|P_{(-\infty,0]}^\perp Y\|^2 + \|P_{[-n,\infty)}^\perp P_{(-\infty,0]} Y\|^2 \\ &\quad + \|P_{[-n,0]}^\perp P_{[-n,\infty)} P_{(-\infty,0]} Y\|^2, \end{aligned}$$

and similarly, by induction, for $m \geq 0$

$$\begin{aligned} \|P_{[-n,0]}^\perp Y\|^2 &= \sum_{k=0}^m \|P_{(-\infty,0]}^\perp \{P_{[-n,\infty)} P_{(-\infty,0]}\}^k Y\|^2 \\ (4.2) \quad &\quad + \sum_{k=0}^m \|P_{[-n,\infty)}^\perp \{P_{(-\infty,0]} P_{[-n,\infty)}\}^k P_{(-\infty,0]} Y\|^2 \\ &\quad + \|P_{[-n,0]}^\perp \{P_{[-n,\infty)} P_{(-\infty,0]}\}^{m+1} Y\|^2. \end{aligned}$$

Now what will happen if we let $m \rightarrow \infty$? The following theorem gives an answer.

Theorem 4.1. *If the spectral density $\Delta(\cdot)$ of X satisfies $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$, then for $Y \in H$ and $n \geq 0$*

$$\begin{aligned} \|P_{[-n,0]}^\perp Y\|^2 &= \sum_{k=0}^{\infty} \|P_{(-\infty,0]}^\perp \{P_{[-n,\infty)} P_{(-\infty,0]}\}^k Y\|^2 \\ (4.3) \quad &\quad + \sum_{k=0}^{\infty} \|P_{[-n,\infty)}^\perp \{P_{(-\infty,0]} P_{[-n,\infty)}\}^k P_{(-\infty,0]} Y\|^2. \end{aligned}$$

Proof. It follows from Theorem 3.1 that $H_{(-\infty,0]} \cap H_{[-n,\infty)} = H_{[-n,0]}$. Hence, by Nakano's theorem (cf. Reed-Simon [RS, Ch. VI, Problem 39]), we have

$$\text{s-lim}_{m \rightarrow \infty} \{P_{(-\infty,0]} P_{[-n,\infty)}\}^m = P_{[-n,0]}.$$

This implies that the last term of the right-hand side of (4.2) tends to zero as $m \rightarrow \infty$. Thus the theorem follows. $\square$

The point of Theorem 4.1 is that it enables us to investigate the prediction from a finite part of time via the prediction from an infinite past and that from an infinite future.

The key assumption $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$ above can be given in different ways.

Proposition 4.2. *The following conditions are equivalent:*

- (1) $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$;
- (2) $h^{-1} \in H^{2+}$;
- (3) $\sum_0^{\infty} (a_n)^2 < \infty$.

Proof. The implication (2) $\Leftrightarrow$ (3) follows from the well-known characterization of the space H^{2+} in terms of power series coefficients (cf. [Ru, Thm. 17.12]). On the other hand, since

$$\frac{1}{h(z)} = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\{1/\Delta(\theta)\} d\theta \right\} \quad (|z| < 1),$$

we see that (1) and (2) are equivalent (cf. [Ru, Thm. 17.16]). $\square$

We look at the relation between the condition above and those in §2.

Proposition 4.3. *If (C1) and (A1) hold, then we have $\sum_0^{\infty} |a_n| < \infty$ and hence $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$.*

Proof. Bearing in mind that $\{a_n\}$ is eventually non-negative, we apply the monotone convergence theorem to (2.2). Then it follows that

$$\sum_{n=0}^{\infty} a_n = -\lim_{r \uparrow 1} \left(\sum_{n=0}^{\infty} c_n r^n \right)^{-1} \in (-\infty, 0],$$

which implies $\sum_0^{\infty} |a_n| < \infty$. In particular, we have $\sum_0^{\infty} (a_n)^2 < \infty$, so that, by Proposition 4.2, $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$. $\square$

Now we consider $P_{(-\infty, 0]} X_n$ for $n \geq 1$. We vaguely think of it as a linear combination of $\{X_k : -\infty < k \leq 0\}$. We shall make this point clear to the extent sufficient for our purpose. We put

$$(4.4) \quad b_j^m := \sum_{k=1}^m c_{m-k} a_{k+j} \quad (m \geq 1, \quad j \geq 0).$$

Theorem 4.4. *If $\sum_0^{\infty} |a_k| < \infty$, then for $n \in \mathbb{N}$*

$$(4.5) \quad P_{(-\infty, 0]} X_n = \sum_{j=0}^{\infty} b_j^n X_{-j},$$

the sum converging absolutely in H .

Proof. Recall the spectral representation (3.3). Since $|h(e^{i\theta})|^2 = \Delta(\theta) > 0$ a.e. on $(-\pi, \pi)$, we may put

$$(4.6) \quad \xi_n := (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{in\theta} \left\{ \overline{h(e^{i\theta})} \right\}^{-1} Z(d\theta) \quad (n \in \mathbb{Z}).$$

Then, as is well known, $\{\xi_n : n \in \mathbb{Z}\}$ forms a complete orthonormal system for H such that

$$(4.7) \quad X_n = \sum_{j=-\infty}^n c_{n-j} \xi_j, \quad H_{(-\infty, n]} = H_{(-\infty, n]}(\xi) \quad (\forall n \in \mathbb{Z})$$

where $H_{(-\infty, n]}(\xi)$ is the closed subspace of H spanned by $\{\xi_k : -\infty < k \leq n\}$ (see [Ro, Ch. II]). The representation (4.7) is the so-called *canonical representation* of $\{X_n\}$. It follows that

$$(4.8) \quad \left\| \sum_{k=0}^m a_k X_{n-k} + \xi_n \right\|^2 = \int_{-\pi}^{\pi} |f_m(\theta)|^2 \Delta(\theta) d\theta \quad (m \in \mathbb{N}),$$

where

$$f_m(\theta) := \frac{1}{(2\pi)^{1/2}h(e^{i\theta})} + \sum_{k=0}^m a_k e^{ik\theta} \quad (-\pi < \theta < \pi).$$

By assumption, $h^{-1}(\cdot)$ is in H^{2+} . Hence we have the Fourier series expansion

$$\frac{1}{(2\pi)^{1/2}h(e^{i\theta})} = \sum_{k=0}^{\infty} a_k e^{ik\theta}$$

in $L^2((-\pi, \pi), d\theta)$, which yields $f_m(\theta) = -\sum_{k=m+1}^{\infty} a_k e^{ik\theta}$. The condition $\sum_0^{\infty} |a_k| < \infty$ implies that $f_m(\theta)$ tends boundedly to zero as $m \rightarrow \infty$, and so the right-hand side of (4.8) converges to zero as $m \rightarrow \infty$. Thus we obtain the following AR(∞) representation for $\{X_n\}$:

$$(4.9) \quad \sum_{j=-\infty}^n a_{n-j} X_j + \xi_n = 0 \quad (n \in \mathbb{Z}).$$

We set $Y_n := P_{(-\infty, 0]} X_n$ for $n \in \mathbb{N}$. By (4.9), the sequence $\{Y_n : n \in \mathbb{N}\}$ is a solution to

$$(4.10) \quad \sum_{m=1}^n a_{n-m} Y_m = -\sum_{j=0}^{\infty} a_{n+j} X_{-j} \quad (n \in \mathbb{N}).$$

On the other hand, by (2.3), we have

$$\begin{aligned} \sum_{m=1}^n a_{n-m} \sum_{j=0}^{\infty} b_j^m X_{-j} &= \sum_{j=0}^{\infty} \left(\sum_{m=1}^n a_{n-m} \sum_{k=1}^m c_{m-k} a_{k+j} \right) X_{-j} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=1}^n a_{k+j} \sum_{p=0}^{n-k} a_{n-k-p} c_p \right) X_{-j} = -\sum_{j=0}^{\infty} a_{n+j} X_{-j}. \end{aligned}$$

This implies that the sequence $\{\sum_{j=0}^{\infty} b_j^n X_{-j} : n \in \mathbb{N}\}$ is also a solution to (4.10). Since $a_0 \neq 0$, the solution to (4.10) is unique. Thus (4.5) follows. $\square$

Since the stationary process $\{X_n\}$ is assumed to be purely non-deterministic, we have $P_{(-\infty, 0]}^{\perp} X_1 \neq 0$; use (4.18) below and the fact $c_0 \neq 0$. So we put

$$(4.11) \quad \epsilon(n) := \frac{\|P_{[-n, 0]}^{\perp} X_1\|^2 - \|P_{(-\infty, 0]}^{\perp} X_1\|^2}{\|P_{(-\infty, 0]}^{\perp} X_1\|^2} \quad (n = 0, 1, \dots).$$

We note that $\epsilon(n) \rightarrow 0$ as $n \rightarrow \infty$ since, by (4.1) and (4.7),

$$\begin{aligned} \|P_{[-n, 0]}^{\perp} X_1\|^2 - \|P_{(-\infty, 0]}^{\perp} X_1\|^2 &= \left\| P_{[-n, 0]}^{\perp} \sum_{j=n+2}^{\infty} c_j \xi_{1-j} \right\|^2 \\ &\leq \left\| \sum_{j=n+2}^{\infty} c_j \xi_{1-j} \right\|^2 = \sum_{j=n+2}^{\infty} (c_j)^2 \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

In §6, we shall give a detailed treatment of the asymptotic behaviour of $\epsilon(n)$ as $n \rightarrow \infty$. Here we prove a representation of $\epsilon(n)$ in terms $\{c_k\}$ and $\{a_k\}$.

Theorem 4.5. *If $\sum_0^{\infty} |a_k| < \infty$, then for $n \in \mathbb{N}$*

$$(4.12) \quad \epsilon(n) = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n, p)^2,$$

where $d_1(n, p) := \sum_{v=0}^{\infty} c_v a_{v+n+2+p}$, and for $k \geq 2$

$$d_k(n, p) := \sum_{m_1=1}^{\infty} a_{n+1+m_1} \sum_{m_2=1}^{\infty} b_{n+m_2}^{m_1} \cdots \sum_{m_{k-1}=1}^{\infty} b_{n+m_{k-1}}^{m_{k-2}} \sum_{m_k=1}^{\infty} b_{n+p+m_k}^{m_{k-1}} c_{m_k-1}.$$

Proof. By Theorem 4.4, we have for $m \geq 1$ and $n \geq 0$

$$P_{(-\infty, 0]} X_m = \sum_{j=1}^{\infty} b_{n+j}^m X_{-n-j} \pmod{H_{[-n, 0]}}.$$

Let S_k and θ be the k -step shift operator and reflection operator on H :

$$S_k(X_m) = X_{m+k}, \quad \theta(X_m) = X_{-m}.$$

Then S_k and θ are Hilbert space automorphisms of H such that $S_k^{-1} = S_{-k}$, $\theta^{-1} = \theta$. In view of the identity $(\theta S_n)^{-1} P_{(-\infty, 0]} (\theta S_n) = P_{[-n, \infty)}$, we have

$$P_{[-n, \infty)} X_{-n-m} = \sum_{j=1}^{\infty} b_{n+j}^m X_j \pmod{H_{[-n, 0]}}.$$

Hence

$$(4.13) \quad \begin{aligned} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1 &= c_0 \sum_{m_1=1}^{\infty} a_{n+1+m_1} \sum_{m_2=1}^{\infty} b_{n+m_2}^{m_1} \\ &\cdots \sum_{m_{r-1}=1}^{\infty} b_{n+m_{r-1}}^{m_{r-2}} \sum_{m_r=1}^{\infty} b_{n+m_r}^{m_{r-1}} X_{m_r} \pmod{H_{[-n, 0]}}, \end{aligned}$$

where $r := 2k$. This and

$$P_{(-\infty, 0]}^{\perp} X_m = \sum_{q=1}^m c_{m-q} \xi_q \quad (m \in \mathbb{N})$$

yield

$$\begin{aligned} &P_{(-\infty, 0]}^{\perp} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1 \\ &= c_0 \sum_{m_1=1}^{\infty} a_{n+1+m_1} \sum_{m_2=1}^{\infty} b_{n+m_2}^{m_1} \cdots \sum_{m_{r-1}=1}^{\infty} b_{n+m_{r-1}}^{m_{r-2}} \sum_{m_r=1}^{\infty} b_{n+m_r}^{m_{r-1}} \sum_{p=1}^{m_r} c_{m_r-p} \xi_p. \end{aligned}$$

Therefore, for $p \geq 0$

$$\left(P_{(-\infty, 0]}^{\perp} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1, \xi_{p+1} \right) = c_0 d_{2k}(n, p),$$

so that

$$(4.14) \quad \begin{aligned} &\|P_{(-\infty, 0]}^{\perp} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1\|^2 \\ &= \sum_{p=0}^{\infty} \left(P_{(-\infty, 0]}^{\perp} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1, \xi_{p+1} \right)^2 = (c_0)^2 \sum_{p=0}^{\infty} d_{2k}(n, p)^2. \end{aligned}$$

