Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric

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ABSTRACT

R. Jordan, D. Kinderlehrer and F. Otto proposed the discrete-time approximation of the Fokker-Planck equation by the variational formulation. It is determined by the Wasserstein metric, an energy functional and the Gibbs-Boltzmann entropy functional. In this paper we study the asymptotic behavior of the dynamical systems which describe their approximation of the Fokker-Planck equation and characterize the limit as a solution to a class of variational problems. We also give a simple approach in a one-dimensional case.

1. Introduction.

Let d denote the Wasserstein metric (or distance) defined by the following (see [23] or [4], [5], [9]): for Borel probability measures P, Q on R^d, put

$$\begin{aligned} d(P,Q) \equiv \inf\{(\int_{\mathbf{R}^d\times\mathbf{R}^d} |x-y|^2\mu(dxdy))^{1/2} : \\ \mu(dx\times\mathbf{R}^d) = P(dx), \mu(\mathbf{R}^d\times dy) = Q(dy)\}. \end{aligned} \tag{1.1}$$

In particular, we put $d(p,q) \equiv d(P,Q)$ when $P(dx) = p(x)dx$ and $Q(dx) = q(x)dx$.

Next we introduce the assumption used by R. Jordan, D. Kinderlehrer and F. Otto in [11].

(A.1). $\Psi \in C^\infty(\mathbf{R}^d; [0,\infty))$ and

$$\sup_{x\in\mathbf{R}^d}\{|\nabla\Psi(x)|/(\Psi(x)+1)\} < \infty.$$

(A.2). $p_0(x)$ is a probability density function on $\mathbf{R}^d$ and the following holds:

$$M(p_0) \equiv \int_{\mathbf{R}^d} |x|^2 p_0(x)dx < \infty, \qquad (1.2).$$

$$F(p_0) \equiv \int_{\mathbf{R}^d} (\log p_0(x) + \Psi(x))p_0(x)dx < \infty. \qquad (1.3).$$

Here we put $\nabla \equiv (\partial/\partial x_i)_{i=1}^d$.

Under (A.1)-(A.2), for $h > 0$, we can define, a sequence of probability density functions $\{p_h^n\}_{n\geq 0}$ on $\mathbf{R}^d$, inductively, by the following: put $p_h^0 = p_0$, and for p_h^n, determine p_h^{n+1} as the minimizer of

$$d(p_h^n, p)^2/2 + hF(p) \qquad (1.4).$$

over all probability density functions p for which $M(p)$ is finite (see [11, Proposition 4.1]).

For a probability density function p on $\mathbf{R}^d$, put

$$E(p) \equiv \int_{\mathbf{R}^d} \Psi(x)p(x)dx, \qquad (1.5).$$

$$S(p) \equiv \int_{\mathbf{R}^d} \log p(x)p(x)dx, \qquad (1.6).$$

and for $h \in (0,1)$, $t \geq 0$ and $x \in \mathbf{R}^d$, put

$$p_h(t,x) \equiv p_h^{[t/h]}(x), \qquad (1.7).$$

where $[r]$ denotes the integer part of $r \in \mathbf{R}$. Then the following is known (see [11], [21] and the references therein for an application to physics).

Theorem 1.1. *([11, Theorem 5.1]). Suppose that (A.1)-(A.2) hold. Then as $h \to 0$, $p_h(T,\cdot)$ converges to $p(T,\cdot)$ weakly in $L^1(\mathbf{R}^d; dx)$, and p_h converges to p strongly in $L^1([0,T] \times \mathbf{R}^d; dtdx)$ for all $T > 0$, where $p(t,x) \in C^\infty((0,\infty) \times \mathbf{R}^d; [0,\infty))$ is the unique solution of*

$$\partial p(t,x)/\partial t = \triangle_x p(t,x) + div_x(\nabla\Psi(x)p(t,x)) \quad (t > 0, x \in \mathbf{R}^d), \qquad (1.8).$$

with an initial condition

$$p(t,\cdot) \to p_0, \quad \text{strongly in } L^1(\mathbf{R}^d; dx), \text{ as } t \to 0, \tag{1.9.}$$

and $M(p(t,\cdot))$, $E(p(t,\cdot))$ and $S(p(t,\cdot))$ belong to $L^\infty([0,T];dt)$ for all $T > 0$. Here we put $\triangle_x \equiv \sum_{i=1}^d \partial^2/\partial x_i^2$, and $div_x(\cdot) \equiv < (\partial/\partial x_i)_{i=1}^d, \cdot >$.

REMARK 1.1. Since $p_h(t,\cdot)$ is a probability density function on $\mathbf{R}^d$, so is $p(t,\cdot)$ for $t \geq 0$.

For $p_h^n(x)$ and $p_h^{n+1}(x)$, there exists a lower semicontinuous convex function $\varphi_h^{n+1}(x)$ such that

$$p_h^n(x)\delta_{\nabla\varphi_h^{n+1}(x)}(dy)dx \tag{1.10.}$$

is the minimizer of $d(p_h^n, p_h^{n+1})$ (see [9], and also [4], [5], [23]). In other word,

$$d(p_h^n, p_h^{n+1})^2 = \int_{\mathbf{R}^d} |x - \nabla\varphi_h^{n+1}(x)|^2 p_h^n(x)dx, \tag{1.11.}$$

and for any $f \in C_b(\mathbf{R}^d; \mathbf{R})$

$$\int_{\mathbf{R}^d} f(\nabla\varphi_h^{n+1}(x))p_h^n(x)dx = \int_{\mathbf{R}^d} f(y)p_h^{n+1}(y)dy. \tag{1.12.}$$

On the probability space $(\mathbf{R}^d, \mathbf{B}(\mathbf{R^d}), P_0(dx) \equiv p_0(x)dx)$, put for $h \in (0,1)$, $t \geq 0$ and $x \in \mathbf{R}^d$,

$$\begin{aligned} X^h(0,x) &= \nabla\varphi_h^0(x) \equiv x, \\ X^h(t,x) &= \nabla\varphi_h^{[t/h]}(X^h(\max([t/h]-1,0)h,x)). \end{aligned} \tag{1.13.}$$

In this paper we first give a stochastic representation for $p(t,x)$ (see Theorem 2.1) from which we give the estimate for $\nabla_x \log p(t,x)$ (see Theorem 2.2) . In the proof, we use exponential estimates on large deviations and the idea in [26] where they gave estimates for the derivatives of the transition probability density functions of diffusion processes (see section 4). By this estimate and an assumption on Ψ (see section 2), we can construct the solution to the following: for $x \in \mathbf{R}^d$,

$$dX(t,x)/dt = -\nabla_x \log p(t, X(t,x)) - \nabla\Psi(X(t,x)) \quad (t>0),$$
$$X(0,x) = x. \tag{1.14}$$

We also show that $X^h(t,x)$ converges to $X(t,x)$, as $h \to 0$. In particular, it can be shown that $P_0^{X(t,\cdot)^{-1}}(dx) = p(t,x)dx$ for $t \geq 0$ (see Theorem 2.3). This is conjecturable by the Euler equation to (1.4). It can be written, formally, as the following: for $n \geq 0$,

$$X^h((n+1)h,x) - X^h(nh,x) \tag{1.15}$$
$$= -h\{\nabla \log p_h^{n+1}(X^h((n+1)h,x)) + \nabla\Psi(X^h((n+1)h,x))\}$$

(see Lemma 5.3 in section 5 for the exact statement of (1.15)).

We also characterize $X(t,x)$ as the solution to a class of variational problems. Theorems 2.3-2.5 in section 2 can be considered as the version of [14] in a case where stochastic processes do not have random time evolution.

Let us introduce [14] to state the problem more precisely (see also [6], [13] and [15]). First we introduce the following notation. For a probability space $(\Omega, \mathbf{B}, P)$, and a right continuous, increasing sub σ-fields $\{\mathbf{B}_t\}_{t\geq 0}$ of $\mathbf{B}$, and a d-dimensional $(\mathbf{B}_t)$-Wiener process $\overline{W}(t)$, and a $(\mathbf{B}_t)$-progressively measurable stochastic process $u(t)$, and a $\mathbf{R}^d$-valued random variable X_o on $(\Omega, \mathbf{B}, P)$ such that $P(X_o \in dx) = p_0(x)dx$ and that is independent of $\{\overline{W}(t), u(t)\}_{t\geq 0}$, put

$$X^u(t) = X_o + \int_0^t u(s)ds + \overline{W}(t) \quad (t \geq 0). \tag{1.16}$$

Put, for $T>0$,

$$A^T \equiv \{\{u(t)\}_{0\leq t\leq T} : P^{X^u(t)^{-1}}(dx) = p(t,x)dx, dt - a.e. \text{ on } [0,T]\}. \tag{1.17}$$

Then the following is known (see also [2], [3], [17], [22] and [28]).

Theorem 1.2. *([14]). Suppose that (1.8) has a weak solution such that*

$$
\begin{aligned}
p(0,x) = p_0(x) \quad (x \in \mathbf{R}^d),& \\
\int_{\mathbf{R}^d} p(t,x)dx = 1 \quad (0 \le t \le T),& \\
\int_0^T \int_{\mathbf{R}^d} |\nabla \Psi(x)|^2 p(t,x)dxdt < \infty,&
\end{aligned}
\qquad (1.18).
$$

for some $T > 0$. Then there exists a measurable function $b(t,x) : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ such that the following stochastic integral equation has a unique weak solution:

$$X(t) = X_o + \int_0^t b(s, X(s))ds + \overline{W}(t) \quad (0 \le t \le T), \qquad (1.19).$$

and that

$$P^{X(t)^{-1}}(dx) = p(t,x)dx \quad (0 \le t \le T). \qquad (1.20).$$

Moreover for any $\{u(t)\}_{0 \le t \le T} \in A^T$

$$E[\int_0^T |b(t, X(t))|^2 dt] \le E[\int_0^T |u(t)|^2 dt], \qquad (1.21).$$

where the equality holds if and only if $u(t) = b(t, X(t))$ $dtdP$-a.e..

$u(t)$ in (1.16) is called the mean forward derivative (see [19] and [20]) as the following holds:

$$u(t) = \lim_{h \to 0} E[\{X^u(t+h) - X^u(t)\}/h | \mathbf{B}_t] \quad dtdP - a.s.. \qquad (1.22).$$

If the stochastic processes under consideration do not have random time evolution, then the mean forward derivatives should be replaced by the time derivatives in (1.21). In this case the similar result to Theorem 1.2 is given in section 2 (see Theorems 2.3-2.4).

$X(t)$ in Theorem 1.2 is a limit, in the relative entropy, of $X^n(t)$ determined by the following (see [14]). Take sequences of rational numbers

$$Q_n \equiv \{0, T, q_i^n\}_{i=1,\cdots,n} \tag{1.23}$$

such that $Q_n \subset Q_{n+1}$ $(n \geq 1)$ and that $\cup_{n\geq 1} Q_n \setminus \{T\}$ is the set of all rational numbers in $[0, T)$. Put

$$A_n^T \equiv \{\{u(t)\}_{0\leq t\leq T} : P^{X^u(t)^{-1}}(dx) = p(t,x)dx(t \in Q_n)\}. \tag{1.24}$$

Then, under the assumption in Theorem 1.2, for any $n \geq 1$, there exists a measurable function $b^n(t,x) : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ such that (1.19) with $b = b^n$ has a unique weak solution $X^n(t)$ and that $\{b^n(t, X^n(t))\}_{0\leq t\leq T} \in A_n^T$ and is the unique minimizer of

$$E[\int_0^T |u(t)|^2 dt] \tag{1.25}$$

over all $\{u(t)\}_{0\leq t\leq T} \in A_n^T$ (see [14] and also [13]).

In a case where the stochastic processes under consideration do not have random time evolution, we consider the analogue of this problem, replacing the mean forward derivatives by the time derivatives in (1.25) (see Theorem 2.5).

Let us consider the one-dimensional case on the convergence of X^h. Put, for $n \geq 0$, $h \in (0,1)$ and $x \in \mathbf{R}$,

$$F_h^n(x) = \int_{(-\infty,x]} p_h^n(y)dy. \tag{1.26}$$

For a distribution function F on $\mathbf{R}$, put

$$F^{-1}(u) \equiv \sup\{x \in \mathbf{R} : F(x) < u\} \quad \text{for } 0 < u < 1 \tag{1.27}$$

(see [18] or [23]-[25]).

The following can be easily proved from Theorem 1.1 (see section 7).

