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Asymptotic behavior of classical solutions to a system of semilinier wave equations in low space dimensions

Hideo Kubo * and Kôji Kubota †

1 Introduction and statement of main results

The initial value problem of semilinear wave equations for small initial data and the related nonlinear scattering theory have been developed by many authors, since the work of F. John [9] was established in 1979. (See for instance [1]–[8], [11]–[35]). In those works, the "basic estimates" of solutions to the following inhomogenious wave equations play an essential role in a both explicit and implicit manner:

$$\partial_t^2 u - \Delta u = F \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{1.1}$$

where $\partial_t = \partial/\partial t$ and $\Delta = \sum_{j=1}^n \partial_j^2$ with $\partial_j = \partial/\partial x_j$ $(j = 1, \cdots, n)$. The aim of this paper is to give a new basic estimate of the solution to (1.1) for the case where $n = 2$ or $n = 3$, which refines previous ones, especially for $n = 2$. (See also [17], [27] and [33]). To state more precisely, we introduce an integral operator $L(F)(x,t)$ as follows:

$$L(F)(x,t) = \frac{1}{\omega_2}\int_{-\infty}^t ds \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} d\rho \int_{|\omega|=1} F(x+\rho\omega, s) dS_\omega \tag{1.2}$$

for $(x,t) \in \mathbf{R}^2 \times \mathbf{R}$ and

$$L(F)(x,t) = \frac{1}{\omega_3}\int_{-\infty}^t (t-s) ds \int_{|\omega|=1} F(x+(t-s)\omega, s) dS_\omega \tag{1.3}$$

for $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$. Note that $L(F)(x,t)$ satisfies (1.1) under a suitable assumption on $F(x,t)$. The main result of this paper is summarized as follows.

Theorem 1. 1 *Let $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ and $r = |x|$. Let $F \in C^0(\mathbf{R}^n \times \mathbf{R})$. Then we have*

$$\begin{aligned} &|L(F)(x,t)|(1+r+|t|)^{\frac{n-1}{2}}/\Phi_n(r,t;\nu) \\ &\le C \sup_{(y,s)\in\mathbf{R}^n\times\mathbf{R}}\{|y|^{\frac{n-1}{2}}(1+|y|+|s|)^{1+\nu}(1+||y|-|s||)^{1+\mu}|F(y,s)|\} \end{aligned} \tag{1.4}$$

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for any $\mu > 0$ and $\nu > 0$. Here we have set

$$\Phi_3(r,t;\nu) = (1+|r-t|)^{-\nu}, \tag{1.5}$$

and

$$\Phi_2(r,t;\nu) = \begin{cases} (1+|r-t|)^{-\nu} & \text{if} \quad -\infty < t \le r, \\ (1+t-r)^{-\frac{1}{2}}(1+t-r)^{[\frac{1}{2}-\nu]_+} & \text{if} \quad r < t. \end{cases} \tag{1.6}$$

In addition, $[a]_+ = \max\{a,0\}$ and $A^{[0]_+} = 1+\log A$.

As an application of Theorem 1. 1 , we consider a system of semilinear wave equations:

$$\partial_t^2 u - \Delta u = |v|^{p-1}v \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{1.7}$$
$$\partial_t^2 v - \Delta v = |u|^{q-1}u \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{1.8}$$

where $1 < p \le q$ and either $n = 2$ or $n = 3$. Concerning the initial value problem of (1.7) and (1.8) for small initial data, [4] proved the existence of global solutions to (1.7) and (1.8) in $\mathbf{R}^n \times (0,\infty)$, provided

$$\Gamma = \Gamma(p,q,n) > 0, \tag{1.9}$$
$$(n+1)/(n-1) < p \le q, \quad \text{i.e.,} \quad 0 < p^* \le q^*, \tag{1.10}$$

where we have set

$$\begin{gathered} \Gamma = \alpha + p\beta, \quad \alpha = pq^* - 1, \quad \beta = qp^* - 1, \\ p^* = \frac{n-1}{2}p - \frac{n+1}{2}, \quad q^* = \frac{n-1}{2}q - \frac{n+1}{2}. \end{gathered} \tag{1.11}$$

On the other hand, when $\Gamma \le 0$, there is a solution which blows up in finite time, even though the initial data are sufficiently small. (See [4], [5] for the case $\Gamma < 0$ and [18], [15] for the case $\Gamma = 0$). In this article, we study asymptotic behavior of classical solutions to (1.7) and (1.8), provided (1.9) and (1.10). To this end, we introduce the following function space $X_{\nu,\kappa}$ for $\nu > 0$, $\kappa > 0$:

$$X_{\nu,\kappa} = \{(u,v) \in C^0(\mathbf{R}^n \times \mathbf{R}) \times C^0(\mathbf{R}^n \times \mathbf{R});\ \|u\|_\nu + \|v\|_\kappa < +\infty\}, \tag{1.12}$$

where the norm $\|\cdot\|_\nu$ is defined by

$$\|u\|_\nu = \sup_{(x,t)\in\mathbf{R}^n\times\mathbf{R}} \{|u(x,t)|(1+r+|t|)^{\frac{n-1}{2}}/\Phi_n(r,|t|;\nu)\}. \tag{1.13}$$

In addition, we set for $\delta > 0$

$$X_{\nu,\kappa}(\delta) = \{(u,v) \in X_{\nu,\kappa} :\ \|u\|_\nu + \|v\|_\kappa \le \delta\}. \tag{1.14}$$

Next, let us denote by $u^-(x,t)$ and $v^-(x,t)$ the solutions to the homogeneous wave equation satisfying the following initial data, respectively:

$$u^-(x,0) = f_1(x), \quad \partial_t u^-(x,0) = g_1(x) \quad \text{in } \mathbf{R}^n, \tag{1.15}$$
$$v^-(x,0) = f_2(x), \quad \partial_t v^-(x,0) = g_2(x) \quad \text{in } \mathbf{R}^n. \tag{1.16}$$

We assume $f_j \in C^3(\mathbf{R}^n)$ and $g_j \in C^2(\mathbf{R}^n)$ $(j = 1, 2)$ satisfy

$$(1.17)\qquad \begin{aligned} &|f_1(x)| \le \varepsilon(1+r)^{-\frac{n-1}{2}-p^*}, \quad |f_2(x)| \le \varepsilon(1+r)^{-\frac{n-1}{2}-q^*},\\ &\sum_{|\gamma|=1} |\partial_x^\gamma f_1(x)| + |g_1(x)| \le \varepsilon(1+r)^{-\frac{n+1}{2}-p^*},\\ &\sum_{|\gamma|=1} |\partial_x^\gamma f_2(x)| + |g_2(x)| \le \varepsilon(1+r)^{-\frac{n+1}{2}-q^*}, \end{aligned}$$

and

$$(1.18)\qquad \begin{aligned} &\sup_{x\in\mathbf{R}^n} \{(1+r)^{\frac{n+1}{2}+p^*}(\sum_{2\le|\gamma|\le 3} |\partial_x^\gamma f_1(x)| + \sum_{1\le|\gamma|\le 2} |\partial_x^\gamma g_1(x)|)\} < +\infty,\\ &\sup_{x\in\mathbf{R}^n} \{(1+r)^{\frac{n+1}{2}+q^*}(\sum_{2\le|\gamma|\le 3} |\partial_x^\gamma f_2(x)| + \sum_{1\le|\gamma|\le 2} |\partial_x^\gamma g_2(x)|)\} < +\infty, \end{aligned}$$

where $\varepsilon > 0$ and $r = |x|$. Then there is a positive constant $C_0 = C_0(p, q, n)$ such that

$$(1.19)\qquad (u^-, v^-) \in X_{p^*,q^*}(C_0\varepsilon), \quad (\partial_x^\gamma u^-, \partial_x^\gamma v^-) \in X_{p^*,q^*} \quad \text{for } |\gamma| \le 2.$$

For the proof, see e.g. [2], [24] for 3-dimensional case and [16], [29], [30] for 2-dimensional case.

It follows from the definitions of Γ, α and β that

$$(1.20)\qquad \beta \le \Gamma/(p+1) \le \alpha \quad \text{for } 1 < p \le q.$$

Here we shall state a part of our result for a special case, namely, we assume

$$(1.21)\qquad \beta > 0, \quad \textit{i.e.}, \quad qp^* > 1.$$

Notice that (1.21) implies (1.9), by (1.20) and that (1.21) is equivalent to (1.9) when $p = q$. Moreover, since $q \ge p$, we have

$$(1.22)\qquad \beta > 0 \quad \text{if} \quad pp^* > 1, \quad \textit{i.e.}, \ p > p_0(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}.$$

The general case will be discussed in Section 5 below.