Similarly, we obtain

$$(4.15) \quad \begin{aligned} &\|P_{[-n, \infty)}^{\perp} \{P_{(-\infty, 0]} P_{[-n, \infty)}\}^k P_{(-\infty, 0]} X_1\|^2 \\ &= \|P_{[-n, \infty)}^{\perp} P_{(-\infty, 0]} \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^k X_1\|^2 = (c_0)^2 \sum_{p=0}^{\infty} d_{2k+1}(n, p)^2. \end{aligned}$$

Since we have $\int_{-\pi}^{\pi} \Delta(\theta)^{-1} d\theta < \infty$ by Proposition 4.2, it follows from Theorem 4.1 that

$$\|P_{[-n,0]}^{\perp} X_1\|^2 = \|P_{(-\infty,0]}^{\perp} X_1\|^2 + (c_0)^2 \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n,p)^2.$$

But

$$(4.16) \quad \|P_{(-\infty,0]}^{\perp} X_1\|^2 = \|c_0 \xi_1\|^2 = (c_0)^2,$$

whence (4.12) follows. $\square$

Theorem 4.6. *We assume (C1) and (A1), and choose $M \in \mathbb{N}$ so that $a_{n+2} \geq 0$ for $n \geq M$. Then, for $d_k(n,p)$ in Theorem 4.5 with $n \geq M$ and $p \geq 0$, we have*

$$(4.17) \quad d_2(n,p) = \sum_{v_2=0}^{\infty} c_{v_2} \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m=0}^{\infty} a_{v_2+m+n+2+p} a_{v_1+m+n+2},$$

and for $k \geq 3$

$$(4.18) \quad \begin{aligned} d_k(n,p) &= \sum_{v_k=0}^{\infty} c_{v_k} \sum_{v_{k-1}=0}^{\infty} c_{v_{k-1}} \\ &\cdots \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m_{k-1}=0}^{\infty} a_{v_k+m_{k-1}+n+2+p} \sum_{m_{k-2}=0}^{\infty} a_{v_{k-1}+m_{k-1}+m_{k-2}+n+2} \\ &\cdots \sum_{m_2=0}^{\infty} a_{v_3+m_3+m_2+n+2} \sum_{m_1=0}^{\infty} a_{v_2+m_2+m_1+n+2} a_{v_1+m_1+n+2}. \end{aligned}$$

Proof. We first note that, by Proposition 4.3, $\sum_0^{\infty} |a_k| < \infty$. By assumption, we can apply the Fubini-Tonelli theorem to exchange the order of sums. In particular, for $n \geq M$

$$\begin{aligned} d_k(n,p) &= \sum_{v_k=0}^{\infty} c_{v_k} \sum_{m_{k-1}=0}^{\infty} b_{n+1+v_k+p}^{m_{k-1}+1} \sum_{m_{k-2}=0}^{\infty} b_{n+1+m_{k-1}}^{m_{k-2}+1} \\ &\cdots \sum_{m_2=0}^{\infty} b_{n+1+m_3}^{m_2+1} \sum_{m_1=0}^{\infty} b_{n+1+m_2}^{m_1+1} a_{m_1+n+2}. \end{aligned}$$

Since $b_{j+1}^{m+1} = \sum_{v=0}^m c_v a_{m-v+j+2}$ for $m \geq 0$ and $j \geq 0$, it follows that

$$\begin{aligned} \sum_{m_1=0}^{\infty} b_{n+1+m_2}^{m_1+1} a_{m_1+n+2} &= \sum_{m_1=0}^{\infty} \left(\sum_{v_1=0}^{m_1} c_{v_1} a_{m_1-v_1+m_2+n+2} \right) a_{m_1+n+2} \\ &= \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m_1=0}^{\infty} a_{m_2+m_1+n+2} a_{v_1+m_1+n+2}. \end{aligned}$$

This gives (4.17). Similarly

$$\begin{aligned}
& \sum_{m_2=0}^{\infty} b_{n+1+m_3}^{m_2+1} \sum_{m_1=0}^{\infty} b_{n+1+m_2}^{m_1+1} a_{m_1+n+2} \\
&= \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m_2=0}^{\infty} \left(\sum_{v_2=0}^{m_2} c_{v_2} a_{m_2-v_2+m_3+n+2} \right) \sum_{m_1=0}^{\infty} a_{m_2+m_1+n+2} a_{v_1+m_1+n+2} \\
&= \sum_{v_2=0}^{\infty} c_{v_2} \sum_{v_1=0}^{\infty} c_{v_1} \sum_{m_2=0}^{\infty} a_{m_3+m_2+n+2} \sum_{m_1=0}^{\infty} a_{v_2+m_2+m_1+n+2} a_{v_1+m_1+n+2}.
\end{aligned}$$

Repeating the same arguments, we arrive at (4.18). $\square$

5. ASYMPTOTIC RELATIONS

The aim of this section is to give the link among the asymptotics for the autocovariance function $\gamma(\cdot)$, spectral density $\Delta(\cdot)$, sequence $\{c_n\}$ of MA(∞) coefficients, and sequence $\{a_n\}$ of AR(∞) coefficients for a purely non-deterministic stationary process $\{X_n\}$. We refer to [I1]–[I6] for related work.

Since the spectral density $\Delta(\cdot)$ of $\{X_n\}$ is an even function, we have

$$(5.1) \quad \gamma(n) = 2 \int_0^\pi \Delta(\theta) \cos(n\theta) d\theta \quad (n \in \mathbb{Z}),$$

$$(5.2) \quad \sum_{n=0}^{\infty} c_n r^n = (2\pi)^{1/2} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} P_r(\theta) \log \Delta(\theta) d\theta \right\} \quad (-1 < r < 1)$$

(recall $P_r(\theta)$ from (3.4)). On the other hand, it follows from (4.6) that

$$(5.3) \quad \gamma(n) = \sum_{m=0}^{\infty} c_{n+m} c_m \quad (n \geq 0).$$

If the sequence $\{c_n\}$ satisfies (C1) and (C2), then the sequence $\{\gamma(n)\}_0^\infty$ is eventually decreasing to zero, and so the Fourier series

$$(5.4) \quad \frac{1}{2\pi} \gamma(0) + \frac{1}{\pi} \sum_{n=1}^{\infty} \gamma(n) \cos(n\theta)$$

converges to a continuous function on $(-\pi, \pi) \setminus \{0\}$ (cf. Zygmund [Z, Ch. I, (2.6)]). Moreover, by Lebesgue's theorem (cf. [Z, Ch. III, (3.9)]), the Fourier series coincides with $\Delta(\theta)$ almost everywhere. So in the sequel we shall identify $\Delta(\theta)$ with the Fourier series (5.4).

To state the result for the boundary case, we recall the notion of Π -variation. For $\ell \in \mathcal{R}_0$, the class Π_ℓ is the class of measurable g , defined on some neighbourhood $[A, \infty)$ of infinity, such that

$$\forall \lambda > 0, \quad \lim_{x \rightarrow \infty} \{g(\lambda x) - g(x)\} / \ell(x) = c \log \lambda$$

with $c \in \mathbb{R}$ called the ℓ -index of g (see [BGT, Ch. 3] for background).

The following theorems are the results for the long-memory processes, boundary case, and intermediate-memory processes (cf. [BD, p. 520]), respectively.

Theorem 5.1. Let $\ell \in \mathcal{R}_0$ and $0 < d < \frac{1}{2}$. We assume (C1) and (C2). Then (2.5) and the following are equivalent:

$$(5.5) \quad \Delta(\theta) \sim \theta^{-2d} \ell(1/\theta) \cdot \frac{1}{2\Gamma(1-2d) \sin(\pi d)} \quad (\theta \rightarrow 0+),$$

$$(5.6) \quad c_n \sim n^{-(1-d)} \left\{ \frac{\ell(n)}{B(d, 1-2d)} \right\}^{1/2} \quad (n \rightarrow \infty).$$

If we further assume (A1), then all imply

$$(5.7) \quad a_n \sim n^{-(1+d)} \left\{ \frac{\ell(n)}{B(d, 1-2d)} \right\}^{-1/2} \frac{d \sin(\pi d)}{\pi} \quad (n \rightarrow \infty).$$

Theorem 5.2. Let $d = 0$, and $\ell \in \mathcal{R}_0$ such that $\int^\infty \ell(s) ds/s = \infty$. We assume (C1) and (C2). Then (2.5) is equivalent to

$$(5.8) \quad \Delta(1/\cdot) \in \Pi_\ell \text{ with } \ell\text{-index } \pi^{-1}.$$

Both imply

$$(5.9) \quad c_n \sim n^{-1} \ell(n) \{2\tilde{\ell}(n)\}^{-1/2} \quad (n \rightarrow \infty).$$

If we further assume (A1), then all imply

$$(5.10) \quad a_n \sim n^{-1} \ell(n) \{2\tilde{\ell}(n)\}^{-3/2} \quad (n \rightarrow \infty).$$

Theorem 5.3. Let $-\infty < d \leq 0$ and $\ell \in \mathcal{R}_0$. We further assume $\int^\infty \ell(s) ds/s < \infty$ if $d = 0$. We also assume (C1) and (C2). Then (2.5) is equivalent to

$$(5.11) \quad c_n \sim \frac{n^{-(1-2d)} \ell(n)}{\{\sum_{k=-\infty}^\infty \gamma(k)\}^{1/2}} \quad (n \rightarrow \infty).$$

If we further assume (A1), then both imply

$$(5.12) \quad a_n \sim \frac{n^{-(1-2d)} \ell(n)}{\{\sum_{k=-\infty}^\infty \gamma(k)\}^{3/2}} \quad (n \rightarrow \infty).$$

In order to prove the theorems above, we shall start by proving the following lemma which link the asymptotic behaviour of $\{c_n\}$ with that of $\{a_n\}$.

Lemma 5.4. Let $\ell \in \mathcal{R}_0$. Let $\{u_n\}_0^\infty$ and $\{v_n\}_0^\infty$ be real sequences which are both eventually decreasing to zero and satisfy the relation

$$(5.13) \quad \left(\sum_{n=0}^\infty u_n z^n \right) \left(\sum_{n=0}^\infty v_n z^n \right) = -1 \quad (|z| < 1).$$

(1) Let $0 < d < 1$. Suppose either $\sum_0^\infty u_n = \infty$ or $\sum_0^\infty v_n = 0$. Then the following are equivalent:

$$(5.14) \quad u_n \sim n^{-(1-d)} \ell(n) \quad (n \rightarrow \infty),$$

$$(5.15) \quad v_n \sim \frac{n^{-(d+1)}}{\ell(n)} \cdot \frac{d \sin(\pi d)}{\pi} \quad (n \rightarrow \infty).$$

(2) We assume $\int^\infty \ell(s) ds/s = \infty$. Suppose either $\sum_0^\infty u_n = \infty$ or $\sum_0^\infty v_n = 0$. Then the following are equivalent:

$$(5.16) \quad u_n \sim n^{-1} \ell(n) \quad (n \rightarrow \infty),$$

$$(5.17) \quad v_n \sim n^{-1} \frac{\ell(n)}{\tilde{\ell}(n)^2} \quad (n \rightarrow \infty).$$

(3) Let $1 \leq p < \infty$. Suppose $\sum_0^\infty u_n$ is finite and non-zero. Then the following are equivalent:

$$(5.18) \quad u_n \sim n^{-p} \ell(n) \quad (n \rightarrow \infty),$$

$$(5.19) \quad v_n \sim \frac{n^{-p} \ell(n)}{(\sum_0^\infty u_k)^2} \quad (n \rightarrow \infty).$$

Proof. (1) By the assumption, we see that $\sum_0^\infty u_n = \infty$ if and only if $\sum_0^\infty v_n = 0$. We set $w_n := \sum_{k=n+1}^\infty v_k$ for $n \geq 0$. Then

$$(5.20) \quad \sum_{n=0}^\infty v_n z^n = (z-1) \sum_{n=0}^\infty w_n z^n,$$

and so

$$(5.21) \quad (1-z) \left(\sum_{n=1}^\infty u_n z^n \right) \left(\sum_{n=1}^\infty w_n z^n \right) = 1 \quad (|z| < 1).$$

By the monotone density theorem (cf. [BGT, §1.7]), (5.15) holds if and only if

$$w_n \sim \frac{n^{-d}}{\ell(n)} \cdot \frac{\sin(\pi d)}{\pi} \quad (n \rightarrow \infty),$$

which, by Karamata's Tauberian theorem for power series (cf. [BGT, Cor. 1.7.3]), is equivalent to

$$\sum_{n=0}^\infty w_n s^n \sim \frac{1}{(1-s)^{1-d} \ell(1/(1-s)) \Gamma(d)} \quad (s \uparrow 1).$$

Now by (5.21) this is equivalent to

$$\sum_{n=0}^\infty u_n s^n \sim (1-s)^{-d} \ell(1/(1-s)) \Gamma(d) \quad (s \uparrow 1),$$

which, again by [BGT, Cor. 1.7.3], is equivalent to (5.14).