Proposition 1.1. *Suppose that (A.1)-(A.2) hold, and that $d = 1$, and that $p(t,x) > 0$ for $t \geq 0$ and $x \in \mathbf{R}$. Then for $n \geq 0$, $h \in (0,1)$ and $x \in \mathbf{R}$,*

$$\nabla\varphi_h^{n+1}(x) = (F_h^{n+1})^{-1}(F_h^n(x)), \tag{1.28}$$

and for $h \in (0,1)$, $t \geq 0$ and $x \in \mathbf{R}$,

$$X^h(t,x) = (F_h^{[t/h]})^{-1}(F_0(x)). \tag{1.29}$$

In particular, for $t > 0$ and $x \in \mathbf{R}$,

$$\lim_{h \to 0} X^h(t,x) = F_t^{-1}(F_0(x)). \tag{1.30}$$

REMARK 1.2. $p_0(x)\delta_{(F_h^{n+1})^{-1}(F^0(x))}(dy)dx$ and $p_0(x)\delta_{(F_t)^{-1}(F_0(x))}(dy)dx$ are the minimizers of $d(p_0, p_h^{n+1})$ and $d(p_0, p(t,\cdot))$, respectively ($n \geq 0$ and $t > 0$). In other word, $X^h((n+1)h, x)$ and $X(t,x)$ are the Monge functions for $d(p_0, p_h^{n+1})$ and $d(p_0, p(t,\cdot))$, respectively (see [23]). This is true only in case $d = 1$ when composites of (monotone) increasing functions is always a derivative of a convex function.

In section 2, we state our result which will be proved in sections 3-6. In section 7, we prove Proposition 1.1 for the sake of completeness.

2. Convergence and characterization of dynamical systems.

In this section we state our main result. Let us recall that $P_0(dx) = p_0(x)dx$. The following is an additional assumption in this paper.

(A.3). $\Psi \in C^4(\mathbf{R}^d; [0,\infty))$ and has bounded second, third and fourth derivatives.

(A.4). $p_0(\cdot)$ is a probability density function on $\mathbf{R}^d$, and is twice continuously differentiable, with bounded derivatives up to the second order.

(A.5).

$$-\infty < -C_1 \equiv \inf_{x \in \mathbf{R}^d} \{(|x|^2 + 1)^{-1} \log p_0(x)\}. \tag{2.1}$$

(A.6).

$$\infty > C_2 \equiv \sup_{x \in \mathbf{R}^d} \{(|x| + 1)^{-1} |\nabla \log p_0(x)|\}. \tag{2.2}$$

REMARK 2.1. (A.3) does not imply (A.1). As an example, consider $\Psi(x) = \exp(-|x|^2)$ which satisfies (A.3) but not (A.1).

For $t \geq 0$ and $y \in \mathbf{R}^d$, let $\{Y(s,(t,y))\}_{s\geq t}$ be the solution to the following stochastic integral equation:

$$Y(s,(t,y)) = y + \int_t^s \nabla\Psi(Y(u,(t,y)))du + 2^{1/2}(W(s) - W(t)), \qquad (2.3).$$

where $W(t)$ denotes a d-dimensional Wiener process. (2.3) has a unique strong solution under (A.3) (see [8] or [12], [27]). We also put, for the sake of simplicity,

$$Y(s,y) \equiv Y(s,(0,y)). \qquad (2.4).$$

It is known that $\{Y(s,(t,y))\}_{s\geq t}$ has the same probability law as that of $\{Y(s,y)\}_{s\geq 0}$ (see [8] or [12], [27]).

The following theorem gives a stochastic representation for $p(t,x)$.

Theorem 2.1. *Suppose that (A.1)-(A.4) hold. Then for any $T > 0$, $p(t,x)$ is continuously differentiable in t and has bounded, continuous derivatives up to the second order in x on $[0,T] \times \mathbf{R}^d$, and for any $t > 0$ and $x \in \mathbf{R}^d$,*

$$p(t,x) = E[p_0(Y(t,x)) \exp(\int_0^t \Delta\Psi(Y(s,x))ds)]. \qquad (2.5).$$

By Theorem 2.1, using estimates on large deviations and the idea in [26], we get the following result.

Theorem 2.2. *Suppose that (A.1)-(A.6) hold. Then for any $T > 0$,*

$$\sup_{x\in\mathbf{R}^d, 0\leq t\leq T} \{(|x|+1)^{-1}|\nabla_x \log p(t,x)|\} < \infty. \qquad (2.6).$$

In particular, (1.14) has a unique solution.

REMARK 2.2. In theorems 2.1 and 2.2, we assumed (A.1)-(A.2) only to use the fact that $p(t,x)$ is a smooth solution to (1.8) with (1.9) and that $p(t,\cdot)$ is a probability density function for $t \geq 0$.

By Theorems 2.1-2.2, we get our main result.

Theorem 2.3. *Suppose that (A.1)-(A.6) hold. Then for any* $T > 0$ *and* $\varepsilon > 0$,

$$\lim_{h \to 0} P_0(\sup_{0 \le t \le T} |X(t,x) - X^h(t,x)| \ge \varepsilon) = 0. \tag{2.7}$$

In particular, for $t \ge 0$,

$$P_0^{X(t,\cdot)^{-1}}(dy) = p(t,y)dy. \tag{2.8}$$

Put, for $T > 0$,

$$\begin{aligned} \tilde{A}^T \equiv \{\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d}; P_0^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx, dt - a.e., \\ S(0,x) = x, P_0(dx) - a.e., \\ S(t,x) \text{ is differentiable in } t,\ dtP_0(dx) - a.e.\}. \end{aligned} \tag{2.9}$$

The following result is a version of (1.21) in the case the stochastic processes under consideration do not have random time evolution.

Theorem 2.4. *Suppose that (A.1)-(A.6) hold. Then for any* $T > 0$ *and any* $\{S(t,x)\}_{0 \le t \le T, x \in \mathbf{R}^d} \in \tilde{A}^T$,

$$E_0[\int_0^T |dX(t,x)/dt|^2 dt] \le E_0[\int_0^T |dS(t,x)/dt|^2 dt], \tag{2.10}$$

where the equality holds if and only if $dS(t,x)/dt = dX(t,x)/dt\ dtP_0(dx)$-a.e..

Let us give some notation before we state the result which is a version of (1.23)-(1.25) in the case the stochastic processes under consideration do not have random time evolution.

For $h \in (0,1)$ and $n \ge 0$, let $\nabla\tilde{\varphi}_h^{n+1}$ be the Monge function for $d(p(nh,\cdot), p((n+1)h,\cdot))$ (see section 1 and also [9]). On the probability space $(\mathbf{R}^d, \mathbf{B}(\mathbf{R^d}), P_0(dx) \equiv p_0(x)dx)$, put for $h \in (0,1)$, $t \ge 0$ and $x \in \mathbf{R}^d$,

$$
\begin{aligned}
\tilde{X}^h(0,x) &= \nabla\tilde{\varphi}_h^0(x) \equiv x, \\
\tilde{X}^h((k+1)h,x) &= \nabla\tilde{\varphi}_h^{k+1}(\tilde{X}^h(kh,x)) \quad (k \geq 0), \\
\tilde{X}^h(t,x) &= \tilde{X}^h(kh,x) + (t - [t/h]) \\
&\qquad \times (\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x))/h.
\end{aligned} \tag{2.11}
$$

Put also for $h \in (0,1)$ and $T > 0$,

$$
\begin{aligned}
\tilde{A}_h^T \equiv \{&\{S(t,x)\}_{0 \leq t \leq T, x \in \mathbf{R}^d} \\
&; P_0^{S(t,\cdot)^{-1}}(dx) = p(t,x)dx(t = 0, h, \cdots, [T/h]h), \\
&S(0,x) = x, P_0(dx) - a.e., \\
&S(t,x) \text{ is differentiable in } t,\ dtP_0(dx) - a.e.\}.
\end{aligned} \tag{2.12}
$$

Then the following holds.

Theorem 2.5. *Suppose that (A.1)-(A.6) hold. Then for any $h \in (0,1)$ and $T \geq h$, $\{\tilde{X}^h(t,x)\}_{0 \leq t \leq T, x \in \mathbf{R}^d}$ is the unique minimizer of*

$$
\int_0^{[T/h]h} E_0[|dS(t,x)/dt|^2]dt \tag{2.13}
$$

over all $\{S(t,x)\}_{0 \leq t \leq T, x \in \mathbf{R}^d} \in \tilde{A}_h^T$, and the following holds: for any $T > 0$ and $\varepsilon > 0$,

$$
\lim_{h \to 0} P_0(\sup_{0 \leq t \leq T} |X(t,x) - \tilde{X}^h(t,x)| \geq \varepsilon) = 0. \tag{2.14}
$$

3. Proof of Theorem 2.1.

In this section we prove Theorem 2.1. The proof is devided into four lemmas.

For a m-dimensional vector function $f(x) = (f^i(x))_{i=1}^m$ $(x \in \mathbf{R}^d)$, put

$$Df(x) \equiv (f(x)/\partial x_i)_{i=1}^d,$$
$$|f|_\infty \equiv \sup_{x\in\mathbf{R}^d} (\sum_{i=1}^m |f^i(x)|^2)^{1/2}. \qquad (3.1).$$

The following lemma will be used in the proof of Lemma 3.3 and also in the next section, and is given the proof for the sake of completeness.

Lemma 3.1. *Suppose that (A.3) holds. Then (2.3) has a unique strong solution, and there exist positive constants C_3 and $\{C(m)\}_{m\geq 1}$ which depends only on $|\nabla\Psi|_\infty$ and $|D^2\Psi|_\infty$ such that for $t \geq 0$ and $y \in \mathbf{R}^d$,*

$$E[|Y(t,y)|^{2m}] \leq C(m)(\sum_{k=1}^m |y|^{2k} + t)\exp(C(m)t) \quad (m \geq 1),$$
$$|\partial Y^i(t,y)/\partial y_j| \leq C_3 \exp(C_3 t), \quad P-a.s. \quad (i,j = 1,\cdots,d). \qquad (3.2).$$

(Proof). Let us briefly sketch the proof. Since $\nabla\Psi$ is globally Lipschitz continuous from (A.3), there exists $C_4 > 0$ such that

$$|\nabla\Psi(x)| \leq |\nabla\Psi(x) - \nabla\Psi(o)| + |\nabla\Psi(o)| \leq C_4(|x|+1). \qquad (3.3).$$

By the following, the proof of the first part is over by induction: for $m \geq 1$,

$$E[|Y(t,y)|^{2m}] \leq (|y|^{2m} + (2md + 4m(m-1) + C_4 m) \qquad (3.4).$$
$$\times \int_0^t E[|Y(s,y)|^{2(m-1)}]ds)\exp(3C_4 mt).$$

This is true, since by the Ito formula (see e.g. [8]) and (3.3),

$$E[|Y(t,y)|^{2m}]$$

$$= |y|^{2m} + 2m \int_0^t E[|Y(s,y)|^{2(m-1)} < Y(s,y), \nabla\Psi(Y(s,y)) >]ds$$
$$+ (2md + 4m(m-1)) \int_0^t E[|Y(s,y)|^{2(m-1)}]ds$$
$$\le |y|^{2m} + (2md + 4m(m-1) + C_4 m) \int_0^t E[|Y(s,y)|^{2(m-1)}]ds$$
$$+ 3C_4 m \int_0^t E[|Y(s,y)|^{2m}]ds,$$

from which we get (3.4) by Gronwall's inequality (see [10]). Here we used the following: for $y \in \mathbf{R}^d$, by (3.3),

$$|y|^{2(m-1)} < y, \nabla\Psi(y) > \le C_4(|y|^{2m} + |y|^{2m-1}),$$
$$|y|^{2m-1} \le (|y|^{2m} + |y|^{2(m-1)})/2.$$

The rest part of the proof is done by (A.3) and Gronwall's inequality, since for $i, j = 1, \cdots d$,

$$\partial Y^i(t,y)/\partial y_j \tag{3.5}$$
$$= \delta_{ij} + \int_0^t \sum_{m=1}^d (\partial^2\Psi(Y(s,y))/\partial y_i \partial y_m) \partial Y^m(s,y)/\partial y_j ds.$$

(see [8, p. 120, Theorem 5.3]). Here we put $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Q. E. D.

For $t \ge 0$ and $y \in \mathbf{R}^d$, put

$$q(t,y) = E[p_0(Y(t,y)) \exp(\int_0^t \Delta\Psi(Y(s,y))ds)]. \tag{3.6}$$

Then the following can be proved in the same way as in [8, Chap. 5, Theorems 5.5 and 6.1] and the proof is omitted.

Lemma 3.2. *Suppose that (A.3)-(A.4) hold. Then for any $T \ge 0$, $q(t,y)$ has bounded, continuous derivatives in y up to the second order, and is continuously differentiable in t on $[0,T] \times \mathbf{R}^d$, and is a solution to (1.8).*

By Lemmas 3.1 and 3.2, we get the following lemma which will play a crucial role in the proof of Theorem 2.1.

