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**Global existence of classical solutions to systems of  
nonlinear wave equations with different speeds of propagation**

**Dedicated to Professor Rentaro Agemi on his 60th birthday**

**Kôji KUBOTA and Kazuyoshi YOKOYAMA**

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# Global existence of classical solutions to systems of nonlinear wave equations with different speeds of propagation

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**Abstract** This paper is concerned with a system of nonlinear wave equations in three space dimensions

$$\partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u), \quad i = 1, 2, \dots, m,$$

where  $0 < c_1 < c_2 < \dots < c_m$ . We prove the global existence of classical solutions to the system with small initial data, provided  $F^i$  satisfy “Null condition”.

## §1. Introduction.

This paper is a continuation of a previous paper [21] on global (in time) existence of small amplitude solutions to a system of quasilinear wave equations with different speeds of propagation. Indeed, the second author studied in [21] the system stated in the above abstract for the case that  $F^i$  depend only on the first and second derivatives  $\partial u, \partial^2 u$  of  $u$ , i.e., that  $F^i = F^i(\partial u, \partial^2 u)$ . The purpose of the present paper is to extend the results of [21] to the case where  $F^i = F^i(u, \partial u, \partial^2 u)$  depend explicitly on  $u$  as well as  $\partial u$  and  $\partial^2 u$ . As a result, we will refine the proof of the main theorem in Klainerman [14] which deals with the case of one speed of propagation, i.e.,  $m = 1$ , in the sense that we do not need to make use of the vector fields  $L_j = x_j \partial_t + t \partial_j$ ,  $j = 1, 2, 3$ , where  $\partial_t = \partial/\partial t$  and  $\partial_j = \partial/\partial x_j$ .

We consider the following Cauchy problem

$$(1.1) \quad \square_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u)$$

$$\text{in } (0, \infty) \times \mathbb{R}^3, \quad i = 1, 2, \dots, m,$$

$$(1.2) \quad u^i(0, x) = \varepsilon f^i(x), \quad \partial_t u^i(0, x) = \varepsilon g^i(x)$$

$$\text{for } x \in \mathbb{R}^3, \quad i = 1, 2, \dots, m,$$

where  $c_1, c_2, \dots, c_m$  are positive constants different each other, say,  $0 < c_1 < c_2 < \dots < c_m$ ,  $f^i$  and  $g^i$  are fixed functions in  $C_0^\infty(\mathbb{R}^3)$  and  $\varepsilon$  is a positive parameter which serves to measure the *amplitude* of the initial values. Here  $u^i = u^i(t, x) = (u_1^i(t, x), u_2^i(t, x), \dots, u_{p_i}^i(t, x))$  is a vector of  $p_i$  real valued functions of  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ ,  $u = (u^1, u^2, \dots, u^m)$  a  $p$ -vector, where  $p = p_1 + p_2 + \dots + p_m$ ,  $\partial = (\partial_t, \partial_x)$ ,  $\partial_t = \partial_0 = \partial/\partial t$ ,  $\partial_x = (\partial_1, \partial_2, \partial_3)$ ,  $\partial_j = \partial/\partial x_j$  for  $j = 1, 2, 3$ ,  $\Delta = \sum_{j=1}^3 \partial_j^2$  and  $\partial^2 u$  stands for the second derivatives of  $u$ . For convenience we will often denote  $\partial u$  and  $\partial^2 u$  by  $u'$  and  $u''$ , respectively. Moreover  $F^i(u, u', u'')$  is a vector of  $p_i$  real valued functions of  $(u, u', u'')$ , belonging to  $C^\infty(\mathbb{R}^{21p})$ , such that

$$(1.3)_1 \quad F^i(0, 0, 0) = 0, \quad i = 1, 2, \dots, m$$

and

$$(1.3)_2 \quad (F^i)'(0, 0, 0) = 0, \quad i = 1, 2, \dots, m.$$

Here and in what follows we denote by  $(F^i)'(u, u', u'')$  the first derivatives of  $F^i$  with respect to  $(u, u', u'')$  and by  $\partial_u F^i(u, u', u'')$  the first derivatives of  $F^i(u, u', u'')$  with respect to  $u$ , and so on.

**Remark.** The hypothesis  $(1.3)_1$  implies that the Cauchy problem (1.1)-(1.2) has a trivial solution  $u = 0$ , provided  $\varepsilon = 0$ . Moreover the condition  $(1.3)_2$  means that the *linear part* of (1.1) coincides with  $\square_i$ . For the smoothness of  $F^i$  it is sufficient to assume that  $F^i(u, u', u'')$  is of class  $C^\infty$  on a neighborhood of  $(u, u', u'') = (0, 0, 0)$ . However such a function  $F^i$  can be always extended to the whole space  $\mathbb{R}^{21p}$  by multiplying a

cutoff function. For the sake of simplicity of description we have therefore assumed that  $F^i \in C^\infty(\mathbb{R}^{21p})$ .

We shall assume without loss of generality that the system (1.1) is quasilinear, i.e., that  $F^i(u, u', u'')$  is linear in  $u''$ , hence one can write

$$(1.4)_1 \quad F^i(u, u', u'') = \sum_{j=1}^m \sum_{a,b=0}^3 A_{ij}^{ab}(u, u') \partial_a \partial_b u^j + B^i(u, u') \\ \text{for } i = 1, 2, \dots, m,$$

where  $A_{ij}^{ab}(u, u')$  is a  $p_i \times p_j$  matrix function of  $(u, u') \in \mathbb{R}^p \times \mathbb{R}^{4p}$  and  $u^j$  are regarded as column vectors. (See for instance Courant and Hilbert [4], chapter I, §7). We suppose that

$$(1.4)_2 \quad A_{ij}^{ba}(u, u') = A_{ij}^{ab}(u, u') \quad \text{for } a, b = 0, 1, 2, 3 \quad \text{and } i, j = 1, 2, \dots, m,$$

since only classical solutions are considered in this paper. Clearly, conditions (1.3) and (1.4)<sub>1</sub> imply that

$$(1.4)_3 \quad A_{ij}^{ab}(0, 0) = 0 \quad \text{for } i, j = 1, 2, \dots, m, \quad a, b = 0, 1, 2, 3$$

and

$$(1.4)_4 \quad B^i(0, 0) = 0, \quad (B^i)'(0, 0) = 0 \quad \text{for } i = 1, 2, \dots, m.$$

In order to assure the existence of local (in time) solutions, we also require that the system (1.1) can be reduced to a symmetric hyperbolic system of the first order. Namely, we assume that the  $p \times p$  matrices

$$(A_{ij}^{ab}(u, u'); i, j = 1, 2, \dots, m)$$

are (real) symmetric for  $a, b = 0, 1, 2, 3$ , where  $A_{ij}^{ab}(u, u')$  are given by (1.4)<sub>1</sub>, i.e., that

$$(1.5) \quad A_{ji}^{ab}(u, u') = {}^t A_{ij}^{ab}(u, u') \quad \text{for } i, j = 1, 2, \dots, m, \quad a, b = 0, 1, 2, 3,$$

where  ${}^tA$  stands for the transposed matrix of  $A$ .

Let  $F^{i,2}(u, u', u'')$  be the quadratic part of Taylor's expansion for  $F^i(u, u', u'')$  about  $(u, u', u'') = (0, 0, 0)$ . Then by the assumptions (1.3) and (1.4)<sub>1</sub> one can write

$$(1.6) \quad F^i(u, u', u'') = F^{i,2}(u, u', u'') + O((|u| + |u'|)^2 (|u| + |u'| + |u''|))$$

for  $(u, u', u'')$  near  $(0, 0, 0)$ .

We shall divide all terms of the homogeneous polynomial  $F^{i,2}(u, u', u'')$  into two group as follows.

$$(1.7) \quad F^{i,2}(u, \partial u, \partial^2 u) = N^i(u^i, \partial u^i, \partial^2 u^i) + R^i(u, \partial u, \partial^2 u),$$

where  $N^i(u^i, \partial u^i, \partial^2 u^i)$  is a homogeneous polynomial only in  $(u^i, \partial u^i, \partial^2 u^i)$  of degree 2, and  $R^i(u, \partial u, \partial^2 u)$  a homogeneous polynomial in  $(u, \partial u, \partial^2 u)$  of degree 2 which is linear in  $(u^i, \partial u^i, \partial^2 u^i)$ , i.e.,  $R^i$  satisfies

$$(1.8) \quad \partial_w^2 R^i(u, \partial u, \partial^2 u) = 0 \quad \text{for } i = 1, 2, \dots, m,$$

where  $w$  stands for  $(u^i, \partial u^i, \partial^2 u^i)$ . Then we assume that for each  $i = 1, 2, \dots, m$  the polynomial  $N^i(u^i, \partial u^i, \partial^2 u^i)$  satisfies the *null condition* in the sense of Christodoulou [2], which is stated as follows.

Let  $\tilde{u}^i, v, w \in \mathbb{R}^{p_i}$  and  $X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$  be numerical vectors. By  $\tilde{N}^i(\tilde{u}^i, v, w; X)$  we denote the polynomial  $N^i(u^i, \partial u^i, \partial^2 u^i)$  with  $u^i, \partial_a u^i$  and  $\partial_b \partial_c u^i$  replaced by  $\tilde{u}^i, X_a v$  and  $X_b X_c w$ , respectively, for  $a, b, c = 0, 1, 2, 3$ . Then the null condition means that for each  $i = 1, 2, \dots, m$  the following identity

$$(1.9)_1 \quad \tilde{N}^i(\tilde{u}^i, v, w; X^i) = 0$$

holds for any vector  $\tilde{u}^i, v, w \in \mathbb{R}^{p_i}$  and  $X^i = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$  satisfying

$$X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2).$$

(See also Klainerman [13], [14], John [9], [10], Katayama [11], and Agemi and Yokoyama [1]).

In particular, the conditions  $(1.4)_1$  and  $(1.9)_1$  imply that  $N^i(u^i, \partial u^i, \partial^2 u^i)$  is of the following form

$$(1.9)_2 \quad N^i(u^i, \partial u^i, \partial^2 u^i) = N^{i0}(\partial u^i, \partial^2 u^i) + \sum_{j=1}^{p_i} u_j^i (\partial A_{ii}^{00} / \partial u_j^i)(0, 0) \square_i u^i,$$

where  $N^{i0}(\partial u^i, \partial^2 u^i) = N^i(0, \partial u^i, \partial^2 u^i)$ . (For the proof see Appendix below). Clearly the condition  $(1.9)_1$  also implies that

$$(1.9)_3 \quad \tilde{N}^{i0}(v, w; X^i) = 0$$

holds for any  $v, w \in \mathbb{R}^{p_i}$  and  $X^i = (X^0, X^1, X^2, X^3) \in \mathbb{R}^4$  satisfying

$$X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2),$$

where  $\tilde{N}^{i0}(v, w; X^i) = \tilde{N}^i(0, v, w; X^i)$ . (For what the condition  $(1.9)_3$  means see the proof of Proposition 2.5 in the next section, in particular (2.48)).

Unfortunately, we must also require that the polynomial  $R^i(u, \partial u, \partial^2 u)$ , given by (1.7), does not explicitly depend on  $u$ , i.e., we assume that

$$(1.10) \quad \partial_u R^i(u, u', u'') = 0 \quad \text{for } i = 1, 2, \dots, m.$$

Then we write

$$R^i = R^i(u', u'') = R^i(\partial u, \partial^2 u).$$

In what follows we call  $N^{i0}(\partial u^i, \partial^2 u^i)$  and  $R^i(\partial u, \partial^2 u)$  “Null form” and “Resonance form” respectively, provided  $(1.9)_3$  and  $(1.10)$  hold.

In the case where the number of speeds of propagation is more than or equal to 2, i.e., where  $m \geq 2$ , we must impose an extra condition on  $F^i$  which may be perhaps removed. This is caused by the fact that the solution  $u^i(t, x)$  of (1.1) has a loss  $\log(1+t)$  of decay rate on the characteristic cone  $|x| = c_i t$ , more precisely, that  $|u^i(t, x)|$  behaves in general like

$$(1+t)^{-1} (1 + \log(1+t))$$

on  $|x| = c_i t$ . (See (3.28) with  $\nu = 0$  and (2.10)<sub>2</sub> below). To overcome this difficulty we assume that  $F^i$  satisfy one of the following three hypotheses (A)<sub>1</sub>, (A)<sub>2</sub> and (A)<sub>3</sub>. (See Corollary 6.8 at the end of Section 6).

The first hypothesis (A)<sub>1</sub> concerns the resonance form  $R^i(\partial u, \partial^2 u)$ , given by (1.7) and (1.10). To state this we write

$$(1.11) \quad R^i(\partial u, \partial^2 u) = \sum_{\substack{1 \leq k \leq m \\ k \neq i}} R^{ik}(\partial u^k, \partial^2 u^k) + \tilde{R}^{i0}(\partial u, \partial^2 u).$$

Here  $R^{ik}(\partial u^k, \partial^2 u^k)$  is a homogeneous polynomial in  $(\partial u^k, \partial^2 u^k)$  of degree 2 and  $\tilde{R}^{i0}(\partial u, \partial^2 u)$  is linear in  $(\partial u^k, \partial^2 u^k)$  for all  $k = 1, 2, \dots, m$ , namely,  $\tilde{R}^{i0}(\partial u, \partial^2 u)$  satisfies

$$(1.12) \quad \partial_{w^k}^2 \tilde{R}^{i0}(\partial u, \partial^2 u) = 0 \quad \text{for all } i, k = 1, 2, \dots, m,$$

where  $w^k$  stands for  $(\partial u^k, \partial^2 u^k)$ . Then we impose the following condition on  $R^{ik}$  which is analogous to (1.9)<sub>3</sub>.

(A)<sub>1</sub> For each  $i, k = 1, 2, \dots, m$  with  $k \neq i$ , the following identity

$$(1.13) \quad \tilde{R}^{ik}(v, w; X^k) = 0$$

holds for any  $v, w \in \mathbb{R}^{p_k}$  and  $X^k = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$  satisfying

$$X_0^2 = c_k^2(X_1^2 + X_2^2 + X_3^2),$$

where  $\tilde{R}^{ik}(v, w; X^k)$  is defined from  $R^{ik}(\partial u^k, \partial^2 u^k)$ , analogously to  $\tilde{N}^{i0}(v, w; X^i)$  with  $i$  replaced by  $k$ .

The second hypothesis (A)<sub>2</sub> concerns the domain of dependence. More precisely, we assume that for each  $i = 1, 2, \dots, m-1$  the equation  $\square_i u^i = F^i$  has the speed of propagation which does not exceed  $c_i$ , say, we impose the following condition on  $F^i$ .

(A)<sub>2</sub> For each  $i = 1, 2, \dots, m-1$ , the function  $F^i(u, u', u'')$  is of the following form.

$$(1.14) \quad F^i(u, \partial u, \partial^2 u) = \sum_{j=1}^i \sum_{|\alpha| \leq 1} C_{ij\alpha}(u, \partial u, \partial^2 u) \partial^\alpha u^j.$$

where  $C_{ij\alpha}(u, u', u'')$  is a  $p_i \times p_j$  matrix function of  $(u, u', u'')$ .

The last hypothesis  $(A)_3$  concerns the cubic part  $F^{i,3}(u, u', u'')$  of Taylor's expansion for  $F^i(u, u', u'')$  about  $(u, u', u'') = (0, 0, 0)$ . We assume that the parts of  $F^{i,3}(u, u', u'')$ , which are homogeneous of degree 2 or 3 with respect to  $u$ , do not explicitly depend on  $u^1, u^2, \dots, u^{m-1}$ . Namely, we require

**(A)<sub>3</sub>** For each  $i = 1, 2, \dots, m$ , the polynomial  $F^{i,3}(u, u', u'')$  is of the following form.

$$(1.15) \quad F^{i,3}(u, \partial u, \partial^2 u) = \sum_{j=1}^m C_j^i(\partial u, \partial^2 u) u^j + D^i(u^m, \partial u, \partial^2 u),$$

where  $C_j^i(u', u'')$  is a  $p_i \times p_j$  matrix function whose elements are homogeneous polynomial in  $(u', u'')$  of degree 2 and  $D^i(u^m, \partial u, \partial^2 u)$  is a homogeneous polynomial in  $(u^m, \partial u, \partial^2 u)$  of degree 3.

**Remark 1.1.** The first hypothesis  $(A)_1$  will guarantee that one can take the parameter  $\nu$  in the weighted norm (2.13) below to be positive, hence no loss of decay rate arise. (See Corollaries 5.10 and 5.15 in Section 5). On the other hand, the hypothesis  $(A)_2$  together with the weighted norm (2.10)<sub>2</sub> will enable us to apply Lindblad [19], Lemma 1.2 (namely, (6.3)<sub>2</sub> below) to deriving a certain energy estimate (i.e., Proposition 6.1 with  $\nu = 0$  in Section 6), even if the loss of decay arises. (See also Lemma 5.1 in Section 5).

We are now in a position to state the main results on global solutions of the Cauchy problem (1.1)-(1.2).

**Theorem 1.2.** Assume that the conditions (1.4), (1.5), (1.9)<sub>1</sub> and (1.10) are fulfilled.

1) Suppose that one of the hypotheses  $(A)_2$  and  $(A)_3$  is satisfied. Let  $k$  be a positive integer with  $k \geq 18$ . Then there are positive constants  $\varepsilon_0$  and  $C$  such that for

any  $0 < \varepsilon \leq \varepsilon_0$  there exists uniquely a solution  $u(t, x) = (u^1(t, x), u^2(t, x), \dots, u^m(t, x)) \in C^\infty([0, \infty) \times \mathbb{R}^3)$  of (1.1)-(1.2) satisfying

(1.16)

$$|u^i(t, x)|(1+r+t) \left( 1 + \log \frac{1+r+c_i t}{1+|r-c_i t|} \right)^{-1} + \sum_{1 \leq |\alpha| \leq k} |\partial^\alpha u^i(t, x)|(1+r) \\ \times (1+|r-c_i t|) \leq C\varepsilon \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^3, \quad i = 1, 2, \dots, m$$

and

$$(1.17) \quad (1+t)^{-1} \|u(t)\|_{L^2} + \sum_{1 \leq |\alpha| \leq k+8} \|\partial^\alpha u(t)\|_{L^2} \leq C\varepsilon \quad \text{for } t \geq 0,$$

where  $r = |x|$  and

$$\|v(t)\|_{L^2} = \left\{ \int_{\mathbb{R}^3} |v(t, x)|^2 dx \right\}^{1/2}.$$

2) Suppose that the hypothesis (A)<sub>1</sub> is satisfied. Let  $k$  be a positive integer with  $k \geq 14$ . Let  $\nu$  be a positive number with  $\nu < 1/2$ . Then there are positive constants  $\varepsilon_0$  and  $C$  such that for any  $0 < \varepsilon \leq \varepsilon_0$  there exists uniquely a solution  $u(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^3)$  of (1.1)-(1.2) satisfying

(1.18)

$$|u^i(t, x)|(1+r+t)(1+|r-c_i t|)^\nu + \sum_{1 \leq |\alpha| \leq k} |\partial^\alpha u^i(t, x)|(1+r) \\ (1+|r-c_i t|)^{1+\nu} \leq C\varepsilon \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^3, \quad i = 1, 2, \dots, m$$

and

$$(1.19) \quad (1+t)^{-1} \|u(t)\|_{L^2} + \sum_{1 \leq |\alpha| \leq k+6} \|\partial^\alpha u(t)\|_{L^2} \leq C\varepsilon \quad \text{for } t \geq 0.$$

Here the constants  $\varepsilon_0$  and  $C$  depend only on  $k, \nu, F^i, f^i$  and  $g^i$ .

**Remark 1.3.** The assumption (1.10) can be somewhat relaxed in a case. Write  $R^i$  analogously to (1.11), i.e.,

$$(1.20) \quad R^i(u, \partial u, \partial^2 u) = \sum_{\substack{1 \leq k \leq m \\ k \neq i}} R^{ik}(u^k, \partial u^k, \partial^2 u^k) + \tilde{R}^{i0}(u, \partial u, \partial^2 u),$$

where  $R^{ik}(u^k, \partial u^k, \partial^2 u^k)$  is a homogeneous polynomial in  $(u^k, \partial u^k, \partial^2 u^k)$  of degree 2 and  $\tilde{R}^{i0}(u, \partial u, \partial^2 u)$  linear in  $(u^k, \partial u^k, \partial^2 u^k)$  for all  $k = 1, 2, \dots, m$ . Assume that  $R^{ik}(u^k, \partial u^k, \partial^2 u^k)$  fulfills the condition  $(1.9)_1$  with  $i$  replaced by  $k$  hence  $R^{ik}(0, \partial u^k, \partial^2 u^k)$  satisfies (1.13) for each  $k = 1, 2, \dots, m$  with  $k \neq i$ . Moreover suppose that (1.10) as well as (1.4), (1.5) and  $(1.9)_1$  holds with  $R^i$  replaced by  $\tilde{R}^{i0}$ . Then we see easily that the conclusions of the part 2) of Theorem 1.2 are still valid.

We shall here recall some of the known results on global solutions of the Cauchy problem (1.1)-(1.2). (For the details see for example [1]-[11], [13]-[16] and [18]-[21]).

First let us consider the case that  $m = 1$  and  $c_1 = 1$ , and write (1.1)-(1.2) as

$$(1.1)' \quad \begin{aligned} \partial_t^2 u - \Delta u &= F(u, \partial u, \partial^2 u) \\ \text{in } (0, \infty) \times \mathbb{R}^3, \quad i &= 1, 2, \dots, m, \end{aligned}$$

$$(1.2)' \quad \begin{aligned} u(0, x) &= \varepsilon f(x), \quad \partial_t u(0, x) = \varepsilon g(x) \\ \text{for } x &\in \mathbb{R}^3, \quad i = 1, 2, \dots, m, \end{aligned}$$

where  $u = (u_1, u_2, \dots, u_p)$ . When  $u$  is scalar (i.e.,  $p = 1$ ), John studied in [6] the case where  $F = F(u) = |u|^q$  with  $q > 1$ , and proved (among other things) that if  $F(u) = u^2$  then nontrivial solutions  $u(t, x)$  of (1.1)'-(1.2)' blow up in finite time for any  $\varepsilon > 0$ , provided  $f$  and  $g$  satisfy a certain positivity condition. (See also [10]). Moreover he obtained in [7] the analogous results for the cases where  $F = 2u_t u_{tt}$  and  $F = (u_t)^2$ . Notice that these examples do not satisfy the null condition  $(1.9)_1$ . As for the general scalar case see for example Lindblad [19] or Li and Zhou [18].

When  $u$  is a vector-valued unknown (i.e.,  $p \geq 1$ ), we write  $(1.4)_1$  as

$$(1.4)'_1 \quad F(u, u', u'') = \sum_{a,b=0}^3 A^{ab}(u, u') \partial_a \partial_b u + B(u, u').$$

Then Christodoulou [3] and Klainerman [14] proved (independently) the global existence for small  $\varepsilon > 0$  by assuming the null condition  $(1.9)_1$ . ([3] deals with the odd space dimensional case). It is also assumed in these works that the matrices  $A^{ab}(u, u')$  in  $(1.4)'_1$  are diagonal. Katayama obtained in [11] the analogous results for the two space dimensional case.

Next let us consider the case in which  $m \geq 1$ . As for the two space dimensional case, where the  $\mathbb{R}^3$  in (1.1) is replaced by  $\mathbb{R}^2$ , the global existence was studied in [16], [1] and [5]. In these works, the following conditions are assumed to hold: The unknowns  $u^1, u^2, \dots, u^m$  are scalar functions, and the nonlinear terms  $F^i$  depend only on  $\partial u$  and  $\partial^2 u$ , i.e.,  $F^i = F^i(\partial u, \partial^2 u)$ . Moreover  $F^i(0, 0) = 0$ ,  $(F^i)'(0, 0) = 0$ ,  $(F^i)''(0, 0) = 0$  and  $F^i(u', u'')$  are of the forms

$$(1.4)''_1 \quad F^i(u', u'') = \sum_{j=1}^m \sum_{a,b=0}^3 A_{ij}^{ab}(u') \partial_a \partial_b u^j + B^i(u') \\ \text{for } i = 1, 2, \dots, m,$$

where  $A_{ij}^{ab}(u')$  and  $B^i(u')$  are scalar functions of  $u'$  such that  $A_{ji}^{ab}(u') = A_{ij}^{ab}(u') = A_{ij}^{ba}(u')$ . Then one can write

$$(1.6)' \quad F^i(u', u'') = F^{i,3}(u', u'') + O(|u'|^3 (|u'| + |u''|)) \\ \text{for } (u', u'') \text{ near } (u', u'') = (0, 0),$$

where  $F^{i,3}(u', u'')$  is the cubic part of Taylor's expansion for  $F^i(u', u'')$  about  $(u', u'') = (0, 0)$ .

Kovalyov studied in [16] the global existence by assuming that  $A_{ij}^{ab}(u') = 0$  in (1.4)''<sub>1</sub> hence  $F^{i,3}(u', u'') = F^{i,3}(u', 0)$  and that  $F^{i,3}(u', 0)$  is quadratic in  $\partial u^k$  for all  $k = 1, 2, \dots, m$ , i.e.,

$$\partial_{w^k}^3 F^{i,3}(u', 0) = 0 \quad \text{for all } k = 1, 2, \dots, m,$$

where  $w^k$  stands for  $\partial u^k$ . Moreover, Agemi and the second author proved in [1] the global existence for small  $\varepsilon > 0$ , where the hypothesis in [16] are relaxed so that  $F^{i,3}(u', u'')$  is quadratic only in  $(\partial u^i, \partial^2 u^i)$  for each  $i = 1, 2, \dots, m$ . Note that, in the works [16] and [1], the angular-momentum operator  $\Omega = x_1 \partial_2 - x_2 \partial_1$  is used in addition to the usual differential operators  $\partial = (\partial_0, \partial_1, \partial_2)$ . Furthermore, Hoshiga and Kubo relaxed in [5] the hypothesis on  $F^{i,3}(u', u'')$  in [1] so that  $N^i(\partial u^i, \partial^2 u^i)$  satisfies the null condition, analogous to (1.9)<sub>3</sub>, where  $N^i(\partial u^i, \partial^2 u^i)$  is the part of  $F^{i,3}(\partial u, \partial^2 u)$ ,

homogeneous in  $(\partial u^i, \partial^2 u^i)$  of degree 3. Notice that, in the work [5], the scaling operator  $S = t\partial_t + r\partial_r$  and the radiation operator  $S_i = \partial_t + c_i\partial_r$ ,  $i = 1, 2, \dots, m$ , are also made use of. (See (2.19) and (2.21) below).

For the three space dimensional case, the second author obtained recently in [21] the results analogous to [5], where  $F^i = F^i(\partial u, \partial^2 u)$  satisfy (1.4), (1.5) and (1.9)<sub>3</sub>, and  $u^1, u^2, \dots, u^m$  are scalar functions. Note that the hypothesis (A)<sub>3</sub> is trivially satisfied in this case, since  $C_j^i(u', u'') = 0$  and  $D^i(u^m, \partial u, \partial^2 u) = D^i(0, \partial u, \partial^2 u)$  in (1.15).

We now return to the Cauchy problem (1.1)-(1.2). The plan of the proof of Theorem 1.2 is rather similar to Hoshiga and Kubo [5] except the difficulty caused by the fact that  $F^i(u, \partial u, \partial^2 u)$  depend explicitly on  $u$ . Indeed, our main tool is certain  $L^\infty - L^\infty$  estimates for solutions of linear wave equations, which are similar to the ones in [21], Proposition 3.1. (See Theorem 3.4 below).

The plan of this paper is as follows. In the next section we define two kinds of weighted  $L^\infty$  norms and then we show that the null condition (1.9)<sub>3</sub> yields nice decay estimates for the null forms  $N^{i0}(\partial u^i, \partial^2 u^i)$ , such as the radiation conditions in John [8]. (See Proposition 2.5 below). It should be pointed out that we do not need to make use of the fact that  $N^{i0}(\partial u^i, \partial^2 u^i)$  is represented as a linear combination of special bilinear forms (i.e.,  $Q(f, g)$  and  $Q_{ab}(f, g)$  with  $a, b = 0, 1, 2, 3$  in the notations of Klainerman [14]), while this fact plays an essential role in the previous works [14], [11], [5] and [21]. The main part of the present paper is section 3, in which we prove the above-mentioned  $L^\infty - L^\infty$  estimates for solutions of linear wave equations. In section 4 we apply the results of Kato [12] and Majda [20] on local solutions of quasilinear symmetric hyperbolic systems to the Cauchy problem (1.1)-(1.2), and then in section 5 we derive certain  $L^\infty - L^2$  estimates for the local solutions, by making use of the results of section 3. After establishing two energy estimates (i.e., Propositions 6.1 and 6.4) in section 6, we prove Theorem 1.2 in section 7.