(2) We set $U(x) := \sum_{n=0}^{[x]} u_n$ for $x \geq 0$ and $:= 0$ for $x < 0$. We write $\hat{U}$ for the Laplace-Stieltjes transform of U :

$$\hat{U}(x) := \int_{[0, \infty)} e^{-tx} dU(t) = \sum_{n=0}^\infty u_n e^{-nx} \quad (x > 0).$$

Similarly we put $V(x) := \sum_{n=0}^{[x]} v_n$ for $x \geq 0$ and $\hat{V}(x) := \sum_{n=0}^\infty v_n e^{-nx}$ for $x > 0$.

First we assume (5.16). Then $\hat{U}(1/\cdot) \in \Pi_\ell$ with ℓ -index 1 by de Haan's theorem (cf. [I5, Thm. 2.3 and 2.4]). On the other hand, since $U(x) \sim \tilde{\ell}(x)$ as $x \rightarrow \infty$, we have $\hat{U}(1/x) \sim \tilde{\ell}(x)$ as $x \rightarrow \infty$ by Karamata's Tauberian theorem ([BGT, Thm. 1.7.1]). We put $\ell_1(x) := \ell(x)/\tilde{\ell}(x)^2$. Then, by (5.13), we have for $\lambda > 0$

$$\begin{aligned} \frac{\hat{V}(1/\lambda x) - \hat{V}(1/x)}{\ell_1(x)} &= \frac{\hat{U}(1/\lambda x) - \hat{U}(1/x)}{\ell(x)} \cdot \frac{\ell(\tilde{\lambda}x)}{\hat{U}(1/\lambda x)} \cdot \frac{\ell(\tilde{x})}{\hat{U}(1/x)} \cdot \frac{\tilde{\ell}(x)}{\tilde{\ell}(\lambda x)} \\ &\rightarrow \log \lambda \quad (x \rightarrow \infty). \end{aligned}$$

Therefore we see that $\hat{V}(1/\cdot) \in \Pi_{\ell_1}$ with ℓ_1 -index 1, which, by de Haan's Tauberian theorem, implies (5.17).

Next we assume (5.17). Let w_n be as in (1). We write, as above, $W(x) = \sum_{n=0}^{[x]} w_n$ for $x \geq 0$, and $\hat{W}(x) = \sum_{n=0}^{\infty} w_n e^{-nx}$ for $x > 0$. Since

$$w_n \sim \int_n^{\infty} \ell_1(t) dt/t = 1/\tilde{\ell}(n) \quad (n \rightarrow \infty),$$

we see that $W(x) \sim x/\ell(x)$ as $x \rightarrow \infty$. By Karamata's Tauberian theorem, $\hat{W}(1/x) \sim x/\tilde{\ell}(x)$ as $x \rightarrow \infty$, so that, by (5.21), $\hat{V}(1/x) \sim 1/\tilde{\ell}(x)$ as $x \rightarrow \infty$. On the other hand, (5.17) implies $\hat{V}(1/\cdot) \in \Pi_{\ell_1}$ with ℓ_1 -index 1. Therefore, from an argument similar to the above, it follows that $\hat{U}(1/\cdot) \in \Pi_{\ell}$ with ℓ -index 1, and so (5.16).

(3) We use an argument similar to that of the proof of [I1, Thm. 4.1]. Since $\sum_0^{\infty} v_n$ is also finite and non-zero, by symmetry it is enough to prove (5.18) $\Rightarrow$ (5.19) only. Set $f(x) := \sum_{n=0}^{\infty} u_n e^{-nx}$ for $x > 0$. Then by (5.13) we have

$$\sum_{n=0}^{\infty} v_n e^{-nx} = -1/f(x) \quad (x > 0).$$

Let $r := [p]$ be the integer part of p . By differentiating the both sides of the above r -times with respect to x , we obtain

$$\sum_{n=1}^{\infty} v_n n^r e^{-nx} = \frac{\sum_{n=1}^{\infty} u_n n^r e^{-nx}}{f(x)^2} + \frac{F_r(x)}{f(x)^{r+1}},$$

where F_r is a polynomial in $\{f^{(m)} : m = 0, 1, \dots, r-1\}$ (cf. [I1, Lem. 3.3]). Since $r - p > -1$ and

$$u_n n^r \sim n^{r-p} \ell(n) \quad (n \rightarrow \infty),$$

it follows that

$$\sum_{n=0}^{\infty} u_n n^r e^{-nx} \sim x^{p-r-1} \ell(1/x) \Gamma(r-p+1) \quad (x \rightarrow 0+)$$

(cf. [I2, Thm. 5.3]). On the other hand, for any $\epsilon > 0$ and $0 \leq m \leq r-1$, we have

$$x^{\epsilon} f^{(m)}(x) \rightarrow 0 \quad (x \rightarrow 0+)$$

(cf. [I2, Lem. 5.5]), and so

$$F_r(x)/\{x^{p-r-1} \ell(1/x)\} \rightarrow 0 \quad (x \rightarrow 0+).$$

Thus

$$\sum_{n=1}^{\infty} v_n n^r e^{-nx} \sim x^{p-r-1} \ell(1/x) \frac{\Gamma(r-p+1)}{(\sum_0^{\infty} u_k)^2} \quad (x \rightarrow 0+).$$

Since the sequence $\{\log(n^r v_n)\}$ is slowly increasing (cf. [BGT, §1.7.6]), it follows from Karamata's Tauberian theorem that

$$v_n n^r \sim n^{r-p} \ell(n) \quad (n \rightarrow \infty)$$

(cf. [I2, Thm. 5.3]). This yields (5.19). □

Proof of Theorem 5.1. We use an argument similar to that of the proof of [I6, Thm. 4.1]. The implication (2.5) $\Leftrightarrow$ (5.5) follows from the Abel-Tauber theorem for Fourier cosine series (cf. [BGT, Cor. 4.10.2]). If we put $g(t) := c_{[t]}$ for $t \geq 0$, then

$\gamma(n) = \int_0^\infty g(t+n)g(t)dt$ for $n \in \mathbb{N}$, and so by [I6, Prop. 4.3] (5.6) implies (2.5). We put

$$(5.22) \quad f(t) := \Delta(2 \arctan t), \quad x(r) := \frac{1-r}{1+r}.$$

Then by the change of variables $\theta = 2 \arctan t$ we have

$$(5.23) \quad \int_{-\infty}^{\infty} \frac{|\log f(t)|}{1+t^2} dt = \frac{1}{2} \int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty,$$

$$(5.24) \quad \int_{-\pi}^{\pi} P_r(\theta) \log \Delta(\theta) d\theta = \int_{-\infty}^{\infty} \frac{2x(r)}{x(r)^2 + t^2} \log f(t) dt \quad (-1 < r < 1).$$

Since (5.5) implies

$$f(t) \sim t^{-2d} \ell(1/t) \cdot \frac{1}{2^{2d+1} \Gamma(1-2d) \sin(\pi d)} \quad (t \rightarrow 0+),$$

we have by [I6, Thm 4.4]

$$\begin{aligned} \sum_{n=0}^{\infty} c_n r^n &= (2\pi)^{1/2} \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2x(r)}{x(r)^2 + t^2} \log f(t) dt \right\} \sim \{2\pi f(x(r))\}^{1/2} \\ &\sim (1-r)^{-d} \ell(1/(1-r))^{1/2} \left\{ \frac{\pi}{\Gamma(1-2d) \sin(\pi d)} \right\}^{1/2} \quad (r \uparrow 1). \end{aligned}$$

Hence by [BGT, Cor. 1.7.3] we obtain (5.6). Finally, the implication (5.6) $\Rightarrow$ (5.7) follows from Lemma 5.4 (1). $\square$

The following lemma will be used in the proof of Theorem 5.2.

Lemma 5.5. *Let c be a positive constant, and let ℓ be a slowly varying function such that $\int_0^\infty \ell(s)ds/s = \infty$. For a positive, even, and measurable function g on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} \frac{|\log g(t)|}{1+t^2} dt < \infty,$$

we set

$$K(x) := \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + t^2} \log g(t) dt \right\} \quad (x > 0).$$

Then $g \in \Pi_\ell$ with ℓ -index c implies $K \in \Pi_{\ell_1}$ with ℓ_1 -index $\sqrt{c}/2$, where $\ell_1(\cdot)$ is defined by $\ell_1(t) := \ell(t)/\tilde{\ell}(t)^{1/2}$.

Compare the proof below with that of [I6, Thm. 5.2].

Proof. In view of de Haan's theorem (cf. [BGT, Th. 4.4]), we have

$$(5.25) \quad g(t) \sim c\tilde{\ell}(t) \quad (t \rightarrow \infty)$$

(see the argument in [BGT, p. 164]). Since

$$K(1/x) = \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + t^2} \log g(1/t) dt \right\} \quad (x > 0),$$

it follows from [I6, Thm. 4.4] that

$$(5.26) \quad K(x) \sim \{c\ell(x)\}^{1/2} \quad (x \rightarrow \infty).$$

We note that $K(x) = \exp A(x)$, where

$$A(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \log g(tx) dt \quad (x > 0).$$

Let $\lambda > 1$. Then by the mean value theorem we have

$$K(\lambda x) - K(x) = \{A(\lambda x) - A(x)\} \exp B_\lambda(x),$$

where $B_\lambda(x)$ is between $A(\lambda x)$ and $A(x)$. Since, by (5.26), $K(\lambda x)/\tilde{\ell}(x)^{1/2}$ and $K(x)/\tilde{\ell}(x)^{1/2}$ both tend to $\sqrt{c}$ as $x \rightarrow \infty$, we see that

$$(5.27) \quad \exp B_\lambda(x) \sim \{c\tilde{\ell}(x)\}^{1/2} \quad (x \rightarrow \infty).$$

Again by the mean value theorem, we have

$$\log g(\lambda xt) - \log g(xt) = \{g(\lambda xt) - g(xt)\}/k_\lambda(x, t),$$

where $k_\lambda(x, t)$ is between $g(\lambda xt)$ and $g(xt)$. By (5.25), $g(x)/g(\lambda x)$ and $g(x)/g(xt)$ both tend to 1 as $x \rightarrow \infty$, whence

$$(5.28) \quad g(x)/k_\lambda(x, t) \rightarrow 1 \quad (x \rightarrow \infty) \quad \forall t > 0.$$

We note that by [BGT, Thm. 1.5.6 and 3.8.6] (Potter-type bounds) there exist positive constants D and M such that

$$|g(\lambda x) - g(x)|/\ell(x) \leq D\lambda \quad (x \geq M),$$

$$\ell(y)/\ell(x) \leq D \max\left((y/x)^{1/4}, (y/x)^{-1/4}\right) \quad (x \geq M, y \geq M),$$

$$g(x)/g(y) \leq D \max\left((y/x)^{1/4}, (y/x)^{-1/4}\right) \quad (x \geq M, y \geq M).$$

Now

$$\frac{A(\lambda x) - A(x)}{\ell(x)/\tilde{\ell}(x)} = \text{I}(x) - \text{II}(x) + \text{III}(x),$$

where

$$\text{I}(x) := \frac{\tilde{\ell}(x)}{\pi\ell(x)} \int_0^M \frac{x}{x^2 + u^2} \log g(\lambda u) du,$$

$$\text{II}(x) := \frac{\tilde{\ell}(x)}{\pi\ell(x)} \int_0^M \frac{x}{x^2 + u^2} \log g(u) du,$$

$$\text{III}(x) := \frac{\tilde{\ell}(x)}{\pi g(x)} \int_0^\infty F_\lambda(x, t) dt$$

with

$$F_\lambda(x, t) := I_{(M/x, \infty)}(t) \cdot \frac{1}{1+t^2} \cdot \frac{\{g(\lambda xt) - g(xt)\}}{\ell(xt)} \cdot \frac{\ell(xt)}{\ell(x)} \cdot \frac{g(x)}{k_\lambda(x, t)}.$$

By (5.28), $F_\lambda(x, t)$ tends to $c(1+t^2)^{-1} \log \lambda$ as $x \rightarrow \infty$ for any $t > 0$. On the other hand, we have for $x \geq M$

$$\begin{aligned} I_{(M/x, \infty)}(t) \frac{g(x)}{k_\lambda(x, t)} &\leq I_{(M/x, \infty)}(t) \max\left(\frac{g(x)}{g(\lambda xt)}, \frac{g(x)}{g(xt)}\right) \\ &\leq D\lambda^{1/4} \max(t^{1/4}, t^{-1/4}), \end{aligned}$$

whence for $x \geq M$ and $t > 0$

$$|F_\lambda(x, t)| \leq D^3 \lambda^{5/4} \frac{\max(t^{1/2}, t^{-1/2})}{1+t^2}.$$

Therefore by the dominated convergence theorem we obtain

$$\text{III}(x) \rightarrow \frac{1}{\pi} \int_0^\infty \frac{1}{1+t^2} dt \cdot \log \lambda = \frac{\log \lambda}{2} \quad (x \rightarrow \infty).$$