Theorem 1. 2 *Let* $n = 2$ *or* $n = 3$. *Suppose that* (1.10), (1.17), (1.18) *and* (1.21) *hold. Then for any* ν *and* κ *satisfying*

$$(1.23)\qquad 1/q < \nu \le p^*, \quad 1/p < \kappa \le q^*,$$

there exists a positive number $\varepsilon_0 = \varepsilon_0(\nu, \kappa, p, q, n, f_j, g_j)$ *such that for any* ε *with* $0 < \varepsilon \le \varepsilon_0$, *there exists a classical solution* (u, v) *of* (1.7) *and* (1.8) *uniquely in* $X_{\nu,\kappa}(2C_0\varepsilon)$ *verifying the following properties:*

$$(1.24)\qquad |||(u-u^-)(t)|||_e \le C\|v\|_\kappa^p(1+|t|)^{-p^*} \quad \textit{for } t \le 0,$$
$$(1.25)\qquad |||(v-v^-)(t)|||_e \le C\|u\|_\nu^q(1+|t|)^{-q^*} \quad \textit{for } t \le 0,$$

where

$$|||u(t)|||_e^2 = \frac{1}{2}\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\}.$$

Moreover, for any $(x,t) \in \mathbf{R}^n \times \mathbf{R}$

$$|u(x,t) - u^-(x,t)| \leq C\|v\|_\kappa^p (1+r+|t|)^{-\frac{n-1}{2}} \Phi_n(r,t;p^*), \tag{1.26}$$
$$|v(x,t) - v^-(x,t)| \leq C\|u\|_\nu^q (1+r+|t|)^{-\frac{n-1}{2}} \Phi_n(r,t;q^*). \tag{1.27}$$

Remark: The existnce of ν and κ satisfying (1.23) follows from the fact that $\alpha \geq \beta > 0$.

The plan of this paper is as follows: In Section 2, we collect some notations. In Section 3, we prepare basic results which are needed to prove Theorem 1. 1 , and we carry out the proof of the theorem in Section 4. In Section 5, we state our results for a system of semilinear wave equations. In Section 6, we establish a priori estimates by making use of Theorem 1. 1 , and we prove Theorems 5. 1 and 5. 2 in Section 7.

2 Notations

In this section we collect some notations which will be used in the sequel. We set

$$a \vee b = \max\{a,b\} \quad \text{for } a,b \in \mathbf{R}. \tag{2.1}$$

In particular, we put

$$[a]_+ = a \vee 0, \quad A^{[0]+} = 1 + \log A. \tag{2.2}$$

Next we define several norms for a real valued function $u(x,t)$:

$$\|u\|_\nu = \sup_{(y,s)\in\mathbf{R}^n\times\mathbf{R}} \{|u(y,s)|(1+|y|+|s|)^{\frac{n-1}{2}}/\Phi_n(|y|,|s|;\nu)\}, \tag{2.3}$$
$$[u]_\nu = \max_{|\gamma|\leq 2} \|\partial_x^\gamma u\|_\nu, \tag{2.4}$$
$$|||u(t)|||_e^2 = \frac{1}{2}\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\}, \tag{2.5}$$

where $\Phi_n(r,t;\nu)$ is given by (1.5) and (1.6).

We set for $\mu > 0$ and $\nu > 0$

$$z_{\mu,\nu}(\lambda,s) = (1+|s|+\lambda)^{1+\nu}(1+||s|-\lambda|)^{1+\mu}, \tag{2.6}$$
$$M_{\mu,\nu}(F)(r,t) = \sup_{(y,s)\in\mathbf{R}^n\times\mathbf{R}} \{|y|^{\frac{n-1}{2}} z_{\mu,\nu}(|y|,s)|F(y,s)|\}. \tag{2.7}$$

3 Preliminaries

In this section we collect some basic results. First one is a fundamental identity concerning the spherical mean. For the proof, see [25], also Lemma 2.3 in [16].

Lemma 3. 1 *Let $b \in C([0,\infty))$, $x \in \mathbf{R}^n \setminus \{0\}$, $n \geq 2$, $r = |x|$ and $\rho > 0$. Then we have*

$$\int_{|\omega|=1} b(|x+\rho\omega|)dS_\omega = 2^{3-n}\omega_{n-1}(r\rho)^{2-n}\int_{|\rho-r|}^{\rho+r} \lambda b(\lambda)h(\lambda,\rho,r)d\lambda, \tag{3.1}$$

where

$$h(\lambda,\rho,r) = (\rho^2-(\lambda-r)^2)^{\frac{n-3}{2}}((\lambda+r)^2-\rho^2)^{\frac{n-3}{2}}. \tag{3.2}$$

Next one is due to Lemmas 1.2 and 1.3 in [17].

Lemma 3. 2 *For $a < b < d$, we set*

$$J(a,b,d) = \int_a^b \frac{d\sigma}{\sqrt{b-\sigma}\sqrt{\sigma-a}\sqrt{d-\sigma}}. \tag{3.3}$$

Then we have

$$J(a,b,d) \leq \frac{\pi}{\sqrt{d-b}} \tag{3.4}$$

and for any $\theta > 0$

$$J(a,b,d) \leq C\left(\frac{d-a}{d-b}\right)^\theta \frac{1}{\sqrt{b-a}}. \tag{3.5}$$

Finally, we prepare the following useful lemma, which is an extended version of Lemmas 1.3 and 1.5 in [17]. This lemma will be repeatedly used in Section 4 below.

Lemma 3. 3 *Let $\kappa > 0$, $0 \leq \gamma < 1$ and $\kappa + \gamma > 1$. Then we have*

$$\int_{|b|}^\infty (1+\sigma)^{-\kappa}(a+\sigma)^{-\gamma}d\sigma \leq C(1+|b|)^{-\kappa-\gamma+1} \quad \text{for } a \geq -|b|. \tag{3.6}$$

Moreover, we have

$$\int_{-\infty}^b (1+|\sigma|)^{-\kappa}(a-\sigma)^{-\gamma}d\sigma \leq C(1+|a|)^{-\gamma}(1+|a|)^{[1-\kappa]_+} \quad \text{for } a \geq b, \tag{3.7}$$

or equivalently,

$$\int_b^\infty (1+|\sigma|)^{-\kappa}(a+\sigma)^{-\gamma}d\sigma \leq C(1+|a|)^{-\gamma}(1+|a|)^{[1-\kappa]_+} \quad \text{for } a \geq -b, \tag{3.8}$$

where C is a constant depending only on κ and γ.

Remark: If $a \le 0$, one can replace the right hand side in (3.7) by that in (3.6).
Proof: Firstly, we show (3.6). Since $(a+\sigma)^{-\gamma} \le (\sigma - |b|)^{-\gamma}$, by integration by parts, we have

$$\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(a+\sigma)^{-\gamma} d\sigma \le \frac{\kappa}{1-\gamma} \int_{|b|}^{\infty} (1+\sigma)^{-\kappa-\gamma} d\sigma,$$

which yields (3.6), because $\kappa + \gamma > 1$.

Seconderly, we show (3.7). Since it holds that for $a \ge b$

$$\int_{-\infty}^{b} (1+|\sigma|)^{-\kappa}(a-\sigma)^{-\gamma} d\sigma \le \int_{-a}^{\infty} (1+|\sigma|)^{-\kappa}(a+\sigma)^{-\gamma} d\sigma,$$

we get (3.7) from (3.6) for the case where $a \le 0$. In the following, we assume $a > 0$. If we set

$$P_1 = \int_{-\infty}^{-a} (1+|\sigma|)^{-\kappa}(a-\sigma)^{-\gamma} d\sigma, \quad P_2 = \int_{-a}^{a} (1+|\sigma|)^{-\kappa}(a-\sigma)^{-\gamma} d\sigma,$$

we see that the left hand side of (3.7) is dominated by $P_1 + P_2$. Since the estimate of P_1 is handled similarly to the case where $a \le 0$, we shall deal with only P_2.

First, when $0 < a \le 1$, it is enough to show that P_2 is bounded. It follows that

$$P_2 \le \int_{-a}^{a} (a-\sigma)^{-\gamma} d\sigma,$$

hence P_2 is bounded, because $\gamma < 1$.

Next, we let $a > 1$. It follows that

$$P_2 \le \left(\frac{a}{2}\right)^{-\gamma} \int_{-a}^{a/2} (1+|\sigma|)^{-\kappa} d\sigma + (1+\frac{a}{2})^{-\kappa} \int_{a/2}^{a} (a-\sigma)^{-\gamma} d\sigma,$$

which implies $P_2 \le C(1+a)^{-\gamma}(1+a)^{[1-\kappa]_+}$, because $a > 1$ and $\gamma < 1$. The proof is complete. □

4 Basic estimates

In this section, we shall prove Theorem 1. 1 . By (2.7), we have

$$\left| \int_{|\omega|=1} F(x+\rho\omega, s) dS_\omega \right| \le M_{\mu,\nu}(F)(r,t) \int_{|\omega|=1} \frac{dS_\omega}{\lambda^{\frac{n-1}{2}} z_{\mu,\nu}(\lambda, s)} \tag{4.1}$$

with $\lambda = |x + \rho\omega|$. Applying Lemma 3. 1 , we get

$$\int_{|\omega|=1} \frac{dS_\omega}{\lambda^{\frac{n-1}{2}} z_{\mu,\nu}(\lambda, s)} = 2^{3-n}\omega_{n-1}(r\rho)^{2-n} \int_{|\rho-r|}^{\rho+r} \frac{\lambda^{\frac{3-n}{2}} h(\lambda,\rho,r)}{z_{\mu,\nu}(\lambda,s)} d\lambda. \tag{4.2}$$

