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**Crossed Burnside Rings (I) :
The Fundamental Theorem**

Dedicated to the 60th birthday of Professor Yukio TSUSHIMA.

Fumihito ODA and Tomoyuki YOSHIDA

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Crossed Burnside Rings (I): The Fundamental Theorem

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Abstract. Let G be a finite group and S a finite G -monoid. A crossed G -set over S is a finite G -set equipped with a G -map into S called a weight function. A crossed Burnside ring $X\Omega(G, S)$ is the Grothendieck ring of the category of crossed G -sets with respect to disjoint unions and tensor products. In this paper, we prove the fundamental theorem of crossed Burnside rings and an idempotent formula at characteristic 0.

1 Introduction

(1.1) Operations compatible with inductions and restrictions. It is well-known that the center $Z(kG)$ of the group algebra kG acts on the cohomology group $H^n(H, M)$ of a subgroup H of G with coefficient in a kG -module M . In fact, for any $z \in Z(kG)$, the map $\mu_z : M \rightarrow M; m \mapsto zm$ is a kG -homomorphism, and so μ_z induces an endomorphism of $H^n(H, M)$. The most remarkable property of this action is that it is compatible with restriction, conjugation and corestriction on the cohomology groups of subgroups of G . On the other hand, the Burnside ring $\Omega(G)$ also acts on the character ring $R(H)$ of any subgroup H of G so that the action is compatible with restriction, conjugation and induction maps. Unfortunately, there exists no nontrivial action of $Z(kG)$ (resp. $R(G)$) on $R(H)$ (resp. $H^n(H, M)$) having the compatibility property..

A G -functor (or often a Mackey functor) is a family $(\mathbf{a}(K))_{K \leq G}$ equipped with three kind of maps, that is, induction, conjugation, restriction, satisfying some axioms. See [Gr 71], [Dr 75] for the detail. As typical examples, the cohomology groups $(H^n(K, M))_{K \leq G}$ and the character rings $(R(K))_{K \leq G}$ are both G -functors. There exists a universal ring which acts on each component $\mathbf{a}(K)$ of any G -functor $\mathbf{a}$ and is compatible with structure maps. In fact, such a ring is constructed as the center of the representation category of G -functors or as the center of the so-called Mackey algebra.

Here, there exists another ring, called the crossed Burnside ring and denoted by $X\Omega(G, G^c)$, which has actions on $\mathbf{a}(K)$ compatible with structure maps for any G -functor $\mathbf{a}$. Consider the operation $[D, s]$, where $D \leq G$ and $s \in C_G(D)$, on $\mathbf{a}(K)$ defined by

$$[D, s] \cdot \alpha := \sum_{KgD} \text{ind}_K^K \text{res}_{gD \cap {}^g s g^{-1} K} \text{con}^{g s g^{-1}}(\alpha), \quad (1.1.a)$$

where KgD runs over all double coset in $K \backslash G / D$ and ${}^g A$ denotes the conjugation gAg^{-1} for any subset or element A of G . After a little long calculation, we know that this action is compatible with all structure maps of the G -functor $\mathbf{a}$. We, furthermore, know that the composition $[D, s] \cdot [E, t]$ of the operations $[D, s]$ and $[E, t]$ is expressed as follows:

$$[D, s] \cdot [E, t] = \sum_{DgE} [D \cap {}^g E, s \cdot {}^g t]. \quad (1.1.b)$$

Thus these operations $[D, s]$'s generate a ring $X\Omega(G, G^c)$, which is called a crossed Burnside ring. This ring contains $Z(\mathbb{Z}[G])$, the center of the group algebra, and the Burnside ring $\Omega(G)$. Of course, there exists a canonical (but

neither injective nor surjective) ring homomorphism from the crossed Burnside ring to the above universal ring. Comparing the center of the Mackey algebra, the crossed Burnside ring is much easier to treat.

(1.2) Crossed G -sets. There is another description of crossed Burnside rings. Let G be a finite group and S a G -monoid. Then an S -crossed G -set X is defined to be a finite G -set X equipped with a weight function $\|\cdot\| : X \rightarrow S$ which is compatible with the action of G . We can define the tensor product $X \otimes Y$ of two crossed G -sets over S . The notions of crossed G -sets and their tensor products become more and more important in the area of representation theory of finite groups [Wi 96], algebraic topology, quantum group theory [Mag 95], and algebraic topological quantum field theory [Ye 92].

The category $G\text{-xset}/S$ becomes a monoidal category by tensor products. The crossed Burnside ring $X\Omega(G, S)$ of G over S is now define to be the Grothendieck ring of $G\text{-xset}/S$ by disjoint unions and tensor products. The purpose of this serial paper is to study the crossed Burnside ring $X\Omega(G, S)$, in particular, we state the fundamental theorem, the idempotent formulas at characteristic 0 and $p > 0$, together with some applications.

The additive version of crossed G -sets is the concept of crossed $\mathbb{C}[G]$ -modules. Here, a crossed G -module is a G -graded $\mathbb{C}[G]$ -module. (Similarly, a crossed G -set can be viewed as a G -graded G -set.) Furthermore, the category of crossed G -modules is represented by the quantum double $D(G)$ of the group algebra $\mathbb{C}[G]$, and so a crossed G -module is nothing but a $D(G)$ -module. Thus our crossed Burnside ring $X\Omega(G, G^c)$ is a non-additive version of Witherspoon's ring $R(D(G))$.

(1.3) Main results. One of the main results of this paper is the fundamental theorem of crossed Burnside ring. We define the crossed ghost ring of G over S by

$$X\tilde{\Omega}(G, S) := \prod_{(H) \in C(G)} \mathbb{Z}[C_S(H)]^{N_G(H)}$$

where (H) in the first product runs over all conjugacy classes of subgroups of G . We define the Burnside homomorphism by

$$\varphi : X\Omega(G, S) \rightarrow X\tilde{\Omega}(G, S); [X] \mapsto \sum_{x \in X^H} \|x\|.$$

Then we have the fundamental theorem of crossed Burnside rings:

Theorem 4.4. *There is an exact sequence of abelian groups:*

$$0 \rightarrow X\Omega(G, S) \xrightarrow{\varphi} X\tilde{\Omega}(G, S) \xrightarrow{\psi} \text{XObs}(G, S) \rightarrow 0.$$

In particular, the Burnside homomorphism φ is injective and has finite cokernel.

In Section 4, we prove the fundamental theorems for the p -local and p -modular crossed Burnside rings, too. In Section 5, we give an idempotent formula at characteristic 0 when the G -monoid S is a normal subgroup of G with G -action by G -conjugation. (We shall study an idempotent formula at characteristic $p > 0$ in the next paper of this series.)

(1.4) Notation. Notation and terminology used in this paper are standard, and refer to any books on finite group theory and representation theory of finite groups [CR 81], [Fe 82], [Be 91], etc. In this paper, G always denotes a finite group and S a finite monoid with left G -action. Here we list some notation we frequently use without any comments:

- NH $:= N_G(H)$, the normalizer of H in G .
- WH $:= W_G(H) := N_G(H)/H$.
- $C_S(H)$ $:= \{s \in S \mid {}^h s = s \ \forall h \in H\}$, the centralizer.
- ${}^g A$ $:= gAg^{-1}$, the conjugate of A by $g \in G$.
- M^G $:= \{m \in M \mid gm = m \ (\forall g \in G)\}$ for a G -set (or G -module).
- $Z(A)$ the center of a ring A .
- $k[S]$ the semigroup algebra of a semigroup S over a commutative ring k .
- $\sharp X, |X|$ the cardinality of a set X .
- G_x the stabilizer subgroup at an element x of a G -set.
- S^c the G -monoid S with G -action defined by G -conjugation for $S \trianglelefteq G$.
- $G\text{-set}$ the category of finite G -sets.
- $C(G)$ the set of conjugacy classes (H) of subgroups H of a group G .

2 The category of crossed G -sets

(2.1) Group action. Let G be a finite group. By a G -set, we mean a set with left G -action. A mapping $f : X \longrightarrow Y$ between two G -sets is called a G -map if it preserves G -action, that is, $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. The set of G -maps between G -sets X, Y is denoted by $\text{Map}_G(X, Y)$. The category of finite G -sets and G -maps is denoted by $G\text{-set}$.

A monoid S is called a **G -monoid** if the group G acts on S as monoid homomorphisms:

$$\begin{aligned} G \times S &\longrightarrow S; (g, s) \longmapsto {}^g s, \\ {}^{gh} s &= g({}^h s), {}^1 s = s; {}^g(st) = {}^g s \cdot {}^g t, {}^g 1 = 1 \quad \text{for } s, t \in S, g, h \in G. \end{aligned}$$

After this, G always denotes a finite group and S a finite G -monoid.

(2.2) Crossed G -sets. A **crossed G -set** X (over S) is a G -set X equipped with a G -map called a **weight function**:

$$\|\cdot\| : X \longrightarrow S; x \longmapsto \|x\|. \quad (2.2.a)$$

So $\|gx\| = {}^g\|x\|$ for $g \in G, x \in X$. If there is no special indication, we always use the symbol $\|x\|$ for the weight of an element x of any crossed G -set. For each $s \in S$, the **s -component** of a crossed G -set X is defined by

$$X[s] := \{x \in X \mid \|x\| = s\}. \quad (2.2.b)$$

Then we have

$$gX[s] = X[{}^gs], \quad g \in G, s \in S. \quad (2.2.c)$$

In particular, the stabilizer G_s at $s \in S$ acts on $X[s]$. Thus any crossed G -set X possesses a S -grading compatible with G -action:

$$X = \coprod_{s \in S} X[s]. \quad (2.2.d)$$

In this paper, we study only *finite* crossed G -sets, that is, crossed G -sets in which the underlying G -sets are finite sets.

(2.3) The category $G\text{-xset}/S$. A **crossed G -map** between two crossed G -sets X and Y is a G -map $f : X \longrightarrow Y$ preserving weights: $\|f(x)\| = \|x\|$ for all $x \in X$. The set of crossed G -maps of X to Y is denoted by

$$\text{XMap}_G(X, Y).$$

Finite crossed G -sets and crossed G -maps make a category

$$G\text{-xset}/S.$$

This category has two kinds of products. The first one is often called an **Hadamard product** which is simply a product in the categorical meaning and is defined by fiber products $X \times_S Y$ over S . The second one called tensor product is much more interesting.

(2.4) Tensor product. The **tensor product** of two crossed G -sets X, Y over S is defined by

$$\begin{aligned} X \otimes Y &:= X \times Y \text{ (as sets),} \\ g(x, y) &:= (gx, gy), \quad \|(x, y)\| := \|x\| \cdot \|y\| \quad (g \in G, x \in X, y \in Y). \end{aligned}$$

Clearly, the construction $(X, Y) \mapsto X \otimes Y$ is functorial in both variables.

The tensor product makes also the category $G\text{-}\mathbf{xset}/S$ into a monoidal category, in which the unit object $I = \{*\}$ is a singleton with weight $\|*\| = 1$ as usual.

Using the notion of tensor products, we can define some algebraic systems as usual. For example, a crossed G -monoid A over S is defined to be a crossed G -set equipped with crossed G -map $\mu : A \otimes A \rightarrow A$ satisfying the associativity. Thus, the G -monoid S itself is trivially a crossed G -monoid over S with weight $\|s\| = s$. A crossed left A -set X is a crossed G -set with left A -action $: A \otimes X \rightarrow X; (a, x) \mapsto ax$. Let A be a crossed G -monoid and X (resp. Y) a crossed right (resp. left) A -set. Then the tensor product $X \otimes_A Y$ over A is defined to be the set of equivalence classes

$$X \otimes_A Y := X \otimes Y / \sim,$$

where $\sim$ is the equivalence relation generated by $(xa, y) \sim (x, ay)$ for all $a \in A, x \in X, y \in Y$.

(2.5) Closeness. The monoidal category $G\text{-}\mathbf{xset}/S$ is closed, that is, the functor

$$(-) \otimes Y : G\text{-}\mathbf{xset}/S \rightarrow G\text{-}\mathbf{xset}/S; X \mapsto X \otimes Y \quad (2.5.a)$$

has a right adjoint functor called an **internal hom-functor**

$$\mathbf{hom}(Y, -) : G\text{-}\mathbf{xset}/S \rightarrow G\text{-}\mathbf{xset}/S. \quad (2.5.b)$$

In fact, this functor is, on objects, defined by

$$\mathbf{hom}(Y, Z) := \coprod_{s \in S} \{s\} \times \{\lambda : Y \rightarrow Z \mid \|\lambda(y)\| = s \cdot \|y\|\}, \quad (2.5.c)$$

where λ is simply a mapping (not a G -mapping in general). The G -action and the weight are given by

$$\begin{aligned} {}^g(s, \lambda) &:= ({}^g s, {}^g \lambda), \quad {}^g \lambda : y \mapsto g\lambda(g^{-1}y), \\ \|(s, \lambda)\| &:= s. \end{aligned}$$

When Y is empty, we put $\mathbf{hom}(Y, Z) := I$ (the unit object).

For crossed G -sets X, Y, Z , the natural bijection

$$\mathbf{XMap}_G(X \otimes Y, Z) \longleftrightarrow \mathbf{XMap}_G(X, \mathbf{hom}(Y, Z)) \quad (2.5.d)$$

is defined by

$$\begin{aligned} (f : X \otimes Y \longrightarrow Z) &\longmapsto (x \longmapsto (\|x\|, \hat{x} : y \longmapsto f(x, y))); \\ (g : X \longrightarrow \mathbf{hom}(Y, Z)) &\longmapsto ((x, y) \longmapsto g_1(x)(y)), \end{aligned}$$

where $g_1(x)$ is defined by $g(x) = (\|x\|, g_1(x))$. Furthermore, the associativity of the tensor product and the uniqueness of a right adjoint functor to a functor imply the natural isomorphisms of crossed G -sets:

$$\mathbf{hom}(X \otimes Y, Z) \cong \mathbf{hom}(X, \mathbf{hom}(Y, Z)), \quad (2.5.e)$$

$$\mathbf{hom}(I, Z) \cong Z. \quad (2.5.f)$$

The first isomorphism is given by

$$\begin{aligned} (s, X \otimes Y \xrightarrow{\lambda} Z) &\longmapsto (s, X \xrightarrow{\hat{\lambda}} \mathbf{hom}(Y, Z)), \quad (2.5.g) \\ \text{where } \hat{\lambda} : x &\longmapsto (s \cdot \|x\|, y \in Y \longmapsto \lambda(x, y)). \end{aligned}$$

Similarly, the right adjoint functor $\mathbf{hom}'(X, -)$ to $X \otimes (-)$ is given by

$$\mathbf{hom}'(X, Z) := \coprod_{s \in S} \{s\} \times \{\lambda : X \longrightarrow Z \mid \|\lambda(x)\| = \|x\| \cdot s\}. \quad (2.5.h)$$

(2.6) Braiding. The monoidal category $G\text{-}\mathbf{xset}/S$ is not braided in general, and the following problem seems still open:

Problem. Characterize the pair (G, S) where $G\text{-}\mathbf{xset}/S$ is symmetric or braiding.

We shall give a necessary condition to this problem as an application of the fundamental theorem in Section 4 (Corollary 4.7). See also Joyal–Street’s paper [JS93, Section 3].

Assume that $G\text{-}\mathbf{xset}/S$ is braided. Then by the uniqueness of the adjoint functors there exists a natural isomorphism:

$$\mathbf{hom}(X, Y) \cong \mathbf{hom}'(X, Y).$$

If S is an abelian G -monoid, then $G\text{-}\mathbf{xset}/S$ is clearly symmetric by the transposition

$$c_{XY} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (y, x). \quad (2.6.a)$$

Next, let S be a normal subgroup of G and let S^c denote the G -set S itself with G -action defined by conjugation:

$$G \times S^c \longrightarrow S^c; (g, s) \longmapsto {}^g s := gsg^{-1}. \quad (2.6.b)$$

In this case, we have a non-symmetric braided monoidal category $G\text{-}\mathbf{xset}/S^c$ as follows:

(2.7) Lemma. *Let S be a normal subgroup of G and we consider crossed G -sets over the G -monoid S^c defined as above. Then the following hold:*

(1) *The monoidal category $G\text{-}\mathbf{xset}/S^c$ is braided by the braiding*

$$c_{XY} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (\|x\|y, x). \quad (2.7.a)$$

The category $G\text{-}\mathbf{xset}/S^c$ is not symmetric by this braiding if S is not trivial.