As for $\text{I}(x)$, we have

$$|\text{I}(x)| \leq \frac{\tilde{\ell}(x)}{\pi x \ell(x)} \int_0^M |\log g(\lambda t)| dt \rightarrow 0 \quad (x \rightarrow \infty).$$

Similarly $\text{II}(x) \rightarrow 0$ as $x \rightarrow \infty$. Thus

$$\frac{A(\lambda x) - A(x)}{\ell(x)/\tilde{\ell}(x)} \rightarrow \frac{\log \lambda}{2} \quad (x \rightarrow \infty).$$

Combining this with (5.27), we obtain

$$\frac{K(\lambda x) - K(x)}{\ell(x)/\tilde{\ell}(x)^{1/2}} \rightarrow \frac{\sqrt{c}}{2} \log \lambda \quad (x \rightarrow \infty).$$

This proves the lemma. $\square$

Proof of Theorem 5.2. By [I4, Thm. 1.3], (2.5) and (5.8) are equivalent. Define f by (5.22), and g by $g(t) := f(1/t)$. Then, by (5.23) and (5.24), we have

$$\int_{-\infty}^\infty \frac{|\log g(t)|}{1+t^2} dt < \infty,$$

$$\sum_{n=0}^\infty c_n \left(\frac{x-1}{x+1} \right)^n = (2\pi)^{1/2} \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^\infty \frac{x}{x^2+t^2} \log g(t) dt \right\} \quad (x > 1).$$

Now we note that (5.8) implies $g \in \Pi_\ell$ with ℓ -index π^{-1} (see [I5, Prop. 2.7]). Hence, by Lemma 5.5 and [I5, Prop. 2.6], $\sum_{n=0}^\infty c_n e^{-n/x} \in \Pi_{\ell_1}$ with ℓ_1 -index $1/\sqrt{2}$, where $\ell_1(x) := \ell(x)/\{\tilde{\ell}(x)\}^{1/2}$. Therefore, by de Haan's Tauberian theorem (cf. [I5, Thm. 2.3 and 2.4]), we obtain (5.9). To complete the proof, we note that, for $C \gg 1$,

$$(5.29) \quad \int_C^x \frac{\ell(s)}{\{2\tilde{\ell}(s)\}^{1/2} s} ds = \{2\tilde{\ell}(t)\}^{1/2} - \{2\tilde{\ell}(C)\}^{1/2} \sim \{2\tilde{\ell}(x)\}^{1/2} \quad (x \rightarrow \infty).$$

Then, by Lemma 5.4(2), (5.9) implies (5.10). $\square$

Proof of Theorem 5.3. From (5.3), it follows that

$$(5.30) \quad \sum_{n=-\infty}^\infty \gamma(n) = 2 \sum_{n=0}^\infty \sum_{m=0}^\infty c_{n+m} c_m - \sum_{m=0}^\infty (c_m)^2 = \left(\sum_{m=0}^\infty c_m \right)^2.$$

By [I2, Lemma 5.7], $\gamma(n) \sim c_n (\sum_{m=0}^\infty c_m)$ as $n \rightarrow \infty$, whence (2.5) and (5.11) are equivalent. On the other hand, by Lemma 5.4 (3), (5.11) implies (5.12). $\square$

6. PROOF OF THE MAIN THEOREM

In this section we shall prove the main theorem (Theorem 2.1). In order to do that, we first give the asymptotic behaviour of $\epsilon(n)$ as $n \rightarrow \infty$ (recall $\epsilon(\cdot)$ from (4.11)). This will be carried out firstly for the long-memory processes, secondly for the boundary case, and thirdly for the intermediate-memory processes. It turns out that the two infinite sums $\sum_{k=1}^{\infty}$ in (4.3) with $Y = X_1$ are negligible as $n \rightarrow \infty$ in the second and third cases but not so in the first case. As a result the proof for the first case is much more difficult than the others.

First we consider the long memory processes. For $0 < d < \frac{1}{2}$, we put

$$A_1 := \left(\frac{d}{\pi}\right)^2 \int_0^\infty \left\{ \int_0^\infty \frac{dv_1}{(v_1)^{1-d}(v_1+1+u)^{1+d}} \right\}^2 du,$$

$$A_2 := \left(\frac{d}{\pi}\right)^4 \int_0^\infty \left\{ \int_0^\infty \frac{dv_2}{(v_2)^{1-d}} \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \int_0^\infty \frac{ds_1}{(v_2+s_1+1+u)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2 du,$$

and for $k \geq 3$

$$A_k := \left(\frac{d}{\pi}\right)^{2k} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_k}{(v_k)^{1-d}} \int_0^\infty \frac{dv_{k-1}}{(v_{k-1})^{1-d}} \cdots \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \int_0^\infty \frac{ds_{k-1}}{(v_k+s_{k-1}+1+u)^{1+d}} \int_0^\infty \frac{ds_{k-2}}{(v_{k-1}+s_{k-1}+s_{k-2}+1)^{1+d}} \cdots \int_0^\infty \frac{ds_2}{(v_3+s_3+s_2+1)^{1+d}} \int_0^\infty \frac{ds_1}{(v_2+s_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2.$$

By the equality

$$(6.1) \quad \int_0^\infty \frac{dv}{(x+v)^{1+d}v^{1-d}} = \frac{1}{xd} \quad (0 < x < \infty),$$

A_k may be expressed more simply as follows: $A_1 = \pi^{-2}$,

$$A_2 = \pi^{-4} \int_0^\infty \left\{ \int_0^\infty \frac{ds_1}{(s_1+1+u)(s_1+1)} \right\}^2 du,$$

and for $k \geq 3$

$$A_k = \pi^{-2k} \int_0^\infty \left\{ \int_0^\infty \frac{ds_{k-1}}{(s_{k-1}+1+u)} \int_0^\infty \frac{ds_{k-2}}{(s_{k-1}+s_{k-2}+1)} \cdots \int_0^\infty \frac{ds_2}{(s_2+s_1+1)} \int_0^\infty \frac{ds_1}{(s_2+s_1+1)(s_1+1)} \right\}^2 du.$$

In particular, we see that A_k does not depend on d . The value of A_k for $k \geq 2$ will be identified by (6.15) below.

Recall $d_k(n, p)$ from Theorems 4.5 and 4.6.

Proposition 6.1. *Let $0 < d < \frac{1}{2}$ and $\ell \in \mathcal{R}_0$. We assume (C1), (C2), and (A1). Then, for $k \in \mathbb{N}$, (2.5) implies*

$$(6.2) \quad \sum_{p=0}^{\infty} \{d_k(n, p)\}^2 \sim n^{-1} A_k \sin^{2k}(\pi d) \quad (n \rightarrow \infty).$$

Proof. We give full details for the case $k = 2$, and describe how to adapt the argument to the general case.

Choose $\delta \in (0, \min(d, \frac{1}{14}))$. Then by Theorem 5.1 and Potter's bound ([BGT, Thm. 1.5.6]), there exists $M > 0$ such that

$$(6.3) \quad 0 \leq \frac{a_n}{a_m} \leq \frac{2}{(n/m)^{1+d-\delta}} \quad (n \geq m \geq M),$$

$$(6.4) \quad 0 \leq \frac{c_n}{c_m} \leq 2 \max\left((n/m)^{-(1-d+\delta)}, (n/m)^{-(1-d-\delta)}\right) \quad (n \geq M, m \geq M).$$

We set, for $x > 0$, $g_0(x) := I_{(0,M)}(x)c_{[x]}$, $g_1(x) := I_{[M,\infty)}(x)c_{[x]}$, and

$$(6.5) \quad G(x) := 2 \max\left(x^{-(1-d+\delta)}, x^{-(1-d-\delta)}\right).$$

Then we have

$$0 \leq \frac{g_1(nv)}{c_n} \leq G(v) \quad (n \geq M, v > 0),$$

$$\frac{g_1(nv)}{c_n} \rightarrow v^{-(1-d)} \quad (n \rightarrow \infty) \quad (\forall v > 0).$$

From Theorem 4.6, it follows that, for n large enough,

$$\begin{aligned} & \sum_{p=0}^{\infty} \{d_2(n-2, p)\}^2 \\ &= \int_0^{\infty} \left\{ \int_0^{\infty} dv_2 c_{[v_2]} \int_0^{\infty} dv_1 c_{[v_1]} \int_0^{\infty} a_{[v_2]+[s_1]+n+[u]} a_{[v_1]+[s_1]+n} ds_1 \right\}^2 du \\ &= n^7 \{a_n c_n\}^4 \int_0^{\infty} \left\{ \int_0^{\infty} dv_2 \frac{c_{[nv_2]}}{c_n} \int_0^{\infty} dv_1 \frac{c_{[nv_1]}}{c_n} \right. \\ & \quad \left. \int_0^{\infty} \frac{a_{[nv_2]+[ns_1]+n+[nu]}}{a_n} \cdot \frac{a_{[nv_1]+[ns_1]+n}}{a_n} ds_1 \right\}^2 du \\ &= n^7 \{a_n c_n\}^4 \cdot \|I_{00}(n) + I_{01}(n) + I_{10}(n) + I_{11}(n)\|^2, \end{aligned}$$

where $\|\cdot\|$ is the norm of $L^2((0, \infty), du)$ and $I_{ij}(n) = I_{ij}(n)(u)$ are defined by

$$\begin{aligned} I_{ij}(n) &:= \int_0^{\infty} dv_2 \frac{g_i(nv_2)}{c_n} \int_0^{\infty} dv_1 \frac{g_j(nv_1)}{c_n} \\ & \quad \int_0^{\infty} \frac{a_{[nv_2]+[ns_1]+n+[nu]}}{a_n} \cdot \frac{a_{[nv_1]+[ns_1]+n}}{a_n} ds_1. \end{aligned}$$

First we consider $I_{11}(n)$. Since

$$v_2 + s_1 + 1 + u \leq \frac{[nv_2] + [ns_1] + n + [nu]}{n} \leq v_2 + s_1 + 1 + \frac{3}{n} + u,$$

we have

$$0 \leq \frac{a_{[nv_2]+[ns_1]+n+[nu]}}{a_n} \leq 2(v_2 + s_1 + 1 + u)^{-(1+d-\delta)} \quad (n \geq M, v_2 > 0, s_1 > 0, u > 0),$$

$$\frac{a_{[nv_2]+[ns_1]+n+[nu]}}{a_n} \rightarrow (v_2 + s_1 + u + 1)^{-(1+d)} \quad (n \rightarrow \infty).$$

Similarly

$$0 \leq \frac{a_{[nv_1]+[ns_1]+n}}{a_n} \leq 2(v_1 + s_1 + 1)^{-(1+d-\delta)} \quad (n \geq M, v_1 > 0, s_1 > 0),$$

$$\frac{a_{[nv_1]+[ns_1]+n}}{a_n} \rightarrow (v_1 + s_1 + 1)^{-(1+d)} \quad (n \rightarrow \infty).$$

Now

$$(6.6) \quad \begin{aligned} & (v_2 + s_1 + u + 1)^{-(1+d-\delta)} (v_1 + s_1 + 1)^{-(1+d-\delta)} \\ &= (v_2 + s_1 + u + 1)^{-(d+2\delta)-4\delta-(1-7\delta)} (v_1 + s_1 + 1)^{-(d+2\delta)-(1-3\delta)} \\ &\leq (v_2 + 1)^{-(d+2\delta)} (v_1 + 1)^{-(d+2\delta)} (s_1 + 1)^{-(1+\delta)} (u + 1)^{-(1-7\delta)}, \end{aligned}$$

and so for $n \geq M$ we have

$$|I_{11}(n)|^2 \leq \frac{2^4}{(1+u)^{2-14\delta}} \times \left\{ \int_0^\infty \frac{G(v_2)}{(1+v_2)^{d+2\delta}} dv_2 \cdot \int_0^\infty \frac{G(v_1)}{(1+v_1)^{d+2\delta}} dv_1 \cdot \int_0^\infty \frac{1}{(1+s_1)^{1+\delta}} ds_1 \right\}^2.$$

Hence, applying the dominated convergence theorem twice, we obtain

$$\|I_{11}(n)\|^2 \rightarrow (\pi/d)^4 A_2 \quad (n \rightarrow \infty).$$

Next we consider the remaining integrals. By the above estimates, we have for $n \geq M$

$$|I_{01}(n)|^2 \leq \frac{1}{(nc_n)^2} \cdot \frac{2^4}{(1+u)^{2-14\delta}} \times \left\{ \int_0^M |g_0(v_2)| dv_2 \cdot \int_0^\infty \frac{G(v_1)}{(1+v_1)^{d+2\delta}} dv_1 \cdot \int_0^\infty \frac{1}{(1+s_1)^{1+\delta}} ds_1 \right\}^2,$$

so that $\|I_{01}(n)\| \rightarrow 0$ as $n \rightarrow \infty$. Similarly, $\|I_{10}(n)\| \rightarrow 0$ and $\|I_{00}(n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Combining,

$$\|I_{00}(n) + I_{01}(n) + I_{10}(n) + I_{11}(n)\|^2 \rightarrow (\pi/d)^4 A_2 \quad (n \rightarrow \infty).$$