Case 1: n = 2. If we set

$$I(r,t) = \frac{2}{\pi}\int_{-\infty}^{t} ds \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} d\rho \int_{|\rho-r|}^{\rho+r} \frac{\lambda^{\frac{1}{2}} h(\lambda,\rho,r)}{z_{\mu,\nu}(\lambda,s)} d\lambda, \tag{4.3}$$

it follows from (1.2), (4.1) and (4.2) with $n = 2$ that

$$|L(F)(x,t)| \leq M_{\mu,\nu}(F)(r,t) \times I(r,t). \tag{4.4}$$

Changing the order of the integrals, we get

$$I(r,t) = I_1(r,t) + I_2(r,t), \tag{4.5}$$

where we have set

$$I_1(r,t) = \int_{-\infty}^{t} ds \int_{|t-s-r|}^{t-s+r} \frac{1}{z_{\mu,\nu}(\lambda,s)} K_1(\lambda,s,r,t)d\lambda, \tag{4.6}$$

$$I_2(r,t) = \int_{-\infty}^{t-r} ds \int_{0}^{t-s-r} \frac{1}{z_{\mu,\nu}(\lambda,s)} K_2(\lambda,s,r,t)d\lambda. \tag{4.7}$$

Here

$$K_1(\lambda,s,r,t) = \frac{2\sqrt{\lambda}}{\pi} \int_{|\lambda-r|}^{t-s} \frac{\rho h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}} d\rho \quad \text{for } |t-s-r| < \lambda < t-s+r, \tag{4.8}$$

$$K_2(\lambda,s,r,t) = \frac{2\sqrt{\lambda}}{\pi} \int_{|\lambda-r|}^{\lambda+r} \frac{\rho h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}} d\rho \quad \text{for } 0 < \lambda < t-s-r. \tag{4.9}$$

We introduce new variables

$$\alpha = \lambda + s \quad \text{and} \quad \beta = \lambda - s. \tag{4.10}$$

If we denote by $I_1^{\pm}$ and $I_2^{\pm}$ the integrals over $\pm s \geq 0$ of I_1 and I_2, respectively, then we have

$$I_1^+ = \frac{1}{2}\chi(t) \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} d\alpha \int_{r-t}^{\alpha} (1+|\beta|)^{-1-\mu} K_1 d\beta, \tag{4.11}$$

$$I_1^- = \frac{1}{2} \int_{t-r}^{t+r} (1+|\alpha|)^{-1-\mu} d\alpha \int_{\alpha \vee |r-t|}^{\infty} (1+\beta)^{-1-\nu} K_1 d\beta, \tag{4.12}$$

and

$$I_2^+ = \frac{1}{2} \int_{0}^{[t-r]_+} (1+\alpha)^{-1-\nu} d\alpha \int_{-\alpha}^{\alpha} (1+|\beta|)^{-1-\mu} K_2 d\beta, \tag{4.13}$$

$$I_2^- = \frac{1}{2} \int_{-\infty}^{t-r} (1+|\alpha|)^{-1-\mu} d\alpha \int_{|\alpha|}^{\infty} (1+\beta)^{-1-\nu} K_2 d\beta, \tag{4.14}$$

where $\chi(t) = 1$ for $t > 0$ and $\chi(t) = 0$ for $t \leq 0$. In addition, we further divide I_2^- into J_1 and J_2 which are defined by

$$J_1 = \frac{1}{2} \int_{0}^{[t-r]_+} (1+\beta)^{-1-\nu} d\beta \int_{-\beta}^{\beta} (1+|\alpha|)^{-1-\mu} K_2 d\alpha, \tag{4.15}$$

$$J_2 = \frac{1}{2} \int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu} d\beta \int_{-\beta}^{t-r} (1+|\alpha|)^{-1-\mu} K_2 d\alpha. \tag{4.16}$$

Then we have

$$I_1(r,t) = I_1^+ + I_1^-, \quad I_2(r,t) = I_2^+ + J_1 + J_2. \tag{4.17}$$

Next we derive several estimates of K_1 and K_2 in the following lemma.

Lemma 4. 1 *Let $t-r<\alpha<t+r$ and $\beta>r-t$ with* (4.10). *Then it holds that*

$$K_1(\lambda,s,r,t) \le \frac{\sqrt{\alpha}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \ge \beta, \tag{4.18}$$

$$K_1(\lambda,s,r,t) \le \frac{\sqrt{\beta}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \le \beta \tag{4.19}$$

and

$$K_1(\lambda,s,r,t) \le \frac{1}{\sqrt{\alpha+r-t}}. \tag{4.20}$$

Moreover, we have for any $\theta>0$

$$K_1(\lambda,s,r,t) \le \frac{C\sqrt{\beta}}{\sqrt{t+r-\alpha}\sqrt{\beta+t-r}}\frac{\beta^\theta}{(\alpha+r-t)^\theta} \quad \text{for } \alpha \le \beta. \tag{4.21}$$

Let $-\beta<\alpha<t-r$. Then it holds that

$$K_2(\lambda,s,r,t) \le \frac{\sqrt{\alpha}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \text{for } \alpha \ge \beta, \tag{4.22}$$

$$K_2(\lambda,s,r,t) \le \frac{\sqrt{\beta}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \text{for } \alpha \le \beta \tag{4.23}$$

and

$$K_2(\lambda,s,r,t) \le \frac{1}{\sqrt{t-r-\alpha}}. \tag{4.24}$$

Moreover, we have for any $\theta>0$

$$K_2(\lambda,s,r,t) \le \frac{C}{\sqrt{r}}\frac{\beta^\theta}{(t-r-\alpha)^\theta} \quad \text{for } \beta > t-r. \tag{4.25}$$

Proof: First we consider K_1. It is easy to see that $t-r<\alpha<t+r$ and $\beta>r-t$ imply $|\lambda-r|<t-s<\lambda+r$. It follows from (4.8), (3.2) and (3.3) that

$$K_1(\lambda,s,r,t) = \frac{1}{\pi}\sqrt{\lambda}J((\lambda-r)^2,(t-s)^2,(\lambda+r)^2). \tag{4.26}$$

By (3.4) and (4.10), we have

$$\begin{aligned} K_1(\lambda,s,r,t) &\le \frac{\sqrt{\lambda}}{\sqrt{(\lambda+r)^2-(t-s)^2}} \\ &= \frac{\sqrt{\lambda}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}}. \end{aligned} \tag{4.27}$$

Noting that

$$\lambda \le \alpha \quad \text{for } \alpha \ge \beta, \quad \lambda \le \beta \quad \text{for } \alpha \le \beta, \tag{4.28}$$

we get (4.18) and (4.19). Moreover, since $\lambda < \beta + t$ for $s < t$, we get (4.20) from (4.27). Furthermore, it follows from (4.26), (3.5) and (4.10) that

$$\begin{aligned} K_1(\lambda, s, r, t) &\leq C\left(\frac{(\lambda+r)^2-(\lambda-r)^2}{(\lambda+r)^2-(t-s)^2}\right)^{\theta} \frac{\sqrt{\lambda}}{\sqrt{(t-s)^2-(\lambda-r)^2}} \\ &= C\left(\frac{4\lambda r}{(\beta+r+t)(\alpha+r-t)}\right)^{\theta} \frac{\sqrt{\lambda}}{\sqrt{t+r-\alpha}\sqrt{\beta+t-r}}, \end{aligned} \tag{4.29}$$

which implies (4.21), by (4.28) and $\beta + r + t > r$ for $s < t$.

Next we consider K_2. It is easy to see that $-\beta < \alpha < t-r$ implies $|\lambda - r| < \lambda + r < t - s$. It follows from (4.9), (3.2) and (3.3) that

$$K_2(\lambda, s, r, t) = \frac{1}{\pi}\sqrt{\lambda} J((\lambda-r)^2, (\lambda+r)^2, (t-s)^2). \tag{4.30}$$

By (3.4) and (4.10), we have

$$\begin{aligned} K_2(\lambda, s, r, t) &\leq \frac{\sqrt{\lambda}}{\sqrt{(t-s)^2-(\lambda+r)^2}} \\ &= \frac{\sqrt{\lambda}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}}. \end{aligned} \tag{4.31}$$

Noting (4.28), we get (4.22) and (4.23). Moreover, since $\lambda \leq \beta + t$ for $s < t$, we get (4.24) from (4.31). Furthermore, it follows from (4.30), (3.5) and (4.10) that

$$\begin{aligned} K_2(\lambda, s, r, t) &\leq C\left(\frac{(t-s)^2-(\lambda-r)^2}{(t-s)^2-(\lambda+r)^2}\right)^{\theta} \frac{\sqrt{\lambda}}{\sqrt{(\lambda+r)^2-(\lambda-r)^2}} \\ &= C\left(\frac{(t-r+\beta)(t+r-\alpha)}{(t-r-\alpha)(t+r+\beta)}\right)^{\theta} \frac{1}{\sqrt{4r}} \end{aligned} \tag{4.32}$$

which yields (4.25), because $t - r < \beta$ and $-\alpha < \beta$. This completes the proof. □

Now we shall prove for $\mu > 0$, $\nu > 0$

$$I(r,t) \leq C(1+r+|t|)^{-\frac{1}{2}}\Phi_2(r,t;\nu) \quad \text{for } r > 0,\ t \in \mathbf{R}, \tag{4.33}$$

by establishing the estimates in Propositions 4. 1 and 4. 2 below. Once we obtain those estimates, it is easy to see that (4.33) follows from them via (4.5), (4.17) and (1.6).