## §2. Preliminaries.

This section is composed of two subsections. The former is a collection of the notations and definitions of weighted  $L^\infty$ -spaces which will be used throughout the present paper. In the latter subsection we will show that the hypothesis  $(1.9)_3$  implies such a *radiation condition* as in John [8]. (See Proposition 2.5 below).

**2.1 Notations.** For convenience we shall introduce the zero speed  $c_0 = 0$  and write as

$$(2.1) \quad 0 = c_0 < c_1 < c_2 < \cdots c_m,$$

when  $c_1, c_2, \dots, c_m$  are the constants in (1.1). As in [5] we also define “neighborhoods” of the characteristic cones  $r = |x| = c_i t$  for  $\square_i$ , by

$$(2.2)_i \quad \Lambda_i = \{(t, r) \in [1, \infty) \times [1, \infty); |r - c_i t| \leq c_{m+1} t\}$$

for  $i = 1, 2, \dots, m$ ,

where

$$c_{m+1} = \frac{1}{3} \min_{1 \leq i \leq m} (c_i - c_{i-1}).$$

In addition we set

$$(2.2)_0 \quad \Lambda_0 = ((0, \infty) \times (0, \infty)) \setminus \bigcup_{i=1}^m \Lambda_i.$$

Note that  $\Lambda_0$  contains a “neighborhood” of  $r = c_0 t = 0$ , say,

$$\{(t, r) \in (0, \infty) \times (0, \infty); r \leq \frac{1}{3} c_1 t\}.$$

Moreover the intersection of  $\Lambda_j$  and  $\Lambda_k$  is empty for  $j \neq k$ , and

$$(2.3) \quad \frac{2}{3} c_i t \leq r \leq \frac{4}{3} c_i t \quad \text{for } (t, r) \in \Lambda_i, \quad i = 1, 2, \dots, m.$$

In addition to the usual differential operators  $\partial = (\partial_0, \partial_1, \partial_2, \partial_3)$ , we also adopt the angular – momentum operators defined by

$$(2.4) \quad \Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \partial_x,$$

where  $x \wedge \partial_x f(x)$  stands for the vector (or cross) product, and the scaling operator defined by

$$(2.5) \quad S = t \partial_t + r \partial_r,$$

where  $r = |x|$  and

$$\partial_r = \frac{1}{r} x \cdot \partial_x = \frac{1}{r} \sum_{j=1}^3 x_j \partial_j.$$

Then the following commutation relations are well known. (It is easy to verify those).

$$(2.6) \quad [S, \square_i] = -2\square_i, \quad [\Omega, \square_i] = 0 \quad \text{for } i = 1, 2, \dots, m,$$

where  $[S, \square_i] = S\square_i - \square_i S$  and so on. In addition we have for  $j = 1, 2, 3$  and  $a = 0, 1, 2, 3$

$$(2.7)_1 \quad [S, \Omega] = 0, \quad [S, \partial_a] = -\partial_a,$$

$$(2.7)_2 \quad [\Omega_j, \partial_j] = 0$$

and

$$(2.7)_3 \quad [\Omega_j, \partial_{j+1}] = -\partial_{j+2}, \quad [\Omega_j, \partial_{j+2}] = \partial_{j+1},$$

where we have set  $\partial_4 = \partial_1$  and  $\partial_5 = \partial_2$ .

For convenience we write

$$(2.8)_1 \quad \Gamma = (\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_7) = (\partial, \Omega, S),$$

namely,

$$(2.8)_2 \quad (\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3) = (\partial_0, \partial_1, \partial_2, \partial_3)$$

and

$$(2.8)_3 \quad (\Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7) = (\Omega, S).$$

For a multi-index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , where  $\alpha_j$  are integers such that  $0 \leq \alpha_j \leq 7$ , and a smooth function  $v(t, x)$ , we set  $|\alpha| = k$ ,

$$(2.9)_1 \quad \Gamma^\alpha v(t, x) = \Gamma_{\alpha_1} \Gamma_{\alpha_2} \cdots \Gamma_{\alpha_k} v(t, x)$$

and

$$(2.9)_2 \quad |v(t, x)|_k = \sum_{|\alpha| \leq k} |\Gamma_\alpha v(t, x)|,$$

where  $k$  is a nonnegative integer.

Let  $v(t, x) \in C^k([0, T] \times \mathbb{R}^3)$  be a function and let  $\nu$  be a real number such that  $\nu \geq 0$ , where  $k$  is a nonnegative integer and  $T$  a positive number or  $T = \infty$ . Then we shall define two weighted norms  $\langle v(t, x) \rangle_{\nu, k}^{(i)}$  and  $[v(t, x)]_{\nu, k}^{(i)}$  of  $v(t, x)$  as follows, where  $i = 1, 2, \dots, m$ .

Let  $c_1, c_2, \dots, c_m$  be the constants in (1.1) or (2.1). Let  $(t, x) \in [0, T] \times \mathbb{R}^3$  with  $r = |x|$ . Then we set

$$(2.10)_1 \quad \langle v(t, x) \rangle_{\nu, k}^{(i)} = (1 + r + t)(1 + |r - c_i t|)^\nu |v(t, x)|_k \quad \text{if } \nu > 0$$

and

$$(2.10)_2 \quad \langle v(t, x) \rangle_{0, k}^{(i)} = (1 + r + c_i t) \left(1 + \log \frac{1 + r + c_i t}{1 + |r - c_i t|}\right)^{-1} |v(t, x)|_k,$$

so that  $|v(t, x)|_k \leq \langle v(t, x) \rangle_{0, k}^{(i)}$ . Moreover we set for  $\nu > 0$

$$(2.11) \quad [v(t, x)]_{\nu, k}^{(i)} = (1 + r)(1 + |r - c_i t|)^\nu |v(t, x)|_k.$$

Let  $u^i(t, x)$  be a smooth solution of the nonlinear wave equation (1.1) or of the linear wave equation

$$(2.12) \quad \square_i u^i = F^i(t, x) \quad \text{in } (0, \infty) \times \mathbb{R}^3,$$

where  $i = 1, 2, \dots, m$  and  $F^i$  is a given function. When  $v(t, x) = u^i(t, x)$  or  $v(t, x) = \partial u^i(t, x)$ , we shall abbreviate (2.10) or (2.11), respectively, as

$$(2.13) \quad \langle u^i(t, x) \rangle_{\nu, k} = \langle u^i(t, x) \rangle_{\nu, k}^{(i)} \quad \text{for } \nu \geq 0,$$

or

$$(2.14) \quad [\partial u^i(t, x)]_{\nu, k} = [\partial u^i(t, x)]_{\nu, k}^{(i)} \quad \text{for } \nu > 0.$$

We also define weighted  $L^\infty$ -norms  $\langle u^i \rangle_{\nu, k, t}$  and  $[\partial u^i]_{\nu, k, t}$  for  $0 \leq t < T$  by

$$(2.15) \quad \langle u^i \rangle_{\nu, k, t} = \sup_{\substack{0 \leq s \leq t \\ x \in \mathbb{R}^3}} \langle u^i(s, x) \rangle_{\nu, k}$$

and

$$(2.16) \quad [\partial u^i]_{\nu, k, t} = \sup_{\substack{0 \leq s \leq t \\ x \in \mathbb{R}^3}} [\partial u^i(s, x)]_{\nu, k}.$$

In addition we write

$$(2.17)_1 \quad \langle u(t, x) \rangle_{\nu, k} = \sum_{i=1}^m \langle u^i(t, x) \rangle_{\nu, k},$$

$$(2.17)_2 \quad \langle u \rangle_{\nu, k, t} = \sum_{i=1}^m \langle u^i \rangle_{\nu, k, t},$$

$$(2.17)_3 \quad [\partial u(t, x)]_{\nu, k} = \sum_{i=1}^m [\partial u^i(t, x)]_{\nu, k}$$

and

$$(2.17)_4 \quad [\partial u]_{\nu, k, t} = \sum_{i=1}^m [\partial u^i]_{\nu, k, t}.$$

**2.2 Radiation conditions.** Following John [8] we shall define *radiation operators*  $R = (R_1, R_2, R_3)$  and  $S_i$  ( $i = 1, 2, \dots, m$ ) by

$$(2.18) \quad R = \partial_x - \frac{x}{r} \partial_r \quad \text{for } x \in \mathbb{R}^3 \quad \text{with } r = |x| > 0$$

and

$$(2.19) \quad S_i = \partial_t + c_i \partial_r \quad \text{for } i = 1, 2, \dots, m,$$

where  $c_1, c_2, \dots, c_m$  are the same constants as in (1.1). Then the following identity is well known.

$$(2.20) \quad R = -\frac{1}{r^2}(x \wedge \Omega).$$

Indeed, this follows from (2.4) and the identity

$$(A \cdot B)C - (A \cdot C)B = -A \wedge (B \wedge C) \quad \text{for } A, B, C \in \mathbb{R}^3,$$

if we take  $A = B = x/r$  and  $C = \partial_x f(x)$ . Moreover it follows from (2.5) that

$$(2.21) \quad S_i = -\frac{r - c_i t}{t} \partial_r + \frac{1}{t} S \quad \text{for } t > 0 \quad \text{and } i = 1, 2, \dots, m.$$

The last identity was adopted in Hoshiga and Kubo [5] (see (3.7) in it).

The following estimates, similar to the radiation conditions in John [8], will play a basic role together with the weighted norms (2.13) and (2.14) in dealing with the null forms  $N^{i0}$ .

**Lemma 2.1.** Let  $u^i(t, x) \in C^2([0, T] \times \mathbb{R}^3)$  be a solution of (1.1), where  $i = 1, 2, \dots, m$  and  $T$  is a positive number or  $T = \infty$ . Let  $(t, r) \in \Lambda_i$ , where  $r = |x|$  and  $\Lambda_i$  is given by (2.2)<sub>i</sub>. Then we have

$$(2.22) \quad \begin{aligned} & |R u^i(t, x)| + |S_i u^i(t, x)| \\ & \leq C \frac{1}{(1+r+t)^2(1+|r-c_i t|)^\nu} (\langle u^i(t, x) \rangle_{\nu,1} + [\partial u^i(t, x)]_{1+\nu,0}) \\ & \quad \text{for } \nu > 0 \end{aligned}$$

and

$$(2.23) \quad \begin{aligned} & |R \partial u^i(t, x)| + |S_i \partial u^i(t, x)| \\ & \leq C \frac{1}{(1+r+t)^2(1+|r-c_i t|)^\nu} [\partial u^i(t, x)]_{1+\nu,1} \quad \text{for } \nu \geq 0, \end{aligned}$$

where we have used the notations (2.13) and (2.14). Moreover for  $\nu = 0$  we have

$$(2.24) \quad \begin{aligned} & |R u^i(t, x)| + |S_i u^i(t, x)| \\ & \leq C \frac{1}{(1+r+t)^2} \left(1 + \log \frac{1+r+c_i t}{1+|r-c_i t|}\right) (\langle u^i(t, x) \rangle_{0,1} + [\partial u^i(t, x)]_{1,0}). \end{aligned}$$

Here  $C$  is a constant depending only on  $c_i$ .

*Proof.* First we shall prove (2.22). Since (2.14) with  $k = 0$  implies

$$(2.25) \quad \begin{aligned} & |(r - c_i t) \partial_r u^i(t, x)| \\ & \leq [\partial u^i(t, x)]_{1+\nu, 0} (1+r)^{-1} (1 + |r - c_i t|)^{-\nu} \quad \text{for } \nu \geq 0, \end{aligned}$$

it follows from (2.21) and (2.13) with  $\nu > 0$  that

$$\begin{aligned} |S_i u^i(t, x)| & \leq \frac{1}{i} \frac{1}{(1 + |r - c_i t|)^\nu} \\ & \times \left( \frac{1}{1+r} [\partial u^i(t, x)]_{1+\nu, 0} + \frac{1}{1+r+t} \langle u^i(t, x) \rangle_{\nu, 1} \right). \end{aligned}$$

Since  $t \geq 1$  for  $(t, r) \in \Lambda_i$  with  $1 \leq i \leq m$ , we get by (2.3)

$$\begin{aligned} |S_i u^i(t, x)| & \leq C(1+r+t)^{-2} (1 + |r - c_i t|)^{-\nu} \\ & \times ([\partial u^i(t, x)]_{1+\nu, 0} + \langle u^i(t, x) \rangle_{\nu, 1}). \end{aligned}$$

Similarly (less hard) we obtain (2.22), making use of (2.20). One can also prove (2.24) analogously, replacing (2.10)<sub>1</sub> by (2.10)<sub>2</sub>.

Next we shall prove (2.23). It follows from (2.21) that

$$|S_i \partial u^i(t, x)| \leq t^{-1} (|(r - c_i t) \partial_r \partial u^i(t, x)| + |S \partial u^i(t, x)|).$$

Therefore we have from (2.25) and (2.14)

$$\begin{aligned} |S_i \partial u^i(t, x)| & \leq \frac{1}{t} \cdot \frac{1}{(1+r)(1 + |r - c_i t|)^\nu} \\ & \times ([\partial^2 u^i(t, x)]_{1+\nu, 0} + \frac{1}{1 + |r - c_i t|} [\partial u^i(t, x)]_{1+\nu, 1}). \end{aligned}$$

Since  $\partial^2 u^i = (\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3) \partial u^i$  according to (2.8)<sub>2</sub>, we thus obtain (2.23), making use of (2.3) and (2.20).  $\square$

**Remark.** As seen from the proof, the estimates (2.22), (2.23) and (2.24) are valid for arbitrary functions  $u^i(t, x) \in C^2([0, T) \times \mathbb{R}^3)$ .

Next we shall derive estimates analogous to Lemma 2.1 for higher derivatives. Namely we deal with  $|Ru^i(t, x)|_k$  and  $|S_i u^i(t, x)|_k$  for  $k \geq 1$ , where we have used the notations  $(2.9)_1$  and  $(2.9)_2$ . To this end we shall examine the commutators of  $\Gamma$  and the radiation operators  $R$  and  $S_i$ .

First we shall state easy commutation relations.

**Lemma 2.2.** Let  $t > 0, x = (x_1, x_2, x_3) \in \mathbb{R}^3$  and  $r = |x| > 0$ . Let  $e^1, e^2, e^3$  be the canonical unit vectors in  $\mathbb{R}^3$ . Then we have the following commutation relations.

$$(2.26)_1 \quad [S, x] = x,$$

$$(2.26)_2 \quad [\Omega, x_j] = x \wedge e^j$$

and

$$(2.26)_3 \quad [\Omega_j, x] = e^j \wedge x$$

for  $j = 1, 2, 3$ , where the relation  $[S, x] = x$  means that

$$S(x f(t, x)) - x S f(t, x) = x f(t, x)$$

for a smooth function  $f(t, x)$ , and so on.

$$(2.27)_1 \quad [\partial_x, r^{-k}] = -k \frac{x}{r^2} r^{-k}$$

$$(2.27)_2 \quad [S, r^{-k}] = -k r^{-k}$$

and

$$(2.27)_3 \quad [\Omega, r^{-k}] = 0,$$

where  $k$  is a positive integer,

$$(2.28)_1 \quad [\partial_r, \partial_x] = \frac{x}{r^2} \partial_r - \frac{1}{r} \partial_x,$$

$$(2.28)_2 \quad [\partial_r, S] = \partial_r,$$

$$(2.28)_3 \quad [\partial_r, \Omega] = 0$$

and

$$(2.28)_4 \quad [\partial_r, \frac{x}{r}] = 0.$$

Moreover for  $i = 1, 2, \dots, m$  we have

$$(2.29)_1 \quad [S, S_i] = -S_i,$$

$$(2.29)_2 \quad [\Omega, S_i] = 0,$$

$$(2.29)_3 \quad [\partial_t, S_i] = 0,$$

and

$$(2.29)_4 \quad [\partial_x, S_i] = c_i [\partial_x, \partial_r].$$

*Proof.* Since

$$(2.30) \quad r\partial_r = \sum_{k=1}^3 x_k \partial_k,$$

we obtain  $(2.26)_1$  from (2.5). It also follows from (2.4) that

$$[\Omega, x_j] = x \wedge (\partial_x x_j),$$

which yields  $(2.26)_2$ .

In what follows we set for convenience

$$(2.31)_1 \quad x_7 = x_4 = x_1, \quad x_5 = x_2, \quad x_6 = x_3,$$

$$(2.31)_2 \quad \partial_7 = \partial_4 = \partial_1, \quad \partial_5 = \partial_2, \quad \partial_6 = \partial_3,$$

$$(2.31)_3 \quad e^7 = e^4 = e^1, \quad e^5 = e^2, \quad e^6 = e^3$$

and

$$(2.31)_4 \quad \Omega_4 = \Omega_1, \quad \Omega_5 = \Omega_2.$$

Then

$$(2.32) \quad e^j \wedge x = x_{j+1} e^{j+2} - x_{j+2} e^{j+1}$$

and

$$(2.33) \quad \Omega_j = x_{j+1} \partial_{j+2} - x_{j+2} \partial_{j+1}$$

hold for  $j = 1, 2, 3$ , because  $e^j \wedge e^{j+1} = e^{j+2}$  with  $(2.31)_3$ . Hence

$$[\Omega_j, x] = x_{j+1} e^{j+2} - x_{j+2} e^{j+1}.$$

Therefore we obtain  $(2.26)_3$  by  $(2.32)$ . One can also prove without difficulty the other relations, using  $(2.30)$  and  $(2.33)$ .  $\square$

Next we shall examine the commutations of  $\Gamma$  and  $R$ .

**Lemma 2.3.** Let  $(t, x)$  and  $e^1, e^2, e^3$  be as in the preceding lemma. Let  $j = 1, 2, 3$ . Then we have

$$(2.34) \quad [S, R] = -R,$$

$$(2.35) \quad [\partial_j, R] = -\frac{2}{r^2} x_j R - \frac{1}{r^2} \{e^j \wedge \Omega + x \wedge (e^j \wedge \partial_x)\},$$

$$(2.36) \quad [\Omega_j, \Omega_{j+1}] = -\Omega_{j+2}$$

with the notation  $(2.31)_4$ , and

$$(2.37) \quad [\Omega_j, R] = -\frac{1}{r^2} \{(e_j \wedge x) \wedge \Omega + x \wedge (e_j \wedge \Omega)\}.$$

*Proof.* First we shall prove  $(2.34)$ . Since  $(2.30)$  yields

$$(2.38) \quad [r \partial_r, x r^{-1}] = 0,$$

it follows from  $(2.5)$  and  $(2.18)$  that

$$[S, R] = [S, \partial_x] - \frac{x}{r} [S, \partial_r].$$

Therefore we obtain (2.34) by (2.7)<sub>1</sub> and (2.28)<sub>2</sub>.

Next we shall prove (2.35). It follows from (2.20) and (2.27)<sub>1</sub> that

$$(2.39) \quad [\partial_j, R] = -\frac{2}{r^2} x_j R - \frac{1}{r^2} [\partial_j, x \wedge \Omega].$$

Moreover, since (2.7)<sub>2</sub> and (2.7)<sub>3</sub> yield

$$[\partial_j, \Omega] = e^j \wedge \partial_x,$$

we have

$$[\partial_j, x \wedge \Omega] = e^j \wedge \Omega + x \wedge (e^j \wedge \partial_x).$$

Therefore we obtain (2.35) from (2.39).

Finally we shall prove (2.36) and (2.37). It follows from (2.33) that

$$[\Omega_j, \Omega_{j+1}] = x_{j+1} \partial_{j+3} - x_{j+3} \partial_{j+1}.$$

Since  $x_{j+1} = x_{j+4}$  and  $\partial_{j+1} = \partial_{j+4}$  by (2.31)<sub>1</sub> and (2.31)<sub>2</sub>, we get (2.36). Moreover it follows from (2.26)<sub>3</sub> that

$$[\Omega_j, x \wedge \Omega] = (e^j \wedge x) \wedge \Omega + x \wedge [\Omega_j, \Omega].$$

By virtue of (2.36) we also get

$$[\Omega_j, \Omega] = e^j \wedge \Omega.$$

Therefore we obtain

$$(2.40) \quad [\Omega_j, x \wedge \Omega] = (e_j \wedge x) \wedge \Omega + x \wedge (e_j \wedge \Omega).$$

Now the desired relation (2.37) follows immediately from (2.20), (2.27)<sub>3</sub> and (2.40). The proof is complete.  $\square$

We are now in a position to deal with  $|Ru^i(t, x)|_k$  and  $|S_i u^i(t, x)|_k$  for  $k \geq 1$ .

**Corollary 2.4.** Let  $i, T$  and  $(t, r)$  be as in Lemma 2.1. Let  $u^i(t, x) \in C^{2+k}([0, T) \times \mathbb{R}^3)$  be a solution of (1.1), where  $k$  is a nonnegative integer. Then for  $\nu \geq 0$  we have

$$(2.41) \quad |R u^i(t, x)|_k + |S_i u^i(t, x)|_k \leq C \Psi_{\nu, k}^i(u^i)(t, x)$$

and

$$(2.42) \quad |R \partial u^i(t, x)|_k + |S_i \partial u^i(t, x)|_k \leq C \frac{1}{(1+r+t)^2(1+|r-c_i t|)^\nu} [\partial u^i(t, x)]_{1+\nu, 1+k},$$

where  $C$  is a constant depending only on  $c_i$  and  $k$ , and  $\Psi_{\nu, k}^i(v)(t, x)$  defined by

$$(2.43)_1 \quad \begin{aligned} & \Psi_{\nu, k}^i(v)(t, x) \\ &= \frac{1}{(1+r+t)^2(1+|r-c_i t|)^\nu} (\langle v(t, x) \rangle_{\nu, 1+k} + [\partial v(t, x)]_{1+\nu, k}) \quad \text{for } \nu > 0 \end{aligned}$$

and

$$(2.43)_2 \quad \begin{aligned} & \Psi_{0, k}^i(v)(t, x) \\ &= \frac{1}{(1+r+t)^2} \left( 1 + \log \frac{1+r+c_i t}{1+|r-c_i t|} \right) \\ & \quad \times (\langle v(t, x) \rangle_{0, 1+k} + [\partial v(t, x)]_{1, k}). \end{aligned}$$

*Proof.* We shall prove only (2.41) with  $\nu > 0$ , since the others can be analogously handled. If  $k = 0$ , then (2.41) coincides with (2.22). Let  $k = 1$ . Then we have only to prove

$$(2.44) \quad |\Gamma R u^i(t, x)| + |\Gamma S^i u^i(t, x)| \leq C \Psi_{\nu, 1}^i(u^i)(t, x).$$

It follows from (2.29)<sub>1</sub> and (2.34) that

$$SS^i = S^i S - S^i, \quad SR = RS - R.$$

Besides, (2.29)<sub>2</sub> and (2.37) yield

$$\begin{aligned} & |\Omega S^i u^i(t, x)| + |\Omega R u^i(t, x)| \\ & \leq |S^i \Omega u^i(t, x)| + |R \Omega u^i(t, x)| + C r^{-1} |\Omega u^i(t, x)|. \end{aligned}$$

Hence we see from Lemmas 2.1, 2.2, (2.35) and (2.3) that (2.44) holds. Similarly we obtain (2.41) for  $k \geq 2$ .  $\square$

Finally we shall show that the null form  $N^{i0}(\partial u^i, \partial^2 u^i)$  satisfies a certain radiation condition.

**Proposition 2.5.** Let  $u^i(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  be a solution of the nonlinear wave equation (1.1), where  $i = 1, 2, \dots, m$  and  $T$  is as in Lemma 2.1. Assume that condition (1.9)<sub>3</sub> holds. Let  $k$  be a nonnegative integer and  $\nu \geq 0$ . Then we have

$$\begin{aligned} (2.45) \quad & |N^{i0}(\partial u^i, \partial^2 u^i)(t, x)|_k \\ & \leq C \frac{1}{(1+r+t)(1+|r-c_i t|)^{1+\nu}} \\ & \quad \times ([\partial u^i(t, x)]_{1+\nu, 1+k} \Psi_{\nu, [k/2]}^i(u^i)(t, x) \\ & \quad + \Psi_{\nu, k}^i(u^i)(t, x) [\partial u^i(t, x)]_{1+\nu, 1+[k/2]}) \\ & \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^3 \text{ with } (t, r) \in \Lambda_i, \end{aligned}$$

where  $r = |x|$ ,  $C$  is a constant depending only on  $c_i, k$  and the coefficients of the polynomial  $N^{i0}$ , and  $\Psi_{\nu, k}^i$  given by (2.43)<sub>1</sub> or (2.43)<sub>2</sub>. Besides,  $[k/2]$  stands for the largest integer which does not exceed  $k/2$  and  $\Lambda_i$  is defined by (2.2)<sub>i</sub>.

*Proof.* Recall that  $N^{i0}(\partial u^i, \partial^2 u^i)$  is a vector of  $p_i$  polynomials in  $(\partial u^i, \partial^2 u^i)$ , say,  $N_1^{i0}(\partial u^i, \partial^2 u^i), N_2^{i0}(\partial u^i, \partial^2 u^i), \dots, N_{p_i}^{i0}(\partial u^i, \partial^2 u^i)$ . Moreover, for every  $k = 1, 2, \dots, p_i$ , the polynomial  $N_k^{i0}(\partial u^i, \partial^2 u^i)$  is homogeneous in  $(\partial u^i, \partial^2 u^i)$  of degree 2 which is linear in  $\partial^2 u^i$ . Clearly it suffices to prove (2.45) for each  $k = 1, 2, \dots, p_i$ .