Lemma 3.3. *Suppose that (A.1)-(A.4) hold. Then for $t \geq 0$ and $x \in \mathbf{R}^d$,*

$$p(t,x) \geq q(t,x). \tag{3.7.}$$

(Proof). For $R > 0$ and $x \in \mathbf{R}^d$, put

$$\sigma_R(x) = \inf\{t > 0 : |Y(t,x)| > R\}. \tag{3.8.}$$

By the Ito formula (see [8]), if $R > |x|$ and $0 < s < t$, then one can show that the following is true:

$$\begin{aligned}
&p(t,x) \qquad\qquad (3.9.)\\
&= E[p(t - \min(\sigma_R(x), s), Y(\min(\sigma_R(x), s), x))\\
&\qquad \times \exp(\int_0^{\min(\sigma_R(x),s)} \Delta\Psi(Y(u,x))du)]\\
&\geq E[p(t-s, Y(s,x)) \exp(\int_0^s \Delta\Psi(Y(u,x))du); \sigma_R(x) \geq t]\\
&\to E[p(0, Y(t,x)) \exp(\int_0^t \Delta\Psi(Y(u,x))du); \sigma_R(x) \geq t] \text{ (as } s \to t)\\
&\to E[p(0, Y(t,x)) \exp(\int_0^t \Delta\Psi(Y(u,x))du)] = q(t,x) \text{ (as } R \to \infty).
\end{aligned}$$

Let us prove that the first convergence in (3.9) is true to complete the proof.

$$\begin{aligned}
&E[p(t-s, Y(s,x)) \exp(\int_0^s \Delta\Psi(Y(u,x))du); \sigma_R(x) \geq t] \qquad (3.10.)\\
&= E[(p(t-s, Y(s,x)) - p(0, Y(s,x)))\\
&\qquad \times \exp(\int_0^s \Delta\Psi(Y(u,x))du); \sigma_R(x) \geq t]\\
&\quad + E[(p(0, Y(s,x)) - p(0, Y(t,x)))\\
&\qquad \times \exp(\int_0^s \Delta\Psi(Y(u,x))du); \sigma_R(x) \geq t]
\end{aligned}$$

$$
\begin{aligned}
&+ E[p(0, Y(t,x))(\exp(\int_0^s \triangle\Psi(Y(u,x))du) \\
&- \exp(\int_0^t \triangle\Psi(Y(u,x))du)); \sigma_R(x) \geq t] \\
&+ E[p(0, Y(t,x)) \exp(\int_0^t \triangle\Psi(Y(u,x))du); \sigma_R(x) \geq t].
\end{aligned}
$$

The second and third part on the right hand side of (3.10) converge to zero, as $s \to t$, by the bounded convergence theorem. The first part on the right hand side of (3.10) can be shown to converge to zero, as below, as $s \to t$, which completes the proof. By Lemma 3.1, (3.3), and the Cameron-Martin-Maruyama-Girsanov formula (see [12]),

$$
\begin{aligned}
&E[|p(t-s, Y(s,x)) - p(0, Y(s,x))| \\
&\quad \times \exp(\int_0^s \triangle\Psi(Y(u,x))du); \sigma_R(x) \geq t] \\
&= E[|p(t-s, x + 2^{1/2}W(s)) - p(0, x + 2^{1/2}W(s))| \\
&\quad \times \exp(\int_0^t < \nabla\Psi(x + 2^{1/2}W(u)), 2^{-1/2}dW(u) > \\
&\quad - \int_0^t |\nabla\Psi(x + 2^{1/2}W(u))|^2 du/4 + \int_0^s \triangle\Psi(x + 2^{1/2}W(u))du) \\
&\quad ; \sup_{0 \leq s \leq t} |x + 2^{1/2}W(u)| \leq R] \\
&\leq \int_{\mathbf{R}^d} |p(t-s, x + 2^{1/2}y) - p(0, x + 2^{1/2}y)| dy \\
&\quad \times (2\pi s)^{-d/2} \exp(\sup_{|z| \leq R} \Psi(z)/2 + t|\triangle\Psi|_\infty/2) \to 0,
\end{aligned}
\tag{3.11}
$$

as $s \to t$, by Theorem 1.1. Here we used the following: by the Ito formula (see [8]),

$$
\begin{aligned}
&\int_0^t < \nabla\Psi(x + 2^{1/2}W(u)), 2^{-1/2}dW(u) > \\
&= \{\Psi(x + 2^{1/2}W(t)) - \Psi(x) - \int_0^t \triangle\Psi(x + 2^{1/2}W(u))du\}/2,
\end{aligned}
$$

and for $y \in \mathbf{R}^d$,

$$
\exp(-|y|^2/(2s) - \Psi(x)) \leq 1,
$$

since $\Psi(x) \geq 0$ by (A.3).

Q. E. D.

The following lemma together with Lemma 3.2 completes the proof of Theorem 2.1.

Lemma 3.4. *Suppose that (A.1)-(A.4) hold. Then for* $t \geq 0$ *and* $x \in \mathbf{R}^d$,

$$p(t,x) = E[p_0(Y(t,x))\exp(\int_0^t \Delta\Psi(Y(s,x))ds)]. \qquad (3.12).$$

(Proof). By Lemma 3.2, $q(t,x)$ is a solution to (1.8) with $q(0,x) = p_0(x)$. Hence for $t \geq 0$,

$$\int_{\mathbf{R}^d} q(t,x)dx = \int_{\mathbf{R}^d} p_0(x)dx = 1, \qquad (3.13).$$

which completes the proof by Remark 1.1, (3.7) and the continuity of p and q.

Let us prove (3.13), for the sake of completeness, to complete the proof. For any smooth function $f(x)$ ($x \in \mathbf{R}$) such that $f(x) = 1$ ($|x| \leq 1$) and that $f(x) = 0$ ($|x| \geq 2$) and $R > 0$,

$$\begin{aligned}
&|\int_{\mathbf{R}^d} f(|x|^2/R^2)q(t,x)dx - \int_{\mathbf{R}^d} f(|x|^2/R^2)p_0(x)dx| \qquad (3.14).\\
&\leq \int_0^t ds \int_{\mathbf{R}^d} |(4|x|^2/R^4)d^2 f(|x|^2/R^2)/dx^2 \\
&\quad + (2/R^2)(df(|x|^2/R^2)/dx)(d- < \nabla\Psi(x), x >)|q(s,x)dx \\
&\leq (|d^2 f/dx^2|_\infty + |df/dx|_\infty) \int_0^t ds \int_{R \leq |x| \leq 2R} (4|x|^2/R^4 \\
&\quad + (2/R^2)(d + C_4(|x|+1)|x|))q(s,x)dx \quad \text{(by (3.3))} \\
&\leq (|d^2 f/dx^2|_\infty + |df/dx|_\infty)(16/R^2 + (2/R^2)(d + 2RC_4(2R+1))) \\
&\quad \times \int_0^t ds \int_{R \leq |x| \leq 2R} q(s,x)dx \to 0 \quad (\text{as } R \to \infty)
\end{aligned}$$

by the bounded convergence theorem, since by Lemma 3.3, for $s \geq 0$,

$$\int_{\mathbf{R}^d} q(s,x)dx \le \int_{\mathbf{R}^d} p(s,x)dx = 1. \qquad (3.15).$$

Since p_0 is a probability density function by (A.4), the proof is over by (3.14).

Q. E. D.

4. Proof of Theorem 2.2.

In this section we prove Theorem 2.2. Roughly speaking, Lemmas 4.2 and 4.6 will play a crucial role in the proof of Theorem 2.2, and Lemma 4.1 and Lemmas 4.3-4.5 will be used in the proof of Lemmas 4.2 and 4.6, respectively.

The following lemma can be obtained by Lemmas 3.1 and 3.4 and will be used in the proof of Lemmas 4.2, 4.4 and 4.5.

Lemma 4.1. *Suppose that (A.1)-(A.5) hold. Then there exists a positive constant C_5 which depends only on $|\nabla\Psi|_\infty$, $|D^2\Psi|_\infty$ and $|p_0|_\infty$ such that for $t \ge 0$ and $x \in \mathbf{R}^d$,*

$$\exp(-C_5(|x|^2 + 1 + t)\exp(C_5 t)) \le p(t,x) \le C_5 \exp(C_5 t). \qquad (4.1).$$

(Proof). By Lemma 3.4,

$$p(t,x) \le |p_0|_\infty \exp(t|\triangle\Psi|_\infty), \qquad (4.2).$$

and by Jensen's inequality (see [1]),

$$\begin{aligned} p(t,x) &\ge \exp(E[\log p_0(Y(t,x)) + \int_0^t \triangle\Psi(Y(s,y))ds]) \qquad (4.3).\\ &\ge \exp(-E[C_1(|Y(t,x)|^2+1)] - t|\triangle\Psi|_\infty) \end{aligned}$$

by (A.5), which completes the proof by Lemma 3.1.

Q. E. D.

For t and T for which $0 \le t < T$ and $z \in \mathbf{R}^d$, let $\{Z^T(s,(t,z))\}_{t\le s\le T}$ be the solution to the following stochastic integral equation: for $s \in [t,T]$,

$$
\begin{aligned}
&Z^T(s,(t,z)) &(4.4).\\
&= z + \int_t^s \{2\nabla_x \log p(t+T-u, Z^T(u,(t,z))) + \nabla\Psi(Z^T(u,(t,z)))\}du\\
&\quad + 2^{1/2}\{W(s) - W(t)\},
\end{aligned}
$$

up to the explosion time (see [27]).

The following lemma shows that (4.4) has a nonexplosive strong solution and will be used in the proof of Lemmas 4.4, 4.6 and of Theorem 2.2.

Lemma 4.2. *Suppose that (A.1)-(A.5) hold. Then for t and T for which $0 \le t < T$, (4.4) has a unique nonexplosive strong solution and there exists a positive constant C_6 which depends only on $|\nabla\Psi|_\infty$, $|D^2\Psi|_\infty$ and $|p_0|_\infty$ such that for $z \in \mathbf{R}^d$,*

$$
\begin{aligned}
&C_6 \exp(C_6 T)(|z|^2 + 1 + T) &(4.5).\\
&\ge \sup_{t\le s\le T} E[|Z^T(s,(t,z))|^2] + E[\int_t^T |\nabla_x \log p(T+t-s, Z^T(s,(t,z)))|^2 ds].
\end{aligned}
$$

(Proof). For $R > 0$, put

$$
\tau_R^T(t,z) = \inf\{\min(s,T) > t : |Z^T(s,(t,z))| > R\}. \qquad (4.6).
$$

Then by Lemma 4.1,

$$
\begin{aligned}
&E[\int_t^{\tau_R^T(t,z)} |\nabla_x \log p(T+t-s, Z^T(s,(t,z)))|^2 ds] &(4.7).\\
&\le \log C_5 + C_5 T + C_5(|z|^2 + 1 + T)\exp(C_5 T) + T|\Delta\Psi|_\infty.
\end{aligned}
$$

This is true, since by the smoothness of $p(t,x)$ (see Theorem 1.1) and the Ito formula (see [8]),

$$
\begin{aligned}
&E[\log p(T+t-\tau_R^T(t,z), Z^T(\tau_R^T(t,z),(t,z)))] - \log p(T,z) \\
&= E[\int_t^{\tau_R^T(t,z)} [-\partial \log p(T+t-s, Z^T(s,(t,z)))/\partial s \\
&\quad + \triangle_x \log p(T+t-s, Z^T(s,(t,z))) \\
&\quad + < 2\nabla_x \log p(T+t-s, Z^T(s,(t,z))) + \nabla\Psi(Z^T(s,(t,z))) \\
&\quad , \nabla_x \log p(T+t-s, Z^T(s,(t,z))) >]ds] \\
&= E[\int_t^{\tau_R^T(t,z)} [-\triangle\Psi(Z^T(s,(t,z))) + |\nabla_x \log p(T+t-s, Z^T(s,(t,z)))|^2]ds].
\end{aligned}
$$

Here we used the following:

$$
\begin{aligned}
&\partial \log p(t,x)/\partial t \\
&= \triangle_x \log p(t,x) + < 2\nabla_x \log p(t,x) + \nabla\Psi(x), \nabla_x \log p(t,x) > \\
&\quad + \triangle\Psi(x) - |\nabla_x \log p(t,x)|^2,
\end{aligned}
\qquad (4.8).
$$

which can be shown by (1.8).

The following also holds: for $s \in [t,T]$,

$$
\begin{aligned}
&E[|Z^T(\min(s,\tau_R^T(t,z)),(t,z))|^2] \\
&\le (E[\int_t^{\tau_R^T(t,z)} |2\nabla_x \log p(T+t-u, Z^T(u,(t,z)))|^2 du] \\
&\quad + |z|^2 + 2(T-t)(d+C_4^2)) \exp(2(C_4^2+1)(s-t)).
\end{aligned}
\qquad (4.9).
$$

This is true, since by the Ito formula (see [8]) and (3.3), for $s \in [t,T]$,

$$
\begin{aligned}
&E[|Z^T(\min(s,\tau_R^T(t,z)),(t,z))|^2] \\
&= |z|^2 + E[\int_t^{\min(s,\tau_R^T(t,z))} [< 2\nabla_x \log p(T+t-u, Z^T(u,(t,z))) \\
&\quad + \nabla\Psi(Z^T(u,(t,z))), 2Z^T(u,(t,z)) > +2d]du]
\end{aligned}
$$

$$\leq |z|^2 + 2(T-t)d + E[\int_t^{\tau_R^T(t,z)} |2\nabla_x \log p(T+t-u, Z^T(u,(t,z)))|^2 du]$$

$$+ 2C_4^2(T-t) + 2(C_4^2+1)E[\int_t^{\min(s,\tau_R^T(t,z))} |Z^T(u,(t,z))|^2 du]$$

$$\leq |z|^2 + 2(T-t)d + E[\int_t^{\tau_R^T(t,z)} |2\nabla_x \log p(T+t-u, Z^T(u,(t,z)))|^2 du]$$

$$+ 2C_4^2(T-t) + 2(C_4^2+1)\int_t^s E[|Z^T(\min(u,\tau_R^T(t,z)),(t,z))|^2]du,$$

from which we get (4.9) by Gronwall's inequality (see [10]). In the first inequality above, we used the following: for x, y, and $z \in \mathbf{R}^d$,

$$< x+y, 2z > \leq |x|^2 + |y|^2 + 2|z|^2,$$

$$(x+y)^2 \leq 2(x^2+y^2).$$

Let $R \to \infty$ in (4.7) and (4.9) and then the proof is over.