Proposition 4. 1 *Let $t - r > 0$ and let $\mu > 0$ and $\nu > 0$. Then we have*

$$I_2^{+} \leq C(1+r+t)^{-\frac{1}{2}}(1+t-r)^{-\frac{1}{2}}(1+t-r)^{[\frac{1}{2}-\nu]_+}, \tag{4.34}$$

$$J_1 \leq C(1+r+t)^{-\frac{1}{2}}(1+t-r)^{-\frac{1}{2}}(1+t-r)^{[\frac{1}{2}-\nu]_+}. \tag{4.35}$$

Proof: First we consider I_2^+. It follows from (4.13) and (4.22) that for $t-r>0$

$$I_2^+ \le \frac{1}{2}\int_0^{t-r}(1+|\alpha|)^{-\frac{1}{2}-\nu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha\int_{-\alpha}^{\infty}(1+|\beta|)^{-1-\mu}(t+r+\beta)^{-\frac{1}{2}}d\beta. \tag{4.36}$$

Using (3.8) as $a=t+r$, $b=-\alpha$, $\kappa=1+\mu$ and $\gamma=1/2$, we have

$$(1+r+t)^{\frac{1}{2}}I_2^+ \;\le\; C\int_{-\infty}^{t-r}(1+|\alpha|)^{-\frac{1}{2}-\nu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha.$$

Using (3.7) as $a=b=t-r$, $\kappa=(1/2)+\nu$ and $\gamma=1/2$, we get (4.34).

Next we consider J_1. It follows from (4.15) and (4.23) that for $t-r>0$

$$\begin{aligned} J_1 \;&\le\; \frac{1}{2}\int_0^{t-r}(1+|\beta|)^{-\frac{1}{2}-\nu}(t+r+\beta)^{-\frac{1}{2}}d\beta\int_{-\beta}^{\beta}(1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha \\ &\le\; C\int_0^{t-r}(1+|\beta|)^{-\frac{1}{2}-\nu}(t+r+\beta)^{-\frac{1}{2}}(t-r-\beta)^{-\frac{1}{2}}d\beta, \end{aligned} \tag{4.37}$$

because $\mu>0$. When $0\le r+t\le 1$, it is enough to show that J_1 is bounded. It follows from (4.37) that

$$J_1 \;\le\; C\int_0^{t-r}\beta^{-\frac{1}{2}}(t-r-\beta)^{-\frac{1}{2}}d\beta,$$

hence J_1 is bounded. When $r+t\ge 1$, it follows from (4.37) that

$$J_1 \;\le\; C(1+t+r)^{-\frac{1}{2}}\int_{-\infty}^{t-r}(1+|\beta|)^{-\frac{1}{2}-\nu}(t-r-\beta)^{-\frac{1}{2}}d\beta.$$

Using (3.7) as $a=b=t-r$, $\kappa=(1/2)+\nu$ and $\gamma=1/2$, we get (4.35). The proof is complete. $\square$

Proposition 4. 2 *Let $\mu>0$ and $\nu>0$. Then we have*

$$I_1^+ \;\le\; C(1+r+|t|)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}, \tag{4.38}$$
$$I_1^- \;\le\; C(1+r+|t|)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}, \tag{4.39}$$
$$J_2 \;\le\; C(1+r+|t|)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}. \tag{4.40}$$

Proof: First we show (4.38). It follows from (4.11) and (4.18) that for $t>0$

$$I_1^+ \le \frac{1}{2}\int_{|t-r|}^{\infty}(1+|\alpha|)^{-\frac{1}{2}-\nu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha\int_{r-t}^{\infty}(1+|\beta|)^{-1-\mu}(t+r+\beta)^{-\frac{1}{2}}d\beta. \tag{4.41}$$

To deal with the α-integral, we use (3.6) as $a=b=r-t$, $\kappa=(1/2)+\nu$ and $\gamma=1/2$. While, to handle the β-integral, we employ (3.8) as $a=t+r$, $b=r-t$, $\kappa=1+\mu$ and $\gamma=1/2$. Then we obtain (4.38), because $t>0$.

Next we shall show (4.39) and (4.40), by proving the following Lemmas 4. 2 and 4. 3 . Indeed, for such (r,t) that $r+t\ge 4$ and $0\le t\le 3r$, the desired estimate follows from Lemma 4. 3 . While for the other case, Lemma 4. 2 yields them, because $1+|r-t|$ is equivalent to $1+r+|t|$ for the case.

Lemma 4. 2 *Let $\mu > 0$ and $\nu > 0$. Then we have*

$$I_1^- \le C(1+|r-t|)^{-\frac{1}{2}-\nu}, \quad J_2 \le C(1+|r-t|)^{-\frac{1}{2}-\nu}. \tag{4.42}$$

Proof: First we consider I_1^-. It follows from (4.12) and (4.20) that

$$\begin{aligned} I_1^- &\le \frac{1}{2}\int_{t-r}^{\infty}(1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha \int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}d\beta \\ &\le \frac{1}{2\nu}(1+|r-t|)^{-\nu}\int_{t-r}^{\infty}(1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\frac{1}{2}}d\alpha. \end{aligned}$$

Using (3.8) as $a = r-t$, $b = t-r$, $\kappa = 1+\mu$ and $\gamma = 1/2$, we get (4.42) for I_1^-.
Next we consider J_2. It follows from (4.15) and (4.24) that

$$\begin{aligned} J_2 &\le \frac{1}{2}\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}d\beta \int_{-\infty}^{t-r}(1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha \\ &\le \frac{1}{2\nu}(1+|r-t|)^{-\nu}\int_{-\infty}^{t-r}(1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\frac{1}{2}}d\alpha. \end{aligned}$$

Using (3.7) as $a = b = t-r$, $\kappa = 1+\mu$ and $\gamma = 1/2$, we get (4.42) for J_2. The proof is complete. □

Lemma 4. 3 *Let $\mu > 0$ and $\nu > 0$. If $r+t \ge 4$ and $0 \le t \le 3r$, then we have*

$$I_1^- \le C(1+r+t)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}, \quad J_2 \le C(1+r+t)^{-\frac{1}{2}}(1+|r-t|)^{-\nu}. \tag{4.43}$$

Proof: First we consider J_2. We choose θ such that $0 < \theta < \min\{\nu, 1/2\}$. Then it follows from (4.15) and (4.25) that

$$\begin{aligned} J_2 &\le \frac{C}{\sqrt{r}}\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu+\theta}d\beta \int_{-\infty}^{t-r}(1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\theta}d\alpha \\ &\le C(1+r+t)^{-\frac{1}{2}}(1+|r-t|)^{-\nu+\theta}\int_{-\infty}^{t-r}(1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\theta}d\alpha, \end{aligned}$$

because $4r \ge r+t \ge 4$. Using (3.7) as $a = b = t-r$, $\kappa = 1+\mu$ and $\gamma = \theta$, we get (4.43) for J_2.

Next we consider I_1^-. It follows from (4.12) and (4.21) that

$$\begin{aligned} I_1^- &\le C\int_{t-r}^{t+r}(1+|\alpha|)^{-1-\mu}(t+r-\alpha)^{-\frac{1}{2}}(\alpha+r-t)^{-\theta}d\alpha \, \times \\ &\qquad \times \int_{|r-t|}^{\infty}(1+\beta)^{-\frac{1}{2}-\nu+\theta}(\beta+t-r)^{-\frac{1}{2}}d\beta \\ &\le C(1+|r-t|)^{-\nu+\theta}\int_{t-r}^{t+r}(1+|\alpha|)^{-1-\mu}(t+r-\alpha)^{-\frac{1}{2}}(\alpha+r-t)^{-\theta}d\alpha, \end{aligned}$$

where we have taken θ to be $0 < \theta < \min\{\nu, 1/2\}$ and we have used (3.6) as $a = b = t-r$, $\kappa = (1/2)+\nu-\theta$ and $\gamma = 1/2$.