In what follows we shall fix  $i = 1, 2, \dots, m$  and  $k = 1, 2, \dots, p_i$ . Besides, for the sake of simplicity of notation, we shall abbreviate as  $p_i = p, u^i = u = (u_1, u_2, \dots, u_p)$

and  $N_k^{i0}(\partial u^i, \partial^2 u^i) = N(\partial u, \partial^2 u)$ . Then one can write

$$(2.46) \quad \begin{aligned} & N(\partial u, \partial^2 u) \\ &= \sum_{j, \ell=1}^p \sum_{a,b,c=0}^3 A_{j\ell}^{abc} (\partial_a u_j) (\partial_b \partial_c u_\ell) + \sum_{j, \ell=1}^p \sum_{a,b=0}^3 B_{j\ell}^{ab} (\partial_a u_j) (\partial_b u_\ell), \end{aligned}$$

where  $\partial_0 = \partial_t$ ,  $A_{j\ell}^{abc}$  and  $B_{j\ell}^{ab}$  are constants, and  $u_j = u_j(t, x)$  scalar functions. With these notations we see that condition  $(1.9)_3$  implies

$$(2.47) \quad \begin{aligned} & \tilde{N}(v, w; X^i) \\ &= \sum_{j, \ell=1}^p \sum_{a,b,c=0}^3 A_{j\ell}^{abc} (X_a v_j) (X_b X_c w_\ell) + \sum_{j, \ell=1}^p \sum_{a,b=0}^3 B_{j\ell}^{ab} (X_a v_j) (X_b v_\ell) = 0 \\ & \text{for any vector } v = (v_1, v_2, \dots, v_p) \in \mathbb{R}^p, \\ & w = (w_1, w_2, \dots, w_p) \in \mathbb{R}^p \text{ and } X^i = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4 \\ & \text{satisfying } X_0 = -c_i \sqrt{X_1^2 + X_2^2 + X_3^2}. \end{aligned}$$

We are now in a position to prove (2.45). Set

$$X_0 = -c_i, (X_1, X_2, X_3) = \frac{1}{r} x = \frac{1}{r} (x_1, x_2, x_3)$$

and

$$v = \partial_r u(t, x), w = \partial_r^2 u(t, x).$$

Then it follows from (2.46) and (2.47) that

$$(2.48) \quad \begin{aligned} & N(\partial u, \partial^2 u)(t, x) \\ &= N(\partial u, \partial^2 u)(t, x) - \tilde{N}(v, w; X^i) \\ &= \sum_{j, \ell=1}^p \sum_{a,b,c=0}^3 A_{j\ell}^{abc} D_{j\ell}^{abc}(t, x) + \sum_{j, \ell=1}^p \sum_{a,b=0}^3 B_{j\ell}^{ab} E_{j\ell}^{ab}(t, x). \end{aligned}$$

Here we have set

$$(2.49) \quad D_{j\ell}^{abc} = (\partial_a u_j) (\partial_b \partial_c u_\ell) - (X_a \partial_r u_j) (X_b X_c \partial_r^2 u_\ell)$$

and

$$(2.50) \quad E_{j\ell}^{ab} = (\partial_a u_j)(\partial_b u_\ell) - (X_a \partial_r u_j)(X_b \partial_r u_\ell),$$

where

$$X_0 = -c_i \quad \text{and} \quad (X_1, X_2, X_3) = x/r.$$

Clearly it is sufficient to prove (2.45) with  $N^{i0}$  replaced by  $D_{j\ell}^{abc}$  or  $E_{j\ell}^{ab}$ . Hence we have only to prove the following

**Lemma 2.6.** Let  $D_{j\ell}^{abc}$  and  $E_{j\ell}^{ab}$  be defined by (2.49) and (2.50) respectively, where  $j, \ell = 1, 2, \dots, p$  and  $a, b, c = 0, 1, 2, 3$ . Let  $(t, r) \in \Lambda_i$ , where  $r = |x|$ ,  $1 \leq i \leq m$  and  $\Lambda_i$  is given by (2.2)<sub>i</sub>. Let  $k$  be a nonnegative integer and  $\nu \geq 0$ . Then we have

$$(2.51) \quad \begin{aligned} |D_{j\ell}^{abc}(t, x)|_k &\leq C(1+r+t)^{-1}(1+|r-c_it|)^{-1-\nu} \\ &\times ([\partial u(t, x)]_{1+\nu, 1+k} \Psi_{\nu, [k/2]}^i(u)(t, x) \\ &+ \Psi_{\nu, k}(u)(t, x)[\partial u(t, x)]_{1+\nu, 1+[k/2]}) \end{aligned}$$

and

$$(2.52) \quad \begin{aligned} |E_{j\ell}^{ab}(t, x)|_k &\leq C(1+r+t)^{-1}(1+|r-c_it|)^{-1-\nu} \\ &\times ([\partial u(t, x)]_{1+\nu, k} \Psi_{\nu, [k/2]}^i(u)(t, x) \\ &+ \Psi_{\nu, k}^i(u)(t, x)[\partial u(t, x)]_{1+\nu, [k/2]}), \end{aligned}$$

where  $u = (u_1, u_2, \dots, u_p)$ .

*Proof.* We shall prove only (2.51), since (2.52) can be handled similarly (but less hard). It follows from (2.49) that

$$(2.53) \quad \begin{aligned} D_{j\ell}^{abc} &= ((\partial_a - X_a \partial_r)u_j) \partial_b \partial_c u_\ell \\ &\quad + X_a (\partial_r u_j) (\partial_b - X_b \partial_r) \partial_c u_\ell \\ &\quad + X_a X_b (\partial_r u_j) \partial_r (\partial_c - X_c \partial_r) u_\ell. \end{aligned}$$

Besides, we have

$$(2.54) \quad \partial_r(\partial_c - X_c \partial_r)u_\ell = (\partial_c - X_c \partial_r)\partial_r u_\ell + [\partial_r, \partial_c]u_\ell,$$

making use of (2.28)<sub>4</sub>. Notice that

$$(2.55)_1 \quad \partial_a - X_a \partial_r = R_a \quad \text{for } a = 1, 2, 3$$

and

$$(2.55)_2 \quad \partial_0 - X_0 \partial_r = S_i$$

according to (2.18) and (2.19).

First suppose that  $k = 0$ . Then we obtain (2.51) from Corollary 2.4, making use of (2.28)<sub>1</sub>, because

$$|\partial_b \partial_c u_\ell(t, x)| \leq (1+r)^{-1}(1+|r-c_it|)^{-1-\nu}[\partial u_\ell(t, x)]_{1+\nu, 1}$$

and (2.3) yields  $r^{-1} \leq C(1+r+t)^{-1}$ .

Next suppose that  $k \geq 1$ . We shall deal with only the first term on the right hand side of (2.53), since the others can be analogously handled. Then by Leibnitz's formula we get

$$\begin{aligned} & |((\partial_a - X_a \partial_r)u_j(t, x))\partial_b \partial_c u_\ell(t, x)|_k \\ & \leq C|(\partial_a - X_a \partial_r)u_j(t, x)|_k |\partial_b \partial_c u_\ell(t, x)|_{[k/2]} \\ & \quad + C|(\partial_a - X_a \partial_r)u_j(t, x)|_{[k/2]} |\partial_b \partial_c u_\ell(t, x)|_k. \end{aligned}$$

Since

$$|\partial_b \partial_c u_\ell(t, x)|_k \leq (1+r)^{-1}(1+|r-c_it|)^{-1-\nu}[\partial u_\ell(t, x)]_{1+\nu, 1+k},$$

we thus obtain (2.51), making use of Corollary 2.4. The proof is complete.

**Remark.** As seen the proof, the estimate (2.45) is valid for any function  $u^i(t, x) \in C^{2+k}([0, T] \times \mathbb{R}^3)$ .

### §3. Weighted $L^\infty - L^\infty$ estimates for linear wave equations.

The purpose of this section is to derive weighted  $L^\infty - L^\infty$  estimates for the inhomogeneous wave equations related to the nonlinear wave equations (1.1). Theorem 3.4 below will play a basic role in the present paper.

We shall first consider the reduced initial value problem for the homogeneous wave equation

$$(3.1) \quad (\partial_t^2 - a^2 \Delta)v = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^3,$$

$$(3.2) \quad v(0, x) = 0, \quad \partial_t v(0, x) = g(x),$$

where  $a$  is a positive constant and  $g \in C^2(\mathbb{R}^3)$ . It is well known that the (unique) solution  $v(t, x) \in C^2([0, \infty) \times \mathbb{R}^3)$  of (3.1)-(3.2) is given by

$$(3.3) \quad v(t, x) = \frac{t}{4\pi} \int_{|\xi|=1} g(x + at\xi) dS_\xi \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^3.$$

Moreover we shall make use of also the following representation of  $v(t, x)$  which is in essence due to John [8], Appendix.

**Lemma 3.1.** Let  $g(x)$  be a continuous function of  $x \in \mathbb{R}^3$ . Let  $v(t, x)$  be given by (3.3). For  $(t, x) \in [0, \infty) \times \mathbb{R}^3$  with  $r = |x| > 0$  we then have

$$(3.4) \quad v(t, x) = \frac{1}{4\pi ar} \int_{|r-at|}^{r+at} \lambda d\lambda \int_0^{2\pi} g(\lambda \Theta_0) d\varphi.$$

Here  $\Theta_0 = \Theta_0(\lambda, \varphi; t, x)$  is a unit vector in  $\mathbb{R}^3$  defined by

$$(3.5) \quad \Theta_0 = (\sin \psi_0 \cos \varphi, \sin \psi_0 \sin \varphi, \cos \psi_0) A(x),$$

where  $A(x)$  is a  $3 \times 3$  orthogonal matrix, whose elements are piecewise continuous functions of  $x/|x|$ , such that  $x = (0, 0, r)A(x)$ , and  $\psi_0 = \psi_0(\lambda, t, r)$  is a function of  $(\lambda, t, r)$  such that

$$(3.6) \quad \cos \psi_0 = \frac{\lambda^2 + r^2 - (at)^2}{2r\lambda}, \quad 0 \leq \psi_0 \leq \pi.$$

In particular, if  $g(x)$  is radially symmetric, i.e., if  $g(x) = g(|x|)$ , we have

$$(3.7) \quad v(t, x) = \frac{1}{2ar} \int_{|r-at|}^{r+at} \lambda g(\lambda) d\lambda.$$

*Proof.* For convenience we set  $g_A(y) = g(yA(x))$  for  $y \in \mathbb{R}^3$ . Then it follows from (3.3) that

$$v(t, x) = \frac{i}{4\pi} \int_{|\xi|=1} g_A(at z_1, at z_2, r + at z_3) dS_\xi,$$

where we have put  $(z_1, z_2, z_3) = \xi {}^t A(x)$ . Switching to polar coordinates, we get

$$(3.8) \quad v(t, x) = \frac{t}{4\pi} \iiint_{\substack{0 \leq \theta \leq \pi \\ 0 \leq \varphi \leq 2\pi}} g_A(\eta_1, \eta_2, \eta_3) \sin \theta d\theta d\varphi,$$

where

$$\eta_1 = at \sin \theta \cos \varphi, \eta_2 = at \sin \theta \sin \varphi \quad \text{and} \quad \eta_3 = r + at \cos \theta.$$

Moreover, introducing a new variable  $\lambda$  instead of  $\theta$ , by

$$(3.9) \quad \lambda = \{r^2 + (at)^2 + 2art \cos \theta\}^{\frac{1}{2}},$$

we obtain (3.4). Indeed, since

$$(3.10) \quad \begin{aligned} (2r\lambda)^2 - \{\lambda^2 + r^2 - (at)^2\}^2 &= \{(\lambda + r)^2 - (at)^2\} \cdot \{(at)^2 - (\lambda - r)^2\} \\ &= \{\lambda^2 - (r - at)^2\} \cdot \{(r + at)^2 - \lambda^2\}, \end{aligned}$$

one can define a function  $\psi_0$  of  $(\lambda, t, r)$  with  $|r - at| \leq \lambda \leq r + at$  so that (3.6) holds. In addition, (3.6) and (3.9) imply that  $\lambda \cos \psi_0 = r + at \cos \theta$ . Moreover, since (3.9) yields

$$\lambda^2 - (r - at)^2 = 2art(1 + \cos \theta)$$

and

$$(r + at)^2 - \lambda^2 = 2art(1 - \cos \theta),$$

it follows from (3.6) and (3.10) that

$$\begin{aligned} (\lambda \sin \psi_0)^2 &= \lambda^2 - (\lambda \cos \psi_0)^2 \\ &= (at \sin \theta)^2. \end{aligned}$$

Thus we obtain (3.4) from (3.8). The proof is complete.  $\square$

In order to derive decay estimates for solutions of the homogeneous or inhomogeneous wave equations, we will make use of the following elementary inequality.

**Lemma 3.2.** Let  $\kappa$  be a real number such that  $1 + \kappa > 0$ . Let  $t > 0$  and  $r > 0$ . Then we have

$$(3.11) \quad \frac{1}{2r} \int_{|r-t|}^{r+t} \frac{1}{(1+\lambda)^{1+\kappa}} d\lambda \leq C \frac{1}{1+r+t} \Phi_\kappa(t, r),$$

where  $C = 3$  if  $\kappa = 0$ ,  $C = \max\{3, \frac{2}{|\kappa|}\}$  if  $\kappa \neq 0$ , and  $\Phi_\kappa(t, r)$  is defined by

$$(3.12) \quad \Phi_\kappa(t, r) = \begin{cases} (1 + |r - t|)^{-\kappa} & \text{if } \kappa > 0, \\ 1 + \log \frac{1+r+t}{1+|r-t|} & \text{if } \kappa = 0, \\ (1 + r + t)^{-\kappa} & \text{if } \kappa < 0. \end{cases}$$

*Proof.* By  $I_\kappa(t, r)$  we shall denote the left hand side of (3.11). First suppose that

$$(3.13) \quad \text{either } t \geq 2r > 0 \text{ or } 0 < r \leq 1.$$

Then we have

$$(3.14) \quad \frac{1}{1 + |r - t|} \leq \frac{3}{1 + r + t}.$$

Hence we get (3.11) with  $C = 3$ , since

$$I_\kappa(t, r) \leq \frac{1}{1 + |r - t|} \cdot \frac{1}{(1 + |r - r|)^\kappa} \frac{1}{2r} \int_{t-r}^{t+r} d\lambda.$$

Next suppose that

$$(3.15) \quad r \geq 1 \quad \text{and} \quad 0 \leq t \leq 2r.$$

Then we have

$$(3.16) \quad \frac{1}{2r} < \frac{2}{1+r+t}.$$

Hence

$$I_{\kappa}(t, r) \leq \frac{2}{1+r+t} \int_{|r-t|}^{r+t} \frac{1}{(1+\lambda)^{1+\kappa}} d\lambda,$$

which yields (3.11).  $\square$

The following decay estimates for solutions of (3.1)-(3.2) are due to Asakura [2].

**Proposition 3.3.** Let  $v(t, x) \in C^{\infty}([0, \infty) \times \mathbb{R}^3)$  be the solution of the reduced Cauchy problem (3.1)-(3.2). Let  $k$  be a nonnegative integer, and let  $t > 0$  and  $x \in \mathbb{R}^3$  with  $r = |x| > 0$ . Assume that

$$(3.17) \quad |\partial_y^{\alpha} g(y)| \leq M_k \frac{1}{(1+|y|)^{2+\rho}} \quad \text{for } y \in \mathbb{R}^3, \quad \text{and} \quad |\alpha| = k$$

where  $\rho$  and  $M_k$  are positive constants. Then we have for  $|\alpha| = k \geq 0$

$$(3.18) \quad |\partial_x^{\alpha} v(t, x)| \leq CM_k \frac{1}{(1+r+t)(1+|r-at|)^{\rho}}$$

and

$$(3.19) \quad |\partial_x^{\alpha} \partial_t v(t, x)| \leq C(M_k + M_{k+1}) \frac{1}{(1+r+t)(1+|r-at|)^{\rho}},$$

where  $C$  is a constant depending only on  $\rho$  and  $a$ .

*Proof.* For the sake of completeness we shall give a simple proof. Without loss of generality, one can assume that  $a = 1$  in (3.1), changing the variable  $t$  by  $\tau = at$ .

First we shall prove (3.18). It follows from (3.3) and (3.17) that

$$|\partial_x^\alpha v(t, x)| \leq M_k \frac{t}{4\pi} \int_{|\xi|=1} (1 + |x + t\xi|)^{-2-\rho} dS_\xi.$$

Hence by (3.7) we get

$$|\partial_x^\alpha v(t, x)| \leq M_k \frac{1}{2r} \int_{|r-t|}^{r+t} (1 + \lambda)^{-1-\rho} d\lambda$$

Therefore we obtain (3.18), making use of Lemma 3.2 with  $\kappa = \rho > 0$ .

Next we shall prove (3.19). It follows from (3.3) and (3.17) that

$$|\partial_x^\alpha \partial_t v(t, x)| \leq (M_k \frac{1}{t} + 3M_{k+1}) \frac{t}{4\pi} \int_{|\xi|=1} \frac{1}{(1 + |x + t\xi|)^{2+\rho}} dS_\xi.$$

Hence we get as above

$$|\partial_x^\alpha \partial_t v(t, x)| \leq M_k I_1(t, r) + 3M_{k+1} I_2(t, r),$$

where

$$I_1(t, r) = \frac{1}{2rt} \int_{|r-t|}^{r+t} \frac{\lambda}{(1 + \lambda)^{2+\rho}} d\lambda$$

and

$$I_2(t, r) = \frac{1}{2r} \int_{|r-t|}^{r+t} \frac{\lambda}{(1 + \lambda)^{2+\rho}} d\lambda.$$

One can deal with the second integral  $I_2(t, r)$  analogously to (3.18). As for the first integral  $I_1(t, r)$ , we have

$$I_1(t, r) \leq \frac{1}{2r} \int_{|r-t|}^{r+t} \frac{1}{(1 + \lambda)^{2+\rho}} d\lambda + \frac{1}{2t} \int_{|r-t|}^{r+t} \frac{1}{(1 + \lambda)^{2+\rho}} d\lambda,$$

since  $\lambda \leq t + r$ . Making use of Lemma 3.2, we get therefore

$$I_1(t, r) \leq \frac{2C}{1 + r + t} \Phi_{1+\rho}(t, r),$$

because  $\Phi_\kappa(t, r)$  is symmetric with respect to  $t$  and  $r$ . Thus we obtain

$$I_1(t, r) \leq 2C \frac{1}{(1+r+t)(1+|r-t|)^{1+\rho}},$$

which yields (3.19).  $\square$

**Remark.** It is well known that the (unique) solution  $w(t, x) \in C^2([0, \infty) \times \mathbb{R}^3)$  of (3.1) with initial data

$$(3.2)' \quad w(0, x) = f(x), \quad \partial_t w(0, x) = 0$$

is given by

$$(3.3)' \quad w(t, x) = \frac{\partial}{\partial t} \left\{ \frac{t}{4\pi} \int_{|\xi|=1} f(x + t\xi) dS_\xi \right\},$$

where  $f \in C^3(\mathbb{R}^3)$ . Therefore Proposition 3.3 implies the analogous decay estimates for  $w(t, x)$ .

Next we shall consider the Cauchy problem for the inhomogeneous wave equations

$$(3.20) \quad \square_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(t, x) \quad \text{in } (0, \infty) \times \mathbb{R}^3, \quad i = 1, 2, \dots, m$$

with zero initial data

$$(3.21) \quad u^i(0, x) = 0, \partial_t u^i(0, x) = 0, \quad i = 1, 2, \dots, m,$$

where  $c_1, c_2, \dots, c_m$ , are the same constants as in (1.1).

It is well known that the (unique) solution  $u^i(t, x) \in C^2([0, \infty) \times \mathbb{R}^3)$  of the Cauchy problem (3.20)-(3.21) is given by

$$(3.22) \quad u^i(t, x) = L_i(F^i)(t, x),$$

provided  $\partial_x^\alpha F^i(t, x)$  is continuous in  $(t, x) \in [0, \infty) \times \mathbb{R}^3$  for  $|\alpha| \leq 2$ , where  $L^i$  is a linear operator defined by

$$L_i(G)(t, x) = \frac{1}{4\pi} \int_0^t (t-s) ds \int_{|\xi|=1} G(s, x + c_i(t-s)\xi) dS_\xi$$

for a continuous function  $G(t, x)$  of  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ . Moreover by virtue of Lemma 3.1 we have

$$(3.23) \quad L_i(G)(t, x) = \frac{1}{4\pi c_i r} \int_0^t ds \int_{|r-c_i(t-s)|}^{r+c_i(t-s)} \lambda d\lambda \int_0^{2\pi} G(s, \lambda \Theta_i) d\varphi,$$

where  $r = |x| > 0$  and  $\Theta_i = \Theta_i(s, \lambda, \varphi; t, x)$  is a unit vector in  $\mathbb{R}^3$  defined by

$$(3.24) \quad \Theta_i = (\sin \psi_i \cos \varphi, \sin \psi_i \sin \varphi, \cos \psi_i) A(x).$$

Here  $A(x)$  is the same matrix as in (3.5), and  $\psi_i = \psi_i(s, \lambda, t, r)$  is a function of  $(s, \lambda, t, r)$  with

$$|r - c_i(t - s)| \leq \lambda \leq r + c_i(t - s)$$

such that  $0 \leq \psi_i \leq \pi$  and

$$(3.25) \quad \cos \psi_i = \frac{\lambda^2 + r^2 - c_i^2(t - s)^2}{2r\lambda}.$$

Note that such a function  $\psi_i$  is well defined, because (3.10) yields

$$(3.26) \quad \begin{aligned} & (2r\lambda)^2 - \{\lambda^2 + r^2 - c_i^2(t - s)^2\}^2 \\ &= \{\lambda^2 - (r - c_i(t - s))^2\} \cdot \{(r + c_i(t - s))^2 - \lambda^2\}. \end{aligned}$$

In order to state weighted  $L^\infty - L^\infty$  estimates for solutions of the Cauchy problem (3.20)-(3.21), we shall introduce a piecewise continuous function  $z_\mu(s, \lambda)$  of  $(s, \lambda) \in [0, \infty) \times [0, \infty)$  defined by

$$(3.27) \quad z_\mu(s, \lambda) = (1 + s + \lambda)^{1+\mu} (1 + |\lambda - c_j s|)^{1-\mu} \quad \text{for } (s, \lambda) \in \Lambda_j, \\ j = 0, 1, 2, \dots, m,$$

where  $\mu$  is a positive number and  $\Lambda_j$  are given by (2.2)<sub>j</sub> or (2.2)<sub>0</sub>. Notice that

$$z_\mu(s, \lambda) = (1 + s + \lambda)^{1+\mu} (1 + \lambda)^{1-\mu} \quad \text{for } (s, \lambda) \in \Lambda_0,$$

since  $c_0 = 0$  according to (2.1).

Now we are in a position to state the main results in the present section.

**Theorem 3.4.** Let  $i = 1, 2, \dots, m$ . Assume that  $F^i(t, x)$  is a continuous function of  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ . Let  $\nu$  and  $\mu$  be real numbers such that  $\nu \geq 0$  and  $\mu > 0$ . For  $(t, x) \in [0, \infty) \times \mathbb{R}^3$  with  $r = |x| > 0$ , we have then

$$(3.28) \quad \langle L_i(F^i)(t, x) \rangle_{\nu, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| z_\mu(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_0.$$

Moreover if  $F_i(t, x) \in C^1([0, \infty) \times \mathbb{R}^3)$  we have

$$(3.29) \quad [L_i(\partial F^i)(t, x)]_{1+\nu, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| z_\mu(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_1.$$

Here

$$(3.30) \quad D^i(t, r) = \{(s, y) \in \mathbb{R} \times \mathbb{R}^3; 0 \leq s \leq t, |r - c_i(t - s)| \leq |y| \leq r + c_i(t - s)\},$$

$C$  is a constant depending only on  $\nu, \mu$  and the  $c_1, c_2, \dots, c_m$  in (2.1), and we have used the notations given by (2.9), (2.10) and (2.11). In addition,

$$(3.31)_1 \quad \begin{aligned} & (1 + t + r)^\rho |L_i(F^i)(t, x)| \\ & \leq C \sup_{(s, y) \in D^i(t, r)} |y| z_\mu(s, |y|) (1 + s + |y|)^{\rho-1} |F^i(s, y)|_0, \end{aligned}$$

holds, provided  $0 < \rho < 1$ . Moreover if  $F^i(t, x) \in C^1([0, \infty) \times \mathbb{R}^3)$  we have

$$(3.31)_2 \quad [L_i(\partial F^i)(t, x)]_{\rho, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| z_{\mu, \rho}^i(s, |y|) |F^i(s, y)|_1,$$

provided  $0 < \rho \leq 1$ , where  $z_{\mu, \rho}^i(s, \lambda)$  is defined by

$$(3.32)_1 \quad \begin{aligned} z_{\mu, \rho}^i(s, \lambda) &= (1 + s + \lambda)^{\rho-1} z_\mu(s, \lambda) \\ &= (1 + s + \lambda)^{\rho+\mu} (1 + |\lambda - c_j s|)^{1-\mu} \\ &\quad \text{for } (s, \lambda) \in \Lambda_j \text{ with } 0 \leq j \leq m \text{ and } j \neq i \end{aligned}$$

and

$$\begin{aligned}
 (3.32)_2 \quad z_{\mu,\rho}^i(s, \lambda) &= (1 + |\lambda - c_i s|)^{\rho-1} z_\mu(s, \lambda) \\
 &= (1 + s + \lambda)^{1+\mu} (1 + |\lambda - c_i s|)^{\rho-\mu} \\
 &\text{for } (s, \lambda) \in \Lambda_i.
 \end{aligned}$$

**Remark.** Let  $\nu = 0$  and  $\rho = 1$ . Then  $(3.31)_2$  coincides with (3.29) according to (2.11), but  $(3.31)_1$  differs from (3.28) by

$$1 + \log \frac{1 + r + c_i t}{1 + |r - c_i t|},$$

because of (2.10)<sub>2</sub>.

We shall here state two useful consequences of (3.28) and (3.29) in the form of corollaries.

**Corollary 3.5.** Let  $i, \nu, \mu$  and  $(t, x)$  be as in the preceding theorem. Assume that  $F^i(t, x) \in C^3([0, \infty) \times \mathbb{R}^3)$ . Then we have

$$(3.29)_1 \quad [\partial_x L_i(F^i)(t, x)]_{1+\nu,0} \leq C \sup_{(s,y) \in D^i(t,r)} |y| z_\mu(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_1$$

and

$$\begin{aligned}
 (3.29)_2 \quad [\partial_t L_i(F_i)(t, x)]_{1+\nu,0} &\leq C \sup_{(y,s) \in D^i(t,r)} |y| z_\mu(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_1 \\
 &\quad + |u_0^i(t, x)| (1 + r) (1 + |r - c_i t|)^{1+\nu},
 \end{aligned}$$

where  $u_0^i(t, x)$  is the solution of the Cauchy problem (3.1)-(3.2) with  $a = c_i$  and  $g(x) = F^i(0, x)$ . Besides, we have used the notation (2.14).

*Proof.* Since  $v(t, x) \equiv \partial_x L_i(F^i)(t, x) \in C^2([0, \infty) \times \mathbb{R}^3)$  is the solution of the Cauchy problem (3.20)-(3.21) with  $F^i$  replaced by  $\partial_x F^i$ , we have from (3.22)

$$v(t, x) = L_i(\partial_x F^i)(t, x).$$

Hence (3.29)<sub>1</sub> immediately follows from (3.29). Moreover  $w(t, x) \equiv \partial_t L_i(F^i)(t, x)$  is the solution of the equation (3.20) with  $F^i$  replaced by  $\partial_t F^i$  which satisfies the initial condition  $w(0, x) = 0$  and  $\partial_t w(0, x) = F^i(0, x)$ . Therefore we get

$$w(t, x) = u_0^i(t, x) + L_i(\partial_t F^i)(t, x).$$

Hence (3.29)<sub>2</sub> follows from (3.29), because of the notation (2.14).  $\square$

**Corollary 3.6.** Let  $i = 1, 2, \dots, m$ . Assume that  $F^i(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$ , where  $T$  is a positive number or  $T = \infty$ . Let  $\nu, \mu$  and  $\rho$  be real numbers such that  $\nu \geq 0$ ,  $\mu > 0$  and  $\rho \geq 0$ . For a nonnegative integer  $k$  and  $(t, x) \in [0, T] \times \mathbb{R}^3$  with  $r = |x| > 0$  we have then

$$\begin{aligned} (3.33)_1 \quad & \langle L_i(F^i)(t, x) \rangle_{\nu, k} (1+t+r)^{-\rho} \\ & \leq C \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} |y| z_\mu(s, |y|) (1+s+|y|)^{\nu-\rho} |F^i(s, y)|_k \\ & \quad + C \sum_{|\alpha| \leq k-1} \sup_{y \in \mathbb{R}^3} |F_\alpha^i(y)| (1+|y|)^{2+\nu} \end{aligned}$$

and

$$\begin{aligned} (3.33)_2 \quad & [\partial L_i(F^i)(t, x)]_{1+\nu, k} (1+t+r)^{-\rho} \\ & \leq C \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} |y| z_\mu(s, |y|) (1+s+|y|)^{\nu-\rho} |F^i(s, y)|_{k+1} \\ & \quad + C \sum_{|\alpha| \leq k} \sup_{y \in \mathbb{R}^3} |F_\alpha^i(y)| (1+|y|)^{3+\nu} \end{aligned}$$

where

$$F_\alpha^i(y) = \Gamma^\alpha F^i(s, y)|_{s=0}$$

and  $C$  is a constant depending only on  $\nu, \mu, \rho, k$  and the  $c_1, c_2, \dots, c_m$  in (2.1). We have also used the notations (2.13) and (2.14).

*Proof.* Since  $1 + |y| + c_i s \leq 1 + r + c_i t$  for  $(s, y) \in D^i(t, r)$ , the estimate (3.33)<sub>1</sub> with  $k = 0$  follows immediately from (3.28). To derive the others we set

$$v_j(t, x) = \Gamma_j L_i(F^i)(t, x) \quad \text{for } j = 0, 1, 2, \dots, 7.$$

where  $\Gamma_j$  are given by (2.8). If  $1 \leq j \leq 7$ , then  $v_j(t, x)$  is the solution of the Cauchy problem (3.20)-(3.21) with  $F^i$  replaced by  $(\Gamma_j + 2\delta_{7j})F^i$ , where  $\delta_{7j} = 0$  for  $j \neq 7$  and  $\delta_{77} = 1$ , because of (2.6). Hence we have from (3.22)

$$v_j(t, x) = L_i((\Gamma_j + 2\delta_{7j})F^i)(t, x).$$

On the other hand, if  $j = 0$ , we have

$$v_0(t, x) = u_0^i(t, x) + L_i(\Gamma_0 F^i)(t, x),$$

where  $u_0^i$  is the same function as in Corollary 3.5. Applying Proposition 3.3 to  $u_0^i$ , we thus obtain (3.33)<sub>1</sub> with  $k = 1$  from (3.28). Similarly one can prove (3.33)<sub>1</sub> and (3.33)<sub>2</sub>, making use of (3.29) and (2.7) also. The proof is complete.  $\square$

**Remark.** We will often make use of Corollary 3.6 in deriving weighted  $L^\infty - L^2$  estimates for local solutions of the nonlinear wave equation (1.1), in the case where  $\rho = 0$  or  $\nu = 0$ .