Q. E. D.

The following lemma will be used in the proof of Lemma 4.4.

Lemma 4.3. *Suppose that (A.3) holds. Then for* $T \in (0, 1/(2C_4))$ *(see (3.3)) and* $y \in \mathbf{R}^d$,

$$\limsup_{R\to\infty} R^{-2} \log P(\sup_{0\leq t\leq T} |Y(t,y)| \geq R) \leq -(1-2C_4T)^2/(16T). \qquad (4.10).$$

(Proof). Put

$$r = (R^2 - |y|^2 - 2Td - 2TC_4R(R+1))/(8TR^2) \qquad (4.11).$$

which is positive for sufficiently large $R > 0$. Then by (3.8), (3.3) and the Ito formula (see[8]),

$$P(\sup_{0\leq t\leq T} |Y(t,y)| \geq R) \qquad (4.12).$$

$$
\begin{aligned}
&= \exp(-rR^2 + r|y|^2)E[\exp(r|Y(\sigma_R(y),y)|^2 - r|y|^2); \sigma_R(y) \le T] \\
&= \exp(-rR^2 + r|y|^2)E[\exp(r[\int_0^{\min(T,\sigma_R(y))} < 2Y(s,y), 2^{1/2}dW(s) > \\
&\quad + \int_0^{\min(T,\sigma_R(y))} (< 2Y(s,y), \nabla\Psi(Y(s,y)) > +2d)ds]); \sigma_R(y) \le T] \\
&\le \exp(-rR^2 + r|y|^2)E[\exp(r2^{3/2} \int_0^{\min(T,\sigma_R(y))} < Y(s,y), dW(s) > \\
&\quad - 4r^2 \int_0^{\min(T,\sigma_R(y))} |Y(s,y)|^2 ds + \int_0^{\min(T,\sigma_R(y))} (4r^2|Y(s,y)|^2 \\
&\quad + r < 2Y(s,y), \nabla\Psi(Y(s,y)) > +2rd)ds)] \\
&\le \exp(-rR^2 + r|y|^2 + T(4r^2R^2 + 2rC_4R(R+1) + 2rd)) \\
&= \exp(-R^2(1 - (|y|^2 + 2Td)/R^2 - 2TC_4(1+1/R))^2/(16T)),
\end{aligned}
$$

which completes the proof. Here we used the following:

$$
\begin{aligned}
1 = E[\exp(r2^{3/2} &\int_0^{\min(T,\sigma_R(y))} < Y(s,y), dW(s) > \\
- 4r^2 &\int_0^{\min(T,\sigma_R(y))} |Y(s,y)|^2 ds)]
\end{aligned}
$$

(see [12]), and

$$
\begin{aligned}
&- rR^2 + r|y|^2 + T(4r^2R^2 + 2rC_4R(R+1) + 2rd) \\
&= 4TR^2(r - (R^2 - |y|^2 - 2TC_4R(R+1) - 2Td)/(8TR^2))^2 \\
&\quad - R^2(1 - (|y|^2 + 2Td)/R^2 - 2TC_4(1+1/R))^2/(16T).
\end{aligned}
$$

Q. E. D.

The following lemma will be used in the proof of lemma 4.6.

Lemma 4.4. *Suppose that (A.1)-(A.5) hold. Then for t and T for which $0 \le t < T$ and $z \in \mathbf{R}^d$, the probability law of $\{Z^T(s,(t,z))\}_{t \le s \le T}$ is absolutely continuous with respect to that of $\{Y(s,(t,z))\}_{t \le s \le T}$ and on $C([t,T];\mathbf{R}^d)$,*

$$
\begin{aligned}
&(dP^{Z^T(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z))) \qquad (4.13). \\
&= [p(t, Y(T,(t,z)))/p(T,z)] \exp(\int_t^T \triangle\Psi(Y(s,(t,z)))ds).
\end{aligned}
$$

Moreover if $T-t<1/(2C_4)$, then

$$\limsup_{R\to\infty} R^{-2}\log P(\sup_{t\le s\le T}|Z^T(s,(t,z))|\ge R) \qquad (4.14).$$
$$\le -(1-2C_4(T-t))^2/(16(T-t)).$$

(Proof). First we prove (4.13). By Lemma 4.2, $P^{Z^T(\cdot,(t,z))^{-1}}$ is absolutely continuous with respect to $P^{Y(\cdot,(t,z))^{-1}}$ on $C([t,T];\mathbf{R}^d)$, and

$$(dP^{Z^T(\cdot,(t,z))^{-1}}/dP^{Y(\cdot,(t,z))^{-1}})(Y(\cdot,(t,z))) \qquad (4.15).$$
$$= \exp(2^{1/2}\int_t^T <\nabla_x \log p(T+t-s,Y(s,(t,z))),dW(s)>$$
$$-\int_t^T |\nabla_x \log p(T+t-s,Y(s,(t,z)))|^2 ds)$$

on $C([t,T];\mathbf{R}^d)$ (see [12, Chap. 7]). (4.15) and (4.16) below imply (4.13): by the Ito formula (see [8]),

$$\log p(t,Y(T,(t,z)))-\log p(T,z) \qquad (4.16).$$
$$=\int_t^T [-\partial \log p(T+t-s,Y(s,(t,z)))/\partial s$$
$$+\triangle_x \log p(T+t-s,Y(s,(t,z)))$$
$$+<\nabla\Psi(Y(s,(t,z))),\nabla_x \log p(T+t-s,Y(s,(t,z)))>]ds$$
$$+2^{1/2}\int_t^T <\nabla_x \log p(T+t-s,Y(s,(t,z))),dW(s)>$$
$$=-\int_t^T (\triangle\Psi(Y(s,(t,z)))+|\nabla_x \log p(T+t-s,Y(s,(t,z)))|^2)ds$$
$$+2^{1/2}\int_t^T <\nabla_x \log p(T+t-s,Y(s,(t,z))),dW(s)>$$

(by (4.8)) since $\{Y(s,(t,z))\}_{s\ge t}$ is nonexplosive by Lemma 3.1.

Next we prove (4.14). By (4.13),

$$
\begin{aligned}
&P(\sup_{t\le s\le T}|Z^T(s,(t,z))|\ge R) \qquad\qquad (4.17).\\
&= E[(p(t,Y(T,(t,z)))/p(T,z))\exp(\int_t^T \triangle\Psi(Y(s,(t,z)))ds)\\
&\quad ;\ \sup_{t\le s\le T}|Y(s,(t,z))|\ge R]\\
&\le C_5\exp(C_5t + C_5(|z|^2+1+T)\exp(C_5T) + (T-t)|\triangle\Psi|_\infty)\\
&\quad \times P(\sup_{t\le s\le T}|Y(s,(t,z))|\ge R)
\end{aligned}
$$

by Lemma 4.1. This and Lemma 4.3 completes the proof by the following:

$$
P(\sup_{t\le s\le T}|Y(s,(t,z))|\ge R) = P(\sup_{0\le s\le T-t}|Y(s,z)|\ge R).
$$

Q. E. D.

Put $\partial_i = \partial/\partial_{x_i}$. The following lemma will be used in the proof of Lemma 4.6.

Lemma 4.5. *Suppose that (A.1)-(A.6) hold. Then for any $T>0$,*

$$
\limsup_{R\to\infty} R^{-2}\log\{\sup_{|x|=R,0\le t\le T}|\partial_i \log p(t,x)|\}\le 0. \qquad (4.18).
$$

(Proof). For $t\in[0,T]$ and $y\in\mathbf{R}^d$, by (A.6) (see [8, p.122, Theorem 5.5]),

$$
\begin{aligned}
&|\partial_i \log p(t,y)| \qquad\qquad (4.19).\\
&= |E[\{<\nabla p_0(Y(t,y)),\partial Y(t,y)/\partial y_i> + p_0(Y(t,y))\\
&\quad \times\int_0^t <\nabla(\triangle\Psi(Y(s,y))),\partial Y(s,y)/\partial y_i> ds\}\\
&\quad \times\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]|/p(t,y)\\
&\le E[\{C_2(|Y(t,y)|+1) + t|\nabla(\triangle\Psi)|_\infty\}\sup_{0\le s\le t}|\partial Y(s,y)/\partial y_i|\\
&\quad \times p_0(Y(t,y))\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]/p(t,y)\\
&\le d^{1/2}C_3\exp(C_3t)\{C_2 + t|\nabla(\triangle\Psi)|_\infty\\
&+ C_2E[|Y(t,y)|p_0(Y(t,y))\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]/p(t,y)\}
\end{aligned}
$$

by Lemma 3.1. We only have to consider the second part on the last part of (4.19): for $m \in \mathbf{N}$, by Hölder's inequality

$$\begin{aligned}
&E[|Y(t,y)|p_0(Y(t,y))\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]/p(t,y) \qquad (4.20).\\
&\le E[|Y(t,y)|^{2m}p_0(Y(t,y))\exp(\int_0^t \triangle\Psi(Y(s,y))ds)]^{1/(2m)}p(t,y)^{-1/(2m)}\\
&\le \{|p_0|_\infty \exp(t|\triangle\Psi|_\infty)\}^{1/(2m)}E[|Y(t,y)|^{2m}]^{1/(2m)}\\
&\quad \times \exp(C_5(|y|^2+t+1)\exp(C_5 t)/(2m))
\end{aligned}$$

by Lemma 4.1. By Lemma 3.1, (4.19) and (4.20),

$$\begin{aligned}
&\limsup_{R\to\infty} R^{-2}\log\{\sup_{|x|=R, 0\le t\le T}|\partial_i \log p(t,x)|\} \qquad (4.21).\\
&\le C_5\exp(C_5 T)/(2m) \to 0
\end{aligned}$$

as $m \to \infty$.

Q. E. D.

The following lemma will be used in the proof of Theorem 2.2.

Lemma 4.6. *Suppose that (A.1)-(A.6) hold and that (2.6) holds with $T = T_0$ for some $T_0 \ge 0$. Then for $T \in (T_0, T_0 + 1/(2C_4))$ and $z \in \mathbf{R}^d$,*

$$\begin{aligned}
&\lim_{R\to\infty} E[\partial_i \log p(T+T_0-\tau_R^T(T_0,z), Z^T(\tau_R^T(T_0,z),(T_0,z)))] \qquad (4.22).\\
&= E[\partial_i \log p(T_0, Z^T(T,(T_0,z)))].
\end{aligned}$$

(Proof). For $T \in (T_0, T_0 + 1/(2C_4))$ (see (3.3)) and $z \in \mathbf{R}^d$, by (4.6),

$$\begin{aligned}
&E[\partial_i \log p(T+T_0-\tau_R^T(T_0,z), Z^T(\tau_R^T(T_0,z),(T_0,z)))] \qquad (4.23).\\
&= E[\partial_i \log p(T_0, Z^T(T,(T_0,z))); \tau_R^T(T_0,z) = T]\\
&\quad + E[\partial_i \log p(T+T_0-\tau_R^T(T_0,z), Z^T(\tau_R^T(T_0,z),(T_0,z)))\\
&\qquad ; \tau_R^T(T_0,z) < T].
\end{aligned}$$

On the first part on the right hand side of (4.23), we have the following:

$$\lim_{R\to\infty} E[\partial_i \log p(T_0, Z^T(T,(T_0,z))); \tau_R^T(T_0,z)=T] \tag{4.24.}$$
$$= E[\partial_i \log p(T_0, Z^T(T,(T_0,z)))],$$

by Lemma 4.2 and the assumption on induction.

On the second part on the right hand side of (4.23), we have the following:

$$E[|\partial_i \log p(T+T_0-\tau_R^T(T_0,z), Z^T(\tau_R^T(T_0,z),(T_0,z)))| \tag{4.25.}$$
$$;\tau_R^T(T_0,z)<T]$$
$$\le \sup_{|x|=R, 0\le t\le T} |\partial_i \log p(t,x)| P(\tau_R^T(T_0,z)<T)$$
$$\to 0, \quad \text{as } R\to\infty,$$

by Lemmas 4.4 and 4.5.

Q. E. D.

Finally we prove Theorem 2.2.

Proof of Theorem 2.2.