(2) *There is a natural isomorphism:*

$$\mathbf{hom}(Y, Z) \xrightarrow{\sim} \mathbf{hom}'(Y, Z); (s, \lambda) \longmapsto (s, y \longmapsto \|y\| \cdot \lambda(y)). \quad (2.7.b)$$

PROOF. Easy. See [FY 89, Theorem 4.2.2]. Since

$$c_{YX}^{-1} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (y, \|y\|^{-1}x),$$

we have $c_{YX}^{-1} \neq c_{XY}$ in general, and so $G\text{-}\mathbf{xset}$ is not symmetric for any nontrivial normal subgroup S . $\square$

(2.8) Groupoid. A groupoid is a category in which each morphism is an isomorphism. From a G -monoid S , we can construct a groupoid (G, S) in which an object is an element of S and a morphism from s to t is a triplet of the form (t, g, s) such that $t = {}^g s$. The composition is defined in the multiplication of G . If X is a crossed G -set over S , then we have a S -grading:

$$\begin{aligned} X[s] &:= \{x \in X \mid \|x\| = s\}, \quad s \in S, \\ \widehat{g} : X[s] &\longrightarrow X[{}^g s]; x \longmapsto gx, \quad g \in G. \end{aligned} \quad (2.8.a)$$

Furthermore, the assignment $X[-] : s \longmapsto X[s], (t, g, s) \longmapsto \widehat{g}$ is a functor from the groupoid (G, S) to $\mathbf{set}$. Conversely, given a functor $X[-]$ on (G, S) , a crossed G -set is constructed by the disjoint union $\coprod_{s \in S} X[s]$.

Viewing a crossed G -set as a functor on (G, S) , the tensor product of crossed G -sets X and Y is written as

$$(X \otimes Y)[s] = \coprod_{uv=s} X[u] \times Y[v]. \quad (2.8.b)$$

(2.9) Remark. When S is a normal subgroup G itself of G , the functor from (G, G^c) to the category $\mathbf{Vec}_C$ of C -vector spaces is viewed as a module over $D(G)$, where $D(G)$ is the quantum double of the group algebra $\mathbb{C}[G]$. Thus a crossed “ G -module” is nothing but a $D(G)$ -module. Furthermore, the $\mathbb{C}$ -coefficient category algebra of the groupoid (G, G^c) is Morita equivalent to $D(G)$. See [Wi 96, Section 2], [Ma 95] for $D(G)$ -modules as G -graded G -modules.

(2.10) The category of elements. Here is another viewpoint to the concept of crossed G -sets. Let

$$F : G\text{-set} \longrightarrow \mathbf{mon}$$

be a contravariant functor from the category of finite G -sets (or another nice monoidal category) to the category of finite monoids which maps disjoint unions to direct products. Then an **element** (see [MM 94, Definition 1.6.4]) of this functor is a pair (X, s) of a finite G -set X and an element $s \in F(X)$. A morphism $f : (X, s) \longrightarrow (Y, t)$ between elements is defined to be a G -map $f : X \longrightarrow Y$ such that $F(f)(t) = s$. Thus we have a category $\mathbf{Elem}(G\text{-set}, F)$ of elements of F . This category is a monoidal category with tensor product:

$$(X, s) \otimes (Y, t) := (X \times Y, F(\pi_1)[s] \cdot F(\pi_2)(t)), \quad (2.10.a)$$

where π_i is the i -th projection on $X \times Y$.

As is shown in the next lemma, $\mathbf{Elem}(G\text{-set}, F)$ is a generalization of the category of crossed G -sets. One of the author would like to propose a “monomial category” as the name of $\mathbf{Elem}(G\text{-set}, F)$ after monomial representations of finite groups. In fact, for the contravariant functor $X \longmapsto \text{Ext}_{\mathbb{Z}G}^0(\mathbb{Z}X, \mathbb{C}^*)$ (and so $G/H \longmapsto \hat{H}$, the group of linear characters), we have the category of monomial representations, and then taking its Grothendieck ring, we have the ring $R_+(G)$, the ring of monomial characters, which was applied to the explicit formula of Brauer induction theorem. We can develop a theory of the monomial category and its Grothendieck ring which is a partial generalization of the theory of crossed G -sets and crossed Burnside rings to a limited extent, e.g., induction-restriction maps, the fundamental theorem, idempotent formula.

(2.11) Lemma. *Let S be a finite G -monoid. Then there is an equivalence of monoidal categories:*

$$G\text{-xset}/S \cong \mathbf{Elem}(G\text{-set}, \text{Map}_G(-, S)), \quad (2.11.a)$$

where $\text{Map}_G(-, S) : G\text{-set} \longrightarrow \mathbf{mon}$ is a Hom-functor in $G\text{-set}$ and the monoid structure on $\text{Map}_G(X, S)$ is given by the element-wise multiplication: $(\alpha \cdot \beta)(x) := \alpha(x) \cdot \beta(x)$.

PROOF. Easy. □

(2.12) Transitive crossed G -sets. For a subgroup D of the group G and an element s of $C_S(D)$, the centralizer of D in S , the homogeneous G -set G/D becomes a crossed G -set with the weight function $\|gD\| := {}^g s$. We denote this crossed G -set by $(G/D)_s$.

A crossed G -set X is called **transitive** if the underlying G -set X is a transitive G -set. Such a transitive crossed G -set is isomorphic to $(G/D)_s$ for some $D \leq G$, $s \in C_S(D)$.

(2.13) Lemma. *The following statements hold:*

$$(1) (G/D)_s \cong (G/E)_t \iff D = {}^g E (:= gEg^{-1}), s = {}^g t \quad \text{for } \exists g \in G.$$

(2) Any crossed G -map from $(G/D)_s$ to $(G/K)_t$ has the form

$$(G/D)_s \xrightarrow{\hat{g}} (G/E)_t; uD \mapsto ugE, \quad (2.13.a)$$

where $D \subseteq {}^g E$, $s = {}^g t$.

(3) $\text{Aut}(G/D)_s = \text{End}(G/D)_s \cong (WD)_s$, where

$$(WD)_s := \{nD \in N_G(D)/D \mid {}^n s = s\}. \quad (2.13.b)$$

(4) For a crossed G -set X ,

$$\text{XMap}_G((G/D)_s, X) \cong X[s]^D := \{x \in X^D \mid \|x\| = s\}, \quad (2.13.c)$$

where X^D is the set of D -fixed points of X . In particular,

$$\text{XMap}_G((G/1)_s, X) \cong X[s] (= \{x \in X \mid \|x\| = s\}). \quad (2.13.d)$$

(5) $\text{XMap}_G((G/D)_s, (G/E)_t) \cong \{gE \in G/E \mid D \subseteq {}^g E, s = {}^g t\}$. In particular, if D is G -conjugate to no subgroup of E , then there is no crossed G -map from $(G/D)_s$ to $(G/E)_t$. Furthermore, if D is G -conjugate to E , then any crossed G -map from $(G/D)_s$ to $(G/E)_t$ is an isomorphism.

(6) Any crossed G -set X is uniquely decomposed into a direct sum of transitive crossed G -subsets.

$$(7) (G/D)_s \otimes (G/E)_t \cong \coprod_{DgE \in D \backslash G/E} (G/D \cap {}^g E)_{s \cdot {}^g t}.$$

PROOF. Easy. □

3 Crossed Burnside rings

(3.1) Crossed Burnside ring $X\Omega(G, S)$. The crossed Burnside ring $X\Omega(G, S)$ is the Grothendieck ring of the category $G\text{-xset}/S$ with respect

to disjoint unions and tensor products, and so this ring is, as abelian group, generated by the isomorphism classes $[X]$ of crossed G -sets with relations:

$$[X + Y] = [X] + [Y]. \quad (3.1.a)$$

The multiplication is defined by the tensor product of crossed G -sets:

$$[X] \cdot [Y] := [X \otimes Y]. \quad (3.1.b)$$

Since any crossed G -set is a disjoint union of transitive crossed G -sets, $X\Omega(G, S)$ has the following free $\mathbb{Z}$ -basis as an abelian group:

$$\{[D, s] \mid D \leq G, s \in C_S(D)\}, \quad ([D, s] := [(G/D)_s]). \quad (3.1.c)$$

By Lemma 2.13, we have the relation

$$[D, s] = [E, t] \iff E = {}^g D, t = {}^g s \quad \text{for } \exists g \in G \quad (3.1.d)$$

and the multiplication defined by

$$[D, s] \cdot [E, t] = \sum_{DgE \in D \backslash G/E} [D \cap {}^g E, s \cdot {}^g t]. \quad (3.1.e)$$

The identity element is $[G, 1]$.

The crossed Burnside ring is not commutative in general (see Corollary 4.6). When $S = 1$, the crossed Burnside ring is the ordinary **Burnside ring** $\Omega(G)$, that is, the Grothendieck ring of G -set with respect to disjoint unions and Cartesian products. On the other hand, when $G = 1$, the crossed Burnside ring is the semigroup algebra $\mathbb{Z}S$.

(3.2) Proposition. *The rank of the free abelian groups $X\Omega(G, S)$ is equal to*

$$\sum_{(K) \in C(G)} \frac{|C_S(K)|}{|N_G(K)|} \cdot |K| \left(\sum_{x \in K/K'} \frac{1}{|x|} \right) \quad (3.2.a)$$

where $C(G)$ is the set of all conjugacy classes (K) of subgroups K of G and K' denotes the commutator subgroup of K .

PROOF. The crossed Burnside ring $X\Omega(G, S)$ has a basis $[H, s]$, $H \leq G, s \in C_S(H)$. Since

$$[H, s] = [K, t] \iff K = {}^g H, t = {}^g s \quad (\exists g \in G),$$

we have

$$\text{rank}_{\mathbb{Z}} X\Omega(G, S) = \sum_{(H) \in C(G)} |C_S(H) / \sim_{NH}|, \quad (3.2.b)$$

where $\sim_{NH}$ means the equivalence relation defined by $N_G(H)$ -conjugation. We now apply the Cauchy-Frobenius lemma (see [Yo 90b]) to calculate the number of orbits:

$$\begin{aligned} |C_S(H) / \sim_{NH}| &= \frac{1}{|N_G(H)|} \sum_{nH \in NH} |C_S(H)^{\langle n \rangle H}| \\ &= \frac{1}{|N_G(H)|} \sum_{n \in N_G(H)} |C_S(\langle n \rangle H)| \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{(H) \in C(G)} |C_S(H) / \sim_{NH}| &= \sum_{(H) \in C(G)} \frac{1}{|N_G(H)|} \sum_{n \in N_G(H)} |C_S(\langle n \rangle H)| \\ &= \frac{1}{|G|} \sum_{H \leq G} \sum_{n \in N_G(H)} |C_S(\langle n \rangle H)| \\ &= \frac{1}{|G|} \sum_{H \triangleleft K \leq G} |C_S(K)| \cdot \#\{n \in N_G(H) \mid K = \langle n \rangle H\} \\ &= \frac{1}{|G|} \sum_{\substack{H \triangleleft K \leq G \\ K/H: \text{cyclic}}} |C_S(K)| \cdot |H| \cdot \varphi(|K/H|) \\ &= \frac{1}{|G|} \sum_{K \leq G} |C_S(K)| \sum_{\substack{H \triangleleft K \leq G \\ K/H: \text{cyclic}}} |H| \varphi(|K/H|) \\ &= \sum_{(K) \in C(G)} \frac{|C_S(K)|}{|N_G(K)|} \cdot |K| \cdot \sum_{\substack{H \triangleleft K \\ K/H: \text{cyclic}}} \frac{\varphi(|K/H|)}{|K/H|} \end{aligned}$$

Here φ denotes the Euler function. Moreover, we have

$$\sum_{\substack{H \triangleleft K \\ K/H: \text{cyclic}}} \frac{\varphi(|K/H|)}{|K/H|} = \sum_{\substack{C \leq K/K' \\ C: \text{cyclic}}} \frac{\varphi(|C|)}{|C|} \quad (3.2.c)$$

$$= \sum_{x \in K/K'} \frac{1}{|x|}. \quad (3.2.d)$$

Hence

$$\text{rank} X\Omega(G, S) = \sum_{(K) \in C(G)} \frac{|C_S(K)|}{|N_G(K)|} \cdot |K| \left(\sum_{x \in K/K'} \frac{1}{|x|} \right)$$

The proposition is proved. $\square$

(3.3) Remark. The summation like $\sum 1/|x|$ on an abelian group K/K' appears also in the following equation:

$$\zeta(z) \sum_{x \in G} \frac{1}{|x|^z} = \sum_{n=1} \frac{\sharp\{x \in G \mid x^n = 1\}}{n^z}, \quad (3.3.a)$$

where G is a finite group, z is a complex number with its real part greater than 1 and $\zeta(z)$ is the Riemann zeta function.

(3.4) Changing the G -monoids. Let $\varphi : S \longrightarrow T$ be a G -monoid homomorphism. Then φ induces a functor

$$\varphi_! : G\text{-xset}/S \longrightarrow G\text{-xset}/T; X \longmapsto X, \quad (3.4.a)$$

where X was viewed as a crossed G -set over T with weight function $x \longmapsto \varphi(\|x\|)$. Clearly, $\varphi_!$ maps a transitive crossed G -set $(G/D)_s$ to $(G/D)_{\varphi(s)}$. The functor $\varphi_!$ preserves coproducts and tensor products, and so it induces a ring homomorphism

$$\varphi_! : X\Omega(G, S) \longrightarrow X\Omega(G, T); [X] \longrightarrow [X]. \quad (3.4.b)$$

Let $\Omega(G)$ be the Burnside ring of G (see 3.1). Then the trivial G -monoid homomorphisms $\eta : 1 \longrightarrow S$ and $\epsilon : S \longrightarrow 1$ induce the following ring homomorphisms:

$$\eta_! : \Omega(G) \longrightarrow X\Omega(G, S); [G/D] \longmapsto [D, 1] \quad (3.4.c)$$

$$\epsilon_! : X\Omega(G, S) \longrightarrow \Omega(G); [D, s] \longmapsto [G/D]. \quad (3.4.d)$$

Clearly, $\epsilon_! \circ \eta_!$ is an identity map. Thus we can view the Burnside ring $\Omega(G)$ as a subring of $X\Omega(G, S)$ through the injection $\eta_!$, and so $X\Omega(G, S)$ is an $\Omega(G)$ -algebra and

$$X\Omega(G, S) = \Omega(G) \oplus \text{Ker}(\eta_!). \quad (3.4.e)$$

4 The fundamental theorem

(4.1) Ghost rings. As before, let G be a finite group and S a finite G -monoid. For any subgroup H of G , the centralizer of H in S

$$C_S(H) := \{s \in S \mid {}^h s = s \quad \forall h \in H\} \quad (4.1.a)$$

is a submonoid of S , on which $WH := N_G(H)/H$ acts as monoid automorphisms. The semigroup algebra $\mathbb{Z}[S]$ of S is a permutation G -module with G -basis S . Clearly, $\mathbb{Z}[C_S(H)]$ is a subring of $\mathbb{Z}[S]$ with WH -action.