In order to deal with the resonance form  $R^i(\partial u, \partial^2 u)$  given in (1.7) with (1.10) and (1.11), we will need also the following

**Theorem 3.7.** Let  $i, F^i, \nu, \mu$  and  $(t, x)$  be as in Theorem 3.4. Then we have

$$(3.34)_1 \quad \langle L_i(F^i)(t, x) \rangle_{\nu, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| \tilde{z}_\mu^k(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_0$$

and

$$(3.34)_2 \quad \begin{aligned} & (1 + r + t)^\rho |L_i(F^i)(t, x)| \\ & \leq C \sup_{(s, y) \in D^i(t, r)} |y| \tilde{z}_\mu^k(s, |y|) (1 + s + |y|)^{\rho-1} |F^i(s, y)|_0 \end{aligned}$$

for  $k = 1, 2, \dots, m$  with  $k \neq i$ , provided  $0 < \rho < 1$ , where  $\tilde{z}_\mu^k(s, \lambda)$  is defined by

$$(3.34)_3 \quad \begin{aligned} \tilde{z}_\mu^k(s, \lambda) &= z_\mu(s, \lambda) (1 + \lambda + s)^{-\mu} (1 + |\lambda - c_k s|)^{2\mu} \\ &= (1 + \lambda + s) (1 + |\lambda - c_k s|)^{1+\mu} \quad \text{for } (s, \lambda) \in \Lambda_k \end{aligned}$$

and

$$(3.34)_4 \quad \tilde{z}_\mu^k(s, \lambda) = z_\mu(s, \lambda) \quad \text{for } (s, \lambda) \in \Lambda_j \quad \text{with } 0 \leq j \leq m \quad \text{and } j \neq k.$$

Moreover if  $F^i(t, x) \in C^1([0, \infty) \times \mathbb{R}^3)$  we have

$$(3.35)_1 \quad [L_i(\partial F^i)(t, x)]_{\rho+\nu, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| \tilde{z}_{\mu, \rho}^k(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_1$$

for  $1 \leq k \leq m$  with  $k \neq i$ , provided  $0 < \rho \leq 1$ , where  $\tilde{z}_{\mu, \rho}^k(s, \lambda)$  is defined by

$$(3.35)_2 \quad \begin{aligned} \tilde{z}_{\mu, \rho}^k(s, \lambda) &= (1 + \lambda + s)^{\rho-1} \tilde{z}_\mu^k(s, \lambda) \\ &= (1 + \lambda + s)^\rho (1 + |\lambda - c_k s|)^{1+\mu} \\ &\quad \text{for } (s, \lambda) \in \Lambda_k \end{aligned}$$

and

$$(3.35)_3 \quad \begin{aligned} \tilde{z}_{\mu, \rho}^k(s, \lambda) &= z_{\mu, \rho}^i(s, \lambda) \\ &\quad \text{for } (s, \lambda) \in \Lambda_j \quad \text{with } 0 \leq j \leq m \quad \text{and } j \neq k. \end{aligned}$$

Here  $C$  is a constant depending only on  $\nu, \mu, \rho$  and the  $c_1, c_2, \dots, c_m$  in (2.1), and we have used the same notations as in Theorem 3.4.

**Remark.** We will only once make use of (3.35)<sub>1</sub> with  $\nu = 0$  and  $\rho = 1$ , in treating the  $R^{ik}(\partial u^k, \partial^2 u^k)$  with  $1 \leq k \leq m$  and  $k \neq i$  in (1.11).

**Remark.** The function  $\tilde{z}_\mu^k(s, \lambda)$  given by (3.34)<sub>3</sub> is in essence the same one as the  $z_{\bar{\mu}, \bar{\nu}}(s, \lambda)$  in (3.3) of [21], when  $\bar{\nu} = 1$  and  $\bar{\mu} = 1 + \mu$ . Here  $z_{\bar{\mu}, \bar{\nu}}(s, \lambda) = (1 + s + \lambda)^{\bar{\nu}} (1 + |\lambda - cs|)^{\bar{\mu}}$  for all  $(s, \lambda) \in [0, \infty) \times [0, \infty)$ , where  $\bar{\nu} > 0$ ,  $\bar{\mu} \geq 1$  and  $c \geq 0$ .

The rest of the present section will be devoted to prove Theorems 3.4 and 3.7. The lengthy proof of Theorem 3.4 is broken up into a sequence of lemmas, however one can prove Theorem 3.7 by modifying a little the proof of Theorem 3.4.

In what follows we assume that the parameters  $\nu, \mu, \rho$  and the variables  $t, x$  are as in Theorem 3.4, unless otherwise stated. By  $C$  we will denote the various constants

which depend only on  $\nu, \mu, \rho$  and the  $c_1, c_2, \dots, c_m$  in (2.1). Moreover we shall assume without loss of generality that  $c_i = 1$ , by changing the variable  $t$  by  $\tau = c_i t$ .

We start with

**Lemma 3.8.** The estimates (3.28) and (3.31)<sub>1</sub> hold, provided  $0 < \rho < 1$ .

*Proof.* First we shall prove (3.28). Note that

$$(3.36) \quad 1 + s + \lambda \geq 1 + |r - t| \quad \text{for } s \geq 0 \quad \text{and} \quad \lambda \geq |r + s - t|.$$

Let  $\nu \geq 0$ . Then it follows from (3.23) and (3.30) that

$$(3.37) \quad \begin{aligned} & |L_i(F^i)(t, x)| \\ & \leq I(t, r) \sup_{(s, y) \in \tilde{D}^i(t, r)} |y| z_\mu(s, |y|) (1 + s + |y|)^\nu |F^i(s, y)|_0, \end{aligned}$$

where  $I(t, r)$  is defined by

$$(3.38) \quad I(t, r) = \frac{1}{2r} \iint_{\tilde{D}^i} \frac{ds d\lambda}{z_\mu(s, \lambda) (1 + s + \lambda)^\nu},$$

with

$$\tilde{D}^i = \{(s, \lambda) \in \mathbb{R}^2; 0 \leq s \leq t, |r + s - t| \leq \lambda \leq r + t - s\}.$$

Here we shall divide the domain of integration into  $m + 1$  domains  $\tilde{D}^i \cap \Lambda_j$ ,  $j = 0, 1, 2, \dots, m$ , because of (3.27). Namely, we set

$$(3.38)_j \quad I_j(t, r) = \frac{1}{2r} \iint_{\tilde{D}^i \cap \Lambda_j} \frac{ds d\lambda}{z_\mu(s, \lambda) (1 + s + \lambda)^\nu},$$

so that

$$I(t, r) = \sum_{j=0}^m I_j(t, r).$$

Moreover, on each domain  $\tilde{D}^i \cap \Lambda_j$  we shall introduce new variables by

$$(3.39) \quad \alpha = \lambda + s, \quad \beta = \lambda - c_j s \quad \text{for } (s, \lambda) \in \Lambda_j.$$

Note that

$$(3.40) \quad -c_j \alpha \leq \beta \leq \alpha \quad \text{for } (s, \lambda) \in \Lambda_j, \quad j = 0, 1, 2, \dots, m$$

and

$$(3.41) \quad \beta = \lambda, \quad \alpha - \beta = s \quad \text{for } (s, \lambda) \in \Lambda_0$$

Then it follows from (3.27), (3.36) and (3.38)<sub>j</sub> that

$$(3.42) \quad \begin{aligned} I_j(t, r) &\leq \frac{1}{2r} \int_{|r-t|}^{r+t} (1+\alpha)^{-1-\mu-\nu} d\alpha \int_{-c_j \alpha}^{\alpha} (1+|\beta|)^{-1+\mu} d\beta \\ &\leq C \frac{1}{2r} \int_{|r-t|}^{r+t} (1+\alpha)^{-1-\nu} d\alpha, \end{aligned}$$

since  $\mu > 0$ . Making use of Lemma 3.2 we thus obtain (3.28) from (3.37) and (3.38).

Next we shall prove (3.31)<sub>1</sub>. We replace (3.37) and (3.38)<sub>j</sub> by

$$(3.37)' \quad |L_i(F^i)(t, x)| \leq I(t, r) \sup_{(s, y) \in D^i(t, r)} |y| z_\mu(s, \lambda) (1+s+\lambda)^{\rho-1} |F^i(s, y)|_0$$

and

$$(3.38)'_j \quad I_j(t, r) = \frac{1}{2r} \iint_{\tilde{D}^i \cap \Lambda_j} \frac{(1+s+\lambda)^{1-\rho}}{z_\mu(s, \lambda)} ds d\lambda.$$

Then (3.42) becomes

$$(3.42)' \quad I_j(t, r) \leq C \frac{1}{2r} \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} d\alpha.$$

Hence we obtain (3.31)<sub>1</sub> by Lemma 3.2 with  $\kappa = \rho - 1$ . The proof is complete.  $\square$

Next we shall prove (3.29) and (3.31)<sub>2</sub>. Notice that (3.29) is implied by (3.31)<sub>2</sub> with  $\rho = 1$ , since  $\nu \geq 0$  and  $1+s+|y| \geq 1+|r-t|$  for  $(s, y) \in D^i(t, r)$ . Thus we have only to prove (3.31)<sub>2</sub>.

We first deal with the case of small  $r > 0$ .

**Lemma 3.9.** Let  $0 < r \leq 1$ . Then we have (3.31)<sub>2</sub>, provided  $0 < \rho \leq 1$ .

*Proof.* It follows from (2.11) that

$$[L_i(\partial F^i)(t, x)]_{\rho, 0}^{(i)} \leq 2(1 + r + t)^\rho |L_i(\partial F^i)(t, x)|_0$$

for  $0 < r \leq 1$  and  $\rho > 0$ . Therefore if  $0 < \rho < 1$  we get by virtue of (3.31)<sub>1</sub>

$$[L_i(\partial F^i)(t, x)]_{\rho, 0}^{(i)} \leq C \sup_{(s, y) \in D^i(t, r)} |y| z_\mu(s, |y|) (1 + s + |y|)^{\rho-1} |\partial F^i(s, y)|_0,$$

which implies (3.31)<sub>2</sub>. If  $\rho = 1$ , we obtain the same estimate from (3.28) for  $\nu = 0$  together with (3.14), because of (2.10)<sub>2</sub>.  $\square$

In what follows we shall assume that

$$(3.43) \quad r = |x| \geq 1.$$

Then we will often make use of an elementary inequality (3.45) below. Notice that the condition  $|r + s - t| \leq \lambda$  in (3.38) implies  $s + r - t \leq \lambda$ , which can be written as

$$(1 - c_j)\alpha + (1 + c_j)(r - t) \leq 2\beta$$

with the notation (3.39). Bearing this in mind, we set

$$(3.44) \quad \beta_j = \beta_j(\alpha) = \frac{1}{2} \{ (1 - c_j)\alpha + (1 + c_j)(r - t) \}$$

for  $j = 0, 1, 2, \dots, m$  and  $\alpha \geq 0$ , where  $c_0, c_1, c_2, \dots, c_m$  are given in (2.1). Then we obtain the following

**Lemma 3.10.** Let  $\rho > 0$  and  $\mu \geq 0$ . Assume that  $c_j \neq c_i = 1$ . Then we have

$$(3.45) \quad \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-\mu} (1 + |\beta_j|)^{-1+\mu} d\alpha \leq C(1 + |r - t|)^{-\rho} \chi_\mu(r + t),$$

where  $\chi_\mu(s) = 1$  if  $\mu > 0$ ,  $\chi_\mu(s) = 1 + \log(1+s)$  if  $\mu = 0$ , and  $C$  is a constant depending only on  $\rho, \mu$  and  $c_j$ .

*Proof.* By  $I$  we shall denote the left hand side of (3.45). First suppose that  $\mu > 0$ . Then, since

$$(1 + |\beta_j|)^{-1+\mu} = \frac{1}{\mu} \frac{2}{1 - c_j} (\operatorname{sgn} \beta_j) \frac{\partial}{\partial \alpha} (1 + |\beta_j|)^\mu,$$

we have

$$I \leq C(1 + |r - t|)^{-\rho} + C \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-1} d\alpha,$$

noting that (3.44) yields

$$(3.46) \quad |\beta_j| \leq (1 + c_j)\alpha \quad \text{for } \alpha \geq |r - t|.$$

Thus we obtain (3.45).

Next suppose that  $\mu = 0$ . Then we have

$$\begin{aligned} I &\leq C(1 + |r - t|)^{-\rho} \log(1 + (1 + c_j)(r + t)) \\ &\quad + C \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-1} \log(1 + |\beta_j|) d\alpha. \end{aligned}$$

Therefore we obtain (3.45). The proof is complete.  $\square$

The estimate (3.45) will be used also in the following forms.

**Corollary 3.11.** Let  $\rho > 0$  and  $\mu \geq 0$ . Then we have

$$(3.47) \quad \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-\mu} (1 + \alpha + r - t)^{-1+\mu} d\alpha \leq C(1 + |r - t|)^{-\rho} \quad \text{if } \mu > 0,$$

$$(3.48) \quad \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-\frac{1}{2}-\mu} (1 + \alpha + r - t)^{-\frac{1}{2}+\mu} d\alpha \leq C(1 + |r - t|)^{-\rho},$$

$$(3.49) \quad \int_{|r-t|}^{r+t} (1 + \alpha)^{-\rho-\frac{1}{2}} (1 + \alpha + r - t)^{-\frac{1}{2}} d\alpha \leq C(1 + |r - t|)^{-\rho}$$

and

$$(3.50) \quad \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}-\mu} (1+|\beta_j|)^{-\frac{1}{2}+\mu} d\alpha \leq C(1+|r-t|)^{-\rho} \quad \text{if } c_j \neq c_i = 1.$$

*Proof.* Noting that (3.44) with  $j = 0$  yields

$$(3.51) \quad \beta_0 = \frac{1}{2}(\alpha + r - t) \geq 0 \quad \text{for } \alpha \geq t - r,$$

we have (3.47) from (3.45) with  $j = 0$ . Hence (3.48) and (3.49) follows, if we replace  $\mu$  in (3.47) by  $\mu + (1/2)$  or  $1/2$  respectively. Besides, we get (3.50), replacing  $\mu$  in (3.45) by  $\mu + (1/2)$ .  $\square$

We will also make use of the following elementary inequalities.

**Lemma 3.12.** Let  $\rho > 0$ . Then we have

$$(3.52) \quad \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} (1+\alpha+r-t)^{-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \leq \pi(1+|r-t|)^{-\rho}$$

and

$$(3.53) \quad \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \leq \sqrt{2} \pi(1+|r-t|)^{-\rho}.$$

*Proof.* By  $I$  we denote the left hand side of (3.52). Then we have

$$I \leq \frac{1}{(1+|r-t|)^\rho} \int_{|t-r|}^{t+r} (\alpha+r-t)^{-\frac{1}{2}} (t+r-\alpha)^{-\frac{1}{2}} d\alpha,$$

which yields (3.52), since

$$(3.54) \quad \int_a^b (\alpha-a)^{-\frac{1}{2}} (b-\alpha)^{-\frac{1}{2}} d\alpha = \pi \quad \text{for } a < b.$$

We also obtain (3.53) easily from (3.52), since  $1 + \alpha + r - t \leq 2(1 + \alpha)$  for  $\alpha \geq |r - t|$ .  $\square$

Finally we shall derive another elementary inequality which refines such calculation for the  $\beta$ -integral as in the proof of Lemma 3.8.

**Lemma 3.13.** Let  $\kappa$  be a real number and let  $|r - t| \leq \alpha \leq r + t$ . Set

$$(3.55) \quad I_j = I_j(\alpha) = \int_{\beta_j}^{\alpha} (1 + \beta - \beta_j)^{-\frac{1}{2}} (1 + |\beta|)^{-1+\kappa} d\beta$$

for  $j = 0, 1, 2, \dots, m$ , where  $\beta_j = \beta_j(\alpha)$  is given by (3.44).

If  $\beta_j \geq 0$  (for example, if  $j = 0$ ), we have

$$(3.56) \quad I_j \leq \begin{cases} C(1 + \beta_j)^{-\frac{1}{2}+\kappa} & \text{for } \kappa < \frac{1}{2}, \\ C(1 + \alpha)^{[\kappa-\frac{1}{2}]_+} & \text{for } \kappa \geq \frac{1}{2}, \end{cases}$$

where we have set

$$(3.57)_1 \quad [a]_+ = \max\{a, 0\} \quad \text{for } a \in \mathbb{R}$$

and

$$(3.57)_2 \quad (1 + s)^{[0]_+} = 1 + \log(1 + s) \quad \text{for } s \geq 0.$$

Moreover, if  $1 \leq j \leq m$ , we have

$$(3.58) \quad I_j \leq \begin{cases} C(1 + |\beta_j|)^{-\frac{1}{2}} (1 + |\beta_j|)^{[\kappa]_+} & \text{for } \kappa < \frac{1}{2}, \\ C(1 + \alpha)^{[\kappa-\frac{1}{2}]_+} & \text{for } \kappa \geq \frac{1}{2}. \end{cases}$$

*Proof.* First we shall prove (3.56). Note that (3.44) implies

$$(3.59) \quad \beta_j \leq \alpha \quad \text{for } \alpha \geq r - t.$$

Let  $\beta_j \geq 0$ . First suppose that  $\kappa \leq 1$ . Then we have from (3.55)

$$I_j \leq 2(1 + \alpha)^{-\frac{1}{2}+\kappa} + 2(1 - \kappa) \int_{\beta_j}^{\alpha} (1 + \beta)^{-\frac{3}{2}+\kappa} d\beta,$$

integrating by parts, since  $1 + \beta - \beta_j \leq 1 + \beta$ . Hence we get (3.56) by (3.59). Conversely suppose that  $\kappa > 1$ . Then we have as above

$$I_j \leq 2(1 + \alpha)^{-\frac{1}{2} + \kappa},$$

which implies (3.56).

Next we shall prove (3.58). Since (3.56) implies (3.58) if  $\beta_j \geq 0$ , we shall assume that  $\beta_j < 0$ . Note that (3.44) implies

$$(3.60) \quad \beta_j \geq -c_j \alpha \quad \text{for } \alpha \geq t - r,$$

hence

$$0 < -\beta_j/c_j \leq \alpha \quad \text{for } j \geq 1.$$

Therefore one can divide  $I_j$  into three integrals as follows.

$$\begin{aligned} I_j &= \int_{\beta_j}^{\beta_j/2} + \int_{\beta_j/2}^{-\beta_j/c_j} + \int_{-\beta_j/c_j}^{\alpha} \\ &\equiv I_{j1} + I_{j2} + I_{j3}. \end{aligned}$$

For the first term we have

$$\begin{aligned} I_{j1} &\leq C(1 + |\beta_j|)^{-1 + \kappa} \int_{\beta_j}^{\beta_j/2} (1 + \beta - \beta_j)^{-\frac{1}{2}} d\beta \\ &\leq C(1 + |\beta_j|)^{-\frac{1}{2} + \kappa}, \end{aligned}$$

which yields (3.58) for  $I_j = I_{j1}$ , if we use (3.46) when  $\kappa > \frac{1}{2}$ .

For the second term it follows that

$$I_{j2} \leq C(1 + |\beta_j|)^{-\frac{1}{2}} \int_{\beta_j/2}^{-\beta_j/c_j} (1 + |\beta|)^{-1 + \kappa} d\beta,$$

since

$$1 + \beta - \beta_j \geq (1 + |\beta_j|)/2 \quad \text{for } \beta \geq \beta_j/2.$$

Hence we obtain (3.58) for  $I_j = I_{j2}$ , using (3.46) when  $\kappa > 1/2$ .

As for the last term, since  $\beta_j < 0$ , we have

$$I_{j3} \leq \int_{|\beta_j|/c_j}^{\alpha} (1 + |\beta|)^{-\frac{3}{2} + \kappa} d\beta.$$

Therefore we easily obtain (3.58) for  $I_j = I_{j3}$ . Thus we prove the lemma.  $\square$

Now we are in a position to prove (3.31)<sub>2</sub> with  $c_i = 1$ , which is written as

$$(3.61) \quad \begin{aligned} & (1+r)(1+|r-t|)^{\rho} |L_i(\partial F^i)(t, x)| \\ & \leq C \sup_{(s,y) \in D^i(t,r)} |y| z_{\mu,\rho}^i(s, |y|) |F^i(s, y)|_1 \end{aligned}$$

according to (2.11). We start with the following reduction which has been essentially proved in Yokoyama [21].

**Lemma 3.14.** Let (3.43) hold. Then we have

$$(3.62) \quad (1+r) |L^i(\partial F_i)(t, x)| \leq C(I_0 + I_1 + I_2) \sup_{(s,y) \in D^i(t,y)} |y| z(s, |y|) |F^i(s, y)|_1.$$

Here

$$(3.63)_1 \quad z(s, \lambda) = z_{\mu,\rho}^i(s, \lambda),$$

$$(3.63)_2 \quad I_0 = I_0(t, r) = \int_{\partial D} \frac{1}{z(s, \lambda)} d\sigma$$

with  $\partial D$  boundary of  $D$ ,

$$(3.63)_3 \quad I_1 = I_1(t, r) = \iint_{D_1} \frac{1}{z(s, \lambda) \sqrt{1 + \lambda - \lambda_1}} \left( \frac{1}{\sqrt{1 + \lambda}} + \frac{1}{\sqrt{1 + \lambda_2 - \lambda}} \right) ds d\lambda$$

and

$$(3.63)_4 \quad I_2 = I_2(t, r) = \iint_{D_2} \frac{1}{z(s, \lambda) \sqrt{1 + \lambda}} \left( \frac{1}{\sqrt{1 + \lambda - \lambda_1}} + \frac{1}{\sqrt{1 + \lambda_2 - \lambda}} \right) ds d\lambda,$$

where

$$(3.64) \quad \lambda_1 = |r + s - t|, \quad \lambda_2 = r + t - s,$$

$$(3.65)_1 \quad D = \{(s, \lambda) \in \mathbb{R}^2; 0 \leq s \leq t, 1 + \lambda_1 \leq \lambda \leq \lambda_2 - 1\},$$

$$(3.65)_2 \quad D_1 = \{(s, \lambda) \in D; s \leq t - r\}$$

and

$$(3.65)_3 \quad D_2 = \{(s, \lambda) \in D; s \geq t - r\}.$$

*Proof.* We shall refer to the opening of the proof of Proposition 3.1 in [21]. Since the expression (3.8) or (3.10) in it is the same one as our (3.23) with  $c_i = 1$ , we find by virtue of the identity (3.17) through the estimate (3.25) in [21] that

$$(3.66) \quad \begin{aligned} & (1+r)|L_i(\partial F^i)(t, x)| \\ & \leq C(I_0 + I_3 + I_4) \sup_{(s, y) \in D^i(t, r)} |y|z(s, |y|)|F^i(s, y)|_1. \end{aligned}$$

Here

$$I_3 = \iint_D \frac{1}{(1+\lambda)z(s, \lambda)} ds d\lambda$$

and

$$I_4 = \iint_D \frac{1}{z(s, \lambda)} \left( \sup_{0 \leq \varphi \leq 2\pi} |\partial \Theta_i| \right) ds d\lambda,$$

where  $\Theta_i = \Theta_i(s, \lambda, \varphi; t, x)$  is given by (3.24), and  $\partial \Theta_i = (\partial_s \Theta_i, \partial_\lambda \Theta_i)$ . Moreover, by virtue of the estimates (3.34) and (3.35) in [21] for  $|\partial \Theta_i|$ , we get

$$I_4 \leq C(I_1 + I_2).$$

Clearly we have  $I_3 \leq I_1 + I_2$ . Thus we obtain (3.62) from (3.66). The proof is complete.  $\square$

In view of (3.61) and (3.62) we have only to prove

$$(3.67)_k \quad I_k = I_k(t, r) \leq C(1 + |r - t|)^{-\rho} \quad \text{for } k = 0, 1, 2.$$

In what follows we shall divide each domain of integration in (3.63)<sub>2</sub>, (3.63)<sub>3</sub> or (3.63)<sub>4</sub> into  $m + 1$  domains, analogously to the proof of Lemma 3.8. Besides, we will often use, without notice, the notation (3.39) for  $(s, \lambda) \in \Lambda_j$ ,  $j = 0, 1, 2, \dots, m$ , where  $\Lambda_j$  are given by (2.2)<sub>i</sub> or (2.2)<sub>0</sub>. Then it follows from (3.27), (3.32), (3.39) and (3.63)<sub>1</sub> that

$$(3.68)_1 \quad \begin{aligned} z(s, \lambda) &= (1 + \alpha)^{\rho+\mu} (1 + |\beta|)^{1-\mu} \\ &\text{for } (s, \lambda) \in \Lambda_j \text{ with } 0 \leq j \leq m \text{ and } j \neq i \end{aligned}$$

and

$$(3.68)_2 \quad z(s, \lambda) = (1 + \alpha)^{1+\mu} (1 + |\beta|)^{\rho-\mu} \quad \text{for } (s, \lambda) \in \Lambda_i.$$

First we shall deal with  $I_0$  given by (3.63)<sub>2</sub> with (3.65)<sub>1</sub>.

**Lemma 3.15.** Set

$$(3.69) \quad I_{0j} = I_{0j}(t, r) = \int_{(\partial D) \cap \Lambda_j} \frac{1}{z(s, \lambda)} d\sigma$$

for  $j = 0, 1, 2, \dots, m$ . Then we have

$$(3.70) \quad |I_{0j}(t, r)| \leq C(1 + |r - t|)^{-\rho}$$

for  $0 \leq j \leq m$ , provided  $0 < \rho < 1 + \mu$ .

*Proof.* We see from (3.32), (3.64) and (3.65)<sub>1</sub> that in the proof one can replace two segments  $\lambda = \lambda_1 + 1$  and  $\lambda = \lambda_2 - 1$  of the boundary of  $D$  by  $\lambda = \lambda_1$  and  $\lambda = \lambda_2$  respectively, because  $|a|/2 \leq |1 + a| \leq 1 + |a|$  for a real number  $a$  with  $|a| \geq 2$ .