Suppose that (2.6) holds for $T=T_0\ge 0$. Then for $z\in\mathbf{R}^d$ and $T\in(T_0, T_0+1/(2C_4))$ (see (3.3)), by the Ito formula,

$$E[\partial_i \log p(T+T_0-\tau_R^T(T_0,z), Z^T(\tau_R^T(T_0,z),(T_0,z)))] \tag{4.26.}$$
$$-\partial_i \log p(T,z)$$
$$= E[\int_{T_0}^{\tau_R^T(T_0,z)} (-\partial[\partial_i \log p(T+T_0-u, Z^T(u,(T_0,z)))]/\partial u$$
$$+\triangle_x[\partial_i \log p(T+T_0-u, Z^T(u,(T_0,z)))]$$
$$+<2\nabla_x \log p(T+T_0-u, Z^T(u,(T_0,z)))+\nabla\Psi(Z^T(u,(T_0,z)))$$
$$,\nabla_x[\partial_i \log p(T+T_0-u, Z^T(u,(T_0,z)))]>)du]$$
$$= -E[\int_{T_0}^{\tau_R^T(T_0,z)} [\partial_i\triangle\Psi(Z^T(u,(T_0,z)))+<\partial_i\nabla\Psi(Z^T(u,(T_0,z)))$$
$$,\nabla_x \log p(T+T_0-u, Z^T(u,(T_0,z)))>]du],$$

since $p(t,x)$ is smooth by Theorem 1.1, and since

$$\begin{aligned}
&\partial[\partial_i \log p(t,x)]/\partial t \qquad\qquad (4.27).\\
&= \triangle_x[\partial_i \log p(t,x)] + < 2\nabla_x \log p(t,x) + \nabla\Psi(x), \nabla_x[\partial_i \log p(t,x)] > \\
&\quad + \partial_i \triangle\Psi(x) + < \partial_i \nabla\Psi(x), \nabla_x \log p(t,x) >
\end{aligned}$$

from (4.8). Let $R \to \infty$ in (4.26). Then by (A.3), Lemmas 4.2 and 4.6,

$$\begin{aligned}
&E[\partial_i \log p(T_0, Z^T(T,(T_0,z)))] - \partial_i \log p(T,z) \qquad\qquad (4.28).\\
&= -E[\int_{T_0}^{T} [\partial_i \triangle\Psi(Z^T(u,(T_0,z))) + < \partial_i \nabla\Psi(Z^T(u,(T_0,z))) \\
&\quad , \nabla_x \log p(T+T_0-u, Z^T(u,(T_0,z))) >] du].
\end{aligned}$$

(4.28) and Lemmas 3.1 and 4.2 show that (2.6) is true for $T \in (T_0, T_0 + 1/(2C_4))$ by (A.3) and the assumption of induction. Inductively, one can show that (2.6) is true.

Q. E. D.

5. Proof of Theorem 2.3.

In this section, we prove Theorem 2.3. The proof will be done by Lemmas 5.1, 5.2, 5.4, and 5.8. Lemmas 5.3 and 5.5-5.7 will be used to prove Lemmas 5.4 and 5.8, respectively. Throughout this section, we assume that $h \in (0,1)$ and fix $T > 0$.

The following lemma will be used in the proof of Lemma 5.7 and of Theorem 2.3.

Lemma 5.1. *(see [11, p. 12, (45)]) Suppose that (A.1)-(A.2) hold. Then the following holds:*

$$\sup_{0 \le h \le 1} \sum_{k=0}^{[T/h]} E_0[|X^h((k+1)h, x) - X^h(kh, x)|^2]/h < \infty. \tag{5.1}$$

Put, for $h \in (0,1)$, $t \ge 0$ and $x \in \mathbf{R}^d$,

$$\begin{aligned}
\overline{X}^h(t,x) &= X^h([t/h]h, x) \\
&\qquad + (t - [t/h]h)\{X^h(([t/h]+1)h, x) - X^h([t/h]h, x)\}/h, \\
b(t,x) &\equiv -\nabla_x \log p(t,x) - \nabla\Psi(x), \\
C(b,R) &\equiv \sup\{|b(s,x) - b(s,y)|/|x-y| : 0 \le s \le T, \\
&\qquad x \ne y, |x|, |y| \le R\}, \quad (R > 0), \\
C(b) &\equiv \sup\{|b(t,x)/(|x|+1) : 0 \le t \le T, x \in \mathbf{R}^d\}
\end{aligned} \tag{5.2}$$

(see Theorems 2.1 and 2.2).

The following lemma will be used in the proof of Theorem 2.3.

Lemma 5.2. *Suppose that (A.1)-(A.6) hold. For $R_1 > 0$, suppose that*

$$|x| = |X(0,x)| = |\overline{X}^h(0,x)| < R_1, \tag{5.3}$$

and that

$$\sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h < R_1. \tag{5.4.}$$

Then for $R > 0$ for which

$$R > \max((R_1 + C(b)(T+1))\exp(C(b)(T+1)), R_1 + ((T+1)R_1)^{1/2}), \tag{5.5.}$$

the following holds: for $t \in [0,T]$,

$$\begin{aligned} &|X(t,x) - \overline{X}^h(t,x)| \qquad (5.6). \\ &\leq (T\sup\{|b(s,y) - b(s+h,z)| : 0 \leq s \leq T, |y|, |z| \leq R, |y-z|^2 \leq hR_1\} \\ &\quad + \int_0^{[T/h]h} |b(s+h, \overline{X}^h(([s/h]+1)h,x)) \\ &\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h|ds \\ &\quad + h\sup\{|b(s+h,z)| : [T/h]h \leq s \leq T, |z| \leq R\} \\ &\quad + (hR_1)^{1/2})\exp(tC(b,R)). \end{aligned}$$

(Proof). We first show that

$$\sup_{0 \leq t \leq ([T/h]+1)h, |x| \leq R_1} \max(|X(t,x)|, |\overline{X}^h(t,x)|) \leq R. \tag{5.7.}$$

For $t \in [0, ([T/h]+1)h]$, by (5.2),

$$|X(t,x)| = |x + \int_0^t b(s, X(s,x))ds| \leq |x| + \int_0^t C(b)(|X(s,x)|+1)ds,$$

from which we obtain the following: by Gronwall's inequality (see [10]) and (5.3),

$$|X(t,x)| \leq (|x| + C(b)t)\exp(C(b)t) \leq (R_1 + C(b)(T+1))\exp(C(b)(T+1)),$$

and by (5.4),

$$
\begin{aligned}
|\overline{X}^h(t,x)| &= |x + \int_0^t (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/hds| \\
&\le |x| + \sum_{k=0}^{[t/h]} |X^h((k+1)h,x) - X^h(kh,x)| \\
&\le |x| + [(([t/h]+1)\sum_{k=0}^{[t/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2]^{1/2} \\
&\le R_1 + ((T+1)R_1)^{1/2}.
\end{aligned}
$$

By (5.7), we can show that (5.6) is true: for $t \in [0,T]$,

$$
\begin{aligned}
&|X(t,x) - \overline{X}^h(t,x)| \qquad (5.8). \\
&= |\int_0^t (b(s,X(s,x)) - b(s,\overline{X}^h(s,x)))ds \\
&\quad + \int_0^t (b(s,\overline{X}^h(s,x)) - b(s+h,\overline{X}^h(([s/h]+1)h,x)))ds \\
&\quad + \int_0^t (b(s+h,\overline{X}^h(([s/h]+1)h,x)) \\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h)ds| \\
&\le \int_0^t C(b,R)|X(s,x) - \overline{X}^h(s,x)|ds \\
&\quad + t\sup\{|b(s,y) - b(s+h,z)|; 0 \le s \le T, |y|, |z| \le R, |y-z|^2 \le hR_1\} \\
&\quad + \int_0^{[T/h]h} |b(s+h,\overline{X}^h(([s/h]+1)h,x)) \\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h|ds \\
&\quad + h\sup\{|b(s+h,z)| : [T/h]h \le s \le T, |z| \le R\} \\
&\quad + |X^h(([T/h]+1)h,x) - X^h([T/h]h,x)|,
\end{aligned}
$$

which completes the proof by Gronwall's inequality (see [10]). Here we used the following: for $s \in [0,T]$, by (5.4),

$$
\begin{aligned}
&|\overline{X}^h(s,x) - \overline{X}^h(([s/h]+1)h,x)|^2 \\
&\le |\overline{X}^h([s/h]h,x) - \overline{X}^h(([s/h]+1)h,x)|^2
\end{aligned}
$$

$$\leq \sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2 < hR_1.$$

Q. E. D.

The following lemma will be used in the proof of Lemma 5.4.

Lemma 5.3. *(see [11, p. 11, (40)]) Suppose that (A.1)-(A.2) hold. Then for any* $f \in C_o^\infty(\mathbf{R}^d : \mathbf{R}^d)$ *and* $k \geq 0$,

$$\begin{aligned} &E_0[< f(X^h((k+1)h,x)), X^h((k+1)h,x) - X^h(kh,x) >] \qquad (5.9).\\ &= -hE_0[< \nabla\Psi(X^h((k+1)h,x)), f(X^h((k+1)h,x)) >\\ &\quad - divf(X^h((k+1)h,x))]. \end{aligned}$$

For $R > 0$, take $\phi_R \in C_o^\infty(\mathbf{R}^d : [0,\infty))$ such that

$$\sup_{x\in\mathbf{R}^d} |\nabla\phi_R(x)| \leq 1/R,$$

$$\phi_R(x) = \begin{cases} 1; \text{ if } |x| \leq R, \\ \in [0,1]; \text{ if } R \leq |x| \leq 2R+1, \\ 0; \text{ if } 2R+1 \leq |x|, \end{cases} \qquad (5.10).$$

and put

$$b_R(t,x) = \phi_R(x)b(t,x). \qquad (5.11).$$

The following lemma will be used in the proof of Theorem 2.3.

Lemma 5.4. *Suppose that (A.1)-(A.5) hold. Then for any* $R > 0$, *the following holds.*

$$\begin{aligned} &\lim_{h\to 0} E_0[\int_0^{[T/h]h} |b_R(s+h, \overline{X}^h(([s/h]+1)h,x))|^2 ds] \qquad (5.12).\\ &= \int_0^T ds \int_{\mathbf{R}^d} |b_R(s,y)|^2 p(s,y)dy. \end{aligned}$$

$$\lim_{h\to 0} E_0[\int_0^{[T/h]h} < b_R(s+h,\overline{X}^h(([s/h]+1)h,x)) \tag{5.13.}$$
$$, (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h > ds]$$
$$= \int_0^T ds \int_{\mathbf{R}^d} < b_R(s,y), b(s,y) > p(s,y)dy.$$

(Proof). We first prove (5.12).

$$E_0[\int_0^{[T/h]h} |b_R(s+h,\overline{X}^h(([s/h]+1)h,x))|^2 ds] \tag{5.14.}$$
$$= \int_0^{[T/h]h} ds \int_{\mathbf{R}^d} |b_R(s+h,y)|^2 p_h(s+h,y)dy$$
$$\to \int_0^T ds \int_{\mathbf{R}^d} |b_R(s,y)|^2 p(s,y)dy \quad \text{(as } h\to 0\text{, by Theorem 1.1).}$$

Next we prove (5.13). By Lemma 5.3 and Theorem 1.1,

$$E_0[\int_0^{[T/h]h} < b_R(s+h,\overline{X}^h(([s/h]+1)h,x)) \tag{5.15.}$$
$$, (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h > ds]$$
$$= -\int_0^{[T/h]h} E_0[< \nabla\Psi(X^h(([s/h]+1)h,x))$$
$$, b_R(s+h, X^h(([s/h]+1)h,x)) >$$
$$- div_x b_R(s+h, X^h(([s/h]+1)h,x))]ds$$
$$= -\int_0^{[T/h]h} ds \int_{\mathbf{R}^d} (< \nabla\Psi(y), b_R(s+h,y) >$$
$$- div_y b_R(s+h,y))p_h(s+h,y)dy$$
$$\to -\int_0^T ds \int_{\mathbf{R}^d} (< \nabla\Psi(y), b_R(s,y) >$$
$$- div_y b_R(s,y))p(s,y)dy \quad \text{(as } h\to 0)$$
$$= \int_0^T ds \int_{\mathbf{R}^d} < b(s,y), b_R(s,y) > p(s,y)dy,$$

since $div_y b_R(s,y)$ is bounded and continuous on $[0, T+1]\times\mathbf{R}^d$ by Theorem 2.1 and Lemma 4.1.

Q. E. D.

For $h \in (0,1)$, $k \geq 0$, $s \geq kh$, $x \in \mathbf{R}^d$ and $R > 0$, put

$$\begin{aligned}
\Phi^k_{h,R}(s,x) &= x + (s-kh)b_R(kh,x), \\
D\Phi^k_{h,R}(s,x)(= D_x\Phi^k_{h,R}(s,x)) &= Identity + (s-kh)(\partial b^i_R(kh,x)/\partial x_j)^d_{i,j=1}, \quad (5.16). \\
q^k_{h,R}(x)dx &= (p^k_h(x)dx)^{\Phi^k_{h,R}((k+1)h,\cdot)^{-1}},
\end{aligned}$$

provided that it exists.