Now, the group G acts on the right as ring automorphisms on the product ring

$$\Xi(G) := \prod_{H \leq G} \mathbb{Z}[C_S(H)] \quad (4.1.b)$$

by

$$\xi^g := \left(\sum_{s \in C_S(H)} \xi({}^g H, {}^g s) s \right)_H \quad \text{for} \quad \xi = \left(\sum_{s \in C_S(H)} \xi(H, s) s \right)_H \quad \text{and } g \in G. \quad (4.1.c)$$

The subring consisting of G -fixed elements of $\Xi(G)$ is called the **crossed ghost ring**:

$$\begin{aligned} X\tilde{\Omega}(G, S) &:= \Xi(G)^G \\ &= \left\{ \left(\sum_{s \in C_S(H)} \xi(H, s) s \right)_H \mid \xi({}^g H, {}^g s) = \xi(H, s) \quad \forall g \in G \right\} \end{aligned} \quad (4.1.d)$$

The H -component $\xi(H) = \sum_s \xi(H, s)s \in \mathbb{Z}[C_S(H)]$ of any $\xi \in X\tilde{\Omega}(G, S)$ is fixed by $WH = N_G(H)/H$, and so

$$X\tilde{\Omega}(G, S) = \left(\prod_{H \leq G} \mathbb{Z}[C_S(H)]^{WH} \right)^G \quad (4.1.e)$$

$$\cong \prod_{(H) \in C(G)} \mathbb{Z}[C_S(H)]^{WH} \quad (4.1.f)$$

as rings, where $C(G)$ is the set of G -conjugacy classes (H) 's of subgroups H 's of G . Since $X\Omega(G, S)$ has a basis consisting of the classes of the form $[D, s]$, we have

$$\text{rank}_{\mathbb{Z}} X\Omega(G, S) = \text{rank}_{\mathbb{Z}} X\tilde{\Omega}(G, S). \quad (4.1.g)$$

(4.2) The Burnside homomorphism. For each subgroup H of G and for any crossed G -set X over S , define

$$\varphi_H(X) := \sum_{x \in X^H} \|x\| = \sum_{s \in C_S(H)} |X[s]^H| s \in \mathbb{Z}[C_S(H)]^{WH}, \quad (4.2.a)$$

where $X^H := \{x \in X \mid hx = x \ \forall h \in H\}$ as usual. Since the map φ_H is constant on each isomorphism class of crossed G -sets and is additive, it can be extended to an additive map on the Grothendieck ring $X\Omega(G, S)$:

$$\begin{aligned} \varphi_H : X\Omega(G, S) &\longrightarrow \mathbb{Z}[C_S(H)]^{WH} \\ ; \quad [X] &\longrightarrow \sum_{x \in X^H} \|x\| = \sum_{s \in C_S(H)} |X[s]^H| s. \end{aligned} \quad (4.2.b)$$

It easily follows from the definition of the weight on a tensor product that φ_H is a ring homomorphism. At a transitive crossed G -set, the map φ_H has the following value:

$$\begin{aligned} \varphi_H([D, s]) &= \sum_{gD \in (G/D)^H} g_s \\ &= \sum_{t \in S} \sharp\{gD \in (G/D)^H \mid g_s = t\} \cdot t. \end{aligned} \quad (4.2.c)$$

As a special case, we have the ring homomorphism

$$\varphi_1 : X\Omega(G, S) \longrightarrow \mathbb{Z}[S]^G; [X] \longmapsto \sum_{x \in X} \|x\|. \quad (4.2.d)$$

This map is surjective because

$$\varphi_1([C_G(s), s]) = \widehat{s} := \sum_{s' \sim_{NH} s} s' \in \mathbb{Z}[S]. \quad (4.2.e)$$

In particular, $X\Omega(G, G^c)$ has $Z(\mathbb{Z}[G])$ as a quotient ring.

The **Burnside homomorphism** is now defined by

$$\varphi := (\varphi_H) : X\Omega(G, S) \longrightarrow X\tilde{\Omega}(G, S). \quad (4.2.f)$$

We often write the value $\varphi_H(x)$ simply by $x(H)$. Using this notation, we have that for any $g \in G$,

$$x({}^gH) = {}^g(x(H)), \quad (4.2.g)$$

where we extended the action of G on S to the semigroup ring $\mathbb{Z}S$. The Burnside homomorphism φ is, of course, a ring homomorphism.

When S is a trivial monoid 1, we have the Burnside homomorphism

$$\varphi : [X] \longmapsto (|X^H|)_H \quad (4.2.h)$$

from the classical Burnside ring $\Omega(G)$ to the ghost ring $\tilde{\Omega}(G)$ ([Yo 90b]).

(4.3) The group of obstructions. We next define the **group of obstructions** by

$$\text{XObs}(G, S) := \prod_{[H, s]} (\mathbb{Z}/|(WH)_s|\mathbb{Z}), \quad (4.3.a)$$

where $[H, s]$ runs over the isomorphism classes of transitive crossed G -sets, that is, the complete representatives of G -conjugacy classes of pairs (H, s) of $H \leq G$ and $s \in C_S(H)$ under G -action ${}^g(H, s) = ({}^gH, {}^gs)$, and where $(WH)_s$ is the stabilizer at $s \in C_S(H)$ in $WH = N_G(H)/H$. For each $H \leq G$ and $s \in C_S(H)$, define a linear map $\psi_{H,s}$ as follows:

$$\begin{aligned} \psi_{H,s} : X\tilde{\Omega}(G, S) &\longrightarrow \mathbb{Z}/|(WH)_s|\mathbb{Z} \\ : \left(\sum_{s \in C_S(H)} \xi(H, s)s \right) &\longmapsto \sum_{nH \in (WH)_s} \overline{\xi(\langle n \rangle H, s)}. \end{aligned} \quad (4.3.b)$$

Thus we have a linear map called a **CFB-map** after Cauchy-Frobenius lemma or Burnside's lemma:

$$\psi := (\psi_{H,s}) : X\tilde{\Omega}(G, S) \longrightarrow \text{XObs}(G, S). \quad (4.3.c)$$

(4.4) Theorem (The fundamental theorem). *The following sequence of abelian groups is exact:*

$$0 \longrightarrow X\Omega(G, S) \xrightarrow{\varphi} X\tilde{\Omega}(G, S) \xrightarrow{\psi} \text{XObs}(G, S) \longrightarrow 0. \quad (4.4.a)$$

PROOF. Note that the free abelian groups $X\Omega(G, S)$ and $X\tilde{\Omega}(G, S)$ have a same rank by (4.1.g). By the definition of the crossed ghost ring there is an isomorphism of abelian groups:

$$\begin{aligned} f : X\tilde{\Omega}(G, S) &\longrightarrow \prod_{[H, s]} \mathbb{Z} \\ ; \left(\sum_s \xi(H, s)s \right)_H &\longmapsto (\xi(H, s))_{[H, s]} \end{aligned}$$

By Lemma 2.13(4), the $[H, s]$ -component of $f \circ \varphi([X])$ is equal to

$$|X[s]^H| = \sharp\{x \in X^H \mid \|x\| = s\} = |\text{XMap}_G((G/H)_s, X)|.$$

Thus the linear map $f \circ \varphi$ is presented by the matrix(the **table of marks**)

$$M := (|\text{XMap}_G((G/H)_s, (G/K)_t)|)_{[H, s], [K, t]} \quad (4.4.b)$$

indexed by isomorphism classes of transitive crossed G -sets. Since

$$\mathrm{XMap}_G((G/H)_s, (G/K)_t) \neq \emptyset \implies H \subseteq {}^g K \quad (\exists g \in G) \quad (4.4.c)$$

by Lemma 2.13(5), the matrix M is an upper triangular matrix with diagonal constituents $|(WH)_s|$'s after arranging $[H, s]$'s by the order of H 's. Thus the Burnside homomorphism φ is injective and

$$\mathrm{Coker}(\varphi) \cong \mathrm{XObs}(G, S). \quad (4.4.d)$$

Note that ψ is surjective because the matrix corresponding to ψ is conjugate to a triangular matrix. Thus by the universality of the cokernels, we know that it remains only to show that

$$\psi \circ \varphi = 0. \quad (4.4.e)$$

For a crossed G -set X , the H -component of $\varphi(X)$ is given by

$$\varphi_H(X) = \sum_{x \in X^H} \|x\| = \sum_{s \in C_S(H)} |X[s]^H|_s,$$

where $X[s]^H := \{x \in X^H \mid \|x\| = s\}$ as before. Thus

$$\begin{aligned} \psi_{H,s}\varphi(X) &= \sum_{nH \in WH} \varphi(X)(\langle n \rangle H, s) \mod |(WH)_s| \\ &= \sum_{nH \in WH} |X[s]^{\langle n \rangle H}| \mod |(WH)_s|. \end{aligned}$$

Applying the Cauchy-Frobenius lemma to the WH -set $X[s]$, we have

$$\begin{aligned} \psi_{H,s}\varphi(X) &= |(WH)_s| \cdot |(WH)_s \setminus X[s]^H| \\ &\equiv 0 \mod |(WH)_s|, \end{aligned}$$

proving (4.4.e) The theorem is proved. $\square$

(4.5) Corollary. *For two crossed G -sets X, Y over S , the following statements are equivalent :*

- (1) X and Y are isomorphic.
- (2) $\varphi_H(X) = \varphi_H(Y)$ for any subgroup $H \leq G$.
- (3) $|X[s]^H| = |Y[s]^H|$ for any $H \leq G$ and $s \in C_S(H)$.

PROOF. This directly follows from the injectivity of the Burnside homomorphism φ . $\square$

(4.6) Corollary. *Let $\mathbb{K}$ be a commutative ring in which $|G| \cdot 1_K$ is invertible. Then*

$$\mathbb{K} \otimes_{\mathbb{Z}} X\Omega(G, S) \cong \prod_{(H) \in C(G)} \mathbb{K}[C_S(H)]^{WH}. \quad (4.6.a)$$

In particular, $X\Omega(G, S)$ is commutative if and only if $\mathbb{Z}[C_S(H)]^{WH}$ is commutative for any $H \leq G$.

PROOF. The statement (4.6.a) immediately follows from the fundamental theorem and the fact that $K \otimes M = \text{Tor}_1^{\mathbb{Z}}(K, M) = 0$ for any $|G|$ -torsion abelian group M . The remainder directly follows from (4.6.a). $\square$

(4.7) Corollary. *If the category of crossed G -sets is braided by some braiding $c_{X,Y}$, then the WH -fixed point ring $\mathbb{Z}[C_S(H)]^{WH}$ is a commutative ring for any subgroup H of G . Conversely, if $\mathbb{Z}[C_S(H)]^{WH}$ is commutative for any H , then for any crossed G -sets X, Y over S there exists a (non-natural) isomorphism $X \otimes Y \cong Y \otimes X$ of crossed G -sets.*

PROOF. If $G\text{-xset}/S$ is braided, then $X \otimes Y \cong Y \otimes X$ for all X, Y . Furthermore, by Corollary 4.6, the crossed Burnside ring $\mathbb{Q}X\Omega(G, S)$ is commutative if and only if so is the crossed ghost ring $\mathbb{Q}X\tilde{\Omega}(G, S)$. This means that $X \otimes Y \cong Y \otimes X$ for all X, Y if and only if $\mathbb{Z}[C_S(H)]^{WH}$ is commutative for all H . Hence the corollary follows. $\square$

(4.8) Corollary. *Let S be a normal subgroup of G and let G act on $S^c = S$ by conjugation. Then $X\Omega(G, S^c)$ is commutative and is isomorphic to a subring of*

$$X\tilde{\Omega}(G, S^c) \cong \prod_{(H) \in C(G)} \mathbb{Z}[C_S(H)]^{WH} \quad (4.8.a)$$

$$= \prod_{(H) \in C(G)} (Z(\mathbb{Z}[N_G(H)]) \cap \mathbb{Z}[C_S(H)]), \quad (4.8.b)$$

where $Z(\mathbb{Z}[N_G(H)])$ is the center of the group algebra of $N_G(H)$.

PROOF. By the fundamental theorem, it only remains to show

$$\mathbb{Z}[C_S(H)]^{WH} = Z(\mathbb{Z}[N_G(H)]) \cap \mathbb{Z}[C_S(H)].$$

But this is clear because $C_S(H) \subseteq N_G(H)$ holds.

(4.9) The crossed Burnside ring in R -coefficients. Let R be a commutative ring. By the tensoring with R , we have R -versions of the crossed Burnside ring, the crossed ghost ring and the module of obstructions as follows

$$RX\Omega(G, S) := R \otimes_{\mathbb{Z}} X\Omega(G, S); \quad (4.9.a)$$

$$RX\tilde{\Omega}(G, S) := \left(\prod_{H \leq G} R[C_S(H)] \right)^G; \quad (4.9.b)$$

$$RX\text{Obs}(G, S) := \prod_{[H, s]} (R/|(WH)_s|R). \quad (4.9.c)$$

(4.10) Theorem. For any commutative ring R , the following sequence is exact:

$$0 \longrightarrow RA(G, S) \xrightarrow{\text{inc}} RX\Omega(G, S) \xrightarrow{1 \otimes \varphi} RX\tilde{\Omega}(G, S) \xrightarrow{1 \otimes \psi} RX\text{Obs}(G, S) \longrightarrow 0,$$

where $RA(G, S)$ is the ideal of $RX\Omega(G, S)$ defined by

$$RA(G, S) := \left\{ \sum_{[D, s]} a_{[D, s]} [D, s] \mid |(WD)_s| \cdot a_{[D, s]} = 0 \quad \forall [D, s] \right\}. \quad (4.10.a)$$

In particular, if R is $|G|$ -torsion free, then we have an exact sequence:

$$0 \longrightarrow RX\Omega(G, S) \xrightarrow{1 \otimes \varphi} RX\tilde{\Omega}(G, S) \xrightarrow{1 \otimes \psi} RX\text{Obs}(G, S) \longrightarrow 0.$$

PROOF. We only have to show that $RA(G, S)$ is the kernel of $1 \otimes \varphi$. First, let $a = \sum a_{[D, s]} [D, s]$ be an element of $RA(G, S)$. Then we have

$$\begin{aligned} \varphi_H(a) &= \sum_{[D, s]} a_{[D, s]} \varphi_H([D, s]) \\ &= \sum_{[D, s]} a_{[D, s]} \sum_{t \in S} \sharp\{gD \in (G/D)^H \mid {}^g s = t\} t \end{aligned} \quad (4.10.b)$$

by (4.2.c). Note that $(WD)_s$ acts freely on the set $\{gD \in (G/D)^H \mid {}^g s = t\}$ by $(nD, gD) \mapsto gnD$. Thus

$$\sharp\{gD \in (G/D)^H \mid {}^g s = t\} \equiv 0 \pmod{|(WD)_s|}. \quad (4.10.c)$$

Since $|(WD)_s| \cdot a_{[D, s]} = 0$ by the assumption on a , we have that $\varphi_H(a) = 0$, and so a belongs to $\text{Ker}(1 \otimes \varphi)$. Thus $RA(G, S) \subseteq \text{Ker}(1 \otimes \varphi)$.

Conversely, take an element $a = \sum a_{[D,s]}[D, s]$ of $\text{Ker}(1 \otimes \varphi)$. We shall show that $|(WD)_s| \cdot a_{[D,s]} = 0$ for any $[D, s]$ by the induction argument on the inverse order of D . Choose a subgroup H (possibly, $H = G$) satisfying the condition that $|(WD)_s| \cdot a_{[D,s]} = 0$ for any $[D, s]$ with $|D| > |H|$. Then

$$\begin{aligned} 0 &= \varphi_H(a) \\ &= \sum_{[D,s]} a_{[D,s]} \varphi_H([D, s]) \\ &= \sum_{[D,s]} a_{[D,s]} \sum_{t \in S} \sharp\{gD \in (G/D)^H \mid {}^g s = t\} t \end{aligned}$$

(see (4.10.b)). Note that $(G/D)^H \neq \emptyset$ implies $H \subseteq_G D$ and that $\sharp\{gD \in (G/D)^H \mid {}^g s = t\}$ is divisible by $|(WD)_s|$ (see (4.10.c)). Thus it follows from the assumption of induction that all the term at $[D, s]$ except for $[H, s]$ in the above summation vanish, and so

$$0 = \sum_{s \in C_S(H)/\sim_{WH}} a_{[H,s]} \sum_{t \in S} \sharp\{gH \in WH \mid {}^g s = t\} t.$$

Calculating the coefficient of t in the right hand side, we have the required identity:

$$|(WH)_s| \cdot a_{[H,s]} = 0.$$

The theorem is proved. $\square$

(4.11) The p -local crossed Burnside rings. Let $\mathbb{A}_p$ be a local ring of characteristic 0 with residue field $\mathbb{F}$ of characteristic $p > 0$. Clearly, we have

$$\mathbb{A}_p \text{XObs}(G, S) = \prod_{[H,s]} (\mathbb{A}_p / |(WH)_s|_p \mathbb{A}_p), \quad (4.11.a)$$

$$\mathbb{F} \text{XObs}(G, S) = \prod_{[H,s]} (\mathbb{F} / |(WH)_s|_p \mathbb{F}), \quad (4.11.b)$$

$$\mathbb{F} \Lambda(G, S) = \left\{ \sum_{[D,s]} a_{[D,s]} [D, s] \mid a_{[D,s]} = 0 \text{ if } |(WD)_s| \not\equiv 0 \pmod{p} \right\} \quad (4.11.c)$$

See (4.9.c) and (4.10.a). Thus an element of $\text{FXObs}(G, S)$ can be identified with an $\mathbb{F}$ -valued function on the set consisting of $[H, s]$ such that $|(WH)_s| \equiv 0 \pmod{p}$, and so

$$\begin{aligned} \dim(\text{FXObs}(G, S)) &= \dim(\mathbb{F} \Lambda(G, S)) \\ &= \sharp\{[H, s] \mid |(WH)_s| \equiv 0 \pmod{p}\}. \end{aligned} \quad (4.11.d)$$

In the p -local case, another CFB-map is constructed by using the existence of Sylow p -subgroups.

$$\begin{aligned} \psi^{(p)} : \quad \mathbb{A}_p X\tilde{\Omega}(G, S) &\longrightarrow \mathbb{A}_p \text{XObs}(G, S) \\ : \left(\sum_{s \in C_S(H)} \xi(H, s)s \right) &\longmapsto \left(\sum_{nH \in (WH)_{s,p}} \overline{\xi(\langle n \rangle H, s)} \right) \end{aligned} \quad (4.11.e)$$

where $(WH)_{s,p}$ denotes a Sylow p -subgroup of $(WH)_s$. The conjugacy of Sylow p -subgroups implies that the above summation does not depend on the choice of the Sylow p -subgroups of $(WH)_s$.