Let  $(s, \lambda) \in \Lambda_j$ . Then it follows from (3.39) and (3.64) that

$$(3.71)_1 \quad \alpha = r + t \quad \text{for } \lambda = \lambda_2,$$

$$(3.71)_2 \quad \alpha = t - r \quad \text{for } \lambda = \lambda_1 \text{ with } 0 \leq s \leq t - r$$

and

$$(3.71)_3 \quad \beta = \beta_j \quad \text{for} \quad \lambda = \lambda_1 \quad \text{with} \quad s \geq t - r,$$

where  $\beta_j = \beta_j(\alpha)$  is given by (3.44). Hence we find from (3.36) and (3.40) that

$$(3.72) \quad I_{0j} \leq C(A_1 + A_2 + A_3 + A_4),$$

where

$$\begin{aligned} A_1 &= \int_{|r-t|}^{r+t} \left| \frac{1}{z} \right|_{s=0} d\lambda, \\ A_2 &= \int_{-c_j(r+t)}^{r+t} \left| \frac{1}{z} \right|_{\alpha=r+t} d\beta, \\ A_3 &= \int_{-c_j|r-t|}^{|r-t|} \left| \frac{1}{z} \right|_{\alpha=|r-t|} d\beta \end{aligned}$$

and

$$A_4 = \int_{|r-t|}^{r+t} \left| \frac{1}{z} \right|_{\beta=\beta_j} d\alpha.$$

For the first integral we have easily from (3.68) and (3.41)

$$(3.73)_1 \quad A_1 = \int_{|r-t|}^{r+t} (1 + \lambda)^{-\rho-1} d\lambda \leq \rho^{-1} (1 + |r - t|)^{-\rho}.$$

To deal with the others, we first suppose that  $j \neq i$ . Then it follows from (3.68)<sub>1</sub> that

$$\begin{aligned} (3.73)_2 \quad A_2 &= (1 + r + t)^{-\rho-\mu} \int_{-c_j(r+t)}^{r+t} (1 + |\beta|)^{-1+\mu} d\lambda \\ &\leq C(1 + r + t)^{-\rho} \end{aligned}$$

and

$$\begin{aligned} (3.73)_3 \quad A_3 &= (1 + |r - t|)^{-\rho-\mu} \int_{-c_j|r-t|}^{|r-t|} (1 + |\beta|)^{-1+\mu} d\beta \\ &\leq C(1 + |r - t|)^{-\rho}. \end{aligned}$$

As for the last integral we have

$$A_4 = \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+|\beta_j|)^{-1+\mu} d\alpha.$$

By virtue of Lemma 3.10 we obtain

$$(3.73)_4 \quad A_4 \leq C(1+|r-t|)^{-\rho}.$$

Next suppose that  $j = i$ . Then it follows from (3.68)<sub>2</sub> that

$$(3.74)_1 \quad \begin{aligned} A_2 &= (1+r+t)^{-1-\mu} \int_{-c_j(r+t)}^{r+t} (1+|\beta|)^{-\rho+\mu} d\beta \\ &\leq C(1+r+t)^{-\rho} \end{aligned}$$

and

$$(3.74)_2 \quad \begin{aligned} A_3 &= (1+|r+t|)^{-1-\mu} \int_{-c_j(r-t)}^{|r-t|} (1+|\beta|)^{-\rho+\mu} d\beta \\ &\leq C(1+|r-t|)^{-\rho}, \end{aligned}$$

provided  $-\rho + \mu > -1$ . As for the last integral we have

$$A_4 = (1+|r-t|)^{-\rho+\mu} \int_{|r-t|}^{r+t} (1+\alpha)^{-1-\mu} d\alpha,$$

because  $\beta_i = r-t$  according to (3.44) with  $c_i = 1$ . Therefore we get

$$(3.74)_3 \quad A_4 \leq C(1+|r-t|)^{-\rho},$$

since  $\mu > 0$ . Now the desired estimate (3.70) follows immediately from (3.72), (3.73) and (3.74). The proof is complete.  $\square$

Next we shall prove  $(3.67)_1$  and  $(3.67)_2$ , where  $I_1$  and  $I_2$  are given by  $(3.63)_3$  and  $(3.63)_4$  with  $(3.64)$  and  $(3.65)$ . We divide each integral into  $m+1$  terms analogously to the proof of Lemma 3.8. Namely, we set

$(3.75)_1$

$$I_{1j} = I_{1j}(t, r) \\ = \iint_{D_1 \cap \Lambda_j} \frac{1}{z(s, \lambda) \sqrt{1 + \lambda - \lambda_1}} \left( \frac{1}{\sqrt{1 + \lambda}} + \frac{1}{\sqrt{1 + \lambda_2 - \lambda}} \right) ds d\lambda$$

and

$(3.75)_2$

$$I_{2j} = I_{2j}(t, r) \\ = \iint_{D_2 \cap \Lambda_j} \frac{1}{z(s, \lambda) \sqrt{1 + \lambda}} \left( \frac{1}{\sqrt{1 + \lambda - \lambda_1}} + \frac{1}{\sqrt{1 + \lambda_2 - \lambda}} \right) ds d\lambda$$

for  $j = 0, 1, 2, \dots, m$ , so that

$$I_1 = \sum_{j=0}^m I_{1j} \quad \text{and} \quad I_2 = \sum_{j=0}^m I_{2j}.$$

For each  $j = 0, 1, 2, \dots, m$ , it follows from (3.39), (3.44), (3.64) and (3.65) that

$$(3.76)_1 \quad \lambda_2 - \lambda = r + t - \alpha \quad \text{for } (s, \lambda) \in (D_1 \cup D_2) \cap \Lambda_j,$$

$$(3.76)_2 \quad \lambda - \lambda_1 = \alpha + r - t \quad \text{for } (s, \lambda) \in D_1 \cap \Lambda_j$$

and

$$(3.76)_3 \quad (1 + c_j)(\lambda - \lambda_1) = 2(\beta - \beta_j) \quad \text{for } (s, \lambda) \in D_2 \cap \Lambda_j.$$

Moreover we shall divide the integrals  $I_{1j}$  and  $I_{2j}$  into three groups as follows: 1)  $j = 0$ , 2)  $1 \leq j \leq m$  with  $j \neq i$ , and 3)  $j = i$ ,

We start with

**Lemma 3.16.** Let  $j = 0$ . Then we have

$$(3.77) \quad I_{10}(t, r) + I_{20}(t, r) \leq C(1 + |r - t|)^{-\rho},$$

provided  $\rho > 0$ , where  $I_{10}$  and  $I_{20}$  are given by  $(3.75)_1$  and  $(3.75)_2$  respectively.

*Proof.* Let  $j = 0$ . Then it follows from (3.41), (3.51) and (3.65) that

$$(3.78) \quad 2\beta \geq 2\beta_0 = \alpha + r - t \geq 0 \quad \text{for } (s, \lambda) \in (D_1 \cup D_2) \cap \Lambda_0.$$

Indeed, if  $\lambda \geq \lambda_1 = |s + r - t|$ , we have  $\lambda \geq s + r - t$ , which is equivalent to the first inequality of (3.78) according to (3.41) and (3.51).

First we consider  $I_{10}$ . Similarly to (3.42), it follows from  $(3.75)_1$ , (3.41),  $(3.68)_1$ , (3.76) and (3.78) that

$$(3.79)_1 \quad \begin{aligned} I_{10} \leq & C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+\alpha+r-t)^{-\frac{1}{2}} d\alpha \int_{\beta_0}^{\alpha} (1+\beta)^{-\frac{3}{2}+\mu} d\beta \\ & + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+\alpha+r-t)^{-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_0^{\alpha} (1+\beta)^{-1+\mu} d\beta. \end{aligned}$$

One can also assume without loss of generality that  $0 < \mu < 1/2$ , since (3.40) yields

$$(1+\alpha)^{-\mu} \leq C(1+\alpha)^{-\frac{1}{4}}(1+|\beta|)^{-\mu+\frac{1}{4}}$$

if  $\mu \geq 1/2$ . Hence we see from (3.78) that

$$(3.79)_2 \quad \begin{aligned} I_{10} \leq & C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+\alpha+r-t)^{-1+\mu} d\alpha \\ & + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} (1+\alpha+r-t)^{-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha. \end{aligned}$$

Therefore we obtain (3.77) for  $I_{10}$ , making use of (3.47) and (3.52).

Next we consider  $I_{20}$ . Analogously to  $(3.79)_1$  it follows from  $(3.75)_2$  that

$$\begin{aligned} I_{20} \leq & C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} d\alpha \int_{\beta_0}^{\alpha} (1+\beta)^{-\frac{3}{2}+\mu} (1+\beta-\beta_0)^{-\frac{1}{2}} d\beta \\ & + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{\beta_0}^{\alpha} (1+\beta)^{-\frac{3}{2}+\mu} d\beta. \end{aligned}$$

As above one can assume that  $0 < \mu < 1/2$ . Applying (3.56) to the  $\beta$ -integral of the first term, we therefore get by (3.78)

$$\begin{aligned} I_{20} &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+\alpha+r-t)^{-1+\mu} d\alpha \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+r+t-\alpha)^{-\frac{1}{2}} (1+\alpha+r-t)^{-\frac{1}{2}+\mu} d\alpha. \end{aligned}$$

Thus we obtain (3.77) for  $I_{20}$ , making use of (3.47) and (3.52), since

$$(1+\alpha+r-t)^\mu \leq 2^\mu (1+\alpha)^\mu.$$

The proof is complete.  $\square$

Next we deal with the generic case.

**Lemma 3.17.** Let  $1 \leq j \leq m$  and  $j \neq i$ . Then we have

$$(3.80) \quad I_{1j}(t, r) + I_{2j}(t, r) \leq C(1+|r-t|)^{-\rho},$$

provided  $\rho > 0$ .

*Proof.* Note that (2.3) and (3.39) yield

$$(3.81) \quad (1+\lambda)^{-\frac{1}{2}} \leq C(1+\alpha)^{-\frac{1}{2}} \quad \text{for } (s, \lambda) \in \Lambda_j \quad \text{for } 1 \leq j \leq m.$$

Let  $1 \leq j \leq m$  and  $j \neq i$ . Then it follows from (3.75)<sub>1</sub>, (3.68)<sub>1</sub>, (3.76) and (3.40) that

$$\begin{aligned} (3.82) \quad I_{1j}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu-\frac{1}{2}} (1+\alpha+r-t)^{-\frac{1}{2}} d\alpha \int_{-c_j\alpha}^{\alpha} (1+|\beta|)^{-1+\mu} d\beta \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu} (1+\alpha+r-t)^{-\frac{1}{2}} \\ &\quad \times (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{-c_j\alpha}^{\alpha} (1+|\beta|)^{-1+\mu} d\beta. \end{aligned}$$

Hence we have easily

(3.83)

$$\begin{aligned} I_{1j}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+\alpha+r-t)^{-\frac{1}{2}} d\alpha \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} (1+\alpha+r-t)^{-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha. \end{aligned}$$

Therefore we obtain (3.80) for  $I_{1j}$ , making use of (3.49) and (3.52).

Analogously to (3.82) we have from (3.75)<sub>2</sub>

(3.84)

$$\begin{aligned} I_{2j}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu-\frac{1}{2}} d\alpha \int_{\beta_j}^{\alpha} (1+\beta-\beta_j)^{-\frac{1}{2}} (1+|\beta|)^{-1+\mu} d\beta \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{-c_j\alpha}^{\alpha} (1+|\beta|)^{-1+\mu} d\beta. \end{aligned}$$

As explained in the proof of (3.79)<sub>2</sub> one can assume that  $0 < \mu < 1/2$ . Applying (3.58) to the  $\beta$ -integral in the first term, we get

$$\begin{aligned} (3.85) \quad I_{2j}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\mu-\frac{1}{2}} (1+|\beta_j|)^{-\frac{1}{2}+\mu} d\alpha \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha. \end{aligned}$$

Therefore we obtain (3.80) for  $I_{2j}$ , making use of (3.50) and (3.53). The proof is complete.  $\square$

Finally we deal with the critical case.

**Lemma 3.18.** Let  $j = i$ , so that  $c_i = 1$ . Then we have

$$(3.86) \quad I_{1i}(t, r) + I_{2i}(t, r) \leq C(1+|r-t|)^{-\rho},$$

provided  $0 < \rho \leq 1$ .

*Proof.* First we consider  $I_{1i}$ . Analogously to (3.82) it follows from (3.68)<sub>2</sub> instead of (3.68)<sub>1</sub> that

$$\begin{aligned}
 (3.87) \quad I_{1i}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\frac{3}{2}-\mu} (1+\alpha+r-t)^{-\frac{1}{2}} d\alpha \int_{-\alpha}^{\alpha} (1+|\beta|)^{-\rho+\mu} d\beta \\
 &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-1-\mu} (1+\alpha+r-t)^{-\frac{1}{2}} \\
 &\quad \times (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{-\alpha}^{\alpha} (1+|\beta|)^{-\rho+\mu} d\beta.
 \end{aligned}$$

Suppose that

$$(3.88) \quad 0 < \rho < 1 + \mu.$$

Then we have easily

$$\begin{aligned}
 I_{1i}(t, x) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+\alpha+r-t)^{-\frac{1}{2}} d\alpha \\
 &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} (1+\alpha+r-t)^{-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha.
 \end{aligned}$$

Therefore we obtain (3.86) for  $I_{1i}$ , making use of (3.49) and (3.52).

Next we consider  $I_{2i}$ . Analogously to (3.84) we have

$$\begin{aligned}
 (3.89) \quad I_{2i}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\frac{3}{2}-\mu} d\alpha \int_{\beta_i}^{\alpha} (1+\beta-\beta_i)^{-\frac{1}{2}} (1+|\beta|)^{-\rho+\mu} d\beta \\
 &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\frac{3}{2}-\mu} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{-\alpha}^{\alpha} (1+|\beta|)^{-\rho+\mu} d\beta \\
 &\equiv A + B,
 \end{aligned}$$

using (3.68)<sub>2</sub> instead of (3.68)<sub>1</sub>. As for the second term we have easily from (3.88)

$$B \leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha.$$

Hence by (3.53) we get (3.86) for  $B$ .

Finally we shall consider the first term  $A$  on the right hand side of (3.89). Note that  $\beta_i = r - t$  according to (3.44) with  $c_i = 1$ . For convenience we set

$$(3.90) \quad \kappa = 1 - \rho + \mu,$$

so that

$$(3.91) \quad A = C \int_{|r-t|}^{r+t} (1+\alpha)^{-\frac{3}{2}-\mu} d\alpha \int_{\beta_i}^{\alpha} (1+\beta-\beta_i)^{-\frac{1}{2}} (1+|\beta|)^{-1+\kappa} d\beta.$$

Here we shall apply (3.58) to the  $\beta$ -integral. First suppose that  $0 < \rho \leq 1/2$ . Then (3.90) yields

$$\kappa \geq \frac{1}{2} + \mu > \frac{1}{2},$$

hence it follows from the latter inequality of (3.58) that

$$A \leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-1} d\alpha.$$

Therefore we obtain (3.86) for  $A$ .

Next suppose that

$$(3.92) \quad \frac{1}{2} < \rho \leq 1$$

so that

$$\mu \leq \kappa < \frac{1}{2} + \mu$$

according to (3.90). Then one can assume that  $0 < \mu < \rho - (1/2)$  in (3.89) as remarked in the proof of (3.85). Hence (3.90) and (3.88) yield  $0 < \kappa < 1/2$ . Thus we get

$$A \leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\frac{3}{2}-\mu} (1+|\beta_i|)^{\frac{1}{2}-\rho+\mu} d\alpha,$$

making use of (3.58) and (3.91).

Since  $\beta_i = r - t$ , we thus obtain (3.86) for  $A$ . The proof is complete.  $\square$

*End of proof of Theorem 3.4.* We have (3.28) and (3.31)<sub>1</sub> from Lemma 3.8. Let  $0 < \rho \leq 1$  and  $\mu > 0$ . Then from Lemmas 3.9 and 3.14 through 3.18 we obtain (3.61), which yields (3.29) and (3.31)<sub>2</sub>. Thus we prove the theorem.  $\square$

*Proof of Theorem 3.7.* In view of the proof of Theorem 3.4 we have only to modify the calculation over  $\Lambda_k$ , where  $1 \leq k \leq m$  and  $k \neq i$ .

First we shall prove (3.34)<sub>1</sub>. To this end we refer to the proof of Lemma 3.8. Replacing  $z_\mu(s, |y|)$  in (3.37) by  $\tilde{z}_\mu^k(s, |y|)$ , we have

$$I_k(t, r) \leq \frac{1}{2r} \int_{|r-t|}^{r+t} (1+\alpha)^{-1-\nu} d\alpha \int_{-c_k\alpha}^{\alpha} (1+|\beta|)^{-1-\mu} d\beta$$

analogously to the first inequality of (3.42). Since  $\mu > 0$ , we see that the conclusion of (3.42) is still valid. Hence we obtain (3.34)<sub>1</sub>.

Next we shall prove (3.34)<sub>2</sub>. In view of (3.34)<sub>3</sub> one can replace (3.38)<sub>k</sub>' in the proof of (3.31)<sub>1</sub> by

$$I_k(t, r) \leq \frac{1}{2r} \iint_{\tilde{D}^i \cap \Lambda_k} (1+\lambda+s)^{-\rho} (1+|\lambda-c_k s|)^{-1-\mu} ds d\lambda.$$

Therefore (3.42)' is still valid, hence we obtain (3.34)<sub>2</sub>.

Finally we shall prove (3.35)<sub>1</sub>. It is sufficient to do this for  $\nu = 0$ , since  $(1+s+|y|)^\nu \geq (1+|r-t|)^\nu$  for  $\nu \geq 0$  and  $(s, y) \in D^i(t, r)$ . When  $0 < r \leq 1$ , we get (3.35)<sub>1</sub> analogously to the proof of Lemma 3.9, making use of (3.34)<sub>1</sub> and (3.34)<sub>2</sub> instead of (3.28) with  $\nu = 0$  and (3.31)<sub>1</sub> respectively.

In what follows we suppose that (3.43) holds. Then it suffices to show that (3.61) is still valid, even if  $z_{\mu, \rho}^i$  is replaced by  $\tilde{z}_{\mu, \rho}^k$ . To this end we have only to modify the proofs of Lemmas 3.15 and 3.17, where (3.68)<sub>1</sub> changes into

$$(3.93) \quad z(s, \lambda) = (1+\alpha)^\rho (1+|\beta|)^{1+\mu} \quad \text{for } (s, \lambda) \in \Lambda_k.$$

First we consider Lemma 3.15. Then (3.73)<sub>1</sub> is replaced by

$$\begin{aligned} A_1 &= \int_{|r-t|}^{r+t} (1+\lambda)^{-\rho-1-\mu} d\lambda \\ &\leq C(1+|r-t|)^{-\rho-\mu}. \end{aligned}$$

Moreover (3.73)<sub>3</sub> changes into

$$\begin{aligned} A_3 &= (1+|r-t|)^{-\rho} \int_{-c_k|r-t|}^{r+t} (1+|\beta|)^{-1-\mu} d\beta \\ &\leq C(1+|r-t|)^{-\rho}. \end{aligned}$$

Analogously we see that the conclusion of (3.73)<sub>2</sub> is still valid. As for the last integral in (3.72) it follows from (3.93) that

$$\begin{aligned} A_4 &= \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho} (1+|\beta_k|)^{-1-\mu} d\alpha \\ &\leq (1+|r-t|)^{-\rho} \int_{|r-t|}^{r+t} (1+|\beta_k|)^{-1-\mu} d\alpha. \end{aligned}$$

Since  $c_k \neq c_i = 1$ , we see from (3.44) that

$$A_4 \leq C(1+|r-t|)^{-\rho}.$$

Thus we see that Lemma 3.15 is true with (3.68)<sub>1</sub> replaced by (3.93).

Finally we consider Lemma 3.17. Then  $(1+\alpha)^{-\rho-\mu}$  and  $(1+|\beta|)^{-1+\mu}$  in (3.82) are replaced by  $(1+\alpha)^{-\rho}$  and  $(1+|\beta|)^{-1-\mu}$  respectively. Hence we have (3.83). Moreover (3.84) changes into

$$\begin{aligned} I_{2k}(t, r) &\leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} d\alpha \int_{\beta_k}^{\alpha} (1+\beta-\beta_k)^{-\frac{1}{2}} (1+|\beta|)^{-1-\mu} d\beta \\ &\quad + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha \int_{-c_k\alpha}^{\alpha} (1+|\beta|)^{-1-\mu} d\beta. \end{aligned}$$

Applying (3.58) with  $\kappa = -\mu$  to the  $\beta$ -integral in the first term, we see that (3.85) changes into

$$\begin{aligned} I_{2k}(t, r) \leq C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+|\beta_k|)^{-\frac{1}{2}} d\alpha \\ + C \int_{|r-t|}^{r+t} (1+\alpha)^{-\rho-\frac{1}{2}} (1+r+t-\alpha)^{-\frac{1}{2}} d\alpha. \end{aligned}$$

Therefore we obtain (3.80) for  $I_{2k}$ , making use of (3.50) with  $\mu = 0$  and (3.53). Consequently we see that Lemma 3.17 is true with  $(3.68)_1$  replaced by (3.93). Thus we prove the theorem.  $\square$

#### §4. Existence of local solutions and continuous dependence on initial data of the solutions.

In this section, to the Cauchy problem (1.1)-(1.2) we apply the results of Kato [12] on local existence of solutions to quasilinear symmetric hyperbolic systems of the first order. To this end we shall regard a solution  $u = u(t, x)$  of (1.1) as a function of  $t \geq 0$  with values in the Sobolev space  $H^k = H^k(\mathbb{R}^3)$  of  $L^2$ -type, where  $k$  is a positive integer with  $k \geq 4$ , and write  $u(t)$  for  $u(t, \cdot)$ . We also replace the initial conditions (1.2) by

$$(4.1) \quad u(0) = \varphi_0(\varepsilon), \quad \partial_t u(0) = \varphi_1(\varepsilon),$$

where  $\varepsilon > 0$  is a parameter. Then the following theorem is a special case of the results of Kato [12].

**Theorem 4.1.** Assume that conditions (1.4) and (1.5) hold. Let  $\varphi_0(\varepsilon) \in H^k$  and  $\varphi_1(\varepsilon) \in H^{k-1}$  for  $\varepsilon > 0$ , where  $k$  is a positive integer with  $k \geq 4$ . Suppose that

$$(4.2) \quad \lim_{\varepsilon \rightarrow 0} (\|\varphi_0(\varepsilon)\|_{H^k} + \|\varphi_1(\varepsilon)\|_{H^{k-1}}) = 0.$$

Then there are positive constants  $\varepsilon_k$ ,  $\omega_k$  and  $T_k$  such that for any  $\varepsilon \in (0, \varepsilon_k]$  there exists a unique solution  $u(t) = u(t; \varepsilon)$  of the Cauchy problem (1.1)-(4.1) with  $0 \leq t \leq T_k$  satisfying

$$(4.3) \quad u(t) \in C([0, T_k]; H^k) \cap C^1([0, T_k]; H^{k-1}) \cap C^2([0, T_k]; H^{k-2})$$

and

$$(4.4) \quad \|u(t)\|_{H^k} + \|\partial_t u(t)\|_{H^{k-1}} < \omega_k \quad \text{for } t \in [0, T_k].$$

Here  $\varepsilon_k$ ,  $\omega_k$  and  $T_k$  depend only on the derivatives of  $A_{ij}^{ab}(u, u')$  and  $B^i(u, u')$  with respect to  $(u, u')$  of order  $\leq k$  for  $i, j = 1, 2, \dots, m$  and  $a, b = 0, 1, 2, 3$ . Moreover we have

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq T_k} (\|u(t)\|_{H^k} + \|\partial_t u(t)\|_{H^{k-1}}) = 0.$$

*Proof.* First of all we shall modify the  $A_{ij}^{ab}(u, u')$  and  $B^i(u, u')$  in (1.4)<sub>1</sub> outside a neighborhood of  $(u, u') = (0, 0)$  so that conditions (1.4)<sub>2</sub> and (1.5) remain holding as follows. Let  $\chi_\delta(u, u') \in C_0^\infty(\mathbb{R}^{5p})$  be a cutoff function such that  $0 \leq \chi_\delta(u, u') \leq 1$ ,  $\chi_\delta(u, u') = 1$  for  $|u| + |u'| \leq \delta$  and  $\chi_\delta(u, u') = 0$  for  $|u| + |u'| \geq 2\delta$ , where  $\delta$  is a small positive number which will be fixed later. Then we set

$$(4.6) \quad \tilde{A}_{ij}^{ab}(u, u') = \chi_\delta(u, u') A_{ij}^{ab}(u, u'), \quad \tilde{B}^i(u, u') = \chi_\delta(u, u') B^i(u, u') \\ \text{for } i, j = 1, 2, \dots, m \quad \text{and} \quad a, b = 0, 1, 2, 3.$$

Clearly it suffices to prove the theorem for such modified equations (1.1), because of (4.5). Thus (dropping the tildas for notational convenience) one can assume that

$$(4.7) \quad A_{ij}^{ab}(u, u'), B^i(u, u') \in C_0^\infty(\mathbb{R}^{5p}).$$

Next we shall write (1.1) in the form of a symmetric hyperbolic system. To this end we set

$$(4.8) \quad A^{ab}(u, u') = \begin{bmatrix} A_{11}^{ab}(u, u') & \cdots & A_{1m}^{ab}(u, u') \\ \vdots & & \vdots \\ A_{m1}^{ab}(u, u') & \cdots & A_{mm}^{ab}(u, u') \end{bmatrix} \quad \text{for } a, b = 0, 1, 2, 3.$$

Then it follows from (1.5) that  $A^{ab}(u, u')$  are symmetric  $p \times p$  matrices. In addition, (1.4)<sub>2</sub> implies  $A^{ba} = A^{ab}$ . Moreover (1.1) can be written as

$$(4.9) \quad (I_p - A^{00}(u, u'))\partial_t^2 u \\ = \sum_{j=1}^3 \{2A^{0j}(u, u')\partial_j \partial_t u + \sum_{k=1}^3 (\delta_{jk} \Lambda + A^{jk}(u, u'))\partial_j \partial_k u\} + B(u, u'),$$

where  $I_p$  stands for the unit matrix of order  $p$  and we have set

$$(4.10) \quad \Lambda = \begin{bmatrix} c_1^2 I_{p_1} & & & 0 \\ & c_2^2 I_{p_2} & & \\ & & \ddots & \\ 0 & & & c_m^2 I_{p_m} \end{bmatrix}$$

and

$$(4.11) \quad B(u, u') = \begin{bmatrix} B^1(u, u') \\ B^2(u, u') \\ \vdots \\ B^m(u, u') \end{bmatrix}.$$

We also take the  $\delta$  in (4.6) so small as

$$(4.12) \quad I_p - A^{00}(u, u') \geq \frac{1}{2} I_p \quad \text{for } (u, u') \in \mathbb{R}^{5p}.$$

This is possible, since  $A^{00}(0, 0) = 0$  according to (1.4)<sub>3</sub>. Furthermore we put

$$(4.13) \quad v = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \partial_t u \\ \partial_1 u \\ \partial_2 u \\ \partial_3 u \end{bmatrix} = u',$$

$$(4.14) \quad a_0(u, u') = \begin{bmatrix} I_p - A^{00}(u, u') & 0 \\ 0 & I_{3p} \end{bmatrix}$$

and for  $j = 1, 2, 3$

$$(4.15) \quad a_j(u, u') = \begin{bmatrix} 2A^{0j}(u, u') & \delta_{j1}\Lambda + A^{j1}(u, u') & \delta_{j2}\Lambda + A^{j2}(u, u') & \delta_{j3}\Lambda + A^{j3}(u, u') \\ \delta_{j1} I_p & & & \\ \delta_{j2} I_p & 0 & & \\ \delta_{j3} I_p & & & \end{bmatrix}.$$

Then one can write (4.9) as

$$(4.16) \quad a_0(u, u') \partial_t v = \sum_{j=1}^3 a_j(u, u') \partial_j v + b(u, u').$$

where

$$b(u, u') = \begin{bmatrix} B(u, u') \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order to symmetrize the system (4.16) we set

$$(4.17) \quad a(u, u') = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & \Lambda + A^{11}(u, u') & A^{12}(u, u') & A^{13}(u, u') \\ 0 & A^{21}(u, u') & \Lambda + A^{22}(u, u') & A^{23}(u, u') \\ 0 & A^{31}(u, u') & A^{32}(u, u') & \Lambda + A^{33}(u, u') \end{bmatrix}.$$

Then we see from (4.14) and (4.15) that

$$(4.18) \quad a(u, u') a_j(u, u') \quad \text{are symmetric for } j = 0, 1, 2, 3.$$

Moreover we take the  $\delta$  in (4.6) so small as

$$(4.19) \quad a(u, u') a_0(u, u') \geq \tilde{C} I_{4p} \quad \text{for } (u, u') \in \mathbb{R}^{5p},$$

where

$$\tilde{C} = \frac{1}{2} \min \{1, c_1^2\}.$$

This is possible, because

$$\begin{aligned} a(0, 0) a_0(0, 0) &= a(0, 0) \\ &\geq 2\tilde{C} I_{4p} \end{aligned}$$

according to (4.10), (4.14) and (4.17).

Finally we put  $w = \begin{bmatrix} u \\ v \end{bmatrix}$ . Then we have from (4.16) with (4.13)

$$(4.20) \quad \begin{bmatrix} I_p & 0 \\ 0 & aa_0 \end{bmatrix} \partial_t w = \sum_{j=1}^3 \begin{bmatrix} 0 & 0 \\ 0 & aa_j \end{bmatrix} \partial_j w + \begin{bmatrix} v_0 \\ ab \end{bmatrix},$$

which is a symmetric hyperbolic system with respect to the unknown  $w$ , because of (4.18) and (4.19).