The following lemma will be used in the proof of Lemma 5.8.

Lemma 5.5. *Suppose that (A.1)-(A.5) hold. Then for $R > 0$ and $k = 0, \cdots, [T/h]-1$, there exist mappings $\{\Phi^k_{h,R}(s,\cdot)^{-1}\}_{kh \leq s \leq (k+1)h}$ for sufficiently small $h > 0$ depending only on T and R, and the following holds.*

$$\begin{aligned}
&\lim_{h\to 0}\sum_{k=0}^{[T/h]-1} E_0[\log q^k_{h,R}(\Phi^k_{h,R}((k+1)h, X^h(kh,x))) \qquad (5.17). \\
&\quad - \log p^k_h(X^h(kh,x))] \\
&= -\int_0^T ds \int_{\mathbf{R}^d} div_x b_R(s,y)p(s,y)dy.
\end{aligned}$$

$$\begin{aligned}
&\lim_{h\to 0}\sum_{k=0}^{[T/h]-1} E_0[\Psi(\Phi^k_{h,R}((k+1)h, X^h(kh,x))) - \Psi(X^h(kh,x))] \qquad (5.18). \\
&= \int_0^T ds \int_{\mathbf{R}^d} < \nabla\Psi(y), b_R(s,y) > p(s,y)dy.
\end{aligned}$$

(Proof). Take $h \in (0,1)$ sufficiently small so that

$$h\sup\{(\sum_{i,j=1}^d |\partial b^i_R(s,x)/\partial x_j|^2)^{1/2} : 0 \leq s \leq T, x \in \mathbf{R}^d\} < 1, \qquad (5.19).$$

which is possible from Theorem 2.1 and Lemma 4.1. By (5.19), the proof of the first part is trivial (see [10]).

Let us prove (5.17). Since

$$q_{h,R}^k(x) = p_h^k(\Phi_{h,R}^k((k+1)h,\cdot)^{-1}(x))det(D\Phi_{h,R}^k((k+1)h,\cdot)^{-1}(x))$$

for $k = 0,\cdots,[T/h]-1$ and $x \in \mathbf{R}^d$,

$$\begin{aligned}
&E_0[\log q_{h,R}^k(\Phi_{h,R}^k((k+1)h, X^h(kh,x))) - \log p_h^k(X^h(kh,x))] \qquad (5.20).\\
&= E_0[-\log\{det(D\Phi_{h,R}^k((k+1)h, X^h(kh,x)))\}]\\
&= -\int_{kh}^{(k+1)h} E_0[\sum_{\sigma\in S_d} sgn\sigma \sum_{i=1}^d D_x b_R(kh, X^h(kh,x))^{i\sigma(i)}\\
&\quad \times \Pi_{j\neq i} D\Phi_{h,R}^k(s, X^h(kh,x))^{j\sigma(j)}\\
&\quad \times \{det(D\Phi_{h,R}^k(s, X^h(kh,x)))\}^{-1}]ds\\
&= -\int_{kh}^{(k+1)h} \int_{\mathbf{R}^d} \sum_{\sigma\in S_d} sgn\sigma \sum_{i=1}^d D_y b_R(kh,y)^{i\sigma(i)}\\
&\quad \times \Pi_{j\neq i} D\Phi_{h,R}^k(s,y)^{j\sigma(j)}\{det(D\Phi_{h,R}^k(s,y))\}^{-1} p_h(s,y)dyds.
\end{aligned}$$

Here S_d denotes a permutation group on $\{1,\cdots,d\}$. By (5.20), we obtain (5.17) as follows:

$$\begin{aligned}
&\sum_{k=0}^{[T/h]-1} E_0[\log q_{h,R}^k(\Phi_{h,R}^k((k+1)h, X^h(kh,x))) \qquad (5.21).\\
&\quad - \log p_h^k(X^h(kh,x))]\\
&= -\int_0^{[T/h]h} ds \int_{\mathbf{R}^d} \sum_{\sigma\in S_d} sgn\sigma \sum_{i=1}^d D_y b_R([s/h]h,y)^{i\sigma(i)}\\
&\quad \times \Pi_{j\neq i} D\Phi_{h,R}^{[s/h]}(s,y)^{j\sigma(j)}\{det(D\Phi_{h,R}^{[s/h]}(s,y))\}^{-1} p_h(s,y)dy\\
&\to -\int_0^T ds \int_{\mathbf{R}^d} div_y b_R(s,y)p(s,y)dy
\end{aligned}$$

as $h \to 0$, by Theorem 1.1, the smoothness of b_R, and the bounded convergence theorem since $D\Phi_{h,R}^{[s/h]}(s,y)$ is bounded and converges to an identity matrix as $h \to 0$ (see (5.16)).

Next we prove (5.18). For $k = 0, \cdots, [T/h] - 1$,

$$\begin{aligned}
&E_0[\Psi(\Phi^k_{h,R}((k+1)h, X^h(kh,x))) - \Psi(X^h(kh,x))] \qquad (5.22).\\
&= \int_{kh}^{(k+1)h} E_0[< \nabla\Psi(\Phi^k_{h,R}(s, X^h(kh,x))), b_R(kh, X^h(kh,x)) >]ds\\
&= \int_{kh}^{(k+1)h} \int_{\mathbf{R}^d} [< \nabla\Psi(\Phi^k_{h,R}(s,y)), b_R(kh,y) >] p_h(s,y) dy ds.
\end{aligned}$$

By (5.22), we obtain (5.18) as follows:

$$\begin{aligned}
&\sum_{k=0}^{[T/h]-1} E_0[\Psi(\Phi^k_{h,R}((k+1)h, X^h(kh,x))) - \Psi(X^h(kh,x))] \qquad (5.23).\\
&= \int_0^{[T/h]h} ds \int_{\mathbf{R}^d} [< \nabla\Psi(\Phi^{[s/h]}_{h,R}(s,y)), b_R([s/h]h, y) >] p_h(s,y) dy\\
&\to \int_0^T ds \int_{\mathbf{R}^d} < \nabla\Psi(y), b_R(s,y) > p(s,y) dy
\end{aligned}$$

as $h \to 0$, by Theorem 1.1 in the same way as in (5.21).

Q. E. D.

The following lemma will be used in the proof of Lemma 5.7.

Lemma 5.6. *(see [12, p. 6, (15)]) For any $\alpha \in (d/(d+2), 1)$, there exists a positive constant C such that the following holds: for any $R > 0$ and any probability density function ρ on $\mathbf{R}^d$ for which $M(\rho) < \infty$ (see (1.2)),*

$$\begin{aligned}
&\int_{|x|\geq R, \rho(x)<1} |\rho(x)\log\rho(x)| dx \qquad (5.24).\\
&\leq C(R^2+1)^{(-(2+d)\alpha+d)/2}(M(\rho)+1)^\alpha.
\end{aligned}$$

The following lemma will be used in the proof of Lemma 5.8.

Lemma 5.7. *Suppose that (A.1)-(A.2) hold. Then the following holds:*

$$\liminf_{h\to 0} F(p_h^{[T/h]}) \geq F(p(T,\cdot)). \tag{5.25}$$

(Proof).

$$\begin{aligned}
&F(p_h^{[T/h]}) \\
&= \int_{\mathbf{R}^d} [\log p_h^{[T/h]}(x) + \Psi(x)] p_h^{[T/h]}(x)dx \\
&\geq \int_{p_h^{[T/h]}(x)<1,|x|\geq R} p_h^{[T/h]}(x) \log p_h^{[T/h]}(x)dx \\
&\quad + \int_{|x|<R} p_h^{[T/h]}(x) \log p_h^{[T/h]}(x)dx + \int_{|x|<R} \Psi(x) p_h^{[T/h]}(x)dx.
\end{aligned} \tag{5.26}$$

The first integral on the right hand side of (5.26) can be considered as follow: by Lemmas 5.1 and 5.6, and (A.2),

$$\limsup_{R\to\infty} \limsup_{h\to 0} |\int_{p_h^{[T/h]}(x)<1,|x|\geq R} p_h^{[T/h]}(x) \log p_h^{[T/h]}(x)dx| = 0, \tag{5.27}$$

since

$$\begin{aligned}
M(p_h^{[T/h]}) &\leq 2E_0[|X^h([T/h]h,x) - x|^2] + 2E_0[|x|^2] \\
&\leq 2([T/h]) \sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h,x) - X^h(kh,x)|^2] \\
&\quad + 2E_0[|x|^2].
\end{aligned} \tag{5.28}$$

The third integral on the right hand side of (5.26) can be considered as follow: by Theorem 1.1,

$$\begin{aligned}
\lim_{h\to 0} \int_{|x|<R} \Psi(x) p_h^{[T/h]}(x)dx &= \int_{|x|<R} \Psi(x) p(T,x)dx \\
&\to \int_{\mathbf{R}^d} \Psi(x) p(T,x)dx \quad (\text{as } R\to\infty).
\end{aligned} \tag{5.29}$$

The secod integral on the right hand side of (5.26) can be considered as follow: by Jensen's inequality (see [1]) and Theorem 1.1,

$$
\begin{aligned}
&\int_{|x|<R} p_h^{[T/h]}(x)\log p_h^{[T/h]}(x)dx \qquad (5.30).\\
&\geq \int_{|x|<R} p_h^{[T/h]}(x)\log p(T,x)dx\\
&\quad -\int_{|x|<R} p_h^{[T/h]}(x)dx\log(\int_{|x|<R} p(T,x)dx/\int_{|x|<R} p_h^{[T/h]}(x)dx)\\
&\to \int_{|x|<R} p(T,x)\log p(T,x)dx \quad (\text{as } h\to 0)\\
&\to \int_{\mathbf{R}^d} p(T,x)\log p(T,x)dx \quad (\text{as } R\to\infty).
\end{aligned}
$$

Q. E. D.

The following lemma will be used in the proof of Theorem 2.3.

Lemma 5.8. *Suppose that (A.1)-(A.6) hold. Then*

$$
\begin{aligned}
&\limsup_{h\to 0}\sum_{k=0}^{[T/h]-1} E_0[|X^h((k+1)h,x)-X^h(kh,x)|^2]/h \qquad (5.31).\\
&\leq \int_0^T ds\int_{\mathbf{R}^d}|b(s,y)|^2 p(s,y)dy.
\end{aligned}
$$

(Proof). For $k=0,\cdots,[T/h]-1$ and $R>0$,

$$
\begin{aligned}
&E_0[|X^h((k+1)h,x)-X^h(kh,x)|^2]/h = d(p_h^k,p_h^{k+1})^2/h \qquad (5.32).\\
&\leq E_0[|\Phi_{h,R}^k((k+1)h,X^h(kh,x))-X^h(kh,x)|^2/h]\\
&\qquad +2F(q_{h,R}^k)-2F(p_h^{k+1})\\
&=2F(q_{h,R}^k)-2F(p_h^k)+E[h|b_R(kh,X^h(kh,x))|^2]\\
&\qquad -2F(p_h^{k+1})+2F(p_h^k)
\end{aligned}
$$

(see (1.4), (1.10)-(1.13) and (5.16)). By Lemma 5.5,

$$\lim_{h\to 0}\sum_{k=0}^{[T/h]-1}(F(q_{h,R}^k)-F(p_h^k)) \tag{5.33}$$
$$= -\int_0^T ds\int_{\mathbf{R}^d} < b_R(s,y), b(s,y) > p(s,y)dy$$
$$\to -\int_0^T ds\int_{\mathbf{R}^d}|b(s,y)|^2 p(s,y)dy,$$

as $R\to\infty$.