We notice that $\psi^{(p)}$ is differ from $1 \otimes \psi$ in general!! We furthermore define $\varphi^{(p)} := 1 \otimes \varphi$ for the p -local Burnside homomorphism.

(4.12) Theorem. *Let $\mathbb{A}_p$ be a local ring of characteristic 0 with residue field $\mathbb{F}$ of characteristic $p > 0$. Then the following statements hold.*

(1) *The sequence of $\mathbb{A}_p$ -modules*

$$0 \longrightarrow \mathbb{A}_p X\Omega(G, S) \xrightarrow{\varphi^{(p)}} \mathbb{A}_p X\tilde{\Omega}(G, S) \xrightarrow{\psi^{(p)}} \mathbb{A}_p \text{XObs}(G, S) \longrightarrow 0.$$

is exact.

(2) *The sequence of $\mathbb{F}$ -vector spaces*

$$0 \longrightarrow \mathbb{F}\Lambda(G, S) \xrightarrow{\text{inc}} \mathbb{F}X\Omega(G, S) \xrightarrow{\varphi^{(p)}} \mathbb{F}X\tilde{\Omega}(G, S) \xrightarrow{\psi^{(p)}} \mathbb{F}\text{XObs}(G, S) \longrightarrow 0,$$

is exact.

PROOF. (1) Let

$$\mathbb{Z}_{(p)} := \{m/n \in \mathbb{Q} \mid (n, p) = 1\} \subseteq \mathbb{Q} \quad (4.12.a)$$

be the ring of p -local integers. When $\mathbb{A}_p = \mathbb{Z}_{(p)}$, we can prove the exactness of the sequence by the same way as in the proof of the fundamental theorem. In the general case, taking the tensor product of the sequence by $\mathbb{A}_p$ over $\mathbb{Z}_{(p)}$, we have the exactness of the sequence, as required. Here note that $\text{Tor}_1^{\mathbb{Z}_{(p)}}(\mathbb{A}_p, \mathbb{Z}_{(p)}\text{XObs}(G, S))$ vanishes. In fact, it is a $|G|$ -torsion module because so is $\text{XObs}(G, S)$, and on the other hand it is isomorphic to a submodule of the torsion-free $\mathbb{A}_p$ -module $\mathbb{A}_p X\Omega(G, S)$. Hence (1) is proved. (2) follows from (1) and Theorem 4.10. $\square$

(4.13) **Remark.** This theorem means that not only

$$1 \otimes \psi : \mathbb{A}_p X\tilde{\Omega}(G, S) \longrightarrow \mathbb{A}_p \text{XObs}(G, S)$$

but also $\psi^{(p)}$ is a cokernel of $\varphi^{(p)} = 1 \otimes \varphi$, and so there must exist uniquely an automorphism α of $\mathbb{A}_p \text{XObs}(G, S)$ such that $\psi^{(p)} = \alpha \circ (1 \otimes \psi)$. However, the authors do not know the explicit formula for $\alpha(\zeta)$ for $\zeta \in \mathbb{A}_p \text{XObs}(G, S)$.

(4.14) **Lemma.** Let $\mathbb{F}$ be a field of characteristic $p > 0$ which is the residue field of a local field $\mathbb{A}_p$ of characteristic 0. Then the ideal $\mathbb{F}\Lambda(G, S)$ of $\mathbb{F}X\Omega(G, S)$ is contained in the Jacobson radical $J(\mathbb{F}X\Omega(G, S))$. In particular, $\mathbb{F}\Lambda(G, S)$ is a nilpotent ideal of $\mathbb{F}X\Omega(G, S)$.

PROOF. Let $-$ denote the modulo $J(\mathbb{A}_p)$ -reduction. Define an ideal $\mathbb{A}_p\Lambda$ of $\mathbb{A}_p X\Omega(G, S)$ by

$$\begin{aligned} \mathbb{A}_p\Lambda &:= \{a \in \mathbb{A}_p X\Omega(G, S) \mid \bar{a} \in \mathbb{F}\Lambda(G, S)\} \\ &= \left\{ \sum_{[D,s]} a_{[D,s]} [D, s] \mid a_{[D,s]} \in J(\mathbb{A}_p) \text{ if } |(WD)_s| \not\equiv 0 \pmod{p} \right\}, \end{aligned}$$

so that we have $\overline{\mathbb{A}_p\Lambda} = \mathbb{F}\Lambda(G, S)$ by the modulo $J(\mathbb{A}_p)$ -reduction. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{A}_p\Lambda & \xrightarrow{\text{inc}} & \mathbb{A}_p X\Omega(G, S) & \xrightarrow{\varphi^{(p)}} & \mathbb{A}_p X\tilde{\Omega}(G, S) \\ \downarrow - & & \downarrow - & & \downarrow - \\ \mathbb{F}\Lambda(G, S) & \xrightarrow{\text{inc}} & \mathbb{F}X\Omega(G, S) & \xrightarrow{\varphi^{(p)}} & \mathbb{F}X\tilde{\Omega}(G, S). \end{array}$$

We first show that $\varphi^{(p)}(\mathbb{A}_p\Lambda) \subseteq J(\mathbb{A}_p X\tilde{\Omega}(G, S))$. Let $a = \sum a_{[D,s]} [D, s]$ be an element of $\mathbb{A}_p\Lambda$. Then we have

$$\begin{aligned} \varphi_H^{(p)}(a) &= \sum_{[D,s]} a_{[D,s]} \varphi_H([D, s]) \\ &= \sum_{[D,s]} a_{[D,s]} \sum_{t \in S} M([D, s], t) \cdot t \\ &= \sum_{[D,s]} \sum_{t \in S} M([D, s], t) \cdot a_{[D,s]} t, \end{aligned}$$

where

$$M([D, s], t) := \sharp\{gD \in (G/D)^H \mid {}^g s = t\}.$$

See (4.2.c). Since

$$M([D, s], t) \equiv 0 \pmod{|(WD)_s|},$$

and $a \in \mathbb{A}_p \Lambda$, we have that

$$\begin{aligned} |(WD)_s| \equiv 0 \pmod{p} &\implies M([D, s], t) \cdot 1_{\mathbb{A}_p} \in J(\mathbb{A}_p) \\ |(WD)_s| \not\equiv 0 \pmod{p} &\implies a_{[D, s]} \in J(\mathbb{A}_p), \end{aligned}$$

and so, in either case,

$$M([D, s], t) \cdot a_{[D, s]} \in J(\mathbb{A}_p).$$

Thus $\varphi_H^{(p)}(a) \in J(\mathbb{A}_p)[C_S(H)] \subseteq J(\mathbb{A}_p[C_S(H)])$. Now we obtained the required statement $\varphi^{(p)}(a) \in J(\mathbb{A}_p X \tilde{\Omega}(G, S))$.

Since

$$\begin{aligned} \varphi^{(p)}(\mathbb{A}_p \Lambda) &\subseteq J(\mathbb{A}_p X \tilde{\Omega}(G, S)) \cap \varphi^{(p)}(\mathbb{A}_p X \Omega(G, S)) \\ &\subseteq J(\varphi^{(p)}(\mathbb{A}_p X \Omega(G, S))) = \varphi^{(p)}(J(\mathbb{A}_p X \Omega(G, S))), \end{aligned}$$

where we used the facts that $\mathbb{A}_p X \tilde{\Omega}(G, S)$ is finitely generated $\mathbb{A}_p$ -module and that $\varphi^{(p)}$ is injective on $\mathbb{A}_p X \Omega(G, S)$, we have that

$$\mathbb{A}_p \Lambda \subseteq J(\mathbb{A}_p X \Omega(G, S)).$$

Now, the statement of the lemma follows directly from the modulo $J(\mathbb{A}_p)$ -reduction. $\square$

5 Crossed Burnside rings at characteristic 0

(5.1) Möbius functions. The Möbius function $\mu : P \times P \longrightarrow \mathbb{Z}$ of a finite poset P is inductively defined by

$$\mu(x, x) = 1; \mu(x, y) = 0 \quad \text{if } x \not\leq y; \tag{5.1.a}$$

$$\sum_{t \leq y} \mu(x, t) = \sum_{t \geq x} \mu(t, y) = 0 \quad \text{if } x < y. \tag{5.1.b}$$

Define functions $\zeta, \delta : P \times P \longrightarrow \mathbb{Z}$ by $\zeta(x, y) := 1$ if $x \leq y$, and $\zeta(x, y) := 0$ otherwise; $\delta(x, y) := 1$ if $x = y$, and $\delta(x, y) := 0$ otherwise. Then by the definition of the Möbius function, we have that $\zeta * \mu = \mu * \zeta = \delta$, where $*$ denotes the convolution product:

$$(\alpha * \beta)(x, y) := \sum_{x \leq t \leq y} \alpha(x, t) \beta(t, y). \tag{5.1.c}$$

Thus the well-known Möbius inversion formula holds:

$$f * \mu = g \iff f = g * \zeta, \quad (5.1.d)$$

$$\mu * f = g \iff f = \zeta * g, \quad (5.1.e)$$

for functions on P , where $(f * \mu)(x) := \sum_{t \leq x} f(t)\mu(t, x)$, and so on. See Stanley's book for Möbius functions on posets([St 86]).

(5.2) Proposition (Inversion formula). *Let $\mathbb{K}$ be a field of characteristic 0 (or a commutative ring with $|G|^{-1} \in \mathbb{K}$). Let*

$$\xi = \left(\sum_{s \in C_S(H)} \xi(H, s)s \right)_H \in \mathbb{K}X\tilde{\Omega}(G, S). \quad (5.2.a)$$

Then the inverse image of ξ by $1_{\mathbb{K}} \otimes \varphi$ is given by

$$\frac{1}{|G|} \sum_{D \leq H \leq G} \sum_{s \in C_G(H)} |D| \mu(D, H) \xi(H, s)[D, s], \quad (5.2.b)$$

where μ is the Möbius function of the subgroup lattice of G .

PROOF. Let $a \in \mathbb{K} \otimes X\Omega(G, S)$ denote the right hand side of the above formula. Then for any subgroup L of G , we have

$$\begin{aligned} |G|(1 \otimes \varphi)_L(a) &= \sum_{D \leq G} \sum_{(H, s)} |D| \mu(D, H) \xi(H, s) \sum_{gD: L \leq {}^g D} {}^g s \\ &= \sum_{D, H, s} \sum_{g \in G} \mu(D, H) \xi(H, s) \zeta(L, {}^g D) {}^g s \\ &= \sum_{g \in G} \sum_{(H, s)} \sum_{D \leq H} \zeta(g^{-1}Lg, D) \mu(D, H) \xi(H, s) {}^g s \\ &= \sum_{g \in G} \sum_{(H, s)} \delta(g^{-1}Lg, H) \xi(H, s) {}^g s \\ &= \sum_{g \in G} \sum_{s \in C_S(g^{-1}Lg)} \xi(g^{-1}Lg, s) {}^g s \\ &= \sum_{g \in G} \sum_{s: {}^g s \in C_S(L)} \xi(L, {}^g s) {}^g s \\ &= \sum_{g \in G} \sum_{t \in C_S(L)} \xi(L, t) t \\ &= |G| \sum_{s \in C_S(L)} \xi(L, s) s. \end{aligned}$$

Thus $(1 \otimes \varphi)(a) = \xi$, as required. $\square$

(5.3) Corollary (The second fundamental theorem). *Let $\mathbb{K}$ be a field of characteristic 0 (or a commutative ring with $|G|^{-1} \in \mathbb{K}$). Then the following sequence of abelian groups is exact:*

$$0 \longrightarrow X\Omega(G, S) \xrightarrow{\varphi} X\tilde{\Omega}(G, S) \xrightarrow{\psi'} X\text{Obs}(G, S) \longrightarrow 0, \quad (5.3.a)$$

where $\psi = (\psi_{H,s})_{H,s}$ is defined by

$$\psi'_{H,s} : \xi = \left(\sum_{s \in C_G(H)} \xi(H, s)s \right) \mapsto \sum_{L \leq C_G(s)} \overline{\mu(H, L)\xi(L, s)}. \quad (5.3.b)$$

PROOF. First, the linear map ψ' is surjective because it is presented by the upper-triangular matrix of the form $(\mu(H, L))_{H,L}$. Next, by the fundamental theorem (Theorem 4.4), the map φ is injective linear map with cokernel isomorphic to $X\text{Obs}(G, S)$. Finally, we shall prove that $\psi' \circ \varphi = 0$, which completes the proof of the theorem by the universality of cokernels and the finiteness of the cokernel of ψ' . We note that any $\xi \in X\tilde{\Omega}(G, S)$ can be expanded by the inversion formula as follows :

$$\begin{aligned} (1_{\mathbb{Q}} \otimes \varphi)^{-1}(\xi) &= \frac{1}{|G|} \sum_{D \leq H \leq G} \sum_{s \in C_G(H)} |D| \mu(D, H) \xi(H, s) [D, s] \\ &= \sum_{[D, s]} \frac{1}{|(WD)_s|} \left(\sum_{H \leq G_s} \mu(D, H) \xi(H, s) \right) [D, s]. \end{aligned}$$

Thus for any $\xi \in X\tilde{\Omega}(G, S)$,

$$\xi \in \text{Im}(\varphi) \iff \psi'(\xi) = 0,$$

and so $\psi' \circ \varphi = 0$, as required. The theorem is proved. $\square$

(5.4) Some group algebras. In this paragraph, we assume that the G -monoid S is a finite group. Let $\mathbb{K}$ be a field of characteristic 0 such that it is a splitting field of every subgroup of a finite group G . We denote by $\text{Irr}_{\mathbb{K}}(S)$ the set of irreducible characters of a finite group S over $\mathbb{K}$.

We first remember the idempotent formula and the central characters of $Z(\mathbb{K}[S])$. The centrally primitive idempotent associated to an irreducible $\mathbb{K}$ -character χ has the following form:

$$\epsilon_{\theta} = \frac{\chi(1)}{|S|} \sum_{s \in S} \theta(s^{-1})s \in Z(\mathbb{K}[S]). \quad (5.4.a)$$

Furthermore, any central element $z \in Z(\mathbb{K}[S])$ is expressed by the $\mathbb{K}$ -linear combination of e_θ 's as follows:

$$z = \sum_{\theta \in \text{Irr}_{\mathbb{K}}(S)} \omega_\theta(z) e_\theta, \quad (5.4.b)$$

where $\omega_\theta : Z(\mathbb{K}[S]) \rightarrow \mathbb{K}$ is the algebra homomorphism (called central character) associated to θ which has the value at a class sum $\widehat{s}$ as follows:

$$\omega_\theta(\widehat{s}) := \frac{|S| \chi(s)}{|C_S(s)| \theta(1)}, \quad \widehat{s} = \sum_{s' \sim_s} s'. \quad (5.4.c)$$

Thus the primitive idempotent e_θ is characterized by

$$\omega_\chi(e_\theta) = \begin{cases} 1 & \text{if } \theta = \chi \\ 0 & \text{else,} \end{cases} \quad (5.4.d)$$

where χ is an irreducible character of S . Furthermore, $(\omega_\theta)_\theta$ gives an algebra isomorphism from $Z(\mathbb{K}[S])$ to the product ring $\mathbb{K}^r$, where r is the number of conjugacy classes of S .