Note that $(t+r)/2 \geq t-r$ for $t \leq 3r$. Since $t+r-\alpha \geq (t+r)/2$ for $t-r \leq \alpha \leq (t+r)/2$ and $|\alpha| \geq (t+r)/2$ for $(t+r)/2 \leq \alpha \leq t+r$, we have

$$
\begin{aligned}
(4.44)\quad (1+|r-t|)^{\nu-\theta} I_1^- \;&\leq\; C(t+r)^{-\frac{1}{2}} \int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\theta} d\alpha + \\
&\quad + (1+r+t)^{-1-\mu} \int_{t-r}^{t+r} (t+r-\alpha)^{-\frac{1}{2}} (\alpha+r-t)^{-\theta} d\alpha.
\end{aligned}
$$

Using (3.8) as $a = r-t$, $b = t-r$, $\kappa = 1+\mu$ and $\gamma = \theta$, we get

$$
(4.45)\qquad \int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\theta} d\alpha \leq C(1+|r-t|)^{-\theta}.
$$

Moreover, since $\theta < 1/2$, we have

$$
\begin{aligned}
\int_{t-r}^{t+r} (t+r-\alpha)^{-\frac{1}{2}} (\alpha+r-t)^{-\theta} d\alpha \;&\leq\; (2r)^{\frac{1}{2}-\theta} \int_{t-r}^{t+r} (t+r-\alpha)^{-\frac{1}{2}} (\alpha+r-t)^{-\frac{1}{2}} d\alpha \\
&\leq\; C(1+r+t)^{\frac{1}{2}-\theta}.
\end{aligned}
$$

Therefore, we see that the right hand side of (4.44) is dominated by $C(1+r+t)^{-\frac{1}{2}}(1+|r-t|)^{-\theta}$, because $r+t \geq 4$. Hence, we get (4.43) for I_1^-. The proof is complete. □

Case 2: n = 3. If we set

$$
(4.46)\qquad I(r,t) = \frac{1}{2r}\int_{-\infty}^{t} ds \int_{|t-s-r|}^{t-s+r} \frac{1}{z_{\mu,\nu}(\lambda,s)} d\lambda,
$$

it follows from (1.3), (4.1) and (4.2) with $n=3$ and $\rho = t-s$ that

$$
(4.47)\qquad |L(F)(x,t)| \leq M_{\mu,\nu}(F)(r,t) \times I(r,t).
$$

Moreover, if we denote by $I^{\pm}$ the integrals over $\pm s \geq 0$ of I, respectively, then we have

$$
(4.48)\qquad I^+(r,t) = \frac{1}{4r}\chi(t) \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} d\alpha \int_0^{\alpha} (1+|\beta|)^{-1-\mu} d\beta,
$$

$$
(4.49)\qquad I^-(r,t) = \frac{1}{4r} \int_{t-r}^{t+r} (1+|\alpha|)^{-1-\mu} d\alpha \int_{\alpha \vee |r-t|}^{\infty} (1+\beta)^{-1-\nu} d\beta,
$$

where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t \leq 0$. Since

$$
(4.50)\qquad I(r,t) = I^+(r,t) + I^-(r,t),
$$

it suffices to show

$$
(4.51)\qquad I^{\pm}(r,t) \leq C(1+r+|t|)^{-1}(1+|r-t|)^{-\nu}.
$$

To this end, we prepare the following:

Lemma 4.4 *Let $\kappa > 1$. For $r > 0$ and $t > 0$, we have*

$$
(4.52)\qquad \frac{1}{r}\int_{|t-r|}^{t+r} (1+\sigma)^{-\kappa} d\sigma \leq C(1+r+t)^{-1}(1+|r-t|)^{-\kappa+1}.
$$

Moreover, for $r > 0$ and $t \in \mathbf{R}$, we have

$$
(4.53)\qquad \frac{1}{r}\int_{t-r}^{t+r} (1+|\sigma|)^{-\kappa} d\sigma \leq C(1+r+|t|)^{-1}.
$$

Proof: We shall show only (4.53), because (4.52) was handled in a similar fashion. When $r + |t| \geq 1$ and $|t| \leq 2r$, (4.53) follows from the fact that $r^{-1} \leq C(1 + r + |t|)^{-1}$, because $\kappa > 1$. In the other case, we have $(1 + |\sigma|)^{-\kappa} \leq C(1 + r + |t|)^{-\kappa}$ for $t - r \leq \sigma \leq t + r$, hence we get (4.53). This completes the proof. □

Now we estimate $I^{\pm}$. Since $\mu > 0$, we have

$$I^+(r,t) \leq \frac{1}{4\mu r}\chi(t) \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} d\alpha, \tag{4.54}$$

which yields (4.51) for I^+ by (4.52) with $\kappa = 1 + \nu$. Moreover, since $\nu > 0$, we have

$$(1 + |r - t|)^{\nu} I^-(r,t) \leq \frac{C}{r} \int_{t-r}^{t+r} (1 + |\alpha|)^{-1-\mu} d\alpha, \tag{4.55}$$

from which we get (4.51) for I^- by (4.53). This completes the proof of Theorem 1. 1 . □

5 An Application

As we have mentioned in Section 1, we shall consider a system of semilinear wave equations as an application of Theorem 1. 1 :

$$\partial_t^2 u - \Delta u = F(v) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{5.1}$$

$$\partial_t^2 v - \Delta v = G(u) \quad \text{in } \mathbf{R}^n \times \mathbf{R}. \tag{5.2}$$

Suppose that $F \in C^2(\mathbf{R})$ and $G \in C^2(\mathbf{R})$ satisfy

$$F(0) = F'(0) = F''(0) = 0, \quad G(0) = G'(0) = G''(0) = 0, \tag{5.3}$$

and that there are $p > 2$, $q > 2$ and $A > 0$ such that

$$|F''(v_1) - F''(v_2)| \leq \begin{cases} Ap(p-1)|v_1 - v_2|^{p-2} & \text{if} \quad 2 < p < 3, \\ Ap(p-1)|v_1 - v_2|(|v_1| + |v_2|)^{p-3} & \text{if} \quad p \geq 3, \end{cases} \tag{5.4}$$

$$|G''(u_1) - G''(u_2)| \leq \begin{cases} Aq(q-1)|u_1 - u_2|^{q-2} & \text{if} \quad 2 < q < 3, \\ Aq(q-1)|u_1 - u_2|(|u_1| + |u_2|)^{q-3} & \text{if} \quad q \geq 3. \end{cases} \tag{5.5}$$

Remark: Typical examples of F and G are

$$F(v) = |v|^{p-1}v \quad \text{or} \quad F(v) = |v|^p, \tag{5.6}$$

$$G(u) = |u|^{q-1}u \quad \text{or} \quad G(u) = |u|^q. \tag{5.7}$$

Before stating the main result, we prepare the following lemma:

Lemma 5. 1 *Let* (1.10) *hold. Suppose* $\alpha = pq^* - 1 > 0$.
(i) *If* ν *satisfies*

$$0 < \nu \le p^*, \tag{5.8}$$

$$\nu > p^* - \frac{\Gamma}{pq-1} = \frac{1 - p^* + p(1-q^*)}{pq-1}, \tag{5.9}$$

then there is a number κ *verifying*

$$0 < \kappa \le q^*, \tag{5.10}$$

$$\kappa > \frac{1}{p} - \frac{p^* - \nu}{p} \tag{5.11}$$

and

$$\kappa < q^* - 1 + q\nu = \frac{1}{p} + q(\nu - p^* + \frac{\Gamma}{pq}). \tag{5.12}$$

(ii) *Let* ν *satisfy* (5.8) *and* (5.9). *Furthermore, if* $\Gamma > (pq-1)/2$, *we assume*

$$\nu > p^* - \frac{\Gamma}{pq} - \frac{1}{2pq} = \frac{1}{q}(1 + \frac{1}{2p} - q^*). \tag{5.13}$$

Then there is a number κ *verifying* (5.10), (5.11), (5.12) *and*

$$\kappa > \frac{1}{2p}. \tag{5.14}$$

(iii) *If* ν *satisfies* (5.8) *and*

$$\nu > p^* - \frac{\Gamma}{pq} = \frac{1}{q} - \frac{\alpha}{pq} = \frac{1}{q}(1 + \frac{1}{p} - q^*), \tag{5.15}$$

then there is a number κ *verifying* (5.10), (5.12) *and*

$$\kappa > \frac{1}{p}. \tag{5.16}$$

Remark: Note that the conditions (5.9), (5.13) and (5.15) are meaningful only when the right hand sides are positive, by (5.8). Moreover, it is easy to see that (5.15) implies (5.9) and (5.13).
Proof: Firstly, we prove the statement (iii). It suffices to check

$$\frac{1}{p} < q^*, \quad \frac{1}{p} < \frac{1}{p} + q(\nu - p^* + \frac{\Gamma}{pq}). \tag{5.17}$$

By $\alpha > 0$ and (5.15), it is easy to see that (5.17) holds.

Seconderly, we prove the satement (i), by dividing the argument into two cases.
Case 1: $1/q < \nu \le p^*$.
In this case, we have

$$q^* < q^* - 1 + q\nu.$$

Since $q^* > 0$, it is enough to show

$$\frac{1}{p} - \frac{p^* - \nu}{p} < q^*. \tag{5.18}$$

Since

$$q^* - (\frac{1}{p} - \frac{p^* - \nu}{p}) = \frac{1}{p}(\alpha + p^* - \nu),$$

we get (5.18), by $\alpha > 0$ and (5.8).

Case 2: $0 < \nu \leq 1/q$.

In this case, we have

$$q^* - 1 + q\nu \leq q^*. \tag{5.19}$$

Moreover, it holds that

$$\frac{1}{p} - \frac{p^* - \nu}{p} < \frac{1}{p} + q(\nu - p^* + \frac{\Gamma}{pq}). \tag{5.20}$$

Indeed, this is equivalent to

$$p^* - \nu < \frac{\Gamma}{pq - 1},$$

which follows from (5.9). Namely, we obtain by (5.12)

$$\frac{1}{p} - \frac{p^* - \nu}{p} < q^* - 1 + q\nu. \tag{5.21}$$

Note that

$$q^* - 1 + q\nu > 0 \quad \text{if } q^* > 1, \tag{5.22}$$

and that

$$\frac{1}{p} - \frac{p^* - \nu}{p} > 0 \quad \text{if } q^* \leq 1, \tag{5.23}$$

because $p^* \leq q^*$ and $\nu > 0$. Therefore, the statement of (i) follows from (5.19), (5.21), (5.22) and (5.23).