It is easy to see that the following is true: by Theorem 1.1 and the continuity of $b(t,x)$,

$$\sum_{k=0}^{[T/h]-1} E_0[h|b_R(kh, X^h(kh,x))|^2] \tag{5.34}$$
$$= \int_0^{[T/h]h} ds\int_{\mathbf{R}^d}|b_R([s/h]h,y)|^2 p_h(s,y)dy$$
$$\to \int_0^T ds\int_{\mathbf{R}^d}|b_R(s,y)|^2 p(s,y)dy \quad (\text{as } h\to 0)$$
$$\to \int_0^T ds\int_{\mathbf{R}^d}|b(s,y)|^2 p(s,y)dy \quad (\text{as } R\to\infty).$$

By Lemma 5.7, we obtain the following:

$$\limsup_{h\to 0}\sum_{k=0}^{[T/h]-1}\{-F(p_h^{k+1})+F(p_h^k)\} \tag{5.35}$$
$$\le -F(p(T,\cdot))+F(p(0,\cdot)) = \int_0^T ds\int_{\mathbf{R}^d}|b(s,x)|^2 p(s,x)dx.$$

Let us prove the equality in (5.35) to complete the proof.

$$-F(p(T,\cdot))+F(p(0,\cdot)) \tag{5.36}$$
$$= \sum_{k=0}^{[3C_4T]}\{-F(p(\min((k+1)/(3C_4),T),\cdot))+F(p(\min(k/(3C_4),T),\cdot))\}$$

(see (3.3)), and for s and t for which $0 \leq t < s < t + 1/(2C_4)$,

$$-F(p(s,\cdot)) + F(p(t,\cdot)) = \int_t^s du \int_{\mathbf{R}^d} |b(u,x)|^2 p(u,x)dx. \qquad (5.37).$$

This is true, since

$$\int_{\mathbf{R}^d} p(s,z)dzP(Z^s(u,(t,z)) \in dx) = p(t+s-u,x)dx \quad (t \leq u \leq s) \qquad (5.38).$$

by (4.4) (see [7] or [16]), and henceforth

$$\begin{aligned}
&- F(p(s,\cdot)) + F(p(t,\cdot)) \qquad (5.39).\\
&= \int_{\mathbf{R}^d} p(s,z)dzE[\log p(t, Z^s(s,(t,z))) + \Psi(Z^s(s,(t,z)))\\
&\qquad - \log p(s,z) - \Psi(z)]\\
&= \int_t^s du \int_{\mathbf{R}^d} p(t+s-u,x)dx|b(t+s-u,x)|^2.
\end{aligned}$$

Let us explain the reason why the last equality in (5.39) holds. For $z \in \mathbf{R}^d$ and $R > 0$, by the Ito formula (see [8]) and (4.6),

$$\begin{aligned}
&E[\log p(t+s-\tau_R^s(t,z), Z^s(\tau_R^s(t,z),(t,z))) \qquad (5.40).\\
&\qquad + \Psi(Z^s(\tau_R^s(t,z),(t,z))) - \log p(s,z) - \Psi(z)]\\
&= E[\int_t^{\tau_R^s(t,z)} \{-\partial \log p(t+s-u, Z^s(u,(t,z)))/\partial u\\
&\qquad + \triangle_x \log p(t+s-u, Z^s(u,(t,z))) + \triangle\Psi(Z^s(u,(t,z)))\\
&\qquad + < 2\nabla_x \log p(t+s-u, Z^s(u,(t,z))) + \nabla\Psi(Z^s(u,(t,z)))\\
&\qquad , \nabla_x \log p(t+s-u, Z^s(u,(t,z))) + \nabla\Psi(Z^s(u,(t,z))) >\}du]\\
&= E[\int_t^{\tau_R^s(t,z)} |b(t+s-u, Z^s(u,(t,z))|^2 du] \quad \text{(by (4.8))}\\
&\to E[\int_t^s |b(t+s-u, Z^s(u,(t,z))|^2 du] \quad (\text{as } R \to \infty)
\end{aligned}$$

by Lemma 4.2. On the first part of (5.40), we have the following:

$$
\begin{aligned}
&E[\log p(t+s-\tau_R^s(t,z), Z^s(\tau_R^s(t,z),(t,z))) \\
&\qquad + \Psi(Z^s(\tau_R^s(t,z),(t,z)))] \\
&= E[\log p(t, Z^s(s,(t,z))) + \Psi(Z^s(s,(t,z))); \tau_R^s(t,z) = s] \\
&\qquad + E[\log p(t+s-\tau_R^s(t,z), Z^s(\tau_R^s(t,z),(t,z))) \\
&\qquad + \Psi(Z^s(\tau_R^s(t,z),(t,z))); \tau_R^s(t,z) < s],
\end{aligned} \tag{5.41}
$$

and

$$
\begin{aligned}
&E[\log p(t, Z^s(s,(t,z))) + \Psi(Z^s(s,(t,z))); \tau_R^s(t,z) = s] \\
&\to E[\log p(t, Z^s(s,(t,z))) + \Psi(Z^s(s,(t,z)))] \quad (\text{as } R \to \infty)
\end{aligned} \tag{5.42}
$$

by Lemma 4.2, and

$$
\begin{aligned}
&E[|\log p(t+s-\tau_R^s(t,z), Z^s(\tau_R^s(t,z),(t,z))) \\
&\qquad + \Psi(Z^s(\tau_R^s(t,z),(t,z)))|; \tau_R^s(t,z) < s] \\
&\le \sup\{E[|\log p(u,y) + \Psi(y)|; 0 \le u \le s, |y| = R\} P(\tau_R^s(t,z) < s) \\
&\to 0 \quad (\text{as } R \to \infty),
\end{aligned} \tag{5.43}
$$

by Lemma 4.4, since

$$
\sup\{E[|\log p(u,y) + \Psi(y)|; 0 \le u \le s, |y| = R\}/(1+R^2) \tag{5.44}
$$

is bounded in R by Theorem 2.2, (A.3) and the Taylor expansion.

Q. E. D.

Let us finally prove Theorem 2.3.

Proof of Theorem 2.3.

For $R_1 > 0$ and $\varepsilon > (hR_1)^{1/2}$,

$$\begin{aligned}
&P_0(\sup_{0\le t\le T} |X(t,x) - X^h(t,x)| \ge 2\varepsilon) \qquad (5.45).\\
&\le P_0(\sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h \ge R_1)\\
&\quad + P_0(|X(0,x)| = |\overline{X}^h(0,x)| \ge R_1)\\
&\quad + P_0(\sum_{k=0}^{[T/h]} |X^h((k+1)h,x) - X^h(kh,x)|^2/h < R_1,\\
&\quad |X(0,x)| = |\overline{X}^h(0,x)| < R_1, \sup_{0\le t\le T} |X(t,x) - \overline{X}^h(t,x)| \ge \varepsilon).
\end{aligned}$$

This is true, since for $t \in [kh,(k+1)h]$ $(0 \le k \le [T/h])$

$$\begin{aligned}
&|\overline{X}^h(t,x) - X^h(t,x)|\\
&= |(t-kh)[X^h((k+1)h,x) - X^h(kh,x)]/h|\\
&\le |X^h((k+1)h,x) - X^h(kh,x)|\\
&\le \{\sum_{i=0}^{[T/h]} |X^h((i+1)h,x) - X^h(ih,x)|^2\}^{1/2}.
\end{aligned}$$

The first and the second probabilities on the right hand side of (5.45) converge to zero as $h \to 0$ and then $R_1 \to \infty$ by Lemma 5.1 and Chebychev's inequality (see [1]). Let us show that the third probability on the right hand side of (5.45) converges to zero as $h \to 0$. By Lemma 5.2 and Chebychev's inequality (see [1]), we only have to show the following:

$$\begin{aligned}
0 = \lim_{h\to 0} E_0[\int_0^{[T/h]h} &|b(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.46).\\
&- (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h|ds].
\end{aligned}$$

Let us prove (5.46). For $R' > 0$,

$$
\begin{aligned}
&E_0[\int_0^{[T/h]h} |b(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.47). \\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h)|ds] \\
&\le E_0[\int_0^{[T/h]h} |b(s+h, \overline{X}^h(([s/h]+1)h,x)) \\
&\quad - b_{R'}(s+h, \overline{X}^h(([s/h]+1)h,x))|ds] \\
&\quad + (TE_0[\int_0^{[T/h]h} |b_{R'}(s+h, \overline{X}^h(([s/h]+1)h,x)) \\
&\quad - (X^h(([s/h]+1)h,x) - X^h([s/h]h,x))/h)|^2 ds])^{1/2}
\end{aligned}
$$

(see (5.10)-(5.11)).

The first part on the right hand side of (5.47) can be shown to converge to zero as follows: by (5.2) and Chebychev's inequality (see [1]),

$$
\begin{aligned}
&E_0[\int_0^{[T/h]h} |b(s+h, \overline{X}^h(([s/h]+1)h,x)) \qquad (5.48). \\
&\quad - b_{R'}(s+h, \overline{X}^h(([s/h]+1)h,x))|ds] \\
&\le \int_0^T E_0[C(b)(|X^h(([s/h]+1)h,x)|+1) \\
&\quad ; |X^h(([s/h]+1)h,x)| \ge R']ds \\
&\le \int_0^T \int_{\mathbf{R}^d} C(b)(|x|+1)^2/(R'+1) p_h(s+h,x)dxds \\
&\le 2C(b)T(\sup_{0 \le s \le T+h} M(p_h(s,\cdot))+1)/(R'+1),
\end{aligned}
$$

which converges to zero as $h \to 0$ and then $R' \to \infty$ by Lemma 5.1 and (5.28).

By Lemmas 5.4 and 5.8, the second part on the right hand side of (5.47) converges to zero as $h \to 0$ and then $R' \to \infty$.

Q. E. D.

6. Proof of Theorems 2.4 and 2.5.

In this section we prove Theorems 2.4 and 2.5. We fix $T > 0$.

Let us first prove Theorem 2.4.

Proof of Theorem 2.4.

For $\{S(t,x)\}_{0\le t\le T, x\in \mathbf{R}^d} \in \tilde{A}^T$,

$$\begin{aligned} &E_0[\int_0^T |dS(t,x)/dt|^2 dt] \\ &\ge 2E_0[\int_0^T < b(t,S(t,x)), dS(t,x)/dt > dt] - E_0[\int_0^T |b(t,S(t,x))|^2 dt], \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &E_0[\int_0^T < b(t,S(t,x)), dS(t,x)/dt > dt] \\ &= -E_0[\log p(T,S(T,x)) + \Psi(S(T,x)) - \log p(0,S(0,x)) - \Psi(S(0,x))] \\ &\quad + \int_0^T ds E_0[\partial \log p(s,S(s,x))/\partial s] \\ &= -E_0[\log p(T,X(T,x)) + \Psi(X(T,x)) - \log p(0,X(0,x)) - \Psi(X(0,x))] \\ &\quad + \int_0^T ds E_0[\partial \log p(s,X(s,x))/\partial s] \\ &= E_0[\int_0^T < b(t,X(t,x)), dX(t,x)/dt > dt] = E_0[\int_0^T |b(t,X(t,x))|^2 dt] \end{aligned} \tag{6.2}$$

by Theorem 2.3, and

$$\int_0^T E_0[|b(t,S(t,x))|^2]dt = \int_0^T E_0[|b(t,X(t,x))|^2]dt. \tag{6.3}$$

In (6.2), we used the following:

$$\int_0^T ds \int_{\mathbf{R}^d} |\partial \log p(s,y)/\partial s| p(s,y) dy < \infty. \tag{6.4}$$

Let us prove (6.4) to complete the proof. By (4.8), (A.3) and Theorem 2.2, we only have to show the following:

$$\int_0^T ds \int_{\mathbf{R}^d} |\triangle_x \log p(s,x)| p(s,x)dx < \infty, \qquad (6.5).$$

since by Theorem 1.1,

$$\sup_{0 \le s \le T} M(p(s,\cdot)) < \infty. \qquad (6.6).$$

(6.5) can be shown by the following: by (5.38), for $i = 1, \cdots, d$, in the same way as in (4.26),

$$\begin{aligned}
&\int_0^T ds \int_{\mathbf{R}^d} |\partial^2 \log p(s,x)/\partial x_i^2|^2 p(s,x)dx \qquad (6.7).\\
&\le \int_0^T ds \int_{\mathbf{R}^d} |\partial_i \nabla_x \log p(s,x)|^2 p(s,x)dx \\
&= \int_{\mathbf{R}^d} p(T,z)dz E[(\int_0^T < \partial_i \nabla_x \log p(T-t, Z^T(t,(0,z))), dW(t) >)^2] \\
&= \int_{\mathbf{R}^d} p(T,z)dz E[(\partial_i \log p(0, Z^T(T,(0,z))) - \partial_i \log p(T,z) \\
&\quad + \int_0^T [\partial_i \triangle \Psi(Z^T(t,(0,z))) + < \partial_i \nabla \Psi(Z^T(t,(0,z))) \\
&\quad , \nabla_x \log p(T-t, Z^T(t,(0,z))) >]dt)^2] < \infty,
\end{aligned}$$

by (A.3), (A.6), Theorem 2.2, (6.6) and Lemma 4.2.

Q. E. D.

The proof of Theorem 2.5 can be done almost in the same way as in Theorem 2.3. The following lemma plays a similar role to that of Lemma 5.1.

Lemma 6.1. *Suppose that (A.1)-(A.6) hold. Then the following holds: for* $h \in (0,1)$

$$\begin{aligned}
&\sum_{k=0}^{[T/h]} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h \qquad (6.8).\\
&\le \int_0^{T+h} E_0[|b(s,X(s,x))|^2 ds] < \infty.
\end{aligned}$$

(Proof). The proof is done by the following: for any $k = 0, \cdots, [T/h]$,

$$
\begin{aligned}
&E_0[|\tilde{X}^h((k+1)h, x) - \tilde{X}^h(kh, x)|^2] \qquad (6.9).\\
&\le E_0[|X((k+1)h, x) - X(kh, x)|^2] \le h \int_{kh}^{(k+1)h} E_0[|b(s, X(s, x))|^2 ds]
\end{aligned}
$$

(see (2.11)) by Schwartz's inequality.

Q. E. D.

The following lemma can be proved in the same way as in Lemma 5.2 and the proof is omitted.