We next consider the subalgebra $Z(\mathbb{K}[S])^G$ of $Z(\mathbb{K}[S])$ consisting of elements fixed by G . Any primitive idempotent of $Z(\mathbb{K}[S])^G$ has the following form:

$$\tilde{e}_\theta := \sum_{\theta' \sim_G \theta} e_{\theta'} = \frac{\theta(1)}{|G|} \sum_{g \in G} \tilde{\theta}(s^{-1}) s \in Z(\mathbb{K}[S])^G, \quad (5.4.e)$$

where θ is an irreducible $\mathbb{K}$ -character of S and $\tilde{\theta}$ is the summation of all distinct G -conjugates of θ . Clearly, $\tilde{e}_\theta = \tilde{e}_\chi$ if and only if θ and χ belong to a same G -conjugacy class. Any element z of $(Z(\mathbb{K}[S]))^G$ is expressed by the linear combination of $\tilde{e}_\theta$'s as follows:

$$z = \sum_{\theta}' \omega_\theta(z) \tilde{e}_\theta, \quad (5.4.f)$$

where θ runs over a complete representatives of G -conjugacy classes of irreducible characters of S and ω_θ is the central character of $\mathbb{K}[S]$ associated to the irreducible character θ as above (5.4.c). The algebra homomorphism ω_θ on $Z(\mathbb{K}[S])^G$ can be extended to the algebra homomorphism $\omega_{\chi(\theta)}$ on $Z(\mathbb{K}[S \rtimes G])$, where $\chi(\theta)$ is an irreducible constituent of the induced character $\theta^{1S \rtimes G}$.

We note that

$$Z(\mathbb{K}[S])^G = Z(\mathbb{K}[S]^G) \quad (5.4.g)$$

holds if one of the following conditions holds:

- (A) S is a normal subgroup of G .
- (B) S is an abelian group.

Under these conditions, $(\mathbb{K}[S])^G$ is a commutative algebra. Without these conditions, (5.4.g) does not hold in general.

(5.5) Theorem (Idempotent formula). *Assume that the G -monoid S is a finite group. Let $\mathbb{K}$ be a field of characteristic 0 which is a splitting field for all subgroup of G .*

- (1) *For $H \leq G$ and an irreducible $\mathbb{K}$ -character θ of $C_S(H)$, we put*

$$e_{H,\theta} = \frac{\theta(1)}{|N_G(H)||C_S(H)|} \sum_{D \leq H} \sum_{s \in C_S(H)} |D| \mu(D, H) \tilde{\theta}(s^{-1}) [D, s]. \quad (5.5.a)$$

Then $\{e_{H,\theta} \mid (H) \in C(G), \theta \in \text{Irr}_{\mathbb{K}}(C_S(H)) / \sim_{NH}\}$ is a set of orthogonal idempotents of crossed Burnside ring $X\Omega(G, S)$ such that.

$$1 = \sum_{H,\theta} e_{H,\theta}, \quad (5.5.b)$$

where the summation is taken over $(H) \in C(G)$ and $\theta \in \text{Irr}_{\mathbb{K}}(C_S(H)) / \sim_{NH}$.

- (2) *When S is an abelian G -monoid or a normal subgroup of G with G -action defined by G -conjugation, the above idempotents $e_{H,\theta}$'s are all primitive and conversely all primitive idempotent of $X\Omega(G, S)$.*

PROOF. We view the primitive idempotent $\tilde{\epsilon}_{H,\theta}$ of $Z(\mathbb{K}[C_S(H)])^{NH}$ as an element of $\mathbb{K}X\tilde{\Omega}(G, S)$ which has the value at $L \leq G$ as follows:

$$\tilde{\epsilon}_{H,\theta}(L) = \begin{cases} \theta(\tilde{\epsilon}_{H,\theta}) & \text{if } L = {}^g H \\ 0 & \text{else.} \end{cases}$$

Then $\tilde{\epsilon}_{H,\theta}$ is a central idempotent of $\mathbb{K}X\tilde{\Omega}(G, S)$ such that

$$1 = \sum'_{(H,\theta)} e_{H,\theta},$$

where (H, θ) runs over a complete representatives of G -conjugacy classes of $H \leq G$ and $\theta \in \text{Irr}_{\mathbb{K}}(C_S(H))$. Note that under the assumption of (2)

$\tilde{\epsilon}_{H,\theta}$'s are primitive idempotents of $\mathbb{K}X\tilde{\Omega}(G, S)$ and conversely any primitive idempotent of $\mathbb{K}X\tilde{\Omega}(G, S)$ has this form. The (H, s) -component of $\tilde{\epsilon}_{H,\theta}(H)$ is given by

$$\begin{aligned}\tilde{\epsilon}_{H,\theta}(H, s) &= \frac{1}{|C_S(H)|} \sum_{\theta' \sim_{NH} \theta} \theta'(1)\theta'(s^{-1}) \\ &= \frac{\theta(1)}{|C_S(H)|} \tilde{\theta}(s^{-1}).\end{aligned}$$

Thus the theorem follows immediately from the inversion formula (Proposition 5.2) together with (4.6.a) as follows:

$$\begin{aligned}e_{H,\theta} &= \varphi^{-1}(\tilde{\epsilon}_\theta) \\ &= \frac{1}{|G|} \sum_{D \leq H' \sim H} \sum_{s \in C_S(H')} |D| \mu(D, H') \tilde{\epsilon}_{H,\theta}(H', s) [D, s] \\ &= \frac{1}{|N_G(H)|} \sum_{D \leq H} \sum_{s \in C_S(H)} |D| \mu(D, H) \tilde{\epsilon}_{H,\theta}(H, s) [D, s] \\ &= \frac{1}{|N_G(H)|} \sum_{D \leq H} \sum_{s \in C_S(H)} |D| \mu(D, H) \frac{\theta(1)}{|C_S(H)|} \tilde{\theta}(s^{-1}) [D, s] \\ &= \frac{\theta(1)}{|N_G(H)| |C_S(H)|} \sum_{D \leq H} \sum_{s \in C_S(H)} |D| \mu(D, H) \tilde{\theta}(s^{-1}) [D, s].\end{aligned}$$

□.

(5.6) Lemma. *Let S be an abelian G -group or a normal subgroup of G with conjugacy action of G and $\mathbb{K}$ a field of characteristic 0 which is a splitting field for all subgroup of G .*

(1) *The crossed Burnside ring $X\Omega(G, S)$ is a splitting semi-simple $\mathbb{K}$ -algebra, that is, a direct product of some copies of $\mathbb{K}$. Furthermore, the number of direct components is equal to*

$$\dim \mathbb{K}X\Omega(G, S) = \sum_{(H) \in C(G)} |\text{Irr}_{\mathbb{K}}(C_S(H)) / \sim| \quad (5.6.a)$$

$$= \sum_{(H) \in C(G)} |C_S(H) / \sim| \quad (5.6.b)$$

(2) *For a subgroup H of G and an irreducible character θ of $C_S(H)$, the linear map*

$$\omega_{H,\theta} : \mathbb{K}X\Omega(G, S) \longrightarrow \mathbb{K}; [X] \longmapsto \sum_{x \in X^H} \frac{\theta(\|x\|)}{\theta(1)} \quad (5.6.c)$$

is a $\mathbb{K}$ -algebra homomorphism. In particular, $\omega_{H,\theta}([X])$ is an algebraic integer if $\mathbb{K}$ is a subfield of $\mathbb{C}$. Conversely, any $\mathbb{K}$ -algebra homomorphism from $\mathbb{K}X\Omega(G, S)$ to $\mathbb{K}$ has this form.

(3) The above algebra homomorphisms give an algebra isomorphism:

$$\omega := (\omega_{H,\theta}) : \mathbb{K}X\Omega(G, S^c) \longrightarrow \prod_{(H)} \prod_{\theta}' \mathbb{K}, \quad (5.6.d)$$

where (H) runs over all conjugacy classes of subgroups of G and θ runs over a complete set of representatives of NH -conjugacy classes of irreducible characters of $C_S(H)$.

PROOF. (1) follows directly from Corollary 4.6.

(2) The map $\omega_{H,\theta}$ is the composition of the Burnside homomorphism φ_H at H followed by the central character $\tilde{\omega}_{H,\theta}$. In fact, for any crossed G -set $[X]$ over S , we have

$$\begin{aligned} \tilde{\omega}_{H,\theta} \circ \varphi_H([X]) &= \tilde{\omega}_{H,\theta} \left(\sum_{s \in C_S(H)} |X[s]^H| s \right) \\ &= \sum_{s \in C_S(H)} |X[s]^H| \cdot \frac{\theta(s)}{\theta(1)} \\ &= \sum_{x \in X^H} \frac{\theta(\|x\|)}{\theta(1)} \\ &= \omega_{H,\theta}([X]). \end{aligned}$$

In particular, $\omega_{H,\theta}$ is an algebra homomorphism. Note that $\omega_{H,\theta} = \omega_{H',\theta'}$ if and only if there is an element g of G such that $H' = {}^gH, \theta' = {}^g\theta$. Next, for the number on algebra homomorphisms on $\mathbb{K}X\Omega(G, S)$, we have

$$\begin{aligned} \sharp \text{Alg}_{\mathbb{K}}(\mathbb{K}X\Omega(G, S), \mathbb{K}) &= \dim_{\mathbb{K}} \mathbb{K}X\Omega(G, S) \\ &= \sharp \{ \omega_{H,\theta} \mid (H) \in C(G), \theta \in \text{Irr}_{\mathbb{K}}(C_S(H)) \} / \sim. \end{aligned}$$

Thus all algebra homomorphism from $\mathbb{K}X\Omega(G, S)$ to $\mathbb{K}$ has the form $\omega_{H,\theta}$ for a (H, θ) .

(3) This follows directly from the fact that

$$(\omega_{H,\theta}) : \mathbb{K}[C_S(H)]^{N_G(H)} \longrightarrow \mathbb{K}^r$$

is an algebra isomorphism for some r . □

(5.7) Idempotents of the Burnside ring. Let $\varphi : S \longrightarrow T$ be a G -monoid homomorphism. Then φ induces three ring homomorphisms

$$\begin{aligned}\varphi_! & : X\Omega(G, S) \longrightarrow X\Omega(T), \\ \varphi_! & : X\tilde{\Omega}(G, S) \longrightarrow X\tilde{\Omega}(T), \\ \varphi_! & : X\text{Obs}(G, S) \longrightarrow X\text{Obs}(G, T),\end{aligned}$$

and these maps are commutative with the Burnside homomorphisms and CFB-maps. See (3.4).

The crossed Burnside ring over a trivial G -monoid 1 is nothing but the usual Burnside ring $\Omega(G)$ (see (3.1)). A primitive idempotent of $\mathbb{K}\Omega(G) := \mathbb{K} \otimes_{\mathbb{Z}} \Omega(G)$ has the following form:

$$e_H := \frac{1}{|N_G(H)|} \sum_{D \leq H} |D| \mu(D, H) [G/D]. \quad (5.7.a)$$

See [Yo 83]. Let $\eta : \mathbb{K}\Omega(G) \longrightarrow \mathbb{K}X\Omega(G, S)$ be the homomorphism induced from $\eta : 1 \longrightarrow S$. Then η maps e_H to

$$\eta(e_H) = \sum_{\theta} e_{H, \theta}, \quad (5.7.b)$$

where θ runs over $\text{Irr}_{\mathbb{K}}(C_S(H)) / \sim_{NH}$.

(5.8) The Hadamard product on the crossed Burnside ring. The crossed Burnside ring $X\Omega(G, S)$ becomes a commutative ring $X\Omega(G, S)_{\text{Had}}$ with respect to the Hadamard products:

$$[X] * [Y] := [X \times_S Y]. \quad (5.8.a)$$

The identity element with respect to the Hadamard product is $[S]$ with weight function $1_S : S \longrightarrow S$. On the classes of transitive crossed G -sets, we have

$$[D, s] * [E, t] = \sum_{\substack{DgE \in D \backslash G / E \\ : s = {}^g t}} [D \cap {}^g E, s]. \quad (5.8.b)$$

The ring $X\Omega(G, S)_{\text{Had}}$ is the Grothendieck ring of the comma category of $\mathbf{Set}_f^G$ over S , and so it is isomorphic to the product of some usual Burnside rings:

$$X\Omega(G, S)_{\text{Had}} \cong \prod_{s \in S / \sim_G} \Omega(G_s), \quad (5.8.c)$$

where s runs over a complete set of representatives of G -conjugacy classes in S .

The Burnside homomorphism φ preserves not only tensor products but also Hadamard products. In fact, for any crossed G -sets X and Y , we have

$$\varphi_H([X \times_S Y]) = \sum_{s \in C_S(H)} |X[s]^H| \cdot |Y[s]^H| s. \quad (5.8.d)$$

This is nothing but the Hadamard product of $\varphi_H([X])$ and $\varphi_H([Y])$ in the group algebra $\mathbb{Z}[C_S(H)]$.

(5.9) Proposition (Idempotent formula for Hadamard products). *Let $\mathbb{K}$ be a field of characteristic 0 and $\mathbb{A}_p$ a local ring of characteristic 0 and with the residue ring of characteristic $p > 0$.*

(1) *Any primitive idempotent of $\mathbb{K}X\Omega(G, S)_{\text{Had}}$ has the following form:*

$$f_{H,s} = \frac{1}{|(N_G(H))_s|} \sum_{D \leq H} |D| \mu(D, H) [D, s] \quad (5.9.a)$$

for some $H \leq G, s \in C_S(H)$. Furthermore, $f_{H,s} = f_{H',s'}$ if and only if there exists an element g of G such that $H' = {}^g H, s' = {}^g s$.

(2) *Any primitive idempotent of $\mathbb{A}_p X\Omega(G, S)_{\text{Had}}$ has the following form:*

$$f_{Q,s}^{(p)} = \sum_{[H,s]: H^p=Q} f_{H,s}, \quad (5.9.b)$$

where Q is a subgroup of G with no nontrivial p -groups as quotient groups and $s \in C_S(Q)$. The summation is taken over all $N_G(Q)$ -conjugacy classes of subgroups H such that H/Q is a p -group.

(3) *Any primitive idempotent $e_{H,\theta}$ of $\mathbb{K}X\Omega(G, S)$ is presented as follows:*

$$e_{H,\theta} = \frac{\theta(1)}{|C_S(H)|} \sum_{s \in C_S(H)/\sim_{NH}} \tilde{\theta}(s^{-1}) f_{H,s}. \quad (5.9.c)$$

PROOF. (1) and (2) follow from the idempotent formula for the Burnside ring ([Yo 83]) and (5.8.c). (3) easily follows from the idempotent formula for crossed Burnside ring at characteristic 0.

6 Examples

(6.1) Notation. In this section, we give some numerical examples of the crossed Burnside rings of S_3, A_4, A_5 . For each group, we list (A) the conjugacy classes of subgroups and the table of marks for the Burnside ring, (B) Some informations about $k[CH]^{NH}$ for $H \leq G = S_3$, particularly a basis of $\mathbb{K}[C_G(H)]^{N_G(H)}$. (C) the table of marks of the crossed Burnside ring, (D) the multiplication table of the center of group algebra, (E) the multiplication table for the crossed Burnside ring, (F) the idempotent formula of the crossed Burnside ring at characteristic 0, (G) the decompositions of idempotents of the Burnside rings in the crossed Burnside rings, (H) the expression of the transitive crossed G -sets as linear combinations of primitive idempotents of the crossed Burnside ring, (I) p -local idempotent formula for the crossed Burnside ring,

The symbols $C_n, D_{2n}, E_n, A_n, S_n$ denote the groups (or subgroups) isomorphic to a cyclic group of order n , a dihedral group of order $2n$, an elementary abelian group of order n , the alternating group of degree n , the symmetric group of degree n . We denote by G one of the group S_3, A_4, A_5 . Let $C(G)$ be the set of conjugacy classes of subgroups of G . The **table of marks** $M(G)$ for the Burnside ring $\Omega(G)$ is a $C(G) \times C(G)$ -matrix defined by

$$M(G) := (|\text{Map}_G(G/K, G/H)|)_{G/H, (K)}. \quad (6.1.a)$$

We took the transpose of the matrix M in (4.4.b) in order to save the space. Note that

$$|\text{Map}_G(G/H, G/K)| = \sharp\{gK \in G/K \mid H \subseteq {}^gK\}. \quad (6.1.b)$$

By (4.4.c), the matrix $M(G)$ is an lower triangular matrix after arranging the rows and columns by the order of subgroups.