But by Theorem 5.1

$$(6.7) \quad n^7 \{a_n c_n\}^4 \sim \left\{ \frac{d \sin(\pi d)}{\pi} \right\}^4 n^{-1} \quad (n \rightarrow \infty),$$

whence, as asserted in (6.2),

$$\sum_{p=0}^{\infty} \{d_2(n, p)\}^2 \sim n^{-1} A_2 \sin^4(\pi d) \quad (n \rightarrow \infty).$$

For the general case $k \geq 1$, we choose $\delta \in (0, \min\{d, 1/(8k-2)\})$, and instead of (6.6) we use the following estimate:

$$\begin{aligned}
& (v_k + s_{k-1} + u + 1)^{-(1+d-\delta)} (v_{k-1} + s_{k-1} + s_{k-2} + 1)^{-(1+d-\delta)} \\
& \quad \cdots (v_2 + s_2 + s_1 + 1)^{-(1+d-\delta)} (v_1 + s_1 + 1)^{-(1+d-\delta)} \\
& = (v_k + s_{k-1} + u + 1)^{-(d+2\delta)-(4k-4)\delta-\{1-(4k-1)\delta\}} \\
& \quad \times (v_{k-1} + s_{k-1} + s_{k-2} + 1)^{-(d+2\delta)-\{1-(4k-5)\delta\}-(4k-8)\delta} \\
& \quad \cdots (v_2 + s_2 + s_1 + 1)^{-(d+2\delta)-(1-7\delta)-4\delta} (v_1 + s_1 + 1)^{-(d+2\delta)-(1-3\delta)} \\
& \leq (v_k + 1)^{-(d+2\delta)} \times \cdots \times (v_1 + 1)^{-(d+2\delta)} \\
& \quad \times (s_{k-1} + 1)^{-(1+\delta)} \cdots (s_1 + 1)^{-(1+\delta)} (u + 1)^{-1+(4k-1)\delta}.
\end{aligned}$$

Here we note that $\delta < 1/(8k-2)$ implies

$$\int_0^\infty \frac{1}{(u+1)^{2-(8k-2)\delta}} du < \infty.$$

In this way we can prove the assertion for $k \geq 1$ similarly. $\square$

Now in a manner similar to the definition of $d_k(n, p)$, we shall introduce other infinite sums consisting of c_n and a_n , for later use. Thus, under the assumptions (C1) and (A1), we choose $M \in \mathbb{N}$ so that $a_{n+2} \geq 0$ for $n \geq M$, and put for $k \geq 2$, $n \geq M$, and $p \geq 0$

$$e_k(n, p) := \sum_{m_1=1}^\infty a_{n+m_1+1} \sum_{m_2=1}^\infty b_{n+m_2}^{m_1} \cdots \sum_{m_{k-1}=1}^\infty b_{n+m_{k-1}}^{m_{k-2}} \sum_{m_k=1}^\infty b_{n+m_k}^{m_{k-1}} c_{m_k+p}.$$

Then, as in Theorem 4.6, we have

$$e_2(n, p) = \sum_{v_2=0}^\infty c_{v_2+p+1} \sum_{v_1=0}^\infty c_{v_1} \sum_{m_1=0}^\infty a_{v_2+m_1+n+2} a_{v_1+m_1+n+2},$$

and for $k \geq 3$

$$\begin{aligned}
e_k(n, p) &= \sum_{v_k=0}^\infty c_{v_k+p+1} \sum_{v_{k-1}=0}^\infty c_{v_{k-1}} \\
&\quad \cdots \sum_{v_1=0}^\infty c_{v_1} \sum_{m_{k-1}=0}^\infty a_{v_k+m_{k-1}+n+2} \sum_{m_{k-2}=0}^\infty a_{v_{k-1}+m_{k-1}+m_{k-2}+n+2} \\
&\quad \cdots \sum_{m_2=0}^\infty a_{v_3+m_3+m_2+n+2} \sum_{m_1=0}^\infty a_{v_2+m_2+m_1+n+2} a_{v_1+m_1+n+2}.
\end{aligned}$$

For $0 < d < \frac{1}{2}$ and $k \geq 2$, we also define $B_k(d)$, in a manner similar to the definition of A_k , as follows:

$$B_2(d) := \left(\frac{d}{\pi} \right)^4 \int_0^\infty \left\{ \int_0^\infty \frac{dv_2}{(v_2+u)^{1-d}} \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \int_0^\infty \frac{ds_1}{(v_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2 du,$$

and for $k \geq 3$

$$B_k(d) := \left(\frac{d}{\pi}\right)^{2k} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_k}{(v_k+u)^{1-d}} \int_0^\infty \frac{dv_{k-1}}{(v_{k-1})^{1-d}} \cdots \int_0^\infty \frac{dv_1}{(v_1)^{1-d}} \right. \\ \left. \int_0^\infty \frac{ds_{k-1}}{(v_k+s_{k-1}+1)^{1+d}} \int_0^\infty \frac{ds_{k-2}}{(v_{k-1}+s_{k-1}+s_{k-2}+1)^{1+d}} \right. \\ \left. \cdots \int_0^\infty \frac{ds_2}{(v_3+s_3+s_2+1)^{1+d}} \int_0^\infty \frac{ds_1}{(v_2+s_2+s_1+1)^{1+d}(v_1+s_1+1)^{1+d}} \right\}^2.$$

By the equality (6.1), $B_k(d)$ with $k \geq 3$ defined above may be expressed more simply as follows:

$$(6.8) \quad B_k(d) = \frac{d^2}{\pi^{2k}} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_k}{(v_k+u)^{1-d}} \int_0^\infty \frac{ds_{k-1}}{(v_k+s_{k-1}+1)^{1+d}} \right. \\ \left. \int_0^\infty \frac{ds_{k-2}}{(s_{k-1}+s_{k-2}+1)} \cdots \int_0^\infty \frac{ds_2}{(s_3+s_2+1)} \int_0^\infty \frac{ds_1}{(s_2+s_1+1)(s_1+1)} \right\}^2.$$

The following proposition is an analogue of Proposition 6.1 for $\{e_k(n, p)\}$. It will be used, in the proof of Theorem 6.4 below, to estimate the difference of upper and lower bounds.

Proposition 6.2. *Let $0 < d < \frac{1}{2}$ and $\ell \in \mathcal{R}_0$. We assume (C1), (C2), and (A1). Then, for $k \geq 2$, (2.5) implies*

$$(6.9) \quad \sum_{p=0}^\infty \{e_k(n, p)\}^2 \sim n^{-1} B_k(d) \sin^{2k}(\pi d) \quad (n \rightarrow \infty).$$

Proof. For $k \geq 3$ and n large enough, we have

$$\sum_{p=0}^\infty \{e_k(n-2, p)\}^2 = n^{4k-1} \{a_n c_n\}^{2k} \\ \times \int_0^\infty du \left\{ \int_0^\infty dv_k \frac{c_{[nv_k]+[nu]+1}}{c_n} \int_0^\infty dv_{k-1} \frac{c_{[nv_{k-1}]}}{c_n} \cdots \int_0^\infty dv_1 \frac{c_{[nv_1]}}{c_n} \right. \\ \left. \int_0^\infty ds_{k-1} \frac{a_{[nv_k]+[ns_{k-1}]+n}}{a_n} \int_0^\infty ds_{k-2} \frac{a_{[nv_{k-1}]+[ns_{k-1}]+[ns_{k-2}]+n}}{a_n} \right. \\ \left. \cdots \int_0^\infty ds_2 \frac{a_{[nv_3]+[ns_3]+[ns_2]+n}}{a_n} \int_0^\infty \frac{a_{[nv_2]+[ns_2]+[ns_1]+n}}{a_n} \cdot \frac{a_{[nv_1]+[ns_1]+n}}{a_n} ds_1 \right\}^2$$

but for simplicity we give details for the case $k = 2$, and finish the proof with brief notes for the general case.

Take $\delta \in (0, \min(\frac{1}{2} - d, \frac{1}{5}d))$, and M so large that (6.3) and (6.4) both hold. We set $q_0(x) := I_{(0, M)}(x)$ and $q_1(x) := I_{[M, \infty)}(x)$. Then for n large enough we have

$$\sum_{p=0}^\infty \{e_2(n-2, p)\}^2 = n^7 \{a_n c_n\}^4 \cdot \|I_{00}(n) + I_{01}(n) + I_{10}(n) + I_{11}(n)\|^2,$$

where $\|\cdot\|$ is the norm of $L^2((0, \infty), du)$ and $I_{ij}(n) = I_{ij}(n)(u)$ are defined by

$$I_{ij}(n) := \int_0^\infty dv_2 q_i([nu]) \frac{c_{[nv_2]+[nu]+1}}{c_n} \int_0^\infty dv_1 q_j([nv_1]) \frac{c_{[nv_1]}}{c_n} \int_0^\infty \frac{a_{[nv_2]+[ns_1]+n}}{a_n} \cdot \frac{a_{[nv_1]+[ns_1]+n}}{a_n} ds_1.$$

First we consider $I_{11}(n)$. From

$$v_2 + u < \frac{[nv_2] + [nu] + 1}{n} < v_2 + u + \frac{3}{n},$$

we have

$$\begin{aligned} q_1([nu]) \cdot \frac{c_{[nv_2]+[nu]+1}}{c_n} &\leq G(v_2 + u) \quad (n \geq M, u > 0, v_2 > 0), \\ q_1([nu]) \cdot \frac{c_{[nv_2]+[nu]+1}}{c_n} &\rightarrow (v_2 + u)^{-(1-d)} \quad (n \rightarrow \infty) \quad (\forall u > 0, \forall v_2 > 0), \end{aligned}$$

where G is the function defined by (6.5). Since

$$\begin{aligned} (v_2 + s_1 + 1)^{-(1+d-\delta)} (v_1 + s_1 + 1)^{-(1+d-\delta)} \\ (6.10) \quad &= (v_2 + s_1 + 1)^{-(1+d-5\delta)-4\delta} (v_1 + s_1 + 1)^{-(d+2\delta)-(1-3\delta)} \\ &\leq (v_2 + 1)^{-(1+d-5\delta)} (v_1 + 1)^{-(d+2\delta)} (s_1 + 1)^{-(1+\delta)}, \end{aligned}$$

we have for $n \geq M$

$$|I_{11}(n)|^2 \leq 2^4 \left\{ \int_0^\infty \frac{G(u + v_2) dv_2}{(v_2 + 1)^{1+d-5\delta}} \cdot \int_0^\infty \frac{G(v_1) dv_1}{(v_1 + 1)^{d+2\delta}} \cdot \int_0^\infty \frac{ds_1}{(s_1 + 1)^{1+\delta}} \right\}^2.$$

Now

$$(6.11) \quad G(u + v_2) \leq I_{(0,1)}(u)G(v_2) + I_{[1,\infty)}(u)G(u),$$

and so, if we notice that $(a + b)^2 \leq 2(a^2 + b^2)$, then

$$\begin{aligned} |I_{11}(n)|^2 &\leq 2^5 \left\{ \int_0^\infty \frac{G(v_1)}{(v_1 + 1)^{d+2\delta}} dv_1 \cdot \int_0^\infty \frac{1}{(s_1 + 1)^{1+\delta}} ds_1 \right\}^2 \\ &\times \left[\left\{ \int_0^\infty \frac{I_{(0,1)}(u)G(v_2)}{(v_2 + 1)^{1+d-5\delta}} dv_2 \right\}^2 + \left\{ \int_0^\infty \frac{I_{[1,\infty)}(u)G(u)}{(v_2 + 1)^{1+d-5\delta}} dv_2 \right\}^2 \right]. \end{aligned}$$

The right-hand side is in $L^1((0, \infty), du)$, whence, by the dominated convergence theorem, we obtain

$$\|I_{11}(n)\|^2 \rightarrow (\pi/d)^4 B_2(d) \quad (n \rightarrow \infty).$$

We turn to the remaining integrals. We have for $n \geq M$

$$\begin{aligned} |I_{01}(n)|^2 &\leq 2^4 I_{(0,1)}([nu]) \left(\frac{\max c_k}{c_n} \right)^2 \\ &\times \left\{ \int_0^\infty \frac{dv_2}{(v_2 + 1)^{1+d-5\delta}} \cdot \int_0^\infty \frac{G(v_1)}{(v_1 + 1)^{d+2\delta}} dv_1 \cdot \int_0^\infty \frac{ds_1}{(s_1 + 1)^{1+\delta}} \right\}^2. \end{aligned}$$

Hence it follows that $\|I_{01}(n)\| \rightarrow 0$ as $n \rightarrow \infty$. Further, for $n \geq M$, $|I_{10}(n)|^2$ is at most

$$2^4 \left(\frac{\max c_k}{nc_n} \right)^2 \left\{ \int_0^\infty \frac{G(u+v_2)}{(v_2+1)^{1+d-5\delta}} dv_2 \cdot \int_0^M dv_1 \cdot \int_0^\infty \frac{ds_1}{(s_1+1)^{1+\delta}} \right\}^2.$$