Finally, we prove the satement (ii). Since we have assumed (5.8) and (5.9), we see from the statement of (i) that there is a number κ verifying (5.10), (5.11) and (5.12). When either $\Gamma > (pq-1)/2$ or $\Gamma \leq (pq-1)/2$ and $\nu \geq p^* - 1/2$, we have (5.14) by (5.11). Next we consider the case where $\Gamma \leq (pq-1)/2$ and $\nu < p^* - 1/2$. Then we have

$$\frac{1}{p} - \frac{p^* - \nu}{p} < \frac{1}{2p},$$

hence it suffices to show

$$\frac{1}{2p} < q^*, \quad \frac{1}{2p} < \frac{1}{p} + q(\nu - p^* + \frac{\Gamma}{pq}). \tag{5.24}$$

Since $\alpha > 0$, we have $q^* > 1/p$. Moreover, the latter inequality follows from (5.13). This completes the proof of Lemma 5. 1 . $\square$

Theorem 5.1 *Let $n=2$ or $n=3$. Suppose that* (1.9), (1.10), (1.17), (1.18) *and* (5.8) *hold.*

(A) *Let ν and κ satisfy* (5.9) *through* (5.12). *Then there exists a positive number $\varepsilon_0 = \varepsilon_0(\nu,\kappa,p,q,A,n,f_j,g_j)$ such that for any ε with $0<\varepsilon\le\varepsilon_0$, there exists a classical solution (u,v) of* (5.1) *and* (5.2) *uniquely in $X_{\nu,\kappa}(2C_0\varepsilon)$ verifying*

$$|\partial_x^\alpha(u(x,t)-u^-(x,t))| \le C[v]_\kappa^p(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,t;\nu), \tag{5.25}$$

$$|\partial_x^\alpha(v(x,t)-v^-(x,t))| \le C[u]_\nu^q(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,t;\kappa) \tag{5.26}$$

for any $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ and $|\alpha|\le 2$, and

$$|||(u-u^-)(t)|||_e \le C\|v\|_\kappa^p(1+|t|)^{-p^*}\left\{(1+|t|)^{[1-2p\kappa]_+}\right\}^{\frac12} \quad \textit{for } t\le 0, \tag{5.27}$$

$$|||(v-v^-)(t)|||_e \le C\|u\|_\nu^q(1+|t|)^{-q^*}\left\{(1+|t|)^{[1-2q\nu]_+}\right\}^{\frac12} \quad \textit{for } t\le 0, \tag{5.28}$$

where u^- and v^- are the solutions to the homogeneous wave equation satisfying (1.15) *and* (1.16). *In addition, $[\,\cdot\,]_\nu$ and $|||\cdot|||_e$ are defined by* (2.4) *and* (2.5), *respectively.*

(B) *Let ν and κ satisfy* (5.15), (5.10), (5.12) *and* (5.16). *Then we have*

$$|\partial_x^\alpha(u(x,t)-u^-(x,t))| \le C[v]_\kappa^p(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,t;p^*) \tag{5.29}$$

for any $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ and $|\alpha|\le 2$, and

$$|||(u-u^-)(t)|||_e \le C\|v\|_\kappa^p(1+|t|)^{-p^*} \quad \textit{for } t\le 0. \tag{5.30}$$

(C) *Suppose*

$$qp^*>1,\ \textit{i.e.},\ \beta>0. \tag{5.31}$$

Let ν and κ satisfy

$$1/q<\nu\le p^*,\quad 1/p<\kappa\le q^*. \tag{5.32}$$

Then we have (5.29), (5.30),

$$|\partial_x^\alpha(v(x,t)-v^-(x,t))| \le C[u]_\nu^q(1+r+|t|)^{-\frac{n-1}{2}}\Phi_n(r,t;q^*) \tag{5.33}$$

for any $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ and $|\alpha|\le 2$, and

$$|||(v-v^-)(t)|||_e \le C\|u\|_\nu^q(1+|t|)^{-q^*} \quad \textit{for } t\le 0. \tag{5.34}$$

Remark: The existence of such ν and κ as in the part (A) and (B) in Theorem 5. 1 is guaranteed by Lemma 5. 1 . In particular, if we take ν as $\nu=p^*$, we find κ satisfying (5.10) and

$$\frac{1}{p}<\kappa<q^*-1+qp^*=\frac{1}{p}+\frac{\Gamma}{p}, \tag{5.35}$$

by Lemma 5. 1 . Moreover, we have (5.29) and (5.30).

If (5.31) holds, we can find ν such that $1/q<\nu\le p^*$. Moreover, since $\nu>1/q$ imply (5.15), there is a number κ satisfying $1/p<\kappa\le q^*$. Furthermore, Theorem 1. 2 follows from the part (A) and (C) in Theorem 5. 1 .

In both (5.27) and (5.28), the right hand sides decay as $t\to-\infty$. More precisely, we have the following.

Corollary 5. 1 *Let $n = 2$ or $n = 3$. Suppose that* (1.9), (1.10), (1.17) *and* (1.18) *hold. Let ν and κ satisfy* (5.8) *through* (5.12).

If $2p\kappa > 1$, (5.30) *holds. While, if $2p\kappa < 1$, we have*

(5.36) $$|||(u - u^-)(t)|||_e \leq C\|v\|_\kappa^p (1 + |t|)^{-\frac{1}{2}-\nu} \quad \text{for } t \leq 0.$$

Moreover, if $2q\nu > 1$, (5.34) *holds. While, if $2q\nu < 1$, we have*

(5.37) $$|||(v - v^-)(t)|||_e \leq C\|u\|_\nu^q (1 + |t|)^{-\frac{1}{2}-\kappa} \quad \text{for } t \leq 0.$$

Remark: We can find κ satisfying $2p\kappa > 1$ and (5.10) through (5.12), when ν satisfy (5.8), (5.9), and (5.13) if $\Gamma > (pq - 1)/2$, by Lemma 5. 1 .

If we assume

(5.38) $$2qp^* > 1, \ \textit{i.e.}, \ 2\beta + 1 > 0,$$

we can choose ν verifying $2q\nu > 1$, (5.8), (5.9) and (5.13). On the other hand, if (5.38) does not hold, we always have $2q\nu < 1$.

When $2p\kappa < 1$, we must have $p^* > 1/2 + \nu > 1/2$, by (5.11). When $2q\nu < 1$, we must have $q^* > 1/2 + \kappa > 1/2$, by (5.12).

Proof: When $2p\kappa > 1$, (5.30) immediately follows from (5.27). Moreover, when $2p\kappa < 1$, we have $p^* - ([1 - 2p\kappa]_+)/2 > 1/2 + \nu$ by (5.11). Therefore, (5.36) follows from (5.27).

Furthermore, when $2q\nu > 1$, (5.34) immediately follows from (5.28). On the other hand, $2q\nu < 1$ and (5.12) yield $q^* - ([1-2q\nu]_+)/2 > 1/2+\kappa$, hence (5.37) follows from (5.28). This completes the proof. □

The following corollary follows from Theorem 5. 1 and Corollary 5. 1 , by setting $p = q$.

Corollary 5. 2 *Let $n = 2$ or $n = 3$ and let $p = q$. Suppose that $pp^* > 1$,* (5.8), (1.17) *and* (1.18) *hold.*

(A) *If ν and κ satisfy* (5.10),

(5.39) $$\nu > p^* - \frac{pp^* - 1}{p - 1} = \frac{1 - p^*}{p - 1},$$

and

(5.40) $$\frac{1}{p} - \frac{p^* - \nu}{p} < \kappa < p^* - 1 + p\nu,$$

then (5.25) *through* (5.28) *with $p = q$ hold.*

(B) *If ν and κ satisfy* (5.10),

(5.41) $$\nu > p^* - \frac{(p+1)(pp^* - 1)}{p^2} = \frac{1}{p} - \frac{pp^* - 1}{p^2} = \frac{1}{p}(1 + \frac{1}{p} - p^*),$$

and

(5.42) $$\frac{1}{p} < \kappa < p^* - 1 + p\nu,$$

then (5.29) *and* (5.30) *hold.*

(C) *Let ν and κ satisfy* (5.32). *Then* (5.29), (5.30), (5.33) *and* (5.34) *with $p = q$ hold.*

(D) *Let ν and κ satisfy* (5.39), (5.10) *and* (5.40). *If $2p\nu > 1$, we have* (5.30) *and* (5.36). *While, if $2p\nu < 1$, we have* (5.34) *and* (5.37) *with $p = q$.*

Remark: If we take $\kappa = \nu$, (5.40) is equivalent to

$$\nu > \frac{1-p^*}{p-1} = p^* - \frac{pp^*-1}{p-1}. \tag{5.43}$$

Here we would like to compare Corollary 5. 2 with the previous results in [17], [27] and [33], which concerns with the asymptotic behavior of the solution to

$$\partial_t^2 u - \Delta u = F(u) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{5.44}$$

such that

$$|||u(t) - u_0(t)|||_e \to 0 \quad \text{as } t \to -\infty, \tag{5.45}$$

where u_0 is the solutions to the homogeneous wave equation satisfying

$$u_0(x,0) = f(x), \quad \partial_t u_0(x,0) = g(x) \quad \text{in } \mathbf{R}^n. \tag{5.46}$$

Since $u = v$ and $u^- = v^-$ if we choose $f_1 = f_2 = f$ and $g_1 = g_2 = g$, Corollary 5. 2 is a natural extension of those previous work in the following sense: In [24], the parameters ν and κ are taken as $\nu = \kappa = p^*$. In [33], ν and κ are taken in the same way and $\Phi_2(r, |t|; p^*)$ is replaced by

$$(1 + |r - |t||)^{-p^*} \log(2 + |r - |t||) \quad \text{if } p^* \le 1/2.$$

In [17], it is assumed that $p\nu \neq 1$ and $\nu = \kappa$.