Next, let $\mathbb{K}$ be a field of characteristic 0 which is a splitting field for all subgroups of G . Then the **table of marks** $XM(G)$ for the crossed Burnside ring $X\Omega(G, G^c)$ is the matrix defined by

$$XM(G) := (\varphi_K([H, s]))_{[H, s], K}, \quad (6.1.c)$$

where φ_K is the Burnside homomorphism (see 4.2) and

$$\varphi_K([H, s]) = \sum_{t \in G} \sharp\{gH \in (G/H)^K \mid {}^gs = t\} \cdot t \in \mathbb{K}[C_G(K)]^{WK} \quad (6.1.d)$$

See (4.4.c).

Let $e_{H,\theta}$ be the idempotents of $\mathbb{K}X\Omega(G, G^c)$ corresponding to a subgroup H of G and an irreducible character $\theta \in \text{Irr}(C_G(H))$. We simply denote by C the subgroup of G isomorphic to C and by $d, d', \dots$ irreducible characters of $C_G(H)$ of degree d for some subgroup H of G . In particular, $\mathbf{1}$ denotes the principal character of a group. Each class $[D, s]$ of transitive crossed G -sets can be expressed by a linear combination

$$[D, s] = \sum'_{(H,\theta)} \omega_{H,\theta}([D, s]) e_{H,\theta}.$$

Remember that the coefficients $\omega_{H,\theta}([D, s])$'s are algebraic integer (Lemma 5.6(2)).

By (3.4.c), the Burnside ring $\Omega(G)$ is embedded into $X\Omega(G, G^c)$ by the map

$$\eta : [G/D] \longmapsto [D, \mathbf{1}].$$

If e_H is the primitive idempotent of $\mathbb{K}\Omega(G)$ associated to H (see 5.7.a), then by (5.7.a),

$$\eta(e_H) = \sum'_{\theta} e_{H,\theta},$$

where θ runs over a complete representatives of NH -conjugacy classes of irreducible $\mathbb{K}$ -character of $C_S(H)$.

(6.2) S_3 .

(A) The classes of subgroups and the table of marks of $\Omega(G)$ for $G = S_3$. The symmetric group $S_3 = \langle s = (123), t = (12) \rangle$ of degree 3 has three conjugacy classes of subgroups represented by

$$1, C_2 = \langle t \rangle, C_3 = \langle s \rangle, G = S_3.$$

	1	C_2	C_3	S_3
$S_3/1$	6			
S_3/C_2	3	1		
S_3/C_3	2		2	
S_3/S_3	1	1	1	1

(B) Some informations about $k[CH]^{NH}$ for $H \leq G = S_3$.

H	1	C_2	C_3	$G = S_3$
CH	S_3	C_2	C_3	1
NH	S_3	C_2	S_3	S_3
$\dim \mathbb{K}[CH]^{NH}$	3	2	2	1
basis	$1, \widehat{s}, \widehat{t}$	$1, s$	$1, \widehat{s}$	1

Thus $X\Omega(G, G^c)$ is of dimensional 8. In this table, we put $\widehat{s} = s + s^{-1}$, $\widehat{t} = t + st + s^{-1}t$. The are the class sums in the group algebra $\mathbb{K}[G]$.

(C) The table of marks of $X\Omega(G, G^c)$ for $G = S_3$.

	1	C_2	C_3	S_3
$[1, 1]$	$6 \cdot 1$			
$[1, t]$	$2\widehat{t}$			
$[1, s]$	$3\widehat{s}$			
$[C_2, 1]$	$3 \cdot 1$	$1 \cdot 1$		
$[C_2, t]$	$\widehat{t}$	t		
$[C_3, 1]$	$2 \cdot 1$		$2 \cdot 1$	
$[C_3, s]$	$\widehat{s}$		$\widehat{s}$	
$[S_3, 1]$	1	1	1	1

(D) The multiplication table of $Z(\mathbb{K}[S_3])$.

	1	$\widehat{t}$	$\widehat{s}$
1	1	$\widehat{t}$	$\widehat{s}$
$\widehat{t}$	$\widehat{t}$	$3 \cdot 1 + 3\widehat{s}$	$2\widehat{t}$
$\widehat{s}$	$\widehat{s}$	$2\widehat{t}$	$2 \cdot 1 + \widehat{s}$

(E) The multiplication table of $X\Omega(G, G^c)$ for $G = S_3$.

	1	a	b	c	d	e	f	g
$1 = [S_3, 1]$	1	a	b	c	d	e	f	g
$a = [C_3, s]$	a	$a + b$	$2a$	f	e	$e + g$	$2f$	$2e$
$b = [C_3, 1]$	b	$2a$	$2b$	f	g	$2e$	$2f$	$2g$
$c = [C_2, t]$	c	f	f	$d + e$	$c + f$	$3f$	$2e + g$	$3f$
$d = [C_2, 1]$	d	e	g	$c + f$	$d + g$	$3e$	$3f$	$3g$
$e = [1, s]$	e	$e + g$	$2e$	$3f$	$3e$	$3e + 3g$	$6f$	$6e$
$f = [1, t]$	f	$2f$	$2f$	$2e + g$	$3f$	$6f$	$4e + 2g$	$6f$
$g = [1, 1]$	g	$2e$	$2g$	$3f$	$3g$	$6e$	$6f$	$6g$

(F) Primitive idempotents of $\mathbb{K}X\Omega(G, G^c)$ at characteristic 0 for $G = S_3$.

$$\begin{aligned}
e_{S_3,1} &= (1/2)[1, 1] - [C_2, 1] - (1/2)[C_3, 1] + [S_3, 1], \\
e_{C_3,1} &= -(1/18)[1, 1] - (1/9)[1, s] + (1/6)[C_3, 1] + (1/3)[C_3, s], \\
e_{C_3,1'} &= -(1/9)[1, 1] + (1/9)[1, s] + (1/3)[C_3, 1] - (1/3)[C_3, s], \\
e_{C_2,1} &= -(1/4)[1, 1] - (1/4)[1, t] + (1/2)[C_2, 1] + (1/2)[C_2, t], \\
e_{C_2,1'} &= -(1/4)[1, 1] + (1/4)[1, t] + (1/2)[C_2, 1] - (1/2)[C_2, t], \\
e_{1,1} &= (1/36)[1, 1] + (1/18)[1, s] + (1/12)[1, t], \\
e_{1,1'} &= (1/36)[1, 1] + (1/18)[1, s] - (1/12)[1, t], \\
e_{1,2} &= (1/9)[1, 1] - (1/9)[1, s].
\end{aligned}$$

(G) The decompositions of idempotents of $\mathbb{K}\Omega(G)$ in $\mathbb{K}X\Omega(G, G^c)$ for $G = S_3$. The Burnside ring $\mathbb{K}\Omega(S_3)$ has four primitive idempotents

$$e_1, e_{C_2}, e_{C_3}, e_{S_3}.$$

The images of these idempotents by the embedding η is decomposed as follows:

$$\begin{aligned}
\eta(e_{S_3}) &= e_{S_3,1} \\
&= (1/2)[1, 1] - [C_2, 1] - (1/2)[C_3, 1] + [S_3, 1], \\
\eta(e_{C_3}) &= e_{C_3,1} + e_{C_3,1'} \\
&= -(1/6)[1, 1] + (1/2)[C_3, 1], \\
\eta(e_{C_2}) &= e_{C_2,1} + e_{C_2,1'} \\
&= -(1/2)[1, 1] + [C_2, 1] \\
\eta(e_1) &= e_{1,1} + e_{1,1'} + e_{1,2} \\
&= (1/6)[1, 1].
\end{aligned}$$

(H) The expression of the transitive crossed G -sets as linear combinations of primitive idempotents of $X\Omega(G, G^c)$ for $G = S_3$.

$$\begin{aligned}
[1, 1] &= 6e_{1,1} + 6e_{1,1'} + 6e_{1,2}, \\
[1, s] &= 6e_{1,1} + 6e_{1,1'} - 3e_{1,2}, \\
[1, t] &= 6e_{1,1} - 6e_{1,1'}, \\
[C_2, 1] &= 3e_{1,1} + 3e_{1,1'} + 3e_{1,2} + e_{C_2,1} + e_{C_2,1'}, \\
[C_2, t] &= 3e_{1,1} - 3e_{1,1'} + e_{C_2,1} - e_{C_2,1'},
\end{aligned}$$

$$\begin{aligned}
[C_3, 1] &= 2e_{1,1} + 2e_{1,1'} + 2e_{1,2} + 2e_{C_3,1} + 2e_{C_3,1'}, \\
[C_3, s] &= 2e_{1,1} + 2e_{1,1'} - e_{1,2} + 2e_{C_3,1} - e_{C_3,1'}, \\
[S_3, 1] &= e_{1,1} + e_{1,1'} + e_{1,2} + e_{C_2,1} + e_{C_2,1'} + e_{C_3,1} + e_{C_3,1'} + e_{S_3,1}.
\end{aligned}$$

(I) Primitive idempotents of $\mathbb{A}_p X\Omega(G, G^c)$ for $G = S_3$.

(I-1) **Case : $p = 2$.** There are five primitive idempotents.

$$\begin{aligned}
e_{C_3,1} + e_{S_3,1} &= (4/9)[1, 1] - (1/9)[1, s] - [C_2, 1] \\
&\quad - (1/3)[C_3, 1] + (1/3)[C_3, s] + [S_3, 1], \\
e_{C_3,1'} &= -(1/9)[1, 1] + (1/9)[1, s] + (1/3)[C_3, 1] - (1/3)[C_3, s], \\
e_{1,1} + e_{1,1'} + e_{C_2,1} + e_{C_2,1'} &= -(4/9)[1, 1] + (1/9)[1, s] + [C_2, 1], \\
e_{1,2} &= (1/9)[1, 1] - (1/9)[1, s].
\end{aligned}$$

(I-2) **Case : $p = 3$.** There are four primitive idempotents.

$$\begin{aligned}
e_{1,1} + e_{1,1'} + e_{1,2} + e_{C_3,1} + e_{C_3,1'} &= (1/2)[1, 1] + (1/2)[C_3, 1], \\
e_{C_2,1} &= -(1/4)[1, 1] - (1/4)[1, t] + (1/2)[C_2, 1] + (1/2)[C_2, t], \\
e_{C_2,1'} &= -(1/4)[1, 1] + (1/4)[1, t] + (1/2)[C_2, 1] - (1/2)[C_2, t], \\
e_{S_3,1} &= (1/2)[1, 1] - [C_2, 1] - (1/2)[C_3, 1] + [S_3, 1],
\end{aligned}$$

(6.3) A_4 .

(A) The classes of subgroups and the table of marks of $\Omega(G)$ for $G = A_4$. The alternating group $A_4 = \langle s = (123), t = (12)(34) \rangle$ of degree 4 has five conjugacy classes of subgroups represented by

$$1, C_2 = \langle t \rangle, C_3 = \langle s \rangle, E_4 = \{1, t_1, t_2, t_3\}, G = A_4,$$

where

$$t_1 = t, t_2 = sts^{-1} = (14)(23), t_3 = s^{-1}ts = (13)(24).$$

	1	C_2	C_3	E_4	A_4
$A_4/1$	12				
A_4/C_2	6	2			
A_4/C_3	4		1		
A_4/E_4	3	3		3	
A_4/A_4	1	1	1	1	1

(B) Some informations about $k[CH]^{NH}$ for $H \leq G = A_4$.

H	1	C_2	C_3	E_4	$G = A_4$
CH	A_4	E_4	C_3	E_4	1
NH	A_4	E_4	C_3	A_4	A_4
$\dim \mathbb{K}[CH]^{NH}$	4	4	3	2	1
basis	$1, \widehat{t}, \widehat{s}, \widehat{s}'$	$1, t_1, t_2, t_3$	$1, s, s'$	$1, \widehat{t}$	1

Thus $\mathbb{K}X\Omega(G, G^c)$ is of dimensional 14. In this table, we put $s' = s^{-1} = (132)$, $\widehat{t} = t_1 + t_2 + t_3$, $\widehat{s} = (1 + t_1 + t_2 + t_3)s$, $\widehat{s}' = (1 + t_1 + t_2 + t_3)s'$.

(C) The table of marks of $X\Omega(G, G^c)$ for $G = A_4$.

	1	C_2	C_3	E_4	A_4
$[1, 1]$	$12 \cdot 1$				
$[1, \widehat{t}]$	$4\widehat{t}$				
$[1, \widehat{s}]$	$3\widehat{s}$				
$[1, \widehat{s}']$	$3\widehat{s}'$				
$[C_2, 1]$	$6 \cdot 1$	$2 \cdot 1$			
$[C_2, t_1]$	$2\widehat{t}$	$2t_1$			
$[C_2, t_2]$	$2\widehat{t}$	$2t_2$			
$[C_2, t_3]$	$2\widehat{t}$	$2t_3$			
$[C_3, 1]$	$4 \cdot 1$		1		
$[C_3, s]$	$\widehat{s}$		s		
$[C_3, s']$	$\widehat{s}'$		s'		
$[E_4, 1]$	$3 \cdot 1$	$3 \cdot 1$		$3 \cdot 1$	
$[E_4, \widehat{t}]$	$\widehat{t}$	$\widehat{t}$		$\widehat{t}$	
$[A_4, 1]$	1	1	1	1	1

(D) The multiplication table of $Z(\mathbb{K}[A_4])$.

	1	$\widehat{t}$	$\widehat{s}$	$\widehat{s}'$
1	1	$\widehat{t}$	$\widehat{s}$	$\widehat{s}'$
$\widehat{t}$	$\widehat{t}$	$3 \cdot 1 + 2\widehat{t}$	$3\widehat{s}$	$3\widehat{s}'$
$\widehat{s}$	$\widehat{s}$	$3\widehat{t}$	$4\widehat{s}'$	$4 \cdot 1 + 2\widehat{t}$
$\widehat{s}'$	$\widehat{s}'$	$3\widehat{s}'$	$4 \cdot 1 + \widehat{t}$	$4\widehat{s}$

(E) The multiplication table of $X\Omega(G, G^c)$ for $G = S_3$. Omitted.

(F) Primitive idempotents of $\mathbb{K}X\Omega(G, G^c)$ at characteristic 0 for $G = A_4$. There are fourteen primitive idempotents.

$$\begin{aligned}
e_{1,1} &= (1/144)[1,1] + (1/48)[1,t] + (1/36)[1,s] + (1/36)[1,s'], \\
e_{1,1'} &= (1/144)[1,1] + (1/48)[1,t] + (\omega^-/36)[1,s] + (\omega^+/36)[1,s'], \\
e_{1,1''} &= (1/144)[1,1] + (1/48)[1,t] + (\omega^+/36)[1,s] + (\omega^-/36)[1,s'], \\
e_{1,3} &= (1/16)[1,1] - (1/16)[1,t], \\
e_{C_2,1} &= (1/8)[C_2,1] + (1/8)[C_2,t_1] + (1/8)[C_2,t_2] + (1/8)[C_2,t_3] \\
&\quad - (1/16)[1,1] - (3/16)[1,t], \\
e_{C_2,1_1} &= (1/8)[C_2,1] + (1/8)[C_2,t_1] - (1/8)[C_2,t_2] - (1/8)[C_2,t_3] \\
&\quad - (1/16)[1,1] + (1/16)[1,t], \\
e_{C_2,1_2} &= (1/8)[C_2,1] - (1/8)[C_2,t_1] + (1/8)[C_2,t_2] - (1/8)[C_2,t_3] \\
&\quad - (1/16)[1,1] + (1/16)[1,t], \\
e_{C_2,1_3} &= (1/8)[C_2,1] - (1/8)[C_2,t_1] - (1/8)[C_2,t_2] + (1/8)[C_2,t_3] \\
&\quad - (1/16)[1,1] + (1/16)[1,t], \\
e_{C_3,1} &= (1/3)[C_3,1] + (1/3)[C_3,s] + (1/3)[C_3,s'] \\
&\quad - (1/9)[1,1] - (1/9)[1,s] - (1/9)[1,s'], \\
e_{C_3,1^+} &= (1/3)[C_3,1] + (\omega^+/3)[C_3,s] + (\omega^-/3)[C_3,s'] \\
&\quad - (1/9)[1,1] - (\omega^+/9)[1,s] - (\omega^-/9)[1,s'], \\
e_{C_3,1^-} &= (1/3)[C_3,1] + (\omega^-/3)[C_3,s] + (\omega^+/3)[C_3,s'] \\
&\quad - (1/9)[1,1] - (\omega^-/9)[1,s] - (\omega^+/9)[1,s'], \\
e_{E_4,1} &= (1/12)[E_4,1] + (1/4)[E_4,t] - (1/8)[C_2,1] - (1/8)[C_2,t_1] \\
&\quad - (1/8)[C_2,t_2] - (1/8)[C_2,t_3] + (1/24)[1,1] + (1/8)[1,t], \\
e_{E_4,1_1} &= (1/4)[E_4,1] - (1/4)[E_4,t] - (3/8)[C_2,1] + (1/8)[C_2,t_1] \\
&\quad + (1/8)[C_2,t_2] + (1/8)[C_2,t_3] + (1/8)[1,1] - (1/8)[1,t], \\
e_{A_4,1} &= [A_4,1] - (1/3)[E_4,1] - [C_3,1] + (1/3)[1,1].
\end{aligned}$$

Here $\omega^\pm = (-1 \pm \sqrt{3})/2$.