In what follows we fix the  $\delta$  in (4.6) in such a way that both (4.12) and (4.19) hold. Then we shall apply Theorems II, III and IV in Kato [12] to the modified system (4.20). In Theorem IV with  $s = k - 1$  and  $k \geq 4$ , we take  $v_0(x) = 0$ ,  $f_0(t, x) = 0$ ,

$$\Omega = \{(x, w) \in \mathbb{R}^3 \times \mathbb{R}^{5p}; |w| < 1\}$$

and

$$D = \{(u, v) \in H^k \times H^{k-1}; \|u\|_{H^k} + \|v\|_{H^{k-1}} < \omega_k\}$$

with small  $\omega_k > 0$ . Then the hypotheses (5.1), (5.2) with  $s = k$ , (5.3) and (5.4) are fulfilled. Therefore by virtue of Theorem IV we find that the conditions (4.2) through (4.6) in Theorem II and (4.11) in Theorem III hold. Moreover our equation (1.1) has a trivial solution according to (1.3)<sub>1</sub>. Therefore, taking  $u = 0$  in Theorem III, we obtain our Theorem 4.1.

Finally, taking  $\varepsilon_k$  in Theorem 4.1 small, we find from (4.5) and Sobolev's inequality that the solution  $u(t, x)$  of (1.1)-(4.1) satisfies

$$|u(t, x)| + |u'(t, x)| \leq \delta \quad \text{for } (t, x) \in [0, T_k] \times \mathbb{R}^3,$$

where  $\delta$  is the constant in (4.6), hence  $u(t, x)$  is a solution of the original equation (1.1). The proof is complete.  $\square$

For the smoothness of the solution we have the following theorem which is an immediate consequence of Majda [20], Theorem 2.2 or its Corollary 1.

**Theorem 4.2.** Assume that conditions (4.7) and (4.19) as well as (1.4) and (1.5) hold. Let  $u(t, x) \in C^2([0, T] \times \mathbb{R}^3)$  be a solution of the Cauchy problem (1.1)-(4.1) with  $0 \leq t \leq T$ , where  $T$  is a positive number. Suppose that  $\varphi_0(\varepsilon)$  and  $\varphi_1(\varepsilon)$  belong to  $H^k$  for all  $k \geq 4$ . Then we have  $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$ .

## §5. Weighted $L^\infty - L^2$ estimates.

In this section we derive certain weighted  $L^\infty - L^2$  estimates for local solutions of the nonlinear wave equations (1.1). A goal of the present section is Corollary 5.15 below.

We consider the following Cauchy problem

(5.1)

$$\square_i u^i = F^i(u, \partial u, \partial^2 u) \quad \text{in } (0, T) \times \mathbb{R}^3, \quad i = 1, 2, \dots, m,$$

(5.2)

$$u^i(0, x) = \varepsilon f^i(x), \quad \partial_t u^i(0, x) = \varepsilon g^i(x) \quad \text{for } x \in \mathbb{R}^3, \quad i = 1, 2, \dots, m,$$

where  $\square_i$ ,  $F^i$ ,  $\varepsilon$ ,  $f^i$  and  $g^i$  are as in (1.1) or (1.2), and  $T$  is a positive number. Throughout the present section we assume that the conditions (1.4), (1.5), (4.7), (4.12) and (4.19) are fulfilled. Then we see from Theorem 4.1 and 4.2 that the Cauchy problem (5.1)-(5.2) has a solution  $u(t, x) \in C^\infty([0, T) \times \mathbb{R}^3)$  with  $0 < T_4 \leq T$ , provided  $0 < \varepsilon \leq \varepsilon_4$ , where  $\varepsilon_4$  and  $T_4$  are the constants in Theorem 4.1 with  $k = 4$ .

The following lemma on the domain of dependence is well known.

**Lemma 5.1.** Let  $u(t, x) \in C^2([0, T) \times \mathbb{R}^3)$  be a solution of (5.1)-(5.2), where  $u(t, x) = (u^1(t, x), u^2(t, x), \dots, u^m(t, x))$ . Suppose that

$$(5.3) \quad f^i(x) = g^i(x) = 0 \quad \text{for } |x| \geq L, \quad i = 1, 2, \dots, m$$

with some positive number  $L$ . Then we have

$$(5.4) \quad u(t, x) = 0 \quad \text{for } |x| \geq L + c_m t, \quad 0 \leq t < T,$$

where  $c_m$  is the constant in (1.1). Moreover suppose that hypothesis (A)<sub>2</sub> is satisfied. Then we have

$$(5.5) \quad u^i(t, x) = 0 \quad \text{for } |x| \geq L + c_i t, \quad 0 \leq t < T, \quad i = 1, 2, \dots, m.$$

For the proof see for instance John [7], Appendix or [10], Appendix 1.

The following forms of the Sobolev inequality will play a basic role in deriving the  $L^\infty - L^2$  estimates.

**Lemma 5.2.** Let  $v(x) \in C_0^\infty(\mathbb{R}^3)$ . Then we have for  $x \in \mathbb{R}^3$

$$(5.6) \quad |x| \cdot |v(x)|^2 \leq C \sum_{|\alpha| \leq 2} \|\Omega^\alpha \partial_x v\|_{L^2(\mathbb{R}^3)}^2$$

and

$$(5.7) \quad |x|^2 |v(x)|^2 \leq C \sum_{|\alpha| \leq 2} \|\Omega^\alpha v\|_{L^2(\mathbb{R}^3)}^2,$$

where  $\Omega$  is defined by (2.4) and  $C$  a constant independent of  $x$  and  $v$ .

*Proof.* The estimate (5.7) is a special case of Lemma 4.2 in Klainerman and Sideris [15], and (5.6) follows from the proof of the (i) of (2.1) in Klainerman [14].  $\square$

In what follows we set

$$(5.8)_1 \quad \|u(t)\|_k = \left\{ \int_{\mathbb{R}^3} |u(t, x)|_k^2 dx \right\}^{1/2}$$

and

$$(5.8)_2 \quad \|u\|_{k,t} = \sup_{0 \leq s \leq t} \|u(s)\|_k$$

for a nonnegative integer  $k$  and  $t \in [0, T]$ , where we have used the notation (2.9). Then from Lemmas 5.1 and 5.2 we have easily

**Corollary 5.3.** Let  $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  be a solution of (5.1)-(5.2). Then for a nonnegative integer  $k$  and  $(t, x) \in [0, T] \times \mathbb{R}^3$  we have

$$(5.9) \quad \sqrt{r} |u(t, x)|_k \leq C \|\partial u(t)\|_{k+2}$$

and

$$(5.10) \quad r |\partial u(t, x)|_k \leq C \|\partial u(t)\|_{k+2},$$

where  $r = |x|$  and  $C$  is a constant depending only on  $k$ .

In the course of calculation below we will often make use of the following inequalities.

**Lemma 5.4.** Let  $i = 1, 2, \dots, m$  and let  $u^i(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  be a solution of (5.1)-(5.2). Let  $(s, y) \in [0, T] \times \mathbb{R}^3$  and  $(s, \lambda) \in \Lambda_j$  ( $j = 0, 1, 2, \dots, m$ ) with  $\lambda = |y|$ , where  $\Lambda_j$  are defined by (2.2)<sub>i</sub> or (2.2)<sub>0</sub>. Let  $k$  be a nonnegative integer and  $\nu \geq 0$ . Then we have

(5.11)

$$|u^i(s, y)|_k \leq C(1 + \lambda + s)^{-1-\nu} \langle u^i(s, y) \rangle_{\nu, k} \quad \text{if } j \neq i,$$

(5.12)

$$|u^i(s, y)|_k \leq (1 + \lambda + s)^{-1} (1 + |\lambda - c_i s|)^{-\nu} \langle u^i(s, y) \rangle_{\nu, k} \quad \text{if } j = i \quad \text{and } \nu > 0$$

and

$$(5.13) \quad |u^i(s, y)|_k \leq C(1 + \lambda + s)^{-1+\delta} (1 + |\lambda - c_i s|)^{-\delta} \langle u^i(s, y) \rangle_{0, k} \\ \text{if } j = i \quad \text{and } 0 < \delta < 1.$$

Moreover we have

(5.14)

$$|\partial u^i(s, y)|_k \leq C(1 + \lambda + s)^{-2-\nu} [\partial u^i(s, y)]_{1+\nu, k} \quad \text{if } 1 \leq j \leq m \quad \text{and } j \neq i,$$

(5.15)

$$|\partial u^i(s, y)|_k \leq C(1 + \lambda + s)^{-1-\nu} (1 + \lambda)^{-1} [\partial u^i(s, y)]_{1+\nu, k} \quad \text{if } j = 0$$

and

$$(5.16) \quad |\partial u^i(s, y)|_k \leq C(1 + \lambda + s)^{-1} (1 + |\lambda - c_i s|)^{-1-\nu} [\partial u^i(s, y)]_{1+\nu, k} \quad \text{if } j = i.$$

Here  $C$  are constants depending only on the  $c_1, c_2, \dots, c_m$ , the  $L$  in (5.3) and  $\delta$ , and we have used the notations (2.13) and (2.14).

*Proof.* The estimates (5.12) follows immediately from (2.10)<sub>1</sub>. Next we shall prove (5.11). If  $1 \leq j \leq i - 1$ , then it follows from (2.1) and (2.2)<sub>j</sub> that

$$\lambda \leq c_j s + \frac{1}{3}(c_{j+1} - c_j)s \\ \leq c_j s + \frac{1}{3}(c_i - c_j)s.$$

Hence

$$\begin{aligned} |\lambda - c_i s| &= c_i s - \lambda \\ &\geq \frac{2}{3}(c_i - c_j)s. \end{aligned}$$

Therefore we obtain (5.11) by (2.3). If  $i + 1 \leq j \leq m$ , we have as above

$$\lambda - c_i s \geq \frac{2}{3}(c_j - c_i)s,$$

hence (5.11) follows. Similarly one can prove the estimate when  $j = 0$ , because it follows from Lemma 5.1 that (5.4) holds.

To prove (5.13) we note that  $1 + \log x \leq C x^\delta$  for  $x \geq 1$ , where  $C$  is a constant depending on  $\delta$ . Hence we obtain (5.13) from (2.10)<sub>2</sub>. The proofs of (5.14), (5.15) and (5.16) are similar to that of (5.11).  $\square$

Now, we shall return to the Cauchy problem (5.1)-(5.2). In what follows we assume that  $u^i(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  is a solution of (5.1)-(5.2), where  $i = 1, 2, \dots, m$ . Set

$$u_0^i(t, x) = u^i(t, x) - L_i(F^i(u, \partial u, \partial^2 u))(t, x),$$

where  $L_i$  is the linear operator in (3.22). Then we see that  $u_0^i(t, x)$  is a solution of the homogeneous wave equation  $\square_i u^i = 0$  satisfying the initial conditions (5.2). Moreover we find from (2.6) and (2.8) that  $u_\alpha^i(t, x) \equiv \Gamma^\alpha u_0^i(t, x)$  is also a solution of  $\square_i u^i = 0$  satisfying initial conditions of the following forms

$$u_\alpha^i(0, x) = \varepsilon \left( \sum_{|a| \leq |\alpha|} C_a \Gamma^a f^i(x) + \sum_{|b| \leq |\alpha|} D_b \Gamma^b g^i(x) \right)$$

and

$$\partial_t u_\alpha^i(0, x) = \varepsilon \left( \sum_{|a| \leq |\alpha|+1} C'_a \Gamma^a f^i(x) + \sum_{|b| \leq |\alpha|+1} D'_b \Gamma^b g^i(x) \right),$$

where  $C_a, D_b, C'_a$  and  $D'_b$  are constants. Making use of Proposition 3.3 we have therefore

$$(5.17)_1 \quad \langle u^i(t, x) \rangle_{\nu, k} \leq C\varepsilon + \langle L_i(F^i(u, \partial u, \partial^2 u))(t, x) \rangle_{\nu, k}$$

and

$$(5.17)_2 \quad [\partial u^i(t, x)]_{\rho, k} \leq C\varepsilon + [\partial L_i(F^i(u, \partial u, \partial^2 u))(t, x)]_{\rho, k} \\ \text{for } (t, x) \in [0, T] \times \mathbb{R}^3, \quad i = 1, 2, \dots, m.$$

Here  $k$  is a nonnegative integer,  $\nu \geq 0$ ,  $\rho > 0$  and  $C$  is a constant depending only on  $f^i$ ,  $g^i$ ,  $\nu$ ,  $\rho$ ,  $k$  and the  $c_1, c_2, \dots, c_m$ . Thus we must estimate the second terms on the right hand sides of (5.17)<sub>1</sub> and (5.17)<sub>2</sub>.

From now on we assume that conditions (1.9)<sub>2</sub> and (1.10) also hold. Then one can write

$$(5.18) \quad F^i(u, \partial u, \partial^2 u) = N^{i0}(\partial u^i, \partial^2 u^i) + R^i(\partial u, \partial^2 u) + G^i(u) + H^i(u, \partial u, \partial^2 u),$$

where  $N^{i0}(\partial u^i, \partial^2 u^i) = N^i(0, \partial u^i, \partial^2 u^i)$  with the  $N^i$  in (1.7),  $G^i(u)$  is a homogeneous polynomial in  $u$  of degree 3, and  $H^i(u, \partial u, \partial^2 u)$  satisfies the following estimate

$$(5.19) \quad |H^i(u, u', u'')| \leq C\{|u|^2(|u'| + |u''|) + (|u| + |u'|)|u'|(|u'| + |u''|) \\ + (|u| + |u'|)^3(|u| + |u'| + |u''|)\} \quad \text{for } (u, u', u'') \in \mathbb{R}^{21p}.$$

Indeed, it follows from (1.9)<sub>2</sub> and (5.1) that

$$|N^i(u^i, \partial u^i, \partial^2 u^i) - N^{i0}(\partial u^i, \partial^2 u^i)| \leq C|u| \cdot |F^i(u, u', u'')|,$$

hence we get (5.18). For convenience we shall abbreviate (5.18) as

$$(5.20) \quad F^i(t, x) = N^{i0}(t, x) + R^i(t, x) + G^i(t, x) + H^i(t, x),$$

unless otherwise stated. Namely we write  $F^i(t, x) = F^i(u(t, x), \partial u(t, x), \partial^2 u(t, x))$  and so on. Then we find from Corollary 3.6 that

$$(5.21)_1 \quad \langle L_i(F^i)(t, x) \rangle_{\nu, k} (1 + t + r)^{-\rho} \leq \tilde{C} \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} M_{\nu-\rho, k}^\mu(F^i)(s, y) + C\varepsilon$$

and

$$(5.21)_2 \quad \begin{aligned} & [\partial L_i(F^i)(t, x)]_{1+\nu, k} (1+t+r)^{-\rho} \\ & \leq \tilde{C} \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} M_{\nu-\rho, k+1}^\mu(F^i)(s, y) + C\varepsilon \end{aligned}$$

hold for  $(t, x) \in [0, T) \times \mathbb{R}^3$ ,  $0 < \varepsilon \leq 1$  and  $i = 1, 2, \dots, m$ , where  $r = |x|$  and we have set

$$(5.22) \quad M_{\alpha, k}^\mu(F^i)(s, y) = \lambda z_\mu(s, \lambda) (1+s+\lambda)^\alpha |F^i(s, y)|_k$$

with  $\lambda = |y|$ . Here  $\nu \geq 0$ ,  $\rho \geq 0$ ,  $\mu > 0$ ,  $k$  is a nonnegative integer, and  $\tilde{C}$  the same constant as the  $C$  in Corollary 3.6. Indeed, we see from (4.9) together with (4.12) that  $\Gamma^\alpha F^i(u(s, y), \partial u(s, y), \partial^2 u(s, y))$  for  $s = 0$  can be represented in terms of  $\varepsilon f^i(y)$ ,  $\varepsilon g^i(y)$  and their derivatives. Hence we obtain (5.21).

We are now in a position to estimate the last terms of (5.17)<sub>1</sub> and (5.17)<sub>2</sub>. After all we will reduce all the  $L^\infty - L^2$  estimates to the following easy estimates by making use of Theorems 3.4, 3.7 and Lemma 5.13 below.

**Lemma 5.5.** Let  $i, k, \lambda$  and  $(s, y)$  be as in the preceding lemma. Let  $0 \leq s \leq t < T$ .

1) If  $\rho > 0$  and  $0 < \mu \leq \rho$ , we have

$$(5.23) \quad M_{-\rho, k}^\mu(N^{i0})(s, y) + M_{-\rho, k}^\mu(R^i)(s, y) \leq C [\partial u]_{1, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},$$

where we have used the notations (2.15), (2.16), (2.17), (5.8) and (5.22).

2) If  $\rho \geq 1/2$  and  $0 < \mu < 1$ , we have

$$(5.24) \quad M_{-\rho, k}^\mu(G^i)(s, y) \leq C \langle u \rangle_{0, [\frac{k}{2}], t}^2 \|\partial u\|_{k+2, t}.$$

3) If  $0 \leq \nu < 1/2$  and  $0 < \mu < (1/2) - \nu$ , we have

$$(5.25) \quad M_{\nu, k}^\mu(H^i)(s, y) \leq C (\langle u \rangle_{\nu, [\frac{k+1}{2}], t} + [\partial u]_{1, [\frac{k+1}{2}], t}) \|\partial u\|_{k+3, t},$$

provided  $k \geq 1$  and

$$(5.26)_k \quad \langle u \rangle_{0, [\frac{k}{2}], t} + [\partial u]_{1, [\frac{k}{2}], t} \leq 1 \quad \text{for } 0 \leq t < T.$$

Here  $C$  are constants independent of  $s, y, t, T$  and  $u$ .

*Proof.* First we shall prove (5.23) with  $N^{i0} = 0$ . It follows from (5.22) and (3.27) that

$$(5.27) \quad M_{-\rho, k}^\mu(R^i)(s, y) = \lambda(1 + \lambda + s)^{1+\mu-\rho}(1 + |\lambda - c_j s|)^{1-\mu} |R^i(s, y)|_k.$$

Since  $R^i(u', u'')$  is a homogeneous polynomial in  $(u', u'')$  of degree 2, being linear in  $u''$ , we have

$$(5.28) \quad |R^i(s, y)|_k \leq C |\partial u(s, y)|_{[\frac{k+1}{2}]} |\partial u(s, y)|_{k+1},$$

using Leibnitz's formula and noting that  $\partial_a \partial_b u = \Gamma_a \partial_b u$  for  $a = 0, 1, 2, 3$ . Therefore it follows from (5.10), Lemma 5.4 with  $\nu = 0$  and (5.27) that

$$M_{-\rho, k}^\mu(R^i)(s, y) \leq C(1 + \lambda + s)^{\mu-\rho}(1 + |\lambda - c_j s|)^{-\mu} [\partial u]_{1, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},$$

which implies (5.23), provided  $0 < \mu \leq \rho$  and  $N^{i0} = 0$ . Analogously we obtain (5.23).

Next we shall prove (5.24). Since  $G^i(u)$  is a homogeneous polynomial in  $u$  of degree 3, we have

$$(5.29) \quad |G^i(s, y)|_k \leq C |u(s, y)|_{[\frac{k}{2}]}^2 |u(s, y)|_k.$$

Hence it follows from (5.9), (5.11), (5.13) and (5.27) with  $R^i$  replaced by  $G^i$  that

$$\begin{aligned} M_{-\rho, k}^\mu(G^i)(s, y) &\leq C \langle u \rangle_{0, [\frac{k}{2}], t}^2 \|\partial u\|_{k+2, t} (1 + \lambda + s)^{-\rho - \frac{1}{2} + \mu + 2\delta} \\ &\quad \times (1 + |\lambda - c_j s|)^{1-\mu-2\delta}, \end{aligned}$$

which yields (5.24), provided  $\rho \geq 1/2$ ,  $0 < \mu < 1$  and  $2\delta = 1 - \mu$ .

Finally we shall prove (5.25). We deal with only the first term on the right hand side of (5.19), since the others can be handled less hard. Hence we assume that

$$(5.30) \quad |H^i(u, u', u'')| \leq C |u|^2 (|u'| + |u''|) \quad \text{for } (u, u', u'') \in \mathbb{R}^{5p}.$$

Then we have similarly to (5.28)

$$\begin{aligned}
 (5.31) \quad & |H^i(s, y)|_k \\
 & \leq C(|u(s, y)|_{[\frac{k}{2}]} |\partial u(s, y)|_{[\frac{k+1}{2}]} + |\partial u(s, y)|_{[\frac{k}{2}]} |u(s, y)|_{[\frac{k+1}{2}]}) |u(s, y)|_k \\
 & \quad + C|u(s, y)|_{[\frac{k}{2}]} |u(s, y)|_{[\frac{k+1}{2}]} |\partial u(s, y)|_{k+1} \\
 & \equiv H_1(s, y) + H_2(s, y).
 \end{aligned}$$

First suppose that  $\nu = 0$ . For the first term  $H_1$  it follows from (5.9) and Lemma 5.4 that

$$\begin{aligned}
 \sqrt{\lambda} H_1(s, y) & \leq C(1 + \lambda + s)^{-2+2\delta} (1 + |\lambda - c_j s|)^{-1-\delta} (\langle u \rangle_{0, [\frac{k}{2}], t} [\partial u]_{1, [\frac{k+1}{2}], t} \\
 & \quad + [\partial u]_{1, [\frac{k}{2}], t} \langle u \rangle_{0, [\frac{k+1}{2}], t}) \|\partial u\|_{k+2, t}.
 \end{aligned}$$

As for the second term  $H_2$  we have

$$(5.32) \quad \lambda H_2(s, y) \leq C(1 + \lambda + s)^{-2+2\delta} (1 + |\lambda - c_j s|)^{-2\delta} \langle u \rangle_{0, [\frac{k}{2}], t} \langle u \rangle_{0, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},$$

making use of (5.10) instead of (5.9). Therefore by (5.22) with  $\alpha = 0$  and (5.31) we get

$$\begin{aligned}
 M_{0,k}^\mu(H^i)(s, y) & \leq C M_1(s, \lambda) ([\partial u]_{1, [\frac{k+1}{2}], t} + \langle u \rangle_{0, [\frac{k+1}{2}], t}) \|\partial u\|_{k+2, t} \\
 & \quad + C M_2(s, \lambda) \langle u \rangle_{0, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},
 \end{aligned}$$

provided  $k \geq 0$  and (5.26)<sub>k</sub> holds, where we have set

$$M_1(s, \lambda) = (1 + \lambda + s)^{-\frac{1}{2}+\mu+\delta} (1 + |\lambda - c_j s|)^{-\mu-\delta}$$

and

$$M_2(s, \lambda) = (1 + \lambda + s)^{-1+\mu+2\delta} (1 + |\lambda - c_j s|)^{1-\mu-2\delta}.$$

Therefore we obtain (5.25), taking  $2\delta = 1 - 2\mu$ .

Next suppose that  $0 < \nu < 1/2$ . Then one can replace (5.32) by

$$\begin{aligned}
 \lambda H_2(s, y) & \leq C(1 + \lambda + s)^{-2+\delta} (1 + |\lambda - c_j s|)^{-\nu-\delta} \langle u \rangle_{0, [\frac{k}{2}], t} \\
 & \quad \times \langle u \rangle_{\nu, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},
 \end{aligned}$$

making use of (5.11) and (5.12). Therefore we have as above

$$\begin{aligned} M_{\nu,k}^\mu(H^i)(s,y) &\leq C M_1(s,\lambda)([\partial u]_{1, [\frac{k+1}{2}], t} + \langle u \rangle_{0, [\frac{k+1}{2}], t}) \|\partial u\|_{k+2, t} \\ &+ C M_2(s,\lambda) \langle u \rangle_{\nu, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t}, \end{aligned}$$

provided  $k \geq 0$  and (5.26) $_k$  holds, where

$$M_1(s,\lambda) = (1 + \lambda + s)^{-\frac{1}{2} + \nu + \mu + \delta} (1 + |\lambda - c_j s|)^{-\mu - \delta}$$

and

$$M_2(s,\lambda) = (1 + \lambda + s)^{-1 + \nu + \mu + \delta} (1 + |\lambda - c_j s|)^{1 - \nu - \mu - \delta}.$$

Thus we obtain (5.25), taking  $\delta = (1/2) - \nu - \mu$ . The proof is complete.

**Corollary 5.6.** Let  $i = 1, 2, \dots, m$  and  $k$  be a positive integer. Let  $\rho \geq 1/2$  and  $0 < \varepsilon \leq 1$ . Then for  $(t, x) \in [0, T) \times \mathbb{R}^3$  we have

$$\begin{aligned} (5.33)_1 \quad &\langle u^i(t, x) \rangle_{0,k} (1 + t + r)^{-\rho} \\ &\leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+1}{2}], t} + [\partial u]_{1, [\frac{k+1}{2}], t}) \|\partial u\|_{k+3, t} \end{aligned}$$

and

$$\begin{aligned} (5.33)_2 \quad &[\partial u^i(t, x)]_{1,k} (1 + t + r)^{-\rho} \\ &\leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+2}{2}], t} + [\partial u]_{1, [\frac{k+2}{2}], t}) \|\partial u\|_{k+4, t} \end{aligned}$$

provided (5.26) $_k$  and (5.26) $_{k+1}$  hold respectively, where  $r = |x|$  and  $C$  is a constant depending only on the  $C$  in (5.17), (5.21) and the preceding lemma and the  $\tilde{C}$  in (5.21).

*Proof.* Since (2.2) $_i$  and (2.2) $_0$  imply

$$\bigcup_{j=0}^m \Lambda_j = (0, \infty) \times (0, \infty),$$

we easily obtain (5.33) from (5.17), (5.20), (5.21) and Lemma 5.5.  $\square$

**Remark.** The condition  $(5.26)_k$  holds, provided  $\varepsilon$  and  $T$  are chosen approximately according as  $k$ . More precisely, we have

**Lemma 5.7.** Let  $n$  be a positive integer with  $n \geq 4$ . Let  $\nu \geq 0$  and  $0 < \delta \leq 1$ . Then there is a positive constant  $\tilde{\varepsilon}_n$  such that for any  $\varepsilon \in (0, \tilde{\varepsilon}_n]$  the Cauchy problem (5.1)-(5.2) with  $T = T_n$  has a solution  $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  satisfying

$$(5.34) \quad \langle u \rangle_{\nu, n-2, t} + [\partial u]_{1+\nu, n-3, t} \leq \delta \quad \text{for } t \in [0, T],$$

where  $T_n$  is the constant in Theorem 4.1 with  $k = n$  and  $\tilde{\varepsilon}_n$  depends on  $n, \nu, \delta$  and  $T_n$ .

The lemma is an immediate consequence of Theorems 4.1, 4.2, Lemma 5.1 and Sobolev's inequality.

Next we shall derive more sharp estimates for  $N^{i0}$  and  $R^i$  in (5.20), such as (5.25). Namely we deal with  $M_{\nu, k}^\mu(N^{i0})(s, y)$  and  $M_{\nu, k}^\mu(R^i)(s, y)$  for  $\nu \geq 0$ . From now on we suppose that  $i = 1, 2, \dots, m$ ,  $0 < \varepsilon \leq 1$  and  $k$  is a nonnegative integer and that  $y \in \mathbb{R}^3$  and  $0 \leq s \leq t < T$ . Moreover we assume that

$$(5.35) \quad (s, \lambda) \in \Lambda_j \quad \text{for some } 0 \leq j \leq m,$$

where  $\lambda = |y|$  and  $\Lambda_j$  are given by (2.2)<sub>j</sub> or (2.2)<sub>0</sub>. By  $C$  we also denote the various constants independent of  $(s, y)$ ,  $t$ ,  $T$  and the solution  $u$  of (5.1)-(5.2).

We start with

**Lemma 5.8.** Let  $j = 0$  and  $0 \leq \nu < 1/2$ . Then we have

$$(5.36) \quad \begin{aligned} & M_{\nu, k}^\mu(N^{i0})(s, y) + M_{\nu, k}^\mu(R^i)(s, y) \\ & \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+3}{2}], t} + [\partial u]_{1, [\frac{k+3}{2}], t}) \|\partial u\|_{k+5, t}, \end{aligned}$$

provided  $0 < \mu \leq (1/2) - \nu$  and  $(5.26)_{k+2}$  holds.