Hence, by (6.11) and (5.6), $\|I_{10}(n)\| \rightarrow 0$ as $n \rightarrow \infty$. Similarly, $\|I_{00}(n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Combining,

$$\|I_{00}(n) + I_{01}(n) + I_{10}(n) + I_{11}(n)\|^2 \rightarrow (\pi/d)^4 B_2(d) \quad (n \rightarrow \infty),$$

and so by (6.7) we obtain, as asserted,

$$\sum_{p=0}^{\infty} \{e_2(n, p)\}^2 \sim n^{-1} B_2(d) \sin^4(\pi d) \quad (n \rightarrow \infty).$$

For the general case $k \geq 1$, we choose $\delta \in (0, \min\{\frac{1}{2} - d, d/(4k-3)\})$, and instead of (6.10) we use the estimate

$$\begin{aligned} & (v_k + s_{k-1} + 1)^{-(1+d-\delta)} (v_{k-1} + s_{k-1} + s_{k-2} + 1)^{-(1+d-\delta)} \\ & \quad \cdots (v_2 + s_2 + s_1 + 1)^{-(1+d-\delta)} (v_1 + s_1 + 1)^{-(1+d-\delta)} \\ &= (v_k + s_{k-1} + 1)^{-\{1+d-(4k-3)\delta\} - (4k-4)\delta} \\ & \quad \times (v_{k-1} + s_{k-1} + s_{k-2} + 1)^{-(d+2\delta) - \{1-(4k-5)\delta\} - (4k-8)\delta} \\ & \quad \cdots (v_2 + s_2 + s_1 + 1)^{-(d+2\delta) - (1-7\delta) - 4\delta} (v_1 + s_1 + 1)^{-(d+2\delta) - (1-3\delta)} \\ &\leq (v_k + 1)^{-\{1+d-(4k-3)\delta\}} (v_{k-1} + 1)^{-(d+2\delta)} \cdots (v_1 + 1)^{-(d+2\delta)} \\ & \quad \times (s_{k-1} + 1)^{-(1+\delta)} \cdots (s_1 + 1)^{-(1+\delta)}. \end{aligned}$$

We note that $\delta < d/(4k-3)$ implies

$$\int_0^\infty \frac{1}{(v_k + 1)^{1+d-(4k-3)\delta}} dv_k < \infty$$

and that $\delta < \frac{1}{2} - d$ implies $\int_1^\infty G(u)^2 du < \infty$. We also have

$$\int_0^\infty \frac{G(v_k)}{(v_k + 1)^{1+d-(4k-3)\delta}} dv_k < \infty.$$

In this way we can prove the general assertion in a manner similar to the above. $\square$

For $0 < d < \frac{1}{2}$, we define an integral operator K_d on $L^2((0, \infty), du)$ by

$$K_d f(x) := \int_0^\infty k_d(x, y) f(y) dy,$$

where

$$k_d(x, y) := \int_0^\infty \frac{1}{(x+v)^{1-d}(v+y)^{1+d}} dv \quad (x > 0, y > 0).$$

We write $\|K_d\|$ for the operator norm of K_d .

Lemma 6.3. For $0 < d < \frac{1}{2}$, K_d is a bounded operator on $L^2((0, \infty), du)$ such that $\|K_d\| \leq (\pi/d) \tan(\pi d)$.

Proof. Since

$$k_d(x, xy) = \frac{1}{x} \int_0^\infty \frac{1}{(1+s)^{1-d}(s+y)^{1+d}} ds,$$

we have

$$\begin{aligned} \int_0^\infty k_d(x, y)(x/y)^{\frac{1}{2}} dy &= x \int_0^\infty k_d(x, xy)y^{-\frac{1}{2}} dy \\ &= \int_0^\infty \frac{ds}{(s+1)^{1-d}s^{d+\frac{1}{2}}} \cdot \int_0^\infty \frac{dy}{(y+1)^{1+d}y^{\frac{1}{2}}} \\ &= B(\tfrac{1}{2}, \tfrac{1}{2} - d)B(d + \tfrac{1}{2}, \tfrac{1}{2}) = \frac{\pi}{d} \tan(\pi d). \end{aligned}$$

Similarly

$$\int_0^\infty k_d(x, y)(y/x)^{\frac{1}{2}} dx = \frac{\pi}{d} \tan(\pi d).$$

Thus the lemma follows (see Hardy *et al.* [HLP, Thm. 319]). $\square$

Now we are ready to prove the following theorem which give the asymptotic behaviour of $\epsilon(\cdot)$ for long-memory processes. See [IK] where an analogous result for a special continuous-time stationary process is given.

Theorem 6.4. *Let $0 < d < \frac{1}{2}$ and $\ell \in \mathcal{R}_0$. We assume (C1), (C2), and (A1). Then (2.5) implies*

$$(6.12) \quad \epsilon(n) \sim \frac{d^2}{n} \quad (n \rightarrow \infty).$$

Proof. By Proposition 4.3, we have $\sum_0^\infty |a_k| < \infty$. Hence it follows from (4.2) and (4.14)–(4.16) that for $m \geq 1$

$$\begin{aligned} \sum_{k=1}^{2m-1} \sum_{p=0}^\infty \{d_k(n, p)\}^2 &\leq \epsilon(n) \\ &\leq \sum_{k=1}^{2m-1} \sum_{p=0}^\infty \{d_k(n, p)\}^2 + (c_0)^{-2} \|P_{[-n, 0]}^\perp \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^m X_1\|^2. \end{aligned}$$

Let $m \rightarrow \infty$. Then by Proposition 6.1 we have

$$\begin{aligned} \sum_{k=1}^{2m-1} A_k \sin^{2k}(\pi d) &\leq \liminf_{n \rightarrow \infty} \epsilon(n)n \leq \limsup_{n \rightarrow \infty} \epsilon(n)n \\ &\leq \sum_{k=1}^{2m-1} A_k \sin^{2k}(\pi d) + \limsup_{n \rightarrow \infty} \left\{ (c_0)^{-2} n \|P_{[-n, 0]}^\perp \{P_{[-n, \infty)} P_{(-\infty, 0]}\}^m X_1\|^2 \right\}. \end{aligned}$$

Now it follows from (4.13) that

$$\{P_{[-n, \infty)} P_{(-\infty, 0]}\}^m X_1 = c_0 Z(n, m) \pmod{H_{[-n, 0]}},$$

where

$$Z(n, m) := \sum_{j_1=1}^\infty a_{n+j_1+1} \sum_{j_2=1}^\infty b_{n+j_2}^{j_1} \cdots \sum_{j_{2m}=1}^\infty b_{n+j_{2m}}^{j_{2m-1}} X_{j_{2m}}.$$

Let $\{\xi_k\}$ be the complete orthonormal system for H defined by (4.6). Then

$$\begin{aligned}
(c_0)^{-2} \|P_{[-n,0]}^\perp \{P_{[-n,\infty)} P_{(-\infty,0]}\}^m X_1\|^2 &= \|P_{[-n,0]}^\perp Z(n,m)\|^2 \\
&\leq \|Z(n,m)\|^2 = \sum_{p=-\infty}^{\infty} (Z(n,m), \xi_p)^2 \\
&= \sum_{p=-\infty}^0 \left\{ \sum_{j_1=1}^{\infty} a_{n+j_1+1} \sum_{j_2=1}^{\infty} b_{n+j_2}^{j_1} \cdots \sum_{j_{2m}=1}^{\infty} b_{n+j_{2m}}^{j_{2m-1}} c_{j_{2m}-p} \right\}^2 \\
&\quad + \sum_{p=1}^{\infty} \left\{ \sum_{j_1=1}^{\infty} a_{n+j_1+1} \sum_{j_2=1}^{\infty} b_{n+j_2}^{j_1} \cdots \sum_{j_{2m}=p}^{\infty} b_{n+j_{2m}}^{j_{2m-1}} c_{j_{2m}-p} \right\}^2 \\
&= \sum_{p=0}^{\infty} \{e_{2m}(n,p)\}^2 + \sum_{p=0}^{\infty} \{d_{2m}(n,p)\}^2,
\end{aligned}$$

Thus from Propositions 6.1 and 6.2 it follows that

$$\begin{aligned}
(6.13) \quad \sum_{k=1}^{2m-1} A_k \sin^{2k}(\pi d) &\leq \liminf_{n \rightarrow \infty} \epsilon(n)n \leq \limsup_{n \rightarrow \infty} \epsilon(n)n \\
&\leq \sum_{k=1}^{2m} A_k \sin^{2k}(\pi d) + B_{2m}(d) \sin^{4m}(\pi d).
\end{aligned}$$

We are about to estimate the last term. By (6.8), we have for $k \geq 3$

$$\begin{aligned}
B_k(d) &\leq \frac{d^2}{\pi^{2k}} \int_0^\infty du \left\{ \int_0^\infty \frac{dv_k}{(v_k+u)^{1-d}} \int_0^\infty \frac{ds_{k-1}}{(v_k+s_{k-1})^{1+d}} \right. \\
&\quad \left. \int_0^\infty \frac{ds_{k-2}}{(s_{k-1}+s_{k-2})} \cdots \int_0^\infty \frac{ds_2}{(s_3+s_2)} \int_0^\infty \frac{ds_1}{(s_2+s_1)(s_1+1)} \right\}^2 \\
&= \frac{d^2}{\pi^{2k}} \|K_d H^{k-2} f\|^2,
\end{aligned}$$

where $f(x) := 1/(1+x)$, and H is the bounded linear operator on $L^2((0,\infty), du)$ defined by

$$Hg(u) := \int_0^\infty \frac{1}{u+v} g(v) dv.$$

Since, by Hilbert's theorem (cf. [HLP, Thm. 316 and 317]), the operator norm $\|H\|$ of H is equal to π , this and Lemma 6.3 yield

$$B_k(d) \leq \frac{\tan^2(\pi d)}{\pi^2} \quad (k \geq 3).$$

Thus, if we let $m \rightarrow \infty$ in (6.13), then we obtain

$$(6.14) \quad \lim_{n \rightarrow \infty} \epsilon(n)n = \sum_{k=1}^{\infty} A_k \sin^{2k}(\pi d).$$

By Lemma 6.5 below, the right-hand side is equal to d^2 . This completes the proof of the theorem. $\square$

Lemma 6.5. For $|x| < 1$, $\sum_{k=1}^{\infty} A_k x^{2k} = \pi^{-2} \arcsin^2 x$.

Though this is a purely analytic assertion, we shall give a proof based on results for the ARIMA(0, d , 0) processes.

Proof. For $0 < d < \frac{1}{2}$, let $\{Y_n : n \in \mathbb{Z}\}$ be a ARIMA(0, d , 0) process such that $E[Y_0^2] = \Gamma(1 - 2d)/\Gamma^2(1 - d)$ (see Granger-Joyeux [GJ] and Hosking [H]; see also [BD, §13.2]). We denote by $\gamma'(\cdot)$, $\{c'_n\}$, $\{a'_n\}$, $\alpha'(\cdot)$, and $\epsilon'(\cdot)$ the autocovariance function, sequence of MA(∞) coefficients, sequence of AR(∞) coefficients, partial autocorrelation function, and function defined by (4.11), respectively, of $\{Y_n\}$. Then we have

$$a'_n = -\frac{\Gamma(n-d)}{\Gamma(n+1)\Gamma(-d)}, \quad c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)} \quad (n \geq 0),$$

$$\gamma'(n) \sim n^{2d-1} \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \quad (n \rightarrow \infty)$$

(cf. [BD, §13.2]). These imply that $\{Y_n\}$ satisfies (C1), (C2), (A1), and (2.5). Hence it follows from (6.14) that $n\epsilon'(n)$ tends to $\sum_1^\infty A_k \sin^{2k}(\pi d)$ as $n \rightarrow \infty$. On the other hand by [H, Thm. 1] (see also [BD, (13.2.10)]) we have

$$\alpha'(n) = \frac{d}{n-d} \sim \frac{d}{n} \quad (n \rightarrow \infty),$$

which, by (6.22) and (6.23) below, implies

$$\epsilon'(n) \sim \sum_{k=n}^\infty \alpha'(k)^2 \sim \frac{d^2}{n} \quad (n \rightarrow \infty).$$

Thus $\sum_1^\infty A_k \sin^{2k}(\pi d) = d^2$. The lemma follows if we substitute $\pi^{-1} \arcsin x$ with $0 < x < 1$ for d . $\square$

Remark. From Lemma 6.5, it follows that

$$(6.15) \quad A_k = \frac{1}{\pi^2} \cdot \frac{(2k-2)!!}{(2k-1)!!k} \quad (k \in \mathbb{N}).$$

Next we consider the boundary case.