(G) The decompositions of idempotents of $\mathbb{K}\Omega(G)$ in $\mathbb{K}X\Omega(G, G^c)$ for $G = A_4$. The Burnside ring $\mathbb{K}\Omega(A_4)$ has five primitive idempotents:

$$e_1, e_{C_2}, e_{C_3}, e_{E_4}, e_{A_4}.$$

The images of these idempotents by the embedding η_l is decomposed as follows:

$$\begin{aligned}
\eta_l(e_{A_4}) &= e_{A_4,1} \\
&= [A_4,1] - (1/3)[E_4,1] - [C_3,1] + (1/3)[1,1],
\end{aligned}$$

$$\begin{aligned}
\eta(e_{E_4}) &= e_{E_4,1} + e_{E_4,1_1} \\
&= (1/3)[E_4, 1] - (1/2)[C_2, 1] + (1/6)[1, 1], \\
\eta(e_{C_3}) &= e_{C_3,1} + e_{C_3,1^+} + e_{C_3,1^-} \\
&= [C_3, 1] - (1/3)[1, 1], \\
\eta(e_{C_2}) &= e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&= (1/2)[C_2, 1] - (1/4)[1, 1], \\
\eta(e_1) &= e_{1,1} + e_{1,1'} + e_{1,1''} + e_{1,3} \\
&= (1/12)[1, 1].
\end{aligned}$$

(H) The expression of the transitive crossed G -sets as linear combinations of primitive idempotents of $X\Omega(G, G^c)$ for $G = A_4$.

$$\begin{aligned}
[1, 1] &= 12e_{1,1} + 12e_{1,1'} + 12e_{1,1''} + 12e_{1,3}, \\
[1, t] &= 12e_{1,1} + 12e_{1,1'} + 12e_{1,1''} - 4e_{1,3}, \\
[1, s] &= 12e_{1,1} + 12\omega^+e_{1,1'} + 12\omega^-e_{1,1''}, \\
[1, s'] &= 12e_{1,1} + 12\omega^-e_{1,1'} + 12\omega^+e_{1,1''}, \\
[C_2, 1] &= 6e_{1,1} + 6e_{1,1'} + 6e_{1,1''} + 6e_{1,3} + 2e_{C_2,1} + 2e_{C_2,1_1} + 2e_{C_2,1_2} + 2e_{C_2,1_3}, \\
[C_2, t_1] &= 6e_{1,1} + 6e_{1,1'} + 6e_{1,1''} - 2e_{1,3} + 2e_{C_2,1} + 2e_{C_2,1_1} - 2e_{C_2,1_2} - 2e_{C_2,1_3}, \\
[C_2, t_2] &= 6e_{1,1} + 6e_{1,1'} + 6e_{1,1''} - 2e_{1,3} + 2e_{C_2,1} - 2e_{C_2,1_1} + 2e_{C_2,1_2} - 2e_{C_2,1_3}, \\
[C_2, t_3] &= 6e_{1,1} + 6e_{1,1'} + 6e_{1,1''} - 2e_{1,3} + 2e_{C_2,1} - 2e_{C_2,1_1} - 2e_{C_2,1_2} + 2e_{C_2,1_3}, \\
[C_3, 1] &= 4e_{1,1} + 4e_{1,1'} + 4e_{1,1''} + 4e_{1,3} + e_{C_3,1} + e_{C_3,1^+} + e_{C_3,1^-}, \\
[C_3, s] &= 4e_{1,1} + 4\omega^+e_{1,1'} + 4\omega^-e_{1,1''} + e_{C_3,1} + \omega^-e_{C_3,1^+} + \omega^+e_{C_3,1^-}, \\
[C_3, s'] &= 4e_{1,1} + 4\omega^-e_{1,1'} + 4\omega^+e_{1,1''} + e_{C_3,1} + \omega^+e_{C_3,1^+} + \omega^-e_{C_3,1^-}, \\
[E_4, 1] &= 3e_{1,1} + 3e_{1,1'} + 3e_{1,1''} + 3e_{1,3} + 3e_{C_2,1} + 3e_{C_2,1_1} + 3e_{C_2,1_2} + 3e_{C_2,1_3} \\
&\quad + 3e_{E_4,1} + 3e_{E_4,1_1}, \\
[E_4, t] &= 3e_{1,1} + 3e_{1,1'} + 3e_{1,1''} - e_{1,3} + 3e_{C_2,1} - e_{C_2,1_1} - e_{C_2,1_2} - e_{C_2,1_3} \\
&\quad + 3e_{E_4,1} - e_{E_4,1_1}, \\
[A_4, 1] &= e_{1,1} + e_{1,1'} + e_{1,1''} + e_{1,3} + e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&\quad + e_{C_3,1} + e_{C_3,1^+} + e_{C_3,1^-} + e_{E_4,1} + e_{E_4,1_1} + e_{A_4,1}.
\end{aligned}$$

(I) Primitive idempotents of $\mathbb{A}_p X\Omega(G, G^c)$ for $G = A_4$.

(i) **Case : $p = 2$.** There are five primitive idempotents.

$$\begin{aligned}
e_{A_4,1} &= [A_4, 1] - (1/3)[E_4, 1] - [C_3, 1] + (1/3)[1, 1], \\
e_{E_4,1} + e_{E_4,1_1} + e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&\quad + e_{1,1} + e_{1,1'} + e_{1,1''} + e_{1,3} = (1/3)[E_4, 1]
\end{aligned}$$

$$\begin{aligned}
e_{C_3,1} &= (1/3)[C_3, 1] + (1/3)[C_3, s] + (1/3)[C_3, s'] \\
&\quad - (1/9)[1, 1] - (1/9)[1, s] - (1/9)[1, s'], \\
e_{C_3,1+} &= (1/3)[C_3, 1] + (\omega^+/3)[C_3, s] + (\omega^-/3)[C_3, s'] \\
&\quad - (1/9)[1, 1] - (\omega^+/9)[1, s] - (\omega^-/9)[1, s'], \\
e_{C_3,1-} &= (1/3)[C_3, 1] + (\omega^-/3)[C_3, s] + (\omega^+/3)[C_3, s'] \\
&\quad - (1/9)[1, 1] - (\omega^-/9)[1, s] - (\omega^+/9)[1, s'].
\end{aligned}$$

(ii) **Case : $p = 3$.** There are seven primitive idempotents.

$$\begin{aligned}
e_{A_4,1} + e_{E_4,1} + e_{E_4,1_1} &= (1/2)[1, 1] - (1/2)[C_2, 1] + [A_4, 1], \\
e_{C_3,1} + e_{C_3,1+} + e_{C_3,1-} + e_{1,1} + e_{1,1'} + e_{1,1''} \\
&= -(5/16)[1, 1] + (1/16)[1, t] + [C_3, 1], \\
e_{C_2,1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] - (3/16)[1, t], \\
e_{C_2,1_1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,1_2} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] + (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,1_3} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] - (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{1,3} &= (1/16)[1, 1] - (1/16)[1, t].
\end{aligned}$$

(6.4) A_5 . (A) The alternating group A_5 of degree 5 has nine conjugacy classes of subgroups represented by

$$1, C_2, C_3, E_4, C_5, S_3, D_{10}, A_4, A_5 = G.$$

The following is the table of marks of the Burnside ring $\Omega(A_5)$:

	1	C_2	C_3	E_4	C_5	S_3	D_{10}	A_4	A_5
$A_5/1$	60								
A_5/C_2	30	2							
A_5/C_3	20		2						
A_5/E_4	15	3		3					
A_5/C_5	12				2				
A_5/S_3	10	2	1			1			
A_5/D_{10}	6	2			1		1		
A_5/A_4	5	1	2	1				1	
A_5/A_5	1	1	1	1	1	1	1	1	1

(B) Some informations about $k[CH]^{NH}$ for $H \leq G = A_4$.

H	1	C_2	C_3	E_4	C_5	S_3	D_{10}	A_4	A_5
CH	A_5	E_4	C_3	E_4	C_5	1	1	1	1
NH	A_5	E_4	S_3	A_4	D_{10}	S_3	D_{10}	A_4	A_5
$\dim \mathbb{K}[CH]^{NH}$	5	4	2	2	3	1	1	1	1
basis	$1, \hat{t}, \hat{s}, \hat{r}, \hat{r}'$	$1, t_1, t_2, t_3$	$1, s^*$	$1, t^*$	$1, r^*, r'^*$	1	1	1	1

Thus $\mathbb{K}X\Omega(G, G^c)$ is of dimalional 20. In this table, we used the following symbols:

$$\begin{aligned}
t &= (12)(34), s = (123), r = (12345), r' = r^2 = (13524), \\
t_1 &= t = (12)(34), t_2 = (13)(24), t_3 = (14)(23), (E_4 = \{1, t_1, t_2, t_3\}), \\
s^* &= s + s^{-1}, t^* = t_1 + t_2 + t_3, r^* = r + r^{-1}, r'^* = r^2 + r^3, \\
\hat{x} &= \sum_{y \sim x} y \quad (\text{class sum for any } x \in A_5).
\end{aligned}$$

(C) The table of marks of $X\Omega(G, G^c)$ for $G = A_5$. Note that A_5 has five conjugacy classes of elements and they are represented by $1, t, s, r, r'$.

	1	C_2	C_3	E_4	C_5	S_3	D_{10}	A_4	A_5
$[1, 1]$	60								
$[1, t]$	$4\widehat{t}$								
$[1, s]$	$3\widehat{s}$								
$[1, r]$	$5\widehat{r}$								
$[1, r']$	$5\widehat{r}'$								
$[C_2, 1]$	30	2							
$[C_2, t_1]$	$2\widehat{t}$	$2t_1$							
$[C_2, t_2]$	$2\widehat{t}$	$2t_2$							
$[C_2, t_3]$	$2\widehat{t}$	$2t_3$							
$[C_3, 1]$	20		2						
$[C_3, s]$	$\widehat{s}$		s^*						
$[E_4, 1]$	15	3		3					
$[E_4, t]$	$\widehat{t}$	t^*		t^*					
$[C_5, 1]$	12				2				
$[C_5, r]$	$\widehat{r}$				r				
$[C_5, r']$	$\widehat{r}'$				r'				
$[S_3, 1]$	10	2	1			1			
$[D_{10}, 1]$	6	2			1		1		
$[A_4, 1]$	5	1	2	1				1	
$[A_5, 1]$	1	1	1	1	1	1	1	1	1

(D) The multiplication table of $Z(\mathbb{K}[A_5])$.

$$\begin{aligned}
\widehat{t} \cdot \widehat{t} &= 15 + 2\widehat{t} + 3\widehat{s} + 5\widehat{r} + 5\widehat{r}', \\
\widehat{t} \cdot \widehat{s} &= 4\widehat{t} + 6\widehat{s} + 5\widehat{r} + 5\widehat{r}', \\
\widehat{t} \cdot \widehat{r} &= 4\widehat{t} + 3\widehat{s} + 5\widehat{r}', \\
\widehat{t} \cdot \widehat{r}' &= 4\widehat{t} + 3\widehat{s} + 5\widehat{r}, \\
\widehat{s} \cdot \widehat{s} &= 20 + 8\widehat{t} + 7\widehat{s} + 5\widehat{r} + 5\widehat{r}', \\
\widehat{s} \cdot \widehat{r} &= 4\widehat{t} + 3\widehat{s} + 5\widehat{r} + 5\widehat{r}', \\
\widehat{s} \cdot \widehat{r}' &= 4\widehat{t} + 3\widehat{s} + 5\widehat{r} + 5\widehat{r}', \\
\widehat{r} \cdot \widehat{r} &= 12 + 3\widehat{s} + 5\widehat{r} + \widehat{r}', \\
\widehat{r} \cdot \widehat{r}' &= 4\widehat{t} + 3\widehat{s} + \widehat{r} + \widehat{r}', \\
\widehat{r}' \cdot \widehat{r}' &= 12 + 3\widehat{s} + \widehat{r} + 5\widehat{r}'.
\end{aligned}$$

(E) The multiplication table of $X\Omega(G, G^c)$ for $G = A_5$. Omitted.

(F) There are twenty primitive idempotents in $X\Omega(G, G^c)$ for $G = A_5$.

$$\begin{aligned}
e_{1,1} &= (1/3600)[1, 1] + (1/240)[1, t] + (1/180)[1, s] \\
&\quad + (1/300)[1, r] + (1/300)[1, r'].
\end{aligned}$$

$$\begin{aligned}
e_{1,3} &= (1/400)[1, 1] - (1/80)[1, t] + (\epsilon_1^+/100)[1, r] + (\epsilon_1^-/100)[1, r'], \\
e_{1,3'} &= (1/400)[1, 1] - (1/80)[1, t] + (\epsilon_1^-/100)[1, r] + (\epsilon_1^+/100)[1, r'], \\
e_{1,4} &= (1/225)[1, 1] + (1/45)[1, s] - (1/75)[1, r] - (1/75)[1, r'], \\
e_{1,5} &= (1/144)[1, 1] + (1/48)[1, t] - (1/36)[1, s], \\
e_{C_2,1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] - (3/16)[1, t], \\
e_{C_2,11} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,12} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] + (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,13} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] - (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_3,1} &= (1/6)[C_3, 1] + (1/3)[C_3, s] - (1/18)[1, 1] - (1/9)[1, s], \\
e_{C_3,1'} &= (1/3)[C_3, 1] - (1/3)[C_3, s] - (1/9)[1, 1] + (1/9)[1, s], \\
e_{E_4,1} &= (1/12)[E_4, 1] + (1/4)[E_4, t] - (1/8)[C_2, 1] - (1/8)[C_2, t_1] \\
&\quad - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] + (1/24)[1, 1] + (1/8)[1, t], \\
e_{E_4,1'} &= (1/4)[E_4, 1] - (1/4)[E_4, t] - (3/8)[C_2, 1] + (1/8)[C_2, t_1] \\
&\quad + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] + (1/8)[1, 1] - (1/8)[1, t], \\
e_{C_5,1} &= (1/10)[C_5, 1] + (1/5)[C_5, r] + (1/5)[C_5, r'] \\
&\quad - (1/50)[1, 1] - (1/25)[1, r] - (1/25)[1, r'], \\
e_{C_5,1'} &= (1/5)[C_5, 1] + (\epsilon_2^+/5)[C_5, r] + (\epsilon_2^-/5)[C_5, r'] \\
&\quad - (1/25)[1, 1] - (\epsilon_2^+/25)[1, r] - (\epsilon_2^-/25)[1, r'], \\
e_{C_5,1''} &= (1/5)[C_5, 1] + (\epsilon_2^-/5)[C_5, r] + (\epsilon_2^+/5)[C_5, r'] \\
&\quad - (1/25)[1, 1] - (\epsilon_2^-/25)[1, r] - (\epsilon_2^+/25)[1, r'], \\
e_{S_3,1} &= [S_3, 1] - (1/2)[C_3, 1] - [C_2, 1] + (1/2)[1, 1], \\
e_{D_{10},1} &= [D_{10}, 1] - (1/2)[C_5, 1] - [C_2, 1] + (1/2)[1, 1], \\
e_{A_4,1} &= [A_4, 1] - (1/3)[E_4, 1] - [C_3, 1] + (1/3)[1, 1], \\
e_{A_5,1} &= [A_5, 1] - [A_4, 1] - [D_{10}, 1] - [S_3, 1] + [C_3, 1] + 2[C_2, 1] - [1, 1].
\end{aligned}$$