*Proof.* It follows from (5.22) and (3.27) that

$$(5.37) \quad M_{\nu, k}^\mu(F^i)(s, y) = \lambda(1 + \lambda + s)^{1+\mu+\nu}(1 + \lambda)^{1-\mu} |F^i(s, y)|_k,$$

since  $(s, \lambda) \in \Lambda_0$ . Hence from (5.28) and (5.15) we have

$$(5.38) \quad M_{\nu, k}^{\mu}(R^i)(s, y) \leq C(1 + \lambda + s)^{-1+\nu+\mu} [\partial u]_{1, [\frac{k+1}{2}], t} [\partial u(s, y)]_{1, k+1}.$$

Applying (5.33)<sub>2</sub> with  $\rho = 1 - \nu - \mu$  to the last factor, we get (5.36) for  $N^{i0} = 0$ . Since  $N^{i0}$  also satisfies an estimate analogous to (5.28), we thus obtain (5.36).  $\square$

The case where  $1 \leq j \leq m$  in (5.35) is rather complicated. We first deal with the null form  $N^{i0}$  and a special resonance form  $\tilde{R}^{i0}$ , the last term in (1.11) which satisfies (1.12).

**Lemma 5.9.** Let  $1 \leq j \leq m$  and  $0 \leq \nu < 1/2$ . Assume that the condition (1.9)<sub>3</sub> holds. Then we have

$$(5.39) \quad \begin{aligned} & M_{\nu, k}^{\mu}(N^{i0})(s, y) + M_{\nu, k}^{\mu}(\tilde{R}^{i0})(s, y) \\ & \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+3}{2}], t} + [\partial u]_{1, [\frac{k+3}{2}], t}) \|\partial u\|_{k+5, t}, \end{aligned}$$

provided  $0 < \mu < (1/2) - \nu$  and (5.26)<sub>k+2</sub> holds.

*Proof.* It follows from (5.22) and (3.27) that

$$(5.40) \quad M_{\nu, k}^{\mu}(F^i)(s, y) = \lambda(1 + \lambda + s)^{1+\mu+\nu} (1 + |\lambda - c_j s|)^{1-\mu} |F^i(s, y)|_k.$$

First we shall prove (5.39) for  $N^{i0} = 0$ . Similarly to (5.28) it follows from (1.12) that

$$|\tilde{R}^{i0}(s, y)|_k \leq C \sum_{p \neq q} |\partial u^p(s, y)|_{[\frac{k+1}{2}]} |\partial u^q(s, y)|_{k+1}.$$

Therefore from (5.40) we have

$$(5.41)_1 \quad M_{\nu, k}^{\mu}(\tilde{R}^{i0})(s, y) \leq C(1 + \lambda + s)^{-1+\nu+\mu} [\partial u]_{1, [\frac{k+1}{2}], t} [\partial(s, y)]_{1, k+1},$$

making use of (5.14) and (5.16) with  $\nu = 0$ . Hence we get (5.39) for  $N^{i0} = 0$  analogously to (5.36).

Next we shall prove the estimate for  $\tilde{R}^{i0} = 0$ . First suppose that  $1 \leq j \leq m$  and  $j \neq i$ . We have as above

$$|N^{i0}(\partial u^i(s, y), \partial^2 u^i(s, y))|_k \leq C |\partial u^i(s, y)|_{[\frac{k+1}{2}]} |\partial u^i(s, y)|_{k+1}.$$

Since  $j \neq i$ , it follows from (5.14) with  $\nu = 0$  and (5.40) that

$$(5.41)_2 \quad \begin{aligned} M_{\nu, k}^\mu(N^{i0})(s, y) &\leq C(1 + \lambda + s)^{-2+\mu+\nu} \\ &\times (1 + |\lambda - c_j s|)^{1-\mu} [\partial u]_{1, [\frac{k+1}{2}], t} [\partial u(s, y)]_{1, k+1}. \end{aligned}$$

Next suppose that  $j = i$ . Then by virtue of Proposition 2.5 with  $\nu = 0$  we get

$$\begin{aligned} M_{\nu, k}^\mu(N^{i0})(s, y) &\leq C(1 + \lambda + s)^{-1+\nu+\mu+\delta} \\ &\times (1 + |\lambda - c_i s|)^{-\mu-\delta} (\langle u^i(s, y) \rangle_{0, k+1} + [\partial u^i(s, y)]_{1, k+1}) \end{aligned}$$

for any  $\delta > 0$ , provided  $(5.26)_{k+2}$  holds. Thus we obtain (5.39) from (5.41)<sub>2</sub> and Corollary 5.6 with  $\rho = 1 - \nu - \mu - \delta$ , taking  $\delta = (1/2) - \nu - \mu$ . The proof is complete.

Since the condition (1.13) is in essence the same one as  $(1.9)_3$ , we obtain analogously to (5.39)

**Corollary 5.10.** Let  $j, \nu$  and  $\mu$  be as in the preceding lemma. Assume that both  $(1.9)_3$  and  $(A)_1$  hold. Then we have

$$(5.42) \quad \begin{aligned} M_{\nu, k}^\mu(N^{i0})(s, y) + M_{\nu, k}^\mu(R^i)(s, y) \\ \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+3}{2}], t} + [\partial u]_{1, [\frac{k+3}{2}], t}) \|\partial u\|_{k+5, t}, \end{aligned}$$

provided  $(5.26)_{k+2}$  holds.

Next we shall deal with the first term of the right hand side of (1.11), without the condition  $(A)_1$ . For doing so we prepare an auxiliary estimate.

**Lemma 5.11.** Let  $0 < \rho \leq 1/2$ . Assume that  $(1.9)_3$  holds. Then we have for  $(t, x) \in [0, T) \times \mathbb{R}^3$

$$(5.43) \quad [\partial u^i(t, x)]_{\rho, k} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+4}{2}], t} + [\partial u]_{1, [\frac{k+4}{2}], t}) \|\partial u\|_{k+6, t},$$

provided  $(5.26)_{k+3}$  holds.

*Proof.* We have only to estimate the last term of  $(5.17)_2$ . To this end we shall make use of  $(3.31)_2$  in Theorem 3.4. Then analogously to  $(5.21)_2$  we get

$$(5.44) \quad [\partial L_i(F^i)(t, x)]_{\rho, k} \leq C\varepsilon + C \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} \lambda z_{\mu, \rho}^i(s, \lambda) |F^i(s, y)|_{k+1},$$

where  $0 < \mu < 1/2$  and  $z_{\mu, \rho}^i(s, \lambda)$  is given by (3.32).

Let (5.35) hold. If  $0 \leq j \leq m$  and  $j \neq i$ , we have from  $(3.32)_1$  and (5.22)

$$\lambda z_{\mu, \rho}^i(s, \lambda) |F^i(s, y)|_{k+1} = M_{\rho-1, k+1}^\mu(F^i)(s, y).$$

Hence it follows from (5.20) and Lemma 5.5 that

$$(5.45) \quad \lambda z_{\mu, \rho}^i(s, \lambda) |F^i(s, y)|_{k+1} \leq C \left( \langle u \rangle_{0, [\frac{k+2}{2}], t} + [\partial u]_{1, [\frac{k+2}{2}], t} \right) \|\partial u\|_{k+4, t},$$

provided  $(5.26)_{k+1}$  holds, because  $1 - \rho \geq 1/2$ .

Next suppose that  $j = i$ , i.e., that  $(s, \lambda) \in \Lambda_i$ . Then from  $(3.32)_2$ , (5.20) and (5.22) we have

$$(5.46) \quad \begin{aligned} & \lambda z_{\mu, \rho}^i(s, \lambda) |F^i(s, y)|_{k+1} \\ & \leq \lambda(1 + \lambda + s)^{1+\mu} (1 + |\lambda - c_i s|)^{\rho-\mu} |G^i(s, y)|_{k+1} \\ & \quad + M_{0, k+1}^\mu (N^{i0} + R^i + H^i)(s, y). \end{aligned}$$

For the first term on the right hand side it follows from (5.29), (5.9) and (5.13) with  $4\delta = 1 - 2\mu$  that

$$(5.47) \quad \begin{aligned} & \lambda(1 + \lambda + s)^{1+\mu} (1 + |\lambda - c_i s|)^{\rho-\mu} |G^i(s, y)|_{k+1} \\ & \leq C(1 + |\lambda - c_i s|)^{\rho-\frac{1}{2}} \langle u \rangle_{0, [\frac{k+1}{2}], t}^2 \|\partial u\|_{k+3, t}, \end{aligned}$$

because  $0 < 2\mu < 1$ . As for the second term we claim that

$$(5.48) \quad \begin{aligned} & M_{0, k+1}^\mu (N^{i0} + R^i + H^i)(s, y) \\ & \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+4}{2}], t} + [\partial u]_{1, [\frac{k+4}{2}], t}) \|\partial u\|_{k+6, t}, \end{aligned}$$

provided (5.26)<sub>k+3</sub> holds. Indeed, it follows from (5.25) and (5.39) with  $\nu = 0$  that (5.48) holds for  $R^i = 0$ . Moreover analogously to (5.39) for  $N^{i0} = 0$  we obtain (5.48) for  $N^{i0} = H^i = 0$ , because  $R^i(\partial u, \partial u^2)$  is linear in  $(\partial u^i, \partial^2 u^i)$  according to (1.11) and  $(s, \lambda) \in \Lambda_i$  (see also (5.41)<sub>1</sub>).

From (5.46), (5.47) and (5.48) we have for  $(s, \lambda) \in \Lambda_i$

$$\lambda z_{\mu, \rho}^i(s, \lambda) |F^i(s, y)|_{k+1} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+4}{2}], t} + [\partial u]_{1, [\frac{k+4}{2}], t}) \|\partial u\|_{k+6, t},$$

provided  $\rho \leq 1/2$  and (5.26)<sub>k+3</sub> holds. Thus we obtain (5.43) from (5.17)<sub>2</sub>, (5.44) and (5.45). The proof is complete.

We are now in a position to deal with the first term on the right hand side of (1.11).

**Proposition 5.12.** Let  $1 \leq p \leq m$  and  $p \neq i$ . Assume that condition (1.9)<sub>3</sub> holds. Then we have for  $(t, x) \in [0, T) \times \mathbb{R}^3$

$$(5.49)_1 \quad \langle L^i(R^{ip})(t, x) \rangle_{0, k} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+5}{2}], t} + [\partial u]_{1, [\frac{k+5}{2}], t}) \|\partial u\|_{k+7, t}$$

and

$$(5.49)_2 \quad [\partial L_i(R^{ip}(t, x))]_{1, k} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+6}{2}], t} + [\partial u]_{1, [\frac{k+6}{2}], t}) \|\partial u\|_{k+8, t},$$

provided (5.26)<sub>k+4</sub> and (5.26)<sub>k+5</sub> hold respectively, where  $R^{ip} = R^{ip}(\partial u^p, \partial^2 u^p)$ .

*Proof.* First we shall prove (5.49)<sub>1</sub>. To this end we make use of (3.34)<sub>1</sub> in Theorem 3.7 with  $\nu = 0$ . Then analogously to (5.21)<sub>1</sub> we have

$$(5.50) \quad \langle L_i(R^{ip})(t, x) \rangle_{0, k} \leq C\varepsilon + C \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} \lambda \tilde{z}_{\mu}^p(s, \lambda) |R^{ip}(s, y)|_k,$$

where  $0 < \mu \leq 1/2$  and  $\tilde{z}_{\mu}^p(s, \lambda)$  is given by (3.34)<sub>3</sub> and (3.34)<sub>4</sub>.

Let (5.35) hold. First suppose that  $j \neq p$ . Then it follows from (3.34)<sub>4</sub> and (5.22) that

$$\lambda \tilde{z}_{\mu}^p(s, \lambda) |R^{ip}(s, y)|_k = M_{0, k}^{\mu}(R^{ip})(s, y).$$

If  $1 \leq j \leq m$  and  $j \neq p$ , we have

$$(5.51)_1 \quad M_{0,k}^\mu(R^{ip})(s, y) \leq C[\partial u]_{1, [\frac{k+1}{2}], t} \|\partial u\|_{k+3, t},$$

making use of (5.40), (5.28), (5.10) and (5.14), since  $0 < \mu \leq 1$ . If  $j = 0$ , we get analogously to (5.36)

$$(5.51)_2 \quad M_{0,k}^\mu(R^{ip})(s, y) \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+3}{2}], t} + [\partial u]_{1, [\frac{k+3}{2}], t}) \|\partial u\|_{k+5, t},$$

provided (5.26) $_{k+2}$  holds.

Finally suppose that  $j = p$ . Then it follows from (3.34) $_3$ , (5.28) and (5.16) with  $i = p$  that

$$\lambda \tilde{z}_\mu^p(s, \lambda) |R^{ip}(s, y)| \leq C \lambda (1 + |\lambda - c_p s|)^\mu [\partial u^p]_{1, [\frac{k+1}{2}], t} |\partial u^p(s, y)|_{k+1}.$$

In view of (2.14) we have therefore

$$\lambda \tilde{z}_\mu^p(s, \lambda) |R^{ip}(s, y)| \leq C[\partial u]_{1, [\frac{k+1}{2}], t} [\partial u^p(s, y)]_{\mu, k+1}.$$

Taking  $\mu$  so that  $0 < \mu \leq 1/2$  and applying Lemma 5.11 with  $\rho = \mu$  to the last factor, we thus obtain

$$(5.51)_3 \quad \begin{aligned} & \lambda \tilde{z}_\mu^p(s, \lambda) |R^{ip}(s, \lambda)| \\ & \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+5}{2}], t} + [\partial u]_{1, [\frac{k+5}{2}], t}) \|\partial u\|_{k+7, t}, \end{aligned}$$

provided (5.26) $_{k+4}$  holds.

Now the desired estimate (5.49) $_1$  follows from (5.50) and (5.51). Analogously one can prove (5.49) $_2$ , making use of (3.35) $_1$  with  $\nu = 0$  and  $\rho = 1$  instead of (3.34) $_1$ , because of (3.32) $_1$  and (3.32) $_2$ . Thus we prove the proposition.  $\square$

Finally we shall deal with the  $G^i = G^i(u)$  in (5.18) or (5.20). In doing so, the following lemma will play a key role.

**Lemma 5.13.** Let  $0 \leq \nu < 1$ . Then we have

$$(5.52) \quad \sum_{i=1}^m \langle L_i(F^i) \rangle_{\nu, k, t} \leq \sum_{i=1}^m \langle L_i(N^{i0} + R^i + H^i) \rangle_{\nu, k, t} \\ + C\varepsilon + \tilde{C}_k \langle u \rangle_{0, [\frac{k}{2}], t}^2 \sum_{i=1}^m \langle L_i(F^i) \rangle_{\nu, k, t},$$

provided (5.26)<sub>k</sub> holds, where  $\tilde{C}_k$  is a constant depending only on  $k, \nu$ , the  $\tilde{C}$  in (5.21)<sub>1</sub> with  $\rho = 0$ , and the coefficients of the polynomials  $G^1(u), G^2(u), \dots, G^m(u)$ .

*Proof.* It follows from (5.20) that

$$(5.53) \quad \langle L_i(F^i)(t, x) \rangle_{\nu, k} \leq \langle L_i(N^{i0} + R^i + H^i) \rangle_{\nu, k, t} + \langle L_i(G^i)(t, x) \rangle_{\nu, k}$$

for  $(t, x) \in [0, T) \times \mathbb{R}^3$ . Moreover from (5.21)<sub>1</sub> with  $\rho = 0$  and (3.27) we have

$$(5.54) \quad \langle L_i(G^i)(t, x) \rangle_{\nu, k} \leq C\varepsilon \\ + \tilde{C} \sum_{j=0}^m \sup_{\substack{0 \leq s \leq t \\ (s, \lambda) \in \Lambda_j}} \lambda(1 + \lambda + s)^{1+\mu+\nu} (1 + |\lambda - c_j s|)^{1-\mu} |G^i(s, y)|_k,$$

where  $0 < \mu < 1 - \nu$ .

Let (5.35) hold. Then it follows from (5.29), (5.11) and (5.13) that

$$(5.55) \quad |G^i(s, y)|_k \leq C(1 + \lambda + s)^{-2+2\delta} (1 + |\lambda - c_j s|)^{-2\delta} \langle u \rangle_{0, [\frac{k}{2}], t}^2 |u(s, y)|_k,$$

where  $\delta \in (0, 1)$  will be fixed later.

First suppose that  $\nu = 0$ . Then from (5.11), (5.13) and (5.17)<sub>1</sub> we have

$$(5.56)_1 \quad \lambda(1 + \lambda + s)^{1+\mu} (1 + |\lambda - c_j s|)^{1-\mu} |G^i(s, y)|_k \\ \leq C\varepsilon + C'(1 + \lambda + s)^{-1+\mu+3\delta} (1 + |\lambda - c_j s|)^{1-\mu-3\delta} \\ \times \langle u \rangle_{0, [\frac{k}{2}], t}^2 \sum_{p=1}^m \langle L_p(F^p) \rangle_{0, k, t},$$

where  $C''$  is a constant depending only on the  $C$  in (5.13) and (5.55). Therefore, if we take  $3\delta = 1 - \mu$ , it follows from (5.53) and (5.54) that

$$\begin{aligned} & \langle L_i(F^i)(t, x) \rangle_{0, k} \leq C\varepsilon \\ & + \langle L_i(N^{i0} + R^i + H^i) \rangle_{0, k, t} + \tilde{C} C' C'' \langle u \rangle_{0, [\frac{k}{2}], t}^2 \sum_{p=1}^m \langle L_p(F^p) \rangle_{0, k, t} \end{aligned}$$

where  $C''$  is a constant depending only in  $c_j$ . Since the right hand side is increasing with respect to  $t$ , we thus obtain (5.52).

Next suppose that  $0 < \nu < 1$ . Then analogously to (5.56)<sub>1</sub> we get

$$\begin{aligned} (5.56)_2 \quad & \lambda(1 + \lambda + s)^{1+\mu+\nu}(1 + |\lambda - c_j s|)^{1-\mu} |G^i(s, y)|_k \\ & \leq C\varepsilon + C'(1 + \lambda + s)^{-1+\mu+\nu+2\delta}(1 + |\lambda - c_j s|)^{1-\mu-\nu-2\delta} \langle u \rangle_{0, [\frac{k}{2}], t}^2 \sum_{p=1}^m \langle L_p(F^p) \rangle_{\nu, k, t}, \end{aligned}$$

making use of (5.12) instead of (5.13). Taking  $2\delta = 1 - \mu - \nu$ , we thus obtain (5.52) as before. The proof is complete.

By virtue of the above lemma one can reduce the  $L^\infty - L^2$  estimates for  $G^i(u)$  to those for  $N^{i0}$ ,  $R^i$  and  $H^i$ . More precisely we obtain the following

**Corollary 5.14.** Let  $0 \leq \nu < 1$ . Suppose that (5.26)<sub>k</sub> and

$$(5.57)_k \quad \tilde{C}_k \langle u \rangle_{0, [\frac{k}{2}], t}^2 \leq \frac{1}{2} \quad \text{for } 0 \leq t < T$$

hold, where  $\tilde{C}_k$  is the constant in (5.52). Then we have

$$(5.58) \quad \sum_{i=1}^m \langle L_i(F^i) \rangle_{\nu, k, t} \leq C\varepsilon + 2 \sum_{i=1}^m \langle L_i(N^{i0} + R^i + H^i) \rangle_{\nu, k, t}.$$

Moreover we have for  $(t, x) \in [0, T) \times \mathbb{R}^3$

$$(5.59) \quad [\partial L_i(G^i)(t, x)]_{1+\nu, k} \leq C\varepsilon + C \sum_{p=1}^m \langle L_p(N^{p0} + R^p + H^p) \rangle_{\nu, k+1, t},$$

provided (5.26)<sub>k+1</sub> and (5.27)<sub>k+1</sub> hold.

*Proof.* The estimate (5.58) follows immediately from (5.52) and (5.57)<sub>k</sub>. From (5.21)<sub>2</sub> with  $\rho = 0$  we have also

$$[\partial L_i(G^i)(t, x)]_{1+\nu, k} \leq C\varepsilon + \tilde{C} \sup_{\substack{0 \leq s \leq t \\ y \in \mathbb{R}^3}} M_{\nu, k+1}^\mu(G^i)(s, y).$$

Hence we obtain (5.59) by virtue of (5.56) and (5.58).  $\square$

Summarizing the results of the present section we obtain the following

**Corollary 5.15.** Let  $u^i(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  be a solution of the Cauchy problem (5.1)-(5.2). Let  $k$  be a positive integer,  $0 < \varepsilon \leq 1$  and  $(t, x) \in [0, T] \times \mathbb{R}^3$ .

1) Assume that the condition (1.9)<sub>3</sub> is satisfied. Then we have

$$(5.60)_1 \quad \langle u^i(t, x) \rangle_{0, k} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+5}{2}], t} + [\partial u]_{1, [\frac{k+5}{2}], t}) \|\partial u\|_{k+7, t},$$

provided (5.26)<sub>k+4</sub> and (5.57)<sub>k</sub> hold. Moreover we have

$$(5.60)_2 \quad [\partial u^i(t, x)]_{1, k} \leq C\varepsilon + C(\langle u \rangle_{0, [\frac{k+6}{2}], t} + [\partial u]_{1, [\frac{k+6}{2}], t}) \|\partial u\|_{k+8, t},$$

provided (5.26)<sub>k+5</sub> and (5.57)<sub>k+1</sub> hold.

2) Assume that both conditions (1.9)<sub>3</sub> and (A)<sub>1</sub> are satisfied. Let  $0 < \nu < 1/2$ . Then we have

$$(5.61)_1 \quad \langle u^i(t, x) \rangle_{\nu, k} \leq C\varepsilon + C(\langle u \rangle_{\nu, [\frac{k+3}{2}], t} + [\partial u]_{1, [\frac{k+3}{2}], t}) \|\partial u\|_{k+5, t},$$

provided (5.26)<sub>k+2</sub> and (5.57)<sub>k</sub> hold. Moreover we have

$$(5.61)_2 \quad [\partial u^i(t, x)]_{1+\nu, k} \leq C\varepsilon + C(\langle u \rangle_{\nu, [\frac{k+4}{2}], t} + [\partial u]_{1, [\frac{k+4}{2}], t}) \|\partial u\|_{k+6, t},$$

provided (5.26)<sub>k+3</sub> and (5.57)<sub>k+1</sub> hold. Here  $C$  are constants independent of  $T$ ,  $\varepsilon$  and  $u$ .

*Proof.* First suppose that both (1.9)<sub>3</sub> and (1.13) are satisfied. Then we see from (5.17), (5.20), Corollary 5.14, (5.21) with  $\rho = 0$ , (5.25), (5.36) and (5.42) that (5.61)<sub>1</sub>

and  $(5.61)_2$  hold. When only  $(1.9)_3$  is satisfied, we obtain  $(5.60)_1$  and  $(5.60)_2$ , making use of (5.39) and (5.49) instead of (5.42).  $\square$

## §6. Energy estimates

The purpose of this section is to derive two kinds of energy estimates for local solutions of the Cauchy problem (1.1)-(1.2) which will be stated in Proposition 6.1 and 6.4 below. Throughout the present section we suppose that  $u(t, x) = (u^1(t, x), u^2(t, x), \dots, u^m(t, x)) \in C^\infty([0, T] \times \mathbb{R}^3)$  is a solution of the Cauchy problem (5.1)-(5.2) and that  $0 < \varepsilon \leq 1$ . Moreover we assume that conditions (1.4), (1.5), (4.7), (4.19) and (5.18) are fulfilled. By  $C$  we also denote the various constants independent of  $\varepsilon, T$  and  $u(t, x)$ .

First we shall prove the following energy estimate.

**Proposition 6.1.** Let  $\nu$  be a positive number and let  $k$  be a positive integer.

Then we have

$$(6.1)_k \quad \|\partial u(t)\|_k^2 \leq C\varepsilon^2(1+t)^{B\Phi(u)(t; \nu, k+1)} \quad \text{for } 0 \leq t < T,$$

provided

$$(6.2)_k \quad \Phi(u)(t; \nu, k) \leq 1 \quad \text{for } 0 \leq t < T,$$

where

$$\Phi(u)(t; \nu, k) = \langle u \rangle_{\nu, [\frac{k}{2}], t} + [\partial u]_{1, [\frac{k}{2}], t}$$

and  $B$  is a positive constant independent of  $\varepsilon, T$  and  $u(t, x)$ . Besides, we have used the notations (2.17) and (5.8). Moreover assume that the hypothesis either  $(A)_2$  or  $(A)_3$  is satisfied. Then one can take  $\nu = 0$  in  $(6.1)_k$  and  $(6.2)_k$ .

To prove the proposition we will often make use of the following inequalities.

**Lemma 6.2.** Suppose that (5.4) in Lemma 5.1 holds. Then we have

$$(6.3)_1 \quad \int_{\mathbb{R}^3} |u^i(t, x)|^2 dx \leq \frac{1}{2}(L + c_m t)^2 \int_{\mathbb{R}^3} |\partial u^i(t, x)|^2 dx$$

for  $0 \leq t < T$ .

Moreover if  $v(t, x) \in C^1([0, T] \times \mathbb{R}^3)$  satisfies (5.5) we have

$$(6.3)_2 \quad \int_{\mathbb{R}^3} (1 + |r - c_i t|)^{-2} |v(t, x)|_\ell^2 dx \leq C \|\partial v(t)\|_\ell^2 \quad \text{for } 0 \leq t < T,$$

where  $r = |x|$  and  $\ell$  is a nonnegative integer.

*Proof.* The estimate (6.3)<sub>1</sub> is well known. For the proof see for instance Lax and Phillips [17], p.95. The inequality (6.3)<sub>2</sub> follows immediately from Kindblad [19], Lemma 1.2.  $\square$

*Proof of Proposition 6.1.* We shall refer to the proof of Theorem 4.1. Let  $|\alpha| \leq k$ . Then we see from (2.6), (2.7), (2.8) and (5.1) that

$$(6.4)_1 \quad \square_i \Gamma^\alpha u^i = \Gamma^\alpha F^i + \sum_{|\beta| \leq k-1} C_\beta \Gamma^\beta F^i,$$

where  $C_\beta$  are constants. Taking (1.4)<sub>1</sub> into account we shall write as

$$(6.4)_2 \quad \square_i \Gamma^\alpha u^i = \sum_{j=1}^m \sum_{a,b=0}^3 A_{ij}^{ab} \partial_a \partial_b \Gamma^\alpha u^i + B_\alpha^i,$$

where we have set

$$(6.5) \quad B_\alpha^i = \Gamma^\alpha F^i - \sum_{j=1}^m \sum_{a,b=0}^3 A_{ij}^{ab} \partial_a \partial_b \Gamma^\alpha u^j + \sum_{|\beta| \leq k-1} C_\beta \Gamma^\beta F^i.$$

Note that (6.4)<sub>2</sub> is of the same form as (1.1) with (1.4)<sub>1</sub>. Hence analogously to (4.16) we get

$$(6.6) \quad a_0(u, u') \partial_t v_\alpha = \sum_{j=1}^3 a_j(u, u') \partial_j v_\alpha + b_\alpha,$$

putting

$$(6.7)_1 \quad v_\alpha = \partial \Gamma^\alpha u = \begin{bmatrix} \partial_0 \Gamma^\alpha u \\ \partial_1 \Gamma^\alpha u \\ \partial_2 \Gamma^\alpha u \\ \partial_3 \Gamma^\alpha u \end{bmatrix}$$

and

$$(6.7)_2 \quad b_\alpha = \begin{bmatrix} B_\alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_\alpha = \begin{bmatrix} B_\alpha^1 \\ B_\alpha^2 \\ \vdots \\ B_\alpha^m \end{bmatrix}.$$

Here  $a_j(u, u')$  ( $j = 0, 1, 2, 3$ ) are defined by (4.14) and (4.15) with (4.8) and (4.10).

Let  $a(u, u')$  be the  $(4p) \times (4p)$  matrix defined by (4.17). Set

$$(6.8) \quad \tilde{a}_j(u, u') = a(u, u')a_j(u, u') \quad \text{for } j = 0, 1, 2, 3.$$

Then from (6.6) we have

$$(6.9) \quad \tilde{a}_0(u, u')\partial_t v_\alpha = \sum_{j=1}^3 \tilde{a}_j(u, u')\partial_j v_\alpha + a(u, u')b_\alpha.$$

We also see from (4.18) that  $\tilde{a}_j(u, u')$  are symmetric  $(4p) \times (4p)$  matrices. Moreover  $\tilde{a}_0(u, u')$  satisfies (4.19).