At the end of this section, we state asymptotic behavior of the solution (u, v) obtained by Theorem 5. 1 as $t \to +\infty$.

Theorem 5. 2 *Let $n = 2$ or $n = 3$. Suppose that* (1.9), (1.10), (1.17), (1.18) *and* (5.8) *hold. Let u^- and v^- be the solutions to the homogeneous wave equation satisfying* (1.15) *and* (1.16). *For a solution $(u, v) \in X_{\nu,\kappa}$ of* (5.1) *and* (5.2) *verifying* (5.25) *through* (5.28), *we define*

$$\begin{aligned} u^+(x,t) &= u(x,t) - L_1(F(v))(x,t) \quad \textit{in } \mathbf{R}^n \times \mathbf{R}, &(5.47)\\ v^+(x,t) &= v(x,t) - L_1(G(u))(x,t) \quad \textit{in } \mathbf{R}^n \times \mathbf{R}, &(5.48)\end{aligned}$$

where we have set

$$L_1(F)(x,t) = \frac{1}{\omega_2} \int_t^\infty ds \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2 - \rho^2}} d\rho \int_{|\omega|=1} F(x + \rho\omega, s) dS_\omega \tag{5.49}$$

for $(x,t) \in \mathbf{R}^2 \times \mathbf{R}$ and

$$L_1(F)(x,t) = \frac{1}{\omega_3} \int_t^\infty (s-t) ds \int_{|\omega|=1} F(x + (s-t)\omega, s) dS_\omega \tag{5.50}$$

for $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$.

(A) *Let ν and κ satisfy* (5.9) *through* (5.12). *Then u^+ and v^+ be classical solutions to the homogeneous wave equation satisfying*

$$\begin{aligned} |\partial_x^\alpha(u(x,t) - u^+(x,t))| &\le C[v]_\kappa^p (1 + r + |t|)^{-\frac{n-1}{2}} \Phi_n(r, -t; \nu), &(5.51)\\ |\partial_x^\alpha(v(x,t) - v^+(x,t))| &\le C[u]_\nu^q (1 + r + |t|)^{-\frac{n-1}{2}} \Phi_n(r, -t; \kappa) &(5.52)\end{aligned}$$

for any $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ *and* $|\alpha| \le 2$, *and*

$$(5.53) \qquad |||(u-u^+)(t)|||_e \le C\|v\|_\kappa^p (1+|t|)^{-p^*} \left\{(1+|t|)^{[1-2p\kappa]_+}\right\}^{\frac{1}{2}} \quad \textit{for } t \ge 0,$$

$$(5.54) \qquad |||(v-v^+)(t)|||_e \le C\|u\|_\nu^q (1+|t|)^{-q^*} \left\{(1+|t|)^{[1-2q\nu]_+}\right\}^{\frac{1}{2}} \quad \textit{for } t \ge 0.$$

Furthermore, we have

$$(5.55) \qquad |||u^+(t)|||_e = |||u^-(t)|||_e, \quad |||v^+(t)|||_e = |||v^-(t)|||_e \quad \textit{for} \quad t \in \mathbf{R}.$$

(B) *Let* ν *and* κ *satisfy* (5.15), (5.10), (5.12) *and* (5.16). *Then we have*

$$(5.56) \qquad |\partial_x^\alpha(u(x,t)-u^+(x,t))| \le C[v]_\kappa^p (1+r+|t|)^{-\frac{n-1}{2}} \Phi_n(r,-t;p^*)$$

for any $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ *and* $|\alpha| \le 2$, *and*

$$(5.57) \qquad |||(u-u^+)(t)|||_e \le C\|v\|_\kappa^p (1+|t|)^{-p^*} \quad \textit{for } t \ge 0.$$

(C) *Suppose that* (5.31) *holds and that* ν *and* κ *satisfy* (5.32). *Then we have* (5.29), (5.30),

$$(5.58) \qquad |\partial_x^\alpha(v(x,t)-v^+(x,t))| \le C[u]_\nu^q (1+r+|t|)^{-\frac{n-1}{2}} \Phi_n(r,-t;q^*)$$

for any $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ *and* $|\alpha| \le 2$, *and*

$$(5.59) \qquad |||(v-v^+)(t)|||_e \le C\|u\|_\nu^q (1+|t|)^{-q^*} \quad \textit{for } t \ge 0.$$

6 A Priori Estimate

The aim of this section is to prove the following a priori estimates.

Proposition 6. 1 *Let* ν *and* κ *satisfy* (5.8) *through* (5.12). *Assume* (5.3), (5.4) *and* (5.5) *hold. Let* $(u_j, v_j) \in X_{\nu,\kappa}$ $(j = 1, 2)$. *Then we have*

$$(6.1) \qquad \|L(F(v_1)) - L(F(v_2))\|_\nu \le C_1 \|v_1 - v_2\|_\kappa (\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-1},$$

$$(6.2) \qquad \|L(G(u_1)) - L(G(u_2))\|_\kappa \le C_1 \|u_1 - u_2\|_\nu (\|u_1\|_\nu + \|u_2\|_\nu)^{q-1}.$$

Moreover, for $|\gamma| = 1$, *we have*

$$(6.3) \quad \|\partial_x^\gamma(L(F(v_2)) - L(F(v_1)))\|_\nu \le C_2[v_2 - v_1]_\kappa([v_1]_\kappa + [v_2]_\kappa)(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-2},$$

$$(6.4) \quad \|\partial_x^\gamma(L(G(u_2)) - L(G(u_1)))\|_\kappa \le C_2[u_2 - u_1]_\nu([u_1]_\nu + [u_2]_\nu)(\|u_1\|_\nu + \|u_2\|_\nu)^{q-2},$$

where $[\,\cdot\,]_\nu$ *is defined by* (2.4). *Furthermore, for* $|\gamma| = 2$, *we have*

$$(6.5) \quad \|\partial_x^\gamma(L(F(v_2)) - L(F(v_1)))\|_\nu \le C_3[v_2 - v_1]_\kappa([v_1]_\kappa + [v_2]_\kappa)^2(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-3},$$

if $p \geq 3$,

$$\|\partial_x^\gamma(L(F(v_2)) - L(F(v_1)))\|_\nu \leq C_4[v_2 - v_1]_\kappa([v_1]_\kappa + [v_2]_\kappa)(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-2} + C_5\|v_2 - v_1\|_\kappa^{p-2}([v_1]_\kappa + [v_2]_\kappa)^2, \tag{6.6}$$

if $2 < p < 3$,

$$\|\partial_x^\gamma(L(G(u_2)) - L(G(u_1)))\|_\kappa \leq C_3[u_2 - u_1]_\kappa([u_1]_\kappa + [u_2]_\kappa)^2(\|u_1\|_\kappa + \|u_2\|_\kappa)^{q-3}, \tag{6.7}$$

if $q \geq 3$,

$$\|\partial_x^\gamma(L(G(u_2)) - L(G(u_1)))\|_\kappa \leq C_4[u_2 - u_1]_\kappa([u_1]_\kappa + [u_2]_\kappa)(\|u_1\|_\kappa + \|u_2\|_\kappa)^{q-2} + C_5\|u_2 - u_1\|_\kappa^{q-2}([u_1]_\kappa + [u_2]_\kappa)^2, \tag{6.8}$$

if $2 < q < 3$.