Lemma 6.2. *Suppose that (A.1)-(A.6) hold and that $h \in (0,1)$. For $R_1 > 0$, suppose that*

$$|X(0, x)| = |\tilde{X}^h(0, x)| < R_1,$$

and that

$$\sum_{k=0}^{[T/h]} |\tilde{X}^h((k+1)h, x) - \tilde{X}^h(kh, x)|^2/h < R_1.$$

Then for $R > 0$ for which

$$R > \max((R_1 + C(b)(T+1)) \exp(C(b)(T+1)), R_1 + ((T+1)R_1)^{1/2}), \qquad (6.10).$$

the following holds: for $t \in [0, T]$,

$$
\begin{aligned}
&|X(t, x) - \tilde{X}^h(t, x)| \qquad (6.11).\\
&\le \{\int_0^{[T/h]h} |b(s, \tilde{X}^h(s, x)) - (\tilde{X}^h(([s/h]+1)h, x) - \tilde{X}^h([s/h]h, x))/h| ds\\
&\quad + h \sup\{|b(s, z)| : [T/h]h \le s \le T, |z| \le R\} + (hR_1)^{1/2}\} \exp(tC(b, R)).
\end{aligned}
$$

Let us finally prove Theorem 2.5.

Proof of Theorem 2.5.

Let us prove the first part of Theorem 2.5. For $\{S(t,x)\}_{0\le t\le T, x\in \mathbf{R}^d} \in \tilde{A}_h^T$,

$$
\begin{aligned}
&\int_0^{[T/h]h} E_0[|d\tilde{X}^h(t,x)/dt|^2]dt \qquad (6.12).\\
&= \sum_{k=0}^{[T/h]-1} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h\\
&\le \sum_{k=0}^{[T/h]-1} E_0[|S((k+1)h,x) - S(kh,x)|^2]/h\\
&\le \int_0^{[T/h]h} E_0[|dS(t,x)/dt|^2]dt,
\end{aligned}
$$

where the equality holds if and only if $dS(t,x)/dt = d\tilde{X}^h(t,x)/dt \ dtP_0(dx)$-a.e. by definition (see (2.11)).

Let us prove the rest part of Theorem 2.5. In the same way as in (5.45)-(5.48), we only have to show the following:

$$
\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x)) - (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.13).\\
&\quad - \tilde{X}^h([s/h]h,x))/h|^2]ds \to 0
\end{aligned}
$$

as $h \to 0$ and then $R' \to \infty$. Let us prove (6.13).

$$
\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x)) - (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.14).\\
&\quad - \tilde{X}^h([s/h]h,x))/h|^2]ds\\
&= \int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds\\
&\quad - 2\int_0^{[T/h]h} E_0[< b_{R'}(s,\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x)\\
&\quad - \tilde{X}^h([s/h]h,x))/h >]ds\\
&\quad + \sum_{k=0}^{[T/h]} E_0[|\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2]/h,
\end{aligned}
$$

and by Lemma 6.1, we only have to show the following:

$$\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds \to \int_0^T E_0[|b(s,X(s,x))|^2]ds, \qquad (6.15).$$

$$\int_0^{[T/h]h} E_0[< b_{R'}(s,\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.16).$$

$$-\tilde{X}^h([s/h]h,x))/h >]ds \to \int_0^T E_0[|b(s,X(s,x))|^2]ds,$$

as $h \to 0$, and then $R' \to \infty$.

(6.15) can be shown as follows:

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2]ds \qquad (6.17).\\
&= \int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2 - |b_{R'}(s,\tilde{X}^h([s/h]h,x))|^2]ds\\
&\quad + \int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h([s/h]h,x))|^2]ds.
\end{aligned}$$

The first part on the right hand side of (6.17) can be considered as follows:

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h(s,x))|^2 - |b_{R'}(s,\tilde{X}^h([s/h]h,x))|^2]ds \qquad (6.18).\\
&\leq 2 \sup_{0\leq s\leq T} |b_{R'}(s,\cdot)|_\infty \sup_{0\leq s\leq T} |D_z b_{R'}(s,\cdot)|_\infty\\
&\quad \times \int_0^{[T/h]h} E_0[|\tilde{X}^h(s,x) - \tilde{X}^h([s/h]h,x)|]ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|\tilde{X}^h(s,x) - \tilde{X}^h([s/h]h,x)|]ds \qquad (6.19).\\
&\leq hE_0[\sum_{k=0}^{[T/h]-1} |\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|]\\
&\leq h([T/h]E_0[\sum_{k=0}^{[T/h]-1} |\tilde{X}^h((k+1)h,x) - \tilde{X}^h(kh,x)|^2])^{1/2}\\
&\to 0 \quad (\text{as } h \to 0),
\end{aligned}$$

by Lemma 6.1 (see below (5.8)). The second part on the right hand side of (6.17) can be considered as follows: by the continuity of p,

$$\begin{aligned}
&\int_0^{[T/h]h} E_0[|b_{R'}(s,\tilde{X}^h([s/h]h,x))|^2]ds \qquad (6.20).\\
&= \int_0^{[T/h]h}\int_{\mathbf{R}^d} |b_{R'}(s,y)|^2 p([s/h]h,y)dyds\\
&\to \int_0^T\int_{\mathbf{R}^d} |b_{R'}(s,y)|^2 p(s,y)dyds \quad (\text{as } h\to 0)\\
&\to \int_0^T\int_{\mathbf{R}^d} |b(s,y)|^2 p(s,y)dyds \quad (\text{as } R'\to\infty),
\end{aligned}$$

since $P_0^{\tilde{X}^h(kh,\cdot)^{-1}}(dy) = p(kh,y)dy$ for all $k \geq 0$.

Let us prove (6.16). By (5.10)-(5.11),

$$\begin{aligned}
&-\int_0^{[T/h]h} E_0[< b_{R'}(s,\tilde{X}^h(s,x)),(\tilde{X}^h(([s/h]+1)h,x) \qquad (6.21).\\
&\quad - \tilde{X}^h([s/h]h,x))/h >]ds\\
&= E_0[\phi_{R'}(\tilde{X}^h([T/h]h,x))\{\log p([T/h]h,\tilde{X}^h([T/h]h,x))\\
&\quad + \Psi(\tilde{X}^h([T/h]h,x))\} - \phi_{R'}(x)\{\log p(0,x)+\Psi(x)\}]\\
&\quad -\int_0^{[T/h]h} E_0[\phi_{R'}(\tilde{X}^h(s,x))\partial\log p(s,\tilde{X}^h(s,x))/\partial s\\
&\quad + < \nabla\phi_{R'}(\tilde{X}^h(s,x)),(\tilde{X}^h(([s/h]+1)h,x)-\tilde{X}^h([s/h]h,x))/h >\\
&\quad \times \{\log p(s,\tilde{X}^h(s,x))+\Psi(\tilde{X}^h(s,x))\}]ds.
\end{aligned}$$

In the same way as in (6.18),

$$\begin{aligned}
&E_0[\phi_{R'}(\tilde{X}^h([T/h]h,x))\{\log p([T/h]h,\tilde{X}^h([T/h]h,x)) \qquad (6.22).\\
&\quad + \Psi(\tilde{X}^h([T/h]h,x))\} - \phi_{R'}(x)\{\log p(0,x)+\Psi(x)\}]\\
&\quad -\int_0^{[T/h]h} E[\phi_{R'}(\tilde{X}^h(s,x))\partial\log p(s,\tilde{X}^h(s,x))/\partial s]ds\\
&\to E_0[\phi_{R'}(X(T,x))\{\log p(T,X(T,x))+\Psi(X(T,x))\}
\end{aligned}$$

$$
\begin{aligned}
&- \phi_{R'}(x)\{\log p(0,x) + \Psi(x)\}] \\
&- \int_0^T E_0[\phi_{R'}(X(s,x))\partial \log p(s,X(s,x))/\partial s]ds \quad (\text{as } h \to 0) \\
\to &- \int_0^T E[|b(s,X(s,x))|^2]ds \quad (\text{as } R' \to \infty)
\end{aligned}
$$

by Theorem 1.1 and (6.4).

On the last part on the right hand side of (6.21),

$$
\begin{aligned}
&\int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h(s,x)), (\tilde{X}^h(([s/h]+1)h,x) \qquad (6.23). \\
&\quad - \tilde{X}^h([s/h]h,x))/h > \{\log p(s,\tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\}]ds \\
&= \int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h(s,x))\{\log p(s,\tilde{X}^h(s,x)) + \Psi(\tilde{X}^h(s,x))\} \\
&\quad - \nabla\phi_{R'}(\tilde{X}^h([s/h]h,x))\{\log p(s,\tilde{X}^h([s/h]h,x)) + \Psi(\tilde{X}^h([s/h]h,x))\} \\
&\quad , (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds \\
&\quad + \int_0^{[T/h]h} E_0[< \nabla\phi_{R'}(\tilde{X}^h([s/h]h,x))\{\log p(s,\tilde{X}^h([s/h]h,x)) \\
&\quad + \Psi(\tilde{X}^h([s/h]h,x))\}, (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]ds.
\end{aligned}
$$

The first part on the right hand side of (6.23) can be shown to converge to zero as $h \to 0$ in the same way as in (6.18)-(6.19), by Lemma 6.1. On the second part, we consider as follows:

$$
\begin{aligned}
&\int_0^{[T/h]h} |E_0[< \nabla\phi_{R'}(\tilde{X}^h([s/h]h,x))\{\log p(s,\tilde{X}^h([s/h]h,x)) \qquad (6.24). \\
&\quad + \Psi(\tilde{X}^h([s/h]h,x))\}, (\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h >]|ds \\
&\le (\int_0^{[T/h]h} E[|\nabla\phi_{R'}(X([s/h]h,x))\{\log p(s,X([s/h]h,x)) \\
&\quad + \Psi(X([s/h]h,x))\}|^2 ds])^{1/2} \\
&\quad \times (\int_0^{[T/h]h} E[|(\tilde{X}^h(([s/h]+1)h,x) - \tilde{X}^h([s/h]h,x))/h|^2 ds])^{1/2} \\
&\to 0,
\end{aligned}
$$

as $h \to 0$ and $R' \to \infty$ by Lemma 6.1, the continuity of p, and (5.10), since for $y \in \mathbf{R}^d$

$$|\nabla\phi_{R'}(y)\{\log p(s,y)+\Psi(y)\}| \tag{6.25}$$
$$\le I_{[R',2R'+1]}(y)(R')^{-1}(1+(2R'+1)^2)|\log p(s,y)+\Psi(y)|/(1+|y|^2),$$

and since

$$\sup\{(|\log p(s,y)+\Psi(y)|)/(1+|y|^2) : 0\le s\le T, y\in \mathbf{R}^d\}<\infty \tag{6.26}$$

by (A.3), Theorem 2.2 and the Taylor expansion, and since $M(p(t,\cdot))\in L^\infty([0,T];dt)$ by Theorem 1.1.

Q. E. D.

7. Proof of Proposition 1.1.

In this section we prove Proposition 1.1.

Since $F_h^n(X^h(nh,x))$ is uniformly distributed under P_0, (1.28) is true (see [23]).

Inductively one can show that (1.29) is true, since for a distribution function F on $\mathbf{R}$

$$F(F^{-1}(x))=x \quad (0<u<1) \tag{7.1}$$

(see [18, p. 18], or [24]). To prove (1.30), let us first show the following:

$$\limsup_{h\to 0} X^h(t,x)\le F_t^{-1}(F_0(x)). \tag{7.2}$$

Suppose that there exists a sequence of positive numbers $\{h_n\}_{n\ge 1}$ such that $h_n\to 0$ as $n\to\infty$ and that for $n\ge 1$

$$X^{h_n}(t,x)\ge F_t^{-1}(F_0(x))+\delta \tag{7.3}$$

for some $\delta>0$, $t>0$ and $x\in\mathbf{R}$. Then by (1.29), (7.1) and Theorem 1.1,

$$F_0(x) = F_h^{[t/h_n]}(X^{h_n}(t,x)) \geq F_h^{[t/h_n]}(F_t^{-1}(F_0(x)) + \delta) \qquad (7.4).$$
$$\to F_t(F_t^{-1}(F_0(x)) + \delta) > F_0(x),$$

as $n \to \infty$, which is a contradiction.

In the same way as above, we can show that

$$\liminf_{h \to 0} X^h(t,x) \geq F_t^{-1}(F_0(x)). \qquad (7.5).$$

Q. E. D.

REMARK 7.1. It is easy to see, formally, that $X(t,x) \equiv F_t^{-1}(F_0(x))$ is a solution to (1.14). In fact, by (7.1),

$$F_t(X(t,x)) = F_0(x), \qquad (7.6).$$

from which we obtain, formally,

$$\partial F_t(X(t,x))/\partial t + (\partial F_t(X(t,x))/\partial x) dX(t,x)/dt = 0. \qquad (7.7).$$

By (1.8), we formally obtain the following:

$$\partial F_t(x)/\partial t = \partial p(t,x)/\partial x + (d\Psi(x)/dx)p(t,x) \quad (t > 0, x \in \mathbf{R}^d). \qquad (7.8).$$

(7.7)-(7.8) implies that $X(t,x) \equiv F_t^{-1}(F_0(x))$ is a solution to (1.14), under some additional assumption on p to justify (7.7)-(7.8).

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