Here, we used the symbols $\epsilon_1^\pm, \epsilon_2^\pm$ defined as follows:

$$\epsilon_1^\pm = \frac{1 \pm \sqrt{5}}{2}, \quad \epsilon_2^\pm = \frac{-1 \pm \sqrt{5}}{2}.$$

(G) The decompositions of idempotents of $\mathbb{K}\Omega(G)$ in $\mathbb{K}X\Omega(G, G^c)$ for $G = A_5$. The Burnside ring $\mathbb{K}\Omega(A_5)$ has nine primitive idempotents:

$$e_1, e_{C_2}, e_{C_3}, e_{E_4}, e_{C_5}, e_{S_3}, e_{D_{10}}, e_{A_4}, e_{A_5}.$$

The images of these idempotents by the embedding η_l is decomposed as follows:

$$\begin{aligned}
\eta_l(e_{A_5}) &= e_{A_5,1} \\
&= [A_5, 1] - [A_4, 1] - [D_{10}, 1] - [S_3, 1] + [C_3, 1] + 2[C_2, 1] - [1, 1], \\
\eta_l(e_{A_4}) &= e_{A_4,1} \\
&= [A_4, 1] - (1/3)[E_4, 1] - [C_3, 1] + (1/3)[1, 1], \\
\eta_l(e_{D_{10}}) &= e_{D_{10},1} \\
&= [D_{10}, 1] - (1/2)[C_5, 1] - [C_2, 1] + (1/2)[1, 1], \\
\eta_l(e_{S_3}) &= e_{S_3,1} \\
&= [S_3, 1] - (1/2)[C_3, 1] - [C_2, 1] + (1/2)[1, 1], \\
\eta(e_{C_5}) &= e_{C_5,1} + e_{C_5,1'} + e_{C_5,1''} \\
&= (1/2)[C_5, 1] - (1/10)[1, 1], \\
\eta_l(e_{E_4}) &= e_{E_4,1} + e_{E_4,1'} \\
&= (1/3)[E_4, 1] - (1/2)[C_2, 1] + (1/6)[1, 1], \\
\eta(e_{C_3}) &= e_{C_3,1} + e_{C_3,1'} \\
&= (1/2)[C_3, 1] - (1/6)[1, 1], \\
\eta(e_{C_2}) &= e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&= (1/2)[C_2, 1] - (1/4)[1, 1], \\
\eta(e_1) &= e_{1,1} + e_{1,3} + e_{1,3'} + e_{1,4} + e_{1,5} \\
&= (1/60)[1, 1].
\end{aligned}$$

(H) The expression of the transitive crossed G -sets as linear combinations of primitive idempotents of $X\Omega(G, G^c)$ for $G = A_5$.

$$\begin{aligned}
[1, 1] &= 60e_{1,1} + 60e_{1,3} + 60e_{1,3'} + 60e_{1,4} + 60e_{1,5}, \\
[1, t] &= 60e_{1,1} - 20e_{1,3} - 20e_{1,3'} + 12e_{1,5}, \\
[1, s] &= 60e_{1,1} + 15e_{1,4} - 12e_{1,5}, \\
[1, r] &= 60e_{1,1} + 20\epsilon_1^+ e_{1,3} + 20\epsilon_1^- e_{1,3'} - 15e_{1,4}, \\
[1, r'] &= 60e_{1,1} + 20\epsilon_1^- e_{1,3} + 20\epsilon_1^+ e_{1,3'} - 15e_{1,4}, \\
[C_2, 1] &= 30e_{1,1} + 30e_{1,3} + 30e_{1,3'} + 30e_{1,4} + 2e_{1,5} + 2e_{C_2,1} \\
&\quad + 2e_{C_2,1_1} + 2e_{C_2,1_2} + 2e_{C_2,1_3}, \\
[C_2, t_1] &= 30e_{1,1} - 10e_{1,3} - 10e_{1,3'} + 6e_{1,5} + 2e_{C_2,1} \\
&\quad + 2e_{C_2,1_1} - 2e_{C_2,1_2} - 2e_{C_2,1_3}, \\
[C_2, t_2] &= 30e_{1,1} - 10e_{1,3} - 10e_{1,3'} + 6e_{1,5} + 2e_{C_2,1} \\
&\quad - 2e_{C_2,1_1} + 2e_{C_2,1_2} - 2e_{C_2,1_3}, \\
[C_2, t_3] &= 30e_{1,1} - 10e_{1,3} - 10e_{1,3'} + 6e_{1,5} + 2e_{C_2,1} \\
&\quad - 2e_{C_2,1_1} - 2e_{C_2,1_2} + 2e_{C_2,1_3},
\end{aligned}$$

$$\begin{aligned}
[C_3, 1] &= 20e_{1,1} + 20e_{1,3} + 20e_{1,3'} + 20e_{1,4} + 20e_{1,5} + 2e_{C_3,1} + 2e_{C_3,1_1}, \\
[C_3, s] &= 20e_{1,1} + 5e_{1,4} - 4e_{1,5} + 2e_{C_3,1} - e_{C_3,1_1}, \\
[E_4, 1] &= 15e_{1,1} + 15e_{1,3} + 15e_{1,3'} + 15e_{1,4} + 15e_{1,5} \\
&\quad + 3e_{C_2,1} + 3e_{C_2,1_1} + 3e_{C_2,1_2} + 3e_{C_2,1_3} + 3e_{E_4,1} + 3e_{E_4,1'}, \\
[E_4, t] &= 15e_{1,1} - 5e_{1,3} - 5e_{1,3'} + 3e_{1,5} \\
&\quad + 3e_{C_2,1} - e_{C_2,1_1} - e_{C_2,1_2} - e_{C_2,1_3} + 3e_{E_4,1} - e_{E_4,1'}, \\
[C_5, 1] &= 12e_{1,1} + 12e_{1,3} + 12e_{1,3'} + 12e_{1,4} + 12e_{1,5} + 2e_{C_5,1} + 2e_{C_5,1'} + 2e_{C_5,1''}, \\
[C_5, r] &= 12e_{1,1} + 4\epsilon_1^+ e_{1,3} + 4\epsilon_1^- e_{1,3'} - 3e_{1,4} + 2e_{C_5,1} + \epsilon_2^+ e_{C_5,1'} + \epsilon_2^- e_{C_5,1''}, \\
[C_5, r'] &= 12e_{1,1} + 4\epsilon_1^- e_{1,3} + 4\epsilon_1^+ e_{1,3'} - 3e_{1,4} + 2e_{C_5,1} + \epsilon_2^- e_{C_5,1'} + \epsilon_2^+ e_{C_5,1''}, \\
[S_3, 1] &= 10e_{1,1} + 10e_{1,3} + 10e_{1,3'} + 10e_{1,4} + 10e_{1,5} + 2e_{C_2,1} \\
&\quad + 2e_{C_2,1_1} + 2e_{C_2,1_2} + 2e_{C_2,1_3} + e_{C_3,1} + e_{C_3,1_1} + e_{S_3,1}, \\
[D_{10}, 1] &= 6e_{1,1} + 6e_{1,3} + 6e_{1,3'} + 6e_{1,4} + 6e_{1,5} + 2e_{C_2,1} \\
&\quad + 2e_{C_2,1_1} + 2e_{C_2,1_2} + 2e_{C_2,1_3} + e_{C_5,1} + e_{C_5,1'} + e_{C_5,1''} + e_{D_{10},1}, \\
[A_4, 1] &= 5e_{1,1} + 5e_{1,3} + 5e_{1,3'} + 5e_{1,4} + 5e_{1,5} + e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&\quad + 2e_{C_3,1} + 2e_{C_3,1_1} + e_{E_4,1} + e_{E_4,1'} + e_{A_4,1}, \\
[A_5, 1] &= e_{1,1} + e_{1,3} + e_{1,3'} + e_{1,4} + e_{1,5} + e_{C_2,1} + e_{C_2,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&\quad + e_{C_3,1} + e_{C_3,1_1} + e_{E_4,1} + e_{E_4,1'} + e_{C_5,1} + e_{C_5,1'} + e_{C_5,1''} \\
&\quad + e_{S_3,1} + e_{D_{10},1} + e_{A_4,1} + e_{A_5,1}.
\end{aligned}$$

(I) Primitive idempotents of $\mathbb{A}_p X\Omega(G, G^c)$ for $G = A_5$.

(i) **Case : $p = 2$.** There are ten primitive idempotents.

$$\begin{aligned}
e_{A_5,1} &= [A_5, 1] - [A_4, 1] - [D_{10}, 1] - [S_3, 1] + [C_3, 1] + 2[C_2, 1] - [1, 1], \\
e_{A_4,1} &= [A_4, 1] - (1/3)[E_4, 1] - [C_3, 1] + (1/3)[1, 1], \\
e_{D_{10},1} + e_{C_5,1} &= [D_{10}, 1] - (2/5)[C_5, 1] + (1/5)[C_5, r] + (1/5)[C_5, r'] \\
&\quad - [C_2, 1] + (12/25)[1, 1] - (1/25)[1, r] - (1/25)[1, r'], \\
e_{S_3,1} + e_{C_3,1} &= [S_3, 1] - (1/3)[C_3, 1] + (1/3)[C_3, s] - [C_2, 1] \\
&\quad + (4/9)[1, 1] - (1/9)[1, s], \\
e_{C_5,1'} &= (1/5)[C_5, 1] + (\epsilon_2^+/5)[C_5, r] + (\epsilon_2^-/5)[C_5, r'] \\
&\quad - (1/25)[1, 1] - (\epsilon_2^+/25)[1, r] - (\epsilon_2^-/25)[1, r'], \\
e_{C_5,1''} &= (1/5)[C_5, 1] + (\epsilon_2^-/5)[C_5, r] + (\epsilon_2^+/5)[C_5, r'] \\
&\quad - (1/25)[1, 1] - (\epsilon_2^-/25)[1, r] - (\epsilon_2^+/25)[1, r'], \\
e_{E_4,1} + e_{E_4,1'} + e_{C_3,1} + e_{C_3,1_1} + e_{C_2,1_2} + e_{C_2,1_3} \\
&\quad + e_{1,1} + e_{1,3} + e_{1,3'} + e_{1,5} \\
&= (1/3)[E_4, 1] - (16/225)[1, 1] - (1/45)[1, s] + (1/75)[1, r] + (1/75)[1, r'], \\
e_{C_3,1'} &= (1/3)[C_3, 1] - (1/3)[C_3, s] - (1/9)[1, 1] + (1/9)[1, s],
\end{aligned}$$

$$e_{1,4} = (1/225)[1, 1] + (1/45)[1, s] - (1/75)[1, r] - (1/75)[1, r'].$$

(2) **Case : $p = 3$.** There are fifteen primitive idempotents.

$$\begin{aligned} e_{1,1} + e_{1,4} + e_{1,5} + e_{C_3,1} + e_{C_3,1'} &= -(31/200)[1, 1] \\ &\quad + (1/40)[1, t] - (1/100)[1, r] - (1/100)[1, r'] + (1/2)[C_3, 1] \\ e_{1,3} &= (1/400)[1, 1] - (1/80)[1, t] + (\epsilon_1^+/100)[1, r] + (\epsilon_1^-/100)[1, r'], \\ e_{1,3'} &= (1/400)[1, 1] - (1/80)[1, t] + (\epsilon_1^-/100)[1, r] + (\epsilon_1^+/100)[1, r'], \\ e_{C_2,1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\ &\quad - (1/16)[1, 1] - (3/16)[1, t], \\ e_{C_2,1_1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\ &\quad - (1/16)[1, 1] + (1/16)[1, t], \\ e_{C_2,1_2} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] + (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\ &\quad - (1/16)[1, 1] + (1/16)[1, t], \\ e_{C_2,1_3} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] - (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\ &\quad - (1/16)[1, 1] + (1/16)[1, t], \\ e_{E_4,1} + e_{A_4,1} &= [A_4, 1] - (1/4)[E_4, 1] + (1/4)[E_4, t] - [C_3, 1] - (1/8)[C_2, 1] \\ &\quad - (1/8)[C_2, t_1] - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] + (3/8)[1, 1] + (1/8)[1, t], \\ e_{E_4,1'} &= (1/4)[E_4, 1] - (1/4)[E_4, t] - (3/8)[C_2, 1] + (1/8)[C_2, t_1] \\ &\quad + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] + (1/8)[1, 1] - (1/8)[1, t], \\ e_{C_5,1} &= (1/10)[C_5, 1] + (1/5)[C_5, r] + (1/5)[C_5, r'] \\ &\quad - (1/50)[1, 1] - (1/25)[1, r] - (1/25)[1, r'], \\ e_{C_5,1'} &= (1/5)[C_5, 1] + (\epsilon_2^+/5)[C_5, r] + (\epsilon_2^-/5)[C_5, r'] \\ &\quad - (1/25)[1, 1] - (\epsilon_2^+/25)[1, r] - (\epsilon_2^-/25)[1, r'], \\ e_{C_5,1''} &= (1/5)[C_5, 1] + (\epsilon_2^-/5)[C_5, r] + (\epsilon_2^+/5)[C_5, r'] \\ &\quad - (1/25)[1, 1] - (\epsilon_2^-/25)[1, r] - (\epsilon_2^+/25)[1, r'], \\ e_{S_3,1} &= [S_3, 1] - (1/2)[C_3, 1] - [C_2, 1] + (1/2)[1, 1], \\ e_{D_{10},1} &= [D_{10}, 1] - (1/2)[C_5, 1] - [C_2, 1] + (1/2)[1, 1], \\ e_{A_5,1} &= [A_5, 1] - [A_4, 1] - [D_{10}, 1] - [S_3, 1] + [C_3, 1] + 2[C_2, 1] - [1, 1]. \end{aligned}$$

(3) **Case : $p = 5$.** There are fourteen primitive idempotents.

$$\begin{aligned} e_{1,1} + e_{1,3} + e_{1,3'} + e_{1,4} + e_{C_5,1} + e_{C_5,1'} + e_{C_5,1''} \\ = (1/2)[C_5, 1] - (13/144)[1, 1] - (1/48)[1, t] + (1/36)[1, s] \\ e_{1,5} = (1/144)[1, 1] + (1/48)[1, t] - (1/36)[1, s], \end{aligned}$$

$$\begin{aligned}
e_{C_2,1} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] - (3/16)[1, t], \\
e_{C_2,11} &= (1/8)[C_2, 1] + (1/8)[C_2, t_1] - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,12} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] + (1/8)[C_2, t_2] - (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_2,13} &= (1/8)[C_2, 1] - (1/8)[C_2, t_1] - (1/8)[C_2, t_2] + (1/8)[C_2, t_3] \\
&\quad - (1/16)[1, 1] + (1/16)[1, t], \\
e_{C_3,1} &= (1/6)[C_3, 1] + (1/3)[C_3, s] - (1/18)[1, 1] - (1/9)[1, s], \\
e_{C_3,1'} &= (1/3)[C_3, 1] - (1/3)[C_3, s] - (1/9)[1, 1] + (1/9)[1, s], \\
e_{E_4,1} &= (1/12)[E_4, 1] + (1/4)[E_4, t] - (1/8)[C_2, 1] - (1/8)[C_2, t_1] \\
&\quad - (1/8)[C_2, t_2] - (1/8)[C_2, t_3] + (1/24)[1, 1] + (1/8)[1, t], \\
e_{E_4,1'} &= (1/4)[E_4, 1] - (1/4)[E_4, t] - (3/8)[C_2, 1] + (1/8)[C_2, t_1] \\
&\quad + (1/8)[C_2, t_2] + (1/8)[C_2, t_3] + (1/8)[1, 1] - (1/8)[1, t], \\
e_{S_3,1} &= [S_3, 1] - (1/2)[C_3, 1] - [C_2, 1] + (1/2)[1, 1], \\
e_{D_{10},1} &= [D_{10}, 1] - (1/2)[C_5, 1] - [C_2, 1] + (1/2)[1, 1], \\
e_{A_4,1} &= [A_4, 1] - (1/3)[E_4, 1] - [C_3, 1] + (1/3)[1, 1], \\
e_{A_5,1} &= [A_5, 1] - [A_4, 1] - [D_{10}, 1] - [S_3, 1] + [C_3, 1] + 2[C_2, 1] - [1, 1].
\end{aligned}$$

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