Theorem 6.6. *Let $d = 0$, and $\ell \in \mathcal{R}_0$ such that $\int^\infty \ell(s)ds/s = \infty$. We assume (C1), (C2), (A1), and (A2). Then (2.5) implies*

$$(6.16) \quad \epsilon(n) \sim n^{-1} \left\{ \frac{\ell(x)}{2\ell(x)} \right\}^2 \quad (n \rightarrow \infty).$$

Proof. By (4.1) and (4.15), we have the lower bound estimate

$$(6.17) \quad \begin{aligned} \epsilon(n) &\geq (c_0)^{-2} \|P_{[-n, \infty)}^\perp P_{(-\infty, 0]} X_1\|^2 = \sum_{p=0}^\infty \{d_1(n, p)\}^2 \\ &= \sum_{p=0}^\infty \left(\sum_{v=0}^\infty c_v a_{v+n+2+p} \right)^2 = \sum_{p=n+2}^\infty \left(\sum_{v=0}^\infty c_v a_{v+p} \right)^2. \end{aligned}$$

On the other hand, by (4.1) and the equality

$$P_{(-\infty, 0]} X_1 = c_0 \sum_{v=0}^\infty a_{v+1} X_{-v},$$

we have the upper bound estimate

$$\begin{aligned}
\epsilon(n) &= (c_0)^{-2} \left\| P_{[-n,0]}^\perp P_{(-\infty,0]} X_1 \right\|^2 = \left\| P_{[-n,0]} \sum_{v=n+1}^\infty a_{v+1} X_{-v} \right\|^2 \\
(6.18) \quad &\leq \left\| \sum_{v=n+1}^\infty a_{v+1} X_{-v} \right\|^2 = \sum_{v=n+1}^\infty \sum_{m=n+1}^\infty a_{v+1} a_{m+1} \gamma(m-v) \\
&= \gamma(0) \sum_{m=n+1}^\infty (a_{m+1})^2 + 2 \sum_{m=n+1}^\infty a_{m+1} \sum_{v=1}^\infty a_{m+v+1} \gamma(v).
\end{aligned}$$

First we consider the lower bound. By (5.29) and Theorem 5.2,

$$\sum_{u=0}^n c_u \sim \{2\tilde{\ell}(n)\}^{1/2} \quad (n \rightarrow \infty).$$

In particular, $a_{v+p+1} \sum_{u=0}^v c_u \rightarrow 0$ as $v \rightarrow \infty$. Summing by parts,

$$\sum_{v=0}^\infty c_v a_{v+n} = \sum_{v=0}^\infty \left(\sum_{u=0}^v c_u \right) (a_{v+n} - a_{v+n+1}).$$

Now, by (A2), Theorem 5.2, and the monotone density theorem, we have

$$a_n - a_{n+1} \sim n^{-2} \ell(n) \{2\tilde{\ell}(n)\}^{-3/2} \quad (n \rightarrow \infty).$$

Thus it follows from [I6, Prop. 4.3] that

$$(6.19) \quad \sum_{v=0}^\infty c_v a_{v+n} \sim n^{-1} \frac{\ell(n)}{2\tilde{\ell}(n)} \quad (n \rightarrow \infty),$$

whence

$$\sum_{p=n+2}^\infty \left(\sum_{v=0}^\infty c_v a_{v+p} \right)^2 \sim n^{-1} \left\{ \frac{\ell(n)}{2\tilde{\ell}(n)} \right\}^2 \quad (n \rightarrow \infty).$$

Next we consider the upper bound. As in (6.19),

$$\sum_{v=1}^\infty a_{n+v+1} \gamma(v) \sim n^{-1} \frac{\ell(n)}{2^{3/2} \tilde{\ell}(n)^{1/2}} \quad (n \rightarrow \infty),$$

whence

$$2 \sum_{m=n+1}^\infty a_{m+1} \sum_{v=1}^\infty a_{m+v+1} \gamma(v) \sim n^{-1} \left\{ \frac{\ell(n)}{2\tilde{\ell}(n)} \right\}^2 \quad (n \rightarrow \infty).$$

On the other hand, by Theorem 5.2, we have

$$\sum_{m=n+1}^\infty (a_{m+1})^2 \sim n^{-1} \ell(n)^2 \{2\tilde{\ell}(n)\}^{-3} \quad (n \rightarrow \infty).$$

Since $\tilde{\ell}(n) \rightarrow \infty$ as $n \rightarrow \infty$, the latter is negligible in comparison with the former, whence we have

$$\gamma(0) \sum_{m=n+1}^\infty (a_{m+1})^2 + 2 \sum_{m=n+1}^\infty a_{m+1} \sum_{v=1}^\infty a_{m+v+1} \gamma(v) \sim n^{-1} \left\{ \frac{\ell(n)}{2\tilde{\ell}(n)} \right\}^2$$

as $n \rightarrow \infty$. Thus the theorem follows. $\square$

Finally we consider the intermediate-memory processes.

Theorem 6.7. Let $-\infty < d \leq 0$ and $\ell \in \mathcal{R}_0$. We further assume $\int^\infty \ell(s)ds/s < \infty$ if $d = 0$. We also assume (C1), (C2), and (A1). Then (2.5) implies

$$(6.20) \quad \epsilon(n) \sim \frac{n^{4d-1}\ell(n)^2}{(1-4d)\{\sum_{-\infty}^\infty \gamma(k)\}^2} \quad (n \rightarrow \infty).$$

Proof. By Theorem 5.3 as well as [I2, Lem. 5.7] and (5.30), we have

$$\sum_{v=0}^\infty c_v a_{v+n} \sim a_n \left(\sum_{v=0}^\infty c_v \right) \sim \frac{n^{2d-1}\ell(n)}{\sum_{-\infty}^\infty \gamma(k)} \quad (n \rightarrow \infty),$$

so that

$$\sum_{p=n+2}^\infty \left(\sum_{v=0}^\infty c_v a_{v+p} \right)^2 \sim \frac{n^{4d-1}\ell(n)^2}{(1-4d)\{\sum_{-\infty}^\infty \gamma(k)\}^2} \quad (n \rightarrow \infty).$$

On the other hand, by Theorem 5.3 and [I2, Lem. 5.7],

$$\begin{aligned} & \gamma(0) \sum_{m=n+1}^\infty (a_{m+1})^2 + 2 \sum_{m=n+1}^\infty a_{m+1} \sum_{v=1}^\infty a_{m+v+1} \gamma(v) \\ & \sim \left\{ \gamma(0) + 2 \sum_{v=1}^\infty \gamma(v) \right\} \sum_{m=n+1}^\infty (a_{m+1})^2 \quad (n \rightarrow \infty) \\ & \sim \frac{n^{4d-1}\ell(n)^2}{(1-4d)\{\sum_{-\infty}^\infty \gamma(k)\}^2} \quad (n \rightarrow \infty). \end{aligned}$$

Thus the theorem follows from the estimates (6.17) and (6.18). $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 2.1. We put

$$(6.21) \quad \delta(n) := \frac{\|P_{[-n,0]}^\perp X_1\|^2 - \|P_{[-n-1,0]}^\perp X_1\|^2}{\|P_{(-\infty,0]}^\perp X_1\|^2} \quad (n = 1, 2, \dots).$$

Then since $\|P_{[-n,0]}^\perp X_1\| \rightarrow \|P_{(-\infty,0]}^\perp X_1\|$ as $n \rightarrow \infty$, it follows that

$$(6.22) \quad \sum_{k=n}^\infty \delta(k) = \epsilon(n) \quad (n \geq 1).$$

On the other hand, by the Durbin-Levinson algorithm (cf. [BD, (5.2.5)]), we have

$$(6.23) \quad \alpha(n)^2 = \delta(n-2) \frac{\|P_{(-\infty,0]}^\perp X_1\|^2}{\|P_{[-n+2,0]}^\perp X_1\|^2} \sim \delta(n-2) \quad (n \rightarrow \infty).$$

Now it follows from (C1)–(A2) and Theorem 4.6 that, for any $k \geq 1$ and $p \geq 0$, the sequences $\{d_k(n, p) : n = 1, 2, \dots\}$ and $\{d_k(n, p) - d_k(n+1, p) : n = 1, 2, \dots\}$ are both eventually non-negative and decreasing. Since (4.12) implies

$$\delta(n) = \sum_{k=1}^\infty \sum_{p=0}^\infty \{d_k(n, p) + d_k(n+1, p)\} \{d_k(n, p) - d_k(n+1, p)\},$$

the sequence $\{\delta(n)\}$ is also eventually decreasing. Hence, by the monotone density theorem, Theorems 6.4, 6.6 and 6.7 give the asymptotics for $\delta(\cdot)$. Therefore, by (6.23), we obtain (2.7)–(2.9). To complete the proof, it suffices to note that (2.6) is equivalent to (2.7), (2.8) or (2.9) under each assumption on d and $\ell(\cdot)$. $\square$

7. REFLECTION POSITIVITY

In this section, we shall consider the stationary processes with reflection positivity, that is, those with completely monotone autocovariance functions. They have the advantage that relevant series and functions have nice integral representations. In fact, it will be shown below that the sequences $\{c_n\}$ and $\{a_n\}$ of them have such integral representations. These representations in turn imply the conditions (C1)–(A2) immediately.

First, we prove some preliminary analytic results. In what follows, we write $\int_0^1$ for $\int_{[0,1]}$. We put

$$\Sigma := \{\sigma : \sigma \text{ is a non-zero finite Borel measure on } [0, 1)\}.$$

For $\sigma \in \Sigma$, we write

$$\Delta_\sigma(\theta) := \frac{1}{2\pi} \int_0^1 P_r(\theta) \sigma(dr) \quad (-\pi < \theta < \pi),$$

where $P_r(\theta)$ is the Poisson kernel defined by (3.4). Then the function $\Delta_\sigma(\cdot)$ is positive and integrable on $(-\pi, \pi)$. It follows that

$$(7.1) \quad \int_{-\pi}^{\pi} e^{in\theta} \Delta_\sigma(\theta) d\theta = \int_0^1 r^{|n|} \sigma(dr) \quad (n \in \mathbb{Z}),$$

where the convention $0^0 = 1$ is adopted in the integral on the right-hand side.

For a finite Borel measure μ on $[0, 1)$, we write

$$F_\mu(z) := \int_0^1 \frac{1}{1-rz} \mu(dr) \quad (z \in \mathbb{C}, |z| < 1),$$

$$F_\mu(e^{i\theta}) := \int_0^1 \frac{1}{1-re^{i\theta}} \mu(dr) \quad (-\pi < \theta < \pi).$$

We write N for the set of all non-zero Borel measures ν on $[0, 1)$ such that

$$(7.2) \quad \int_0^1 \int_0^1 \frac{1}{1-rs} \nu(dr) \nu(ds) < \infty.$$

If $\nu \in N$, then ν is a finite measure, i.e., $\nu \in \Sigma$. For $\nu \in N$, we define $\sigma = S(\nu) \in \Sigma$ by

$$\sigma(dr) := \left\{ \int_0^1 \frac{1}{1-rs} \nu(ds) \right\} \nu(dr).$$

Then we have

$$(7.3) \quad |F_\nu(e^{i\theta})|^2 = 2\pi \Delta_\sigma(\theta) \quad (-\pi < \theta < \pi),$$

for

$$\frac{1 - (r+s)\cos\theta + rs}{|1-re^{i\theta}|^2 \cdot |1-se^{i\theta}|^2} = \frac{1}{2(1-rs)} \{P_r(\theta) + P_s(\theta)\}.$$

In particular, $\int_{-\pi}^{\pi} |F_\nu(e^{i\theta})|^2 d\theta = 2\pi\sigma([0, 1))$, whence $F_\nu(e^{i\theta}) \in L^2(-\pi, \pi)$ if $\nu \in N$. Moreover, since the real part of $F_\nu(z)$ is positive in $|z| < 1$, $F_\nu(z)$ is an outer function for the space H^{2+} , and so $\log |F_\nu(e^{i\theta})|$ is integrable on $(-\pi, \pi)$ and the following holds:

$$(7.4) \quad F_\nu(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |F_\nu(e^{i\theta})| d\theta \right\} \quad (|z| < 1)$$

(see Duren [Du, Ch. 3, Exercise 1] and [Ru, Thm. 17.16]). If $\sigma = S(\nu)$, then, from (7.3) and (7.4), it follows that $\log \Delta(\cdot)$ is also integrable on $(-\pi, \pi)$ and that

$$(7.5) \quad F_\nu(z) = (2\pi)^{1/2} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |\Delta_\sigma(\theta)| d\theta \right\} \quad (z \in \mathbb{C}, |z| < 1).$$

The following theorem is a discrete analogue of [I3, Thm. 2.5].

Theorem 7.1. *The map S from N to Σ is one-to-one and onto.*

Proof. Step 1. For simplicity, we call a measure σ on $[0, 1]$ to be *simple* if σ is of the form

$$\sigma = \sum_{k=1}^n s_k \delta_{r_k},$$

where $s_k \in (0, \infty)$ and $0 < r_1 < r_2 < \dots < r_n < 1$. In this step, we show that for σ simple there exists a simple measure ν such that $\sigma = S(\nu)$.