Proof: It follows from (5.3), (5.4) and (5.5) that

$$|F(v_1) - F(v_2)| \leq Ap|v_1 - v_2|(|v_1| + |v_2|)^{p-1}, \tag{6.9}$$
$$|G(u_1) - G(u_2)| \leq Aq|u_1 - u_2|(|u_1| + |u_2|)^{q-1}, \tag{6.10}$$
$$|F'(v_1) - F'(v_2)| \leq Ap(p-1)|v_1 - v_2|(|v_1| + |v_2|)^{p-2}, \tag{6.11}$$
$$|G'(u_1) - G'(u_2)| \leq Aq(q-1)|u_1 - u_2|(|u_1| + |u_2|)^{q-2}. \tag{6.12}$$

We also note that for any θ with $0 \leq \theta \leq 1$, we have

$$\|\, |v_1|^\theta |v_2|^{1-\theta}\|_\nu \leq \|v_1\|_\nu^\theta \|v_2\|_\nu^{1-\theta}. \tag{6.13}$$

To begin with, we prove (6.1) and (6.2). If we could show

$$\|L(|v_1|^p)\|_\nu \leq C_1\|v_1\|_\kappa^p, \tag{6.14}$$
$$\|L(|u_1|^q)\|_\kappa \leq C_1\|u_1\|_\nu^q, \tag{6.15}$$

we would obtain (6.1) and (6.2). Indeed, it is easy to see from (6.14), (6.15) and (6.13) that

$$\|L(|v_1|^{\theta p}|v_2|^{(1-\theta)p})\|_\nu \leq C_1\|v_1\|_\kappa^{\theta p}\|v_2\|_\kappa^{(1-\theta)p}, \tag{6.16}$$
$$\|L(|u_1|^{\theta q}|u_2|^{(1-\theta)q})\|_\kappa \leq C_1\|u_1\|_\nu^{\theta q}\|u_2\|_\kappa^{(1-\theta)q}. \tag{6.17}$$

By (6.9) and (6.10), applying (6.16) and (6.17) with $\theta = 1/p$ and $\theta = 1/q$, respectively, we get (6.1) and (6.2). Therefore our task becomes to show (6.14) and (6.15).

It follows from (2.7) and (1.4) with $F = |v_1|^p$ and $F = |u_1|^q$ that for $\mu > 0$

$$|L(|v_1|^p)(x,t)|(1 + r + |t|)^{\frac{n-1}{2}}/\Phi_n(r,t;\nu) \leq C_1 M_{\nu,\mu}(|v_1|^p)(r,t), \tag{6.18}$$
$$|L(|u_1|^q)(x,t)|(1 + r + |t|)^{\frac{n-1}{2}}/\Phi_n(r,t;\kappa) \leq C_1 M_{\kappa,\mu}(|u_1|^q)(r,t). \tag{6.19}$$

Hence, it is enough to show that there is a positive number μ such that

$$M_{\nu,\mu}(|v_1|^p)(r,t) \leq C\|v_1\|_\kappa^p, \quad M_{\kappa,\mu}(|u_1|^q)(r,t) \leq C\|u_1\|_\nu^q, \tag{6.20}$$

because $\Phi_n(r,t;\nu) \le \Phi_n(r,|t|;\nu)$. It follows that

$$\lambda^{\frac{n-1}{2}}(1+\lambda+|s|)^{1+\nu}(1+|\lambda-|s||)^{1+\mu}|v_1(y,s)|^p \le \|v_1\|_\kappa^p \frac{(1+|\lambda-|s||)^{1+\mu-p\kappa}}{(1+\lambda+|s|)^{p^*-\nu}}$$

and

$$\lambda^{\frac{n-1}{2}}(1+\lambda+|s|)^{1+\kappa}(1+|\lambda-|s||)^{1+\mu}|u_1(y,s)|^q \le \|u_1\|_\nu^q \frac{(1+|\lambda-|s||)^{1+\mu-p\nu}}{(1+\lambda+|s|)^{q^*-\kappa}}.$$

By (5.11) and (5.12), we can choose a positive number μ such that

$$\mu < p\kappa - 1 + p^* - \nu, \quad \mu < p\nu - 1 + q^* - \kappa. \tag{6.21}$$

Moreover, by (5.8) and (5.10) together with (2.7), we obtain (6.20).

Furthermore, (6.3) through (6.8) follows from a standard argument, if we note that $\partial_x L(F(v_1)) = L(\partial_x F(v_1))$. We omit further details. □

7 Proof of Theorems 5.1 and 5.2

Firstly we show the part (A) in Theorem 5. 1 . The classical solution (u,v) of (5.1) and (5.2) verifying (5.25) through (5.28) is furnished by a solution of the following system of integral equations:

$$u(x,t) = u^-(x,t) + L(F(v))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{7.1}$$

$$v(x,t) = v^-(x,t) + L(G(u))(x,t) \quad \text{in } \mathbf{R}^n \times \mathbf{R}, \tag{7.2}$$

where u^- and v^- are the solutions to the homogeneous wave equation satisfying (1.15) and (1.16), and $L(F(v))(x,t)$ and $L(G(u))(x,t)$ are given by (1.2) and (1.3) with F replaced by $F(v)$ and $G(u)$. Since we have (1.19) and Proposition 6. 1 , a method of successive approximation is applicable to show the existence of the unique solution (u,v) of (7.1) and (7.2) in $X_{\nu,\kappa}$ for given (u^-,v^-) satisfying (1.19) with sufficiently small $\varepsilon > 0$. (See also [9] and [24]).

Next we prove that the solution (u,v) of (7.1) and (7.2) satisfies (5.25) and (5.26). When $\gamma = 0$, those asymptotic estimates follow from (6.18) through (6.20). Similarly, we obtain them for $|\gamma| = 1, 2$.

Next we prove that the solution (u,v) of (7.1) and (7.2) satisfies (5.27) and (5.28), by making use of

$$|||(u-u^-)(t)|||_e \le (n+1)\int_{-\infty}^t \|F(v)(s)\|_{L^2} ds, \tag{7.3}$$

$$|||(v-v^-)(t)|||_e \le (n+1)\int_{-\infty}^t \|G(u)(s)\|_{L^2} ds. \tag{7.4}$$

Since $-(n-1)p + (n+1) = -2p^* - 2$, it follows that

$$\|F(v)(s)\|_{L^2}^2 \le \omega_n \|v\|_\kappa^{2p} I(s), \tag{7.5}$$

where we have set

$$I(s)=\int_0^\infty (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}dr. \tag{7.6}$$

We shall show

$$I(s)\le C(1+|s|)^{-2p^*-2}(1+|s|)^{[1-2p\kappa]_+}. \tag{7.7}$$

To this end, we divide $I(s)$ into $I_1(s)$ and $I_2(s)$ which are defined by

$$I_1(s) = \int_0^{2|s|}(1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}dr, \tag{7.8}$$

$$I_2(s) = \int_{2|s|}^\infty(1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}dr. \tag{7.9}$$

It is easy to see that for $\kappa>0$

$$I_2(s)\le C(1+|s|)^{-2p^*-1-2p\kappa}. \tag{7.10}$$

While we have

$$(1+|s|)^{2p^*+2}I_1(s)\le C(1+|s|)^{[1-2p\kappa]_+}. \tag{7.11}$$

These estimates yield (7.7).

Now it follows from (7.3), (7.5) and (7.7) that for $t\ge 0$

$$\begin{aligned} |||(u-u^-)(t)|||_e &\le C\|v\|_\kappa^p\int_{-\infty}^t(1+|s|)^{-p^*-1}(1+|s|)^{([1-2p\kappa]_+)/2}ds \\ &= C\|v\|_\kappa^p\int_{|t|}^\infty(1+s)^{-p^*-1}(1+s)^{([1-2p\kappa]_+)/2}ds. \end{aligned} \tag{7.12}$$

If $2p\kappa>1$, we easily have (5.27). When $2p\kappa=1$, by integration by parts, we get (5.27). Moreover, if we notice that (5.11) implies $p^*-(1/2)+p\kappa>(1/2)+\nu>0$, we obtain (5.27).

Since the proof of (5.28) is similar to that of (5.27) if we use (5.12) instead of (5.11), we omit the details. This completes the proof of the part (A).

Secondly we show the part (B) in Theorem 5. 1 . Since (5.30) follows from (5.27) by (5.16), it is enough to show (5.29), By (6.18) with $\nu=p^*$ and (6.19), it suffices to show that there is a positive number μ such that

$$M_{p^*,\mu}(|v_1|^p)(r,t)\le C\|v_1\|_\kappa^p,\quad M_{\kappa,\mu}(|u_1|^q)(r,t)\le C\|u_1\|_\nu^q. \tag{7.13}$$

Indeed, we can choose a positive number μ such that

$$\mu<p\kappa-1,\quad \mu<p\nu-1+q^*-\kappa, \tag{7.14}$$

by (5.16) and (5.12), hence (7.13) holds.

Finally we show the part (C) in Theorem 5. 1 . It is enough to show (5.33), since the others are easily handled. By (6.18) with $\nu=p^*$ and (6.19) with $\kappa=q^*$, it suffices to show that there is a positive number μ such that

$$M_{p^*,\mu}(|v_1|^p)(r,t)\le C\|v_1\|_\kappa^p,\quad M_{q^*,\mu}(|u_1|^q)(r,t)\le C\|u_1\|_\nu^q. \tag{7.15}$$

In fact, we can choose a positive number μ such that

$$\mu < p\kappa - 1, \quad \mu < p\nu - 1, \tag{7.16}$$

by (5.32). This completes the proof of Theorem 5. 1 . □

Next we prove Theorem 5. 2 . If we note that $L_1(F)(x,t) = L(\hat{F})(x,-t)$ with $\hat{F}(x,t) = F(x,-t)$, we obtain the desired estimates from Theorem 5. 1 , except (5.44). For the proof of (5.44), see e.g. the proof of Proposition 8.4 in [14]. We omit further details. This completes the proof. □

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