For a simple measure σ of the above form, we define a polynomial $f(z)$ of degree $2n - 2$ by

$$f(z) := \sum_{k=1}^n \{1 - (r_k)^2\} s_k \prod_{m \neq k} (1 - r_m z)(z - r_m).$$

Then since $f(p_k)f(p_{k+1}) < 0$ ($k = 1, \dots, n-1$), $f(z)$ has a zero q_k in (r_k, r_{k+1}) for $k = 1, \dots, n-1$. Moreover, since $f(1/z) = z^{-2n+2}f(z)$, $1/q_k$ ($k = 1, \dots, n-1$) are also zeros of $f(z)$. Thus $f(z)$ must be of the form

$$f(z) = c \prod_{k=1}^{n-1} (1 - q_k z)(z - q_k)$$

with some positive constant c . Now we define a rational function $F(z)$ by

$$F(z) := \sqrt{c} \frac{\prod_{k=1}^{n-1} (1 - q_k z)}{\prod_{k=1}^n (1 - r_k z)}.$$

Then

$$F(z)F(1/z) = cz \frac{\prod_{k=1}^{n-1} (1 - q_k z)(z - q_k)}{\prod_{k=1}^n (1 - r_k z)(z - r_k)} = \sum_{k=1}^n \frac{\{1 - (r_k)^2\} s_k}{(1 - r_k z)(1 - r_k z^{-1})},$$

so that

$$(7.6) \quad |F(e^{i\theta})|^2 = \lim_{t \rightarrow 1^-} F(te^{i\theta})F(1/te^{i\theta}) = 2\pi \Delta_\sigma(\theta).$$

On the other hand, $F(z)$ has the following partial fraction decomposition:

$$h(z) = \sum_{k=1}^u \frac{m_k}{1 - r_k z},$$

where $m_k \in (0, \infty)$ ($k = 1, \dots, u$). If we write $\nu := \sum_{k=1}^u m_k \delta_{r_k}$, then we have $F(z) = F_\nu(z)$, and so, by (7.3) and (7.6), $\Delta_\sigma(\theta) = \Delta_{\sigma'}(\theta)$ with $\sigma' := S(\nu)$. By (7.1), this implies $\int_0^1 t^n \sigma(dt) = \int_0^1 t^n \sigma'(dt)$ for all $n \in \mathbb{N} \cup \{0\}$. Thus $\sigma = \sigma' = S(\nu)$. *Step 2.* For $\sigma \in \Sigma$, choose a sequence of simple measures σ_n such that $\sigma_n \rightarrow \sigma$ weakly on $[0, 1]$ as $n \rightarrow \infty$. Here we regard σ and σ_n as measures on $[0, 1]$ by

$\sigma(\{1\}) = \sigma_n(\{1\}) = 0$. By Step 1, we have simple measures ν_n such that $S(\nu_n) = \sigma_n$. From (7.1), (7.5), and Jensen's inequality, it follows that

$$\nu_n([0, 1]) = F_\nu(0) = (2\pi)^{1/2} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log |\Delta_{\sigma_n}(\theta)| d\theta \right\} \leq \sqrt{2\pi \sigma_n([0, 1])},$$

whence, by the Helly selection principle, there exists a sequence of integers $n' \rightarrow \infty$ such that $\nu_{n'}$ converges weakly on $[0, 1]$ to a finite measure, ν , say. By (7.3), we have

$$\left| \int_0^1 \frac{1}{1 - re^{i\theta}} \nu_{n'}(dr) \right|^2 = \int_0^1 P_r(\theta) \sigma_{n'}(dr) \quad (-\pi < \theta < \pi).$$

Let $n' \rightarrow \infty$. Then since, for $\theta \neq 0$, the integrands are both bounded and continuous on $[0, 1]$, we have

$$\left| \int_{[0, 1]} \frac{1}{1 - re^{i\theta}} \nu(dr) \right|^2 = 2\pi \Delta_\sigma(\theta) \quad (-\pi < \theta < \pi, \theta \neq 0).$$

The absolute value of the integral on the left-hand side is at least

$$\left| \operatorname{Im} \int_{[0, 1]} \frac{1}{1 - re^{i\theta}} \nu(dr) \right| \geq \frac{|\sin \theta|}{2(1 - \cos \theta)} \nu(\{1\}).$$

Since $\Delta_\sigma \in L^1(-\pi, \pi)$, this implies that $\nu(\{1\}) = 0$. We put

$$\sigma'(dr) := \left\{ \int_0^1 \frac{1}{1 - rs} \nu(ds) \right\} \nu(dr).$$

Then we have

$$\Delta_\sigma(\theta) = \frac{1}{2\pi} \int_0^1 P_r(\theta) \sigma'(dr),$$

and so, as in Step 1, $\int_0^1 t^n \sigma(dt) = \int_0^1 t^n \sigma'(dt)$ for all $n \geq 0$. Consequently, ν is in N , and $\sigma = \sigma' = S(\nu)$. Thus S is onto.

Step 3. It remains to show that the S is one-to-one. By (7.5) we see that F_ν is determined uniquely by σ . Since $F_\nu(z)$ determines ν uniquely, this implies that S is one-to-one. $\square$

Theorem 7.2. *For any $\nu \in N$, there exists a unique triple (b_1, b_2, ρ) consisting of $b_1 \in (0, \infty)$, $b_2 \in [0, \infty)$ and a finite (possibly zero) Borel measure ρ on $[0, 1]$ such that*

$$(7.7) \quad F_\nu(z) \{b_1(1 - z) + b_2(1 + z) + (1 - z^2)F_\rho(z)\} = 1 \quad (z \in \mathbb{C}, |z| < 1).$$

Proof. Let $z \downarrow -1$ or $z \uparrow 1$ in (7.7). Then we obtain

$$(7.8) \quad b_1 = \left\{ \int_0^1 \frac{1}{1 + r} \nu(dr) \right\}^{-1}, \quad b_2 = \left\{ \int_0^1 \frac{1}{1 - r} \nu(dr) \right\}^{-1}$$

($b_2 = 0$ if $\int_0^1 (1 - r)^{-1} \nu(dr) = \infty$). Therefore b_1 and b_2 are both uniquely determined by ν . Since ρ is uniquely determined by F_ρ , it is also uniquely determined by ν .

Now we show the existence. First we assume $\int_0^1 (1-r)^{-1} \nu(dr) < \infty$. Then by [I2, Theorem 3.1(i)] there exist positive constants α_2, β_2 , and a finite Borel measure ρ_2 on $[0, 1)$ such that $\int_0^1 (1+r)^{-1} \rho_2(dr) < 1$ and

$$\frac{\alpha_2}{\sqrt{2\pi}} \left\{ \frac{\beta_2}{2} (1+z) + (1-z) + z(1-z) F_{\rho_2}(z) \right\} F_\nu(z) = 1 \quad (z \in \mathbb{C}, |z| < 1).$$

Since

$$z F_{\rho_2}(z) = - \int_0^1 \frac{1}{1+r} \rho_2(dr) + (1+z) \int_0^1 \frac{1}{(1-rz)(1+r)} \rho_2(dr),$$

it follows that (7.7) holds with

$$b_1 = \frac{\alpha_2}{\sqrt{2\pi}} \left\{ 1 - \int_0^1 \frac{1}{1+r} \rho_2(dr) \right\}, \quad b_2 = \frac{\alpha_2 \beta_2}{2\sqrt{2\pi}}, \quad \rho(dr) = \frac{\alpha_2}{\sqrt{2\pi}(1+r)} \rho_2(dr).$$

Next we assume $\int_0^1 (1-r)^{-1} \nu(dr) = \infty$. We put

$$\nu^{(n)}(dr) := I_{[0, 1-n^{-1}]}(r) \nu(dr) \quad (n \geq M),$$

where we choose M so large that ν_M is not a zero measure. By the result above, there exist $b_1^{(n)} \in (0, \infty)$, $b_2^{(n)} \in (0, \infty)$ and a finite Borel measure $\rho^{(n)}$ on $[0, 1)$ which satisfy (7.7) with $\nu^{(n)}$. By (7.8),

$$\begin{aligned} b_1^{(n)} &= \left\{ \int_0^1 \frac{1}{1+r} \nu^{(n)}(dr) \right\}^{-1} \quad \downarrow \quad b_1 := \left\{ \int_0^1 \frac{1}{1+r} \nu(dr) \right\}^{-1} \quad (n \rightarrow \infty), \\ b_2^{(n)} &= \left\{ \int_0^1 \frac{1}{1-r} \nu^{(n)}(dr) \right\}^{-1} \quad \downarrow \quad 0 \quad (n \rightarrow \infty). \end{aligned}$$

On the other hand, if we let $z = 0$ in (7.7), then we obtain

$$\sup_{n \geq M} \rho^{(n)}([0, 1)) \leq \sup_{n \geq M} \frac{1}{\nu^{(n)}([0, 1))} < \infty.$$

Thus, by a standard argument which involves the Helly selection principle, we see that there exists a finite Borel measure ρ on $[0, 1)$ such that the triple $(b_1, 0, \rho)$ satisfies (7.7) with ν . $\square$

Let us come back to stationary processes. For $\sigma \in \Sigma$, the function $\gamma(\cdot)$ defined by $\gamma(n) := \int_0^1 t^{|n|} \sigma(dt)$ ($n \in \mathbb{Z}$) is non-negative definite by (7.1), whence it is the autocovariance function of a stationary process with spectral density Δ_σ . Thus we come to the following definition of property for a stationary process $\{X_n\}$ with autocovariance function $\gamma(\cdot)$:

$$(RP) \quad \text{there exists } \sigma \in \Sigma \text{ such that } \gamma(n) = \int_0^1 t^{|n|} \sigma(dt) \quad (n \in \mathbb{Z}).$$

This property for $\{X_n\}$ is called *reflection positivity*. Since, for $\sigma \in \Sigma$, $\log \Delta_\sigma(\cdot)$ is integrable on $(-\pi, \pi)$, it follows that such a stationary process is purely non-deterministic.

Now we prove that (RP) implies (C1)–(A2).

Theorem 7.3. *Let $\{X_n\}$ be a stationary process which satisfies (RP). Then there exist two finite Borel measures ν and ρ on $[0, 1)$ such that*

$$(7.9) \quad c_n = \int_0^1 r^n \nu(dr) \quad (n = 0, 1, \dots),$$

$$(7.10) \quad a_n = \int_0^1 r^{n-2} (1 - r^2) \rho(dr) \quad (n = 2, 3, \dots).$$

In particular, $\{X_n\}$ satisfies (C1), (C2), (A1), and (A2).

Proof. Let σ be the measure which appears in (RP). Set $\nu := S^{-1}(\sigma) \in N$. Then, by (7.5), the outer function $h(z)$ of $\{X_n\}$ is equal to $(2\pi)^{-1/2} F_\nu(z)$. Therefore it follows that the MA(∞) coefficients c_n of $\{X_n\}$ are given by (7.9) with this ν . Let (b_1, b_2, ρ) be the triple which is determined by the relation (7.7). Then, for $z \in \mathbb{C}$,

$$-\frac{1}{(2\pi)^{1/2} h(z)} = -\frac{1}{F_\nu(z)} = -b_1(1-z) - b_2(1+z) - (1-z^2)F_\rho(z).$$

Thus (7.10) holds with this ρ . □

Example. Let $-\infty < d < \frac{1}{2}$, and let $\{X_n\}$ be a stationary process with autocovariance function of the form $\gamma(n) = (1 + |n|)^{-(1-2d)}$. Since

$$\frac{1}{(1 + |n|)^{1-2d}} = \int_0^1 t^{|n|} \frac{(-\log t)^{-2d}}{\Gamma(1-2d)} dt \quad (n \in \mathbb{Z}),$$

we see that $\{X_n\}$ satisfies (RP). Let $\alpha(\cdot)$ be the partial autocorrelation function of $\{X_n\}$. Then application of Theorem 2.1 to $\{X_n\}$ yields the following result:

(1) if $0 < d < \frac{1}{2}$, then

$$|\alpha(n)| \sim \frac{d}{n} \quad (n \rightarrow \infty);$$

(2) if $d = 0$, then

$$|\alpha(n)| \sim \frac{1}{2n \log n} \quad (n \rightarrow \infty);$$

(3) if $-\infty < d < 0$, then

$$|\alpha(n)| \sim \frac{n^{2d-1}}{\{2\zeta(1-2d) - 1\}} \quad (n \rightarrow \infty).$$

Here $\zeta(s)$ is the Riemann zeta function. In view of the numerical data obtained by using the Durbin-Levinson algorithm, it seems unnecessary to take the absolute values of $\alpha(n)$ in (1)–(3) above. This observation even suggests a possible improvement of Theorem 2.1.

We have seen that the stationary processes with reflection positivity satisfy the conditions (C1)–(A2). We must, however, point out that it is desirable to remove the assumption (C1) from Theorem 2.1.

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