Now, the equation (6.9) yields

$$(6.10) \quad \begin{aligned} & \partial_t(\tilde{a}_0 v_\alpha \cdot v_\alpha) - (\partial_t \tilde{a}_0)v_\alpha \cdot v_\alpha \\ &= \sum_{j=1}^3 \partial_j(\tilde{a}_j v_\alpha \cdot v_\alpha) - \sum_{j=1}^3 (\partial_j \tilde{a}_j)v_\alpha \cdot v_\alpha + 2ab_\alpha \cdot v_\alpha, \end{aligned}$$

where the dot stands for the scalar product in  $\mathbb{R}^{4p}$ . Moreover it follows from (4.17) and (6.7) that

$$ab_\alpha \cdot v_\alpha = B_\alpha \cdot \partial_t \Gamma^\alpha u.$$

Integrating (6.10) over  $[0, t] \times \mathbb{R}^3$  we thus obtain

$$(6.11) \quad \begin{aligned} E_\alpha(t) - E_\alpha(0) &= 2 \int_0^t ds \int_{\mathbb{R}^3} (B_\alpha \cdot \partial_0 \Gamma^\alpha u)(s, x) dx \\ &+ \int_0^t ds \int_{\mathbb{R}^3} \left( \left( \partial_0 \tilde{a}_0 - \sum_{j=1}^3 \partial_j \tilde{a}_j \right) v_\alpha \cdot v_\alpha \right)(s, x) dx, \end{aligned}$$

where we have set

$$(6.12)_1 \quad E_\alpha(t) = \int_{\mathbb{R}^3} (\tilde{a}_0(u, u')v_\alpha \cdot v_\alpha)(t, x) ds.$$

Notice that (4.19) and (6.8) imply

$$(6.12)_2 \quad E_\alpha(t) \geq \tilde{C} \|v_\alpha(t)\|_0^2,$$

where

$$\tilde{C} = \frac{1}{2} \min\{1, c_1^2\}$$

with the  $c_1$  in (1.1). Thus we must estimate the right hand side of (6.11). As for the last term we have easily

$$(6.13) \quad \int_0^t ds \int_{\mathbb{R}^3} \left| \left( \partial_0 \tilde{a}_0 - \sum_{j=1}^3 \partial_j \tilde{a}_j \right) v_\alpha \cdot v_\alpha \right>(s, x) dx \\ \leq C \int_0^t (1+s)^{-1} [\partial u]_{1,1,s} \|v_\alpha(s)\|_0^2 ds.$$

Indeed, it follows from (4.7) and (6.8) that

$$|\partial_j \tilde{a}_j(u(s, x), \partial u(s, x))| \leq C(|\partial_j u(s, x)| + |\partial_j \partial u(s, x)|) \quad \text{for } j = 0, 1, 2, 3.$$

Hence we get

$$\left| \left( \partial_0 \tilde{a}_0 - \sum_{j=1}^3 \partial_j \tilde{a}_j \right) (s, x) \right| \leq C(1+s+|x|)^{-1} ([\partial u]_{1,0,s} + [\partial^2 u]_{1,0,s}),$$

making use of Lemma 5.4. Therefore we obtain (6.13).

We shall now deal with the first term on the right hand side of (6.11). It follows from (6.7)<sub>2</sub> that

$$(6.14) \quad \int_{\mathbb{R}^3} (B_\alpha \cdot \partial_0 \Gamma^\alpha u)(s, x) dx = \sum_{i=1}^m \int_{\mathbb{R}^3} (B_\alpha^i \cdot \partial_0 \Gamma^\alpha u^i)(s, x) dx,$$

where  $B_\alpha^i$  is defined by (6.5). Moreover we shall divide each integral on the right hand into three terms as follows.

$$(6.15) \quad \int_{\mathbb{R}^3} (B_\alpha^i \cdot \partial_0 \Gamma^\alpha u^i)(s, x) dx = I_1(s) + I_2(s) + I_3(s),$$

where  $I_1(s)$ ,  $I_2(s)$  and  $I_3(s)$  arise, respectively, from  $N^{i0}(\partial u^i, \partial^2 u^i) + R^i(\partial u, \partial^2 u)$  in (5.18), from  $G^i(u)$  and from  $H^i(u, \partial u, \partial^2 u)$ . More precisely,  $I_1(s)$  is obtained by

replacing  $F^i(u, \partial u, \partial^2 u)$  in (6.4)<sub>1</sub> by  $F^{i,2}(0, \partial u, \partial^2 u)$ , where  $F^{i,2}(u, \partial u, \partial^2 u)$  is given by (1.6) and (1.7). Moreover  $I_2(s)$  is obtained by replacing the  $B_\alpha^i$  on the left hand side of (6.15) by

$$(6.16) \quad \Gamma^\alpha G^i(u) + \sum_{|\beta| \leq k-1} C_\beta \Gamma^\beta G^i(u).$$

where  $C_\beta$  are the same constants as in (6.4)<sub>1</sub>.

As for the first integral  $I_1(s)$  it follows from (6.5) that

$$|I_1(s)| \leq C \int_{\mathbb{R}^3} |\partial u(s, x)|_{[\frac{k+1}{2}]} |\partial u(s, x)|_k |\partial_0 \Gamma^\alpha u^i(s, x)| dx.$$

Moreover Lemma 5.4 implies that

$$|\partial u(s, x)|_{[\frac{k+1}{2}]} \leq C(1+s)^{-1} [\partial u]_{1, [\frac{k+1}{2}], s}.$$

Hence we obtain

$$(6.17) \quad |I_1(s)| \leq C(1+s)^{-1} [\partial u]_{1, [\frac{k+1}{2}], s} \|\partial u(s)\|_k^2,$$

applying (2.7) to  $\partial_0 \Gamma^\alpha u^i$ .

Next we shall deal with the second integral  $I_2(s)$ . It follows from (6.16) and (5.29) that

$$(6.18) \quad |I_2(s)| \leq C \int_{\mathbb{R}^3} |u(s, x)|_{[\frac{k}{2}]}^2 |u(s, x)|_k |\partial_0 \Gamma^\alpha u^i(s, x)| dx.$$

First suppose that  $\nu > 0$ . Then it follows from (5.11) and (5.12) that

$$|u(s, x)|_{[\frac{k}{2}]}^2 \leq C(1+s)^{-2} \langle u \rangle_{\nu, [\frac{k}{2}], s}^2.$$

Moreover, making use of (6.3)<sub>1</sub> we have

$$(6.19) \quad \|u(s)\|_k \leq C(1+s) \|\partial u(s)\|_k.$$

Therefore we get by (6.18)

$$(6.20) \quad |I_2(s)| \leq C(1+s)^{-1} \langle u \rangle_{\nu, [\frac{k}{2}], s}^2 \|\partial u(s)\|_k^2.$$

Next we deal with the case where  $\nu = 0$ . From (6.18) we have

$$(6.21) \quad |I_2(s)| \leq C \|\partial u(s)\|_k \{I_4(s)\}^{1/2},$$

where

$$I_4(s) = \int_{\mathbb{R}^3} |u(s, x)|_{[\frac{k}{2}]}^4 |u(s, x)|_k^2 dx.$$

Moreover we shall divide the right hand side into  $m + 1$  integrals, similarly to the proof of Lemma 3.8. Namely we set

$$(6.22) \quad I_4(s) = \sum_{j=0}^m I_{4,j}(s),$$

where

$$I_{4,j}(s) = \int_{\Lambda_j(s)} |u(s, x)|_{[\frac{k}{2}]}^4 |u(s, x)|_k^2 dx$$

with

$$\Lambda_j(s) = \{x \in \mathbb{R}^3; (s, |x|) \in \Lambda_j\}.$$

In what follows we suppose that

$$(6.23) \quad x \in \Lambda_j(s) \quad \text{for some } 0 \leq j \leq m.$$

If  $j = 0$ , it follows from (5.11) that

$$|u(s, x)|_{[\frac{k}{2}]}^4 \leq C(1+s)^{-4} \langle u \rangle_{0, [\frac{k}{2}], s}^4.$$

Hence by (6.19) we get

$$(6.24)_0 \quad I_{4,0}(s) \leq C(1+s)^{-2} \langle u \rangle_{0, [\frac{k}{2}], s}^4 \|\partial u(s)\|_k^2.$$

If  $j = m$ , we see from (5.11) and (5.13) with  $\delta = 1/2$  that

$$\begin{aligned} |u(s, x)|_{[\frac{k}{2}]}^4 &\leq C(1+s)^{-4} \langle u \rangle_{0, [\frac{k}{2}], s}^4 \\ &+ C(1+s)^{-2} (1+|r-c_m s|)^{-2} \langle u \rangle_{0, [\frac{k}{2}], s}^4. \end{aligned}$$

Therefore, making use of (5.4) and (6.3)<sub>2</sub> also, we get

$$(6.24)_m \quad I_{4,m}(s) \leq C(1+s)^{-2} \langle u \rangle_{0, [\frac{k}{2}], s}^4 \|\partial u(s)\|_k^2.$$

If  $1 \leq j \leq m-1$ , we need the following

**Lemma 6.3.** Let  $1 \leq j \leq m-1$ . Suppose that one of the hypotheses (A)<sub>2</sub> and (A)<sub>3</sub> is satisfied. Then we have

$$(6.24)_j \quad I_{4,j}(s) \leq C(1+s)^{-2} \langle u \rangle_{0, [\frac{k}{2}], s}^4 \|\partial u(s)\|_k^2.$$

*Proof.* First suppose that (A)<sub>2</sub> holds. It follows from (5.11) and (5.13) with  $\delta = 1/2$  that

$$\begin{aligned} |u(s, x)|_{[\frac{k}{2}]}^4 &\leq C(1+s)^{-4} \langle u \rangle_{0, [\frac{k}{2}], s}^4 \\ &+ C(1+s)^{-2} (1 + |r - c_j s|)^{-2} \langle u^j(s, y) \rangle_{0, [\frac{k}{2}]}^4. \end{aligned}$$

Therefore similarly to (6.24)<sub>m</sub> we obtain (6.24)<sub>j</sub>, making use of (5.5) instead of (5.4).

Next suppose that (A)<sub>3</sub> holds. Then  $G^i(u)$  is a homogeneous polynomial in  $u^m$  only of degree 3. Hence

$$I_{4,j}(s) = \int_{\Lambda_j(x)} |u^m(s, x)|_{[\frac{k}{2}]}^4 |u^m(s, x)|_k^2 dx.$$

Therefore we get (6.24)<sub>j</sub>, making use of (5.11) and (6.19).  $\square$

From (6.21), (6.22) and (6.24)<sub>j</sub> with  $0 \leq j \leq m$  we obtain

$$(6.25) \quad |I_2(s)| \leq C(1+s)^{-1} \langle u \rangle_{0, [\frac{k}{2}], s}^2 \|\partial u(s)\|_k^2,$$

provided either (A)<sub>2</sub> or (A)<sub>3</sub> holds.

Finally we deal with the last integral  $I_3(s)$  in (6.15). The plan is similar to the proofs of (6.20) and (6.25). Since Lemma 5.4 implies that

$$(6.26) \quad |\partial u(s, x)|_\ell \leq C(1+s)^{-1} [\partial u]_{1, \ell, s} \quad \text{for } (s, x) \in [0, T) \times \mathbb{R}^3, \quad \ell \geq 0,$$

we shall deal with only the first term on the right hand side of (5.19). Thus we assume that (5.30) holds. Then it follows from (6.5) and (6.15) that

$$(6.27) \quad |I_3(s)| \leq C(I_5(s) + I_6(s)),$$

where

$$I_5(s) = \int_{\mathbb{R}^3} (|u(s, x)|_{[\frac{k}{2}]} |\partial u(s, x)|_{[\frac{k+1}{2}]} + |\partial u(s, x)|_{[\frac{k}{2}]} |u(s, x)|_{[\frac{k+1}{2}]}) |u(s, x)|_k |\partial_0 \Gamma^\alpha u^i(s, x)| dx$$

and

$$I_6(s) = \int_{\mathbb{R}^3} |u(s, x)|_{[\frac{k}{2}]} |u(s, x)|_{[\frac{k+1}{2}]} |\partial u(s, x)|_k |\partial_0 \Gamma^\alpha u^i(s, x)| dx.$$

(See also (5.31)). As for the last integral we have easily from (5.11) and (5.13) with  $\delta = 1/2$

$$(6.28) \quad I_6(s) \leq C(1+s)^{-1} \langle u \rangle_{0, [\frac{k}{2}], s} \langle u \rangle_{0, [\frac{k+1}{2}], s} \|\partial u(s)\|_k^2.$$

To deal with the first integral  $I_5(s)$  on the right hand of (6.27) we first suppose that  $\nu > 0$ . Then it follows from (5.11), (5.12), (6.26) and (6.19) that

$$(6.29) \quad I_5(s) \leq C(1+s)^{-1} (\langle u \rangle_{\nu, [\frac{k+1}{2}], s} + [\partial u]_{1, [\frac{k+1}{2}], s}) \|\partial u(s)\|_k^2,$$

provided (6.2)<sub>k</sub> holds.

Next we deal with the case where  $\nu = 0$ . Then analogously to (6.21) we have

$$(6.30) \quad I_5(s) \leq C \|\partial u(s)\|_k \{I_7(s)\}^{1/2},$$

where

$$I_7(s) = \int_{\mathbb{R}^3} (|u(s, x)|_{[\frac{k}{2}]}^2 |\partial u(s, x)|_{[\frac{k+1}{2}]}^2 + |\partial u(s, x)|_{[\frac{k}{2}]}^2 |u(s, x)|_{[\frac{k+1}{2}]}^2) |u(s, x)|_k^2 dx.$$

Moreover analogously to (6.22) we set

$$(6.31) \quad I_7(s) = \sum_{j=0}^m I_{7,j}(s),$$

where  $I_{7,j}(s)$  is the integral over  $\Lambda_j(s)$ . Then similarly to  $(6.24)_0$  and  $(6.24)_m$  we get for  $j = 0$  and  $j = m$

$$(6.32)_j \quad I_{7,j}(s) \leq C(1+s)^{-2} \left( \langle u \rangle_{0, [\frac{k+1}{2}], s}^2 + [\partial u]_{1, [\frac{k+1}{2}], s}^2 \right) \|\partial u(s)\|_k^2,$$

provided  $(6.2)_k$  with  $\nu = 0$  holds, making use of (6.26) also.

Let  $1 \leq j \leq m-1$ . If hypothesis  $(A)_2$  is satisfied, we get  $(6.32)_j$  analogously to Lemma 6.3. Suppose that hypothesis  $(A)_3$  holds. Then we find from (1.15) and (5.30) that

$$\begin{aligned} I_{7,j}(s) &\leq C \int_{\Lambda_j(s)} \left( |u(s, x)|_{[\frac{k}{2}]}^2 |\partial u(s, x)|_{[\frac{k+1}{2}]}^2 \right. \\ &\quad \left. + |\partial u(s, x)|_{[\frac{k}{2}]}^2 |u^m(s, x)|_{[\frac{k+1}{2}]}^2 \right) |u(s, x)|_k^2 dx. \end{aligned}$$

Hence we obtain  $(6.32)_j$ , making use of (5.11), (6.26) and (6.19).

Now it follows from (6.30), (6.31) and  $(6.32)_j$  with  $0 \leq j \leq m$  that

$$(6.33) \quad I_5(s) \leq C(1+s)^{-1} \left( \langle u \rangle_{0, [\frac{k+1}{2}], s} + [\partial u]_{1, [\frac{k+1}{2}], s} \right) \|\partial u(s)\|_k^2,$$

provided  $(6.2)_k$  with  $\nu = 0$  holds and the hypothesis either  $(A)_2$  or  $(A)_3$  is satisfied.

In what follows we assume that either  $\nu > 0$  or one of the hypothesis  $(A)_2$  and  $(A)_3$  is satisfied and that  $(6.2)_k$  holds. Then from (6.27), (6.28), (6.29) and (6.33) we have

$$(6.34) \quad I_3(s) \leq C(1+s)^{-1} \left( \langle u \rangle_{\nu, [\frac{k+1}{2}], s} + [\partial u]_{1, [\frac{k+1}{2}], s} \right) \|\partial u(s)\|_k^2.$$

Consequently we see from (6.14), (6.15), (6.17), (6.20), (6.25) and (6.34) that

$$\begin{aligned} (6.35) \quad & \left| \int_{\mathbb{R}^3} (B_\alpha \cdot \partial_0 \Gamma^\alpha u)(s, x) dx \right| \\ & \leq C(1+s)^{-1} \left( \langle u \rangle_{\nu, [\frac{k+1}{2}], s} + [\partial u]_{1, [\frac{k+1}{2}], s} \right) \|\partial u(s)\|_k^2. \end{aligned}$$

Therefore we find from (6.11), (6.12), (6.13) and  $(6.7)_1$  that

$$\|\partial u(t)\|_k^2 \leq C \sum_{|\alpha| \leq k} E_\alpha(0) + C \int_0^t (1+s)^{-1} \Phi(u)(s; \nu, k+1) \|\partial u(s)\|_k^2 ds,$$

because (2.7) and (6.12)<sub>2</sub> yield

$$\|\partial u(t)\|_k^2 \leq C \sum_{|\alpha| \leq k} E_\alpha(t).$$

Making use of Gronwall's inequality, we thus obtain

$$\|\partial u(t)\|_k^2 \leq C \sum_{|\alpha| \leq k} E_\alpha(0) (1+t)^C \Phi(u)(t; \nu, k+1),$$

which yields (6.1)<sub>k</sub>. Indeed, we have

$$\begin{aligned} & \int_0^t (1+s)^{-1} \Phi(u)(s; \nu, k+1) ds \\ & \leq \Phi(u)(t; \nu, k+1) \int_0^t (1+s)^{-1} ds \\ & = \Phi(u)(t; \nu, k+1) \log(1+t). \end{aligned}$$

Besides, it follows from (6.7)<sub>1</sub>, (6.12)<sub>1</sub>, (4.9), (4.19) and (5.2) that

$$\sum_{|\alpha| \leq k} E_\alpha(0) \leq C\varepsilon^2.$$

Therefore we get (6.1)<sub>k</sub>. Thus we prove Proposition 6.1.

Next we shall derive another energy estimate

**Proposition 6.4.** Assume that the null condition (1.9)<sub>3</sub> is satisfied. Let  $k$  be a positive integer. Suppose that (5.26)<sub>k+2</sub> holds. Then we have

$$(6.36) \quad \|\partial u(t)\|_k^2 \leq C\varepsilon^2 + C \int_0^t (1+s)^{-2+3\delta} \Psi_{k+1}(s; u) [\partial u]_{1, k, s} ds$$

for  $0 \leq t < T$  and  $0 < \delta \leq \frac{1}{4}$ ,

where we have set

$$\Psi_k(s; u) = \langle u \rangle_{0, k, s} + [\partial u]_{1, k, s}.$$

*Proof.* Similarly to (6.10) it follows from (6.4)<sub>1</sub> that

$$\begin{aligned} & \partial_s \int_{\mathbb{R}^3} (|\partial_s \Gamma^\alpha u^i(s, x)|^2 + c_i^2 |\partial_x \Gamma^\alpha u^i(s, x)|^2) dx \\ & = 2 \int_{\mathbb{R}^3} \left( \Gamma^\alpha F^i + \sum_{|\beta| \leq k-1} C_\beta \Gamma^\alpha F^i \right) \cdot \partial_0 \Gamma^\alpha u^i(s, x) dx, \end{aligned}$$

where the dot stands for the scalar product in  $\mathbb{R}^{p_i}$ . Hence we see from (4.9) and (5.2) that

$$(6.37) \quad E_k(u^i)(t) \leq C\varepsilon^2 + C \int_0^t ds \int_{\mathbb{R}^3} (|F^i(s, x)|_k |\partial_0 u^i(s, x)|_k dx$$

for  $i = 1, 2, \dots, m$ ,

setting

$$E_k(u^i)(t) = \int_{\mathbb{R}^3} (|\partial_t u^i(t, x)|_k^2 + c_i^2 |\partial_x u^i(t, x)|_k^2) dx,$$

where  $F^i(s, x) = F^i(u(s, x), \partial u(s, x), \partial^2 u(s, x))$ . Therefore it suffices to prove

$$(6.38) \quad \int_{\mathbb{R}^3} |F^i(s, x)|_k |\partial_0 u^i(s, x)|_k dx \leq C(1+s)^{-2+3\delta} \Psi_{k+1}(s; u) [\partial u]_{1, k, s}$$

for  $0 \leq s \leq t < T$ .

To this end we shall divide the left hand side into  $m+1$  integrals, analogously to (6.22). Set

$$(6.39)_j \quad I_j(s) = \int_{\Lambda_j(s)} |F^i(s, x)|_k |\partial_0 u^i(s, x)|_k dx$$

for  $j = 0, 1, 2, \dots, m$ , where  $\Lambda_j(s)$  are as in (6.22). First we deal with  $I_0(s)$ .

**Lemma 6.5.** Suppose that  $(5.26)_{k+1}$  holds. Then we have

$$(6.40) \quad I_0(s) \leq C(1+s)^{-2+3\delta} \Psi_{k+1}(s; u) [\partial u]_{1, k, s}.$$

*Proof.* From (5.15) with  $\nu = 0$  we have

$$(6.41) \quad |\partial u(s, x)|_k \leq C(1+r)^{-1}(1+r+s)^{-1} [\partial u]_{1, k, s} \quad \text{for } x \in \Lambda_0(s),$$

where  $r = |x|$ . Moreover similarly to (6.15) we shall write as

$$(6.42) \quad I_0(s) = I_{0,1}(s) + I_{0,2}(s) + I_{0,3}(s),$$

where  $I_{0,1}$ ,  $I_{0,2}$  and  $I_{0,3}$  are obtained by replacing the  $F^i(u, \partial u, \partial^2 u)$  in (6.39)<sub>0</sub>, respectively, by  $N^{i0}(\partial u^i, \partial^2 u^i) + R^i(\partial u, \partial^2 u)$ , by  $G^i(u)$  and by  $H^i(u, \partial u, \partial^2 u)$ , according as (5.18).

For the first term  $I_{0,1}(s)$  it follows from (5.28) and (6.41) that

$$I_{0,1}(s) \leq C(1+s)^{-2} [\partial u]_{1, k+1, s} [\partial u^i]_{1, k, s} \int_0^\infty (1+r)^{-2} dr,$$

provided (5.26)<sub>k+1</sub> holds. Hence we get (6.40) with  $\delta = 0$ . For the second term it follows from (5.11), (5.13) and (5.29) that

$$(6.43) \quad |G^i(u(s, x))| \leq C(1+r+s)^{-3+3\delta} (1+|r-c_j s|)^{-3\delta} \langle u \rangle_{0, k, s} \\ \text{for } x \in \Lambda_j(s) \text{ with } 0 \leq j \leq m,$$

provided (5.26)<sub>k</sub> holds. Hence

$$I_{0,2}(s) \leq C(1+s)^{-2+3\delta} \langle u \rangle_{0, k, s} [\partial u^i]_{1, k, s} \int_0^\infty (1+r)^{-1-3\delta} dr.$$

Therefore we obtain (6.40). Similarly one can prove (6.40) with  $\delta = 0$  for  $I_{0,3}(s)$ , because of (5.19). The proof is complete.

Next we deal with the generic case.

**Lemma 6.6.** Let  $1 \leq j \leq m$  and  $j \neq i$ . Suppose that (5.26)<sub>k+1</sub> holds. Then we have

$$(6.44) \quad I_j(s) \leq C(1+s)^{-2+3\delta} \Psi_{k+1}(s; u) [\partial u]_{1, k, s}.$$

*Proof.* It follows from (5.14) with  $\nu = 0$  that

$$(6.45) \quad |\partial u^i(s, x)|_k \leq C(1+r+s)^{-2} [\partial u^i]_{1, k, s} \text{ for } x \in \Lambda_j(s).$$

We also set analogously to (6.42)

$$(6.46) \quad I_j(s) = I_{j,1}(s) + I_{j,2}(s) + I_{j,3}(s).$$

For the first term  $I_{j,1}(s)$  it follows from (5.14) and (5.16) with  $\nu = 0$  that

$$|(N^{i0} + R^i)(s, x)|_k \leq C(1 + r + s)^{-2} (1 + |r - c_j s|)^{-2} [\partial u]_{1, k+1, s},$$

provided (5.26) $_{k+1}$  holds. Hence (6.45) yields

$$I_{j,1}(s) \leq C(1 + s)^{-2} [\partial u]_{1, k+1, s} [\partial u^i]_{1, k, s} \int_{-\infty}^{\infty} (1 + |r - c_j s|)^{-2} dr.$$

Therefore we get (6.44) with  $\delta = 0$ . As for the second and third terms  $I_{j,2}(s)$  and  $I_{j,3}(s)$ , the procedure is analogous to the proof of (6.40), if we make use of (6.43) and (6.45). Hence we omit the details. The proof is complete.

Finally we deal with the critical case where  $j = i$ .

**Lemma 6.7.** Suppose that condition (1.9) $_3$  and (5.26) $_{k+2}$  hold. Then we have (6.44) with  $j = i$ .

*Proof.* It follows from (5.16) with  $\nu = 0$  that

$$(6.47) \quad |\partial u^i(s, x)|_k \leq C(1 + r + s)^{-1} (1 + |r - c_i s|)^{-1} [\partial u^i]_{1, k, s} \quad \text{for } x \in \Lambda_i(s).$$

We shall also keep using (6.46) with  $j = i$ . As for the second and third terms  $I_{i,2}(s)$  and  $I_{i,3}(s)$ , one can prove (6.44) analogously to the preceding lemma, making use of (6.43), (6.47) and (5.19).

For the first term  $I_{i,1}(s)$  it follows from (6.39) $_i$  that

$$(6.48) \quad I_{i,1}(s) \leq J_1(s) + J_2(s),$$

where

$$J_1(s) = \int_{\Lambda_i(s)} |R^i(\partial u, \partial^2 u)(s, x)|_k |\partial_0 u^i(s, x)|_k dx$$

and

$$J_2(s) = \int_{\Lambda_i(s)} |N^{i0}(\partial u^i, \partial^2 u^i)(s, x)|_k |\partial_0 u^i(s, x)|_k dx.$$

Moreover we see from (5.14), (5.16) that

$$\begin{aligned} |R^i(\partial u, \partial^2 u)(s, x)|_k &\leq C(1+r+s)^{-3} \\ &\times (1+|r-c_i s|)^{-1} [\partial u]_{1, [\frac{k+1}{2}], s} [\partial u]_{1, k+1, s} \\ &\text{for } x \in \Lambda_i(s). \end{aligned}$$

Hence (6.47) yields

$$J_1(s) \leq C(1+s)^{-2} [\partial u]_{1, k+1, s} [\partial u^i]_{1, k, s} \int_{-\infty}^{\infty} (1+|r-c_i s|)^{-2} dr,$$

provided (5.26)<sub>k+1</sub> holds. Therefore we get (6.44) for  $J_1(s)$ .

As for  $J_2(s)$  it follows from Proposition 2.5 with  $\nu = 0$  that

$$|N^{i0}(\partial u^i, \partial^2 u^i)(s, x)|_k \leq C(1+r+s)^{-3+\delta} (1+|r-c_i s|)^{-1-\delta} \Psi_{k+1}(s; u),$$

provided (5.26)<sub>k+2</sub> holds. Hence we obtain (6.44) as above. The proof is complete.

*End of proof of Proposition 6.4.* The desired estimate (6.36) follows immediately from (6.37), (6.38), (6.39)<sub>j</sub> and Lemmas 6.5, 6.6 and 6.7.  $\square$

Combining Proposition 6.1 and 6.4 with the results of the preceding section, we obtain a sharp energy estimate.

*Corollary 6.8.* Assume that the null condition (1.9)<sub>3</sub> is satisfied and that one of the hypotheses (A)<sub>1</sub>, (A)<sub>2</sub> and (A)<sub>3</sub> holds. Let  $k$  be a positive integer. Moreover assume that

$$(6.49) \quad B \Phi(u)(t; \nu, k+1) \leq \frac{1}{3} \quad \text{for } 0 \leq t < T$$

holds, where  $B$  is the constant in (6.1)<sub>k</sub>,  $\nu = 0$  if (A)<sub>2</sub> or (A)<sub>3</sub> holds, and  $0 < \nu < 1/2$  if (A)<sub>1</sub> holds.

1) Suppose that (A)<sub>2</sub> or (A)<sub>3</sub> holds. Let  $k \geq 10$ . Then we have

$$(6.50) \quad \|\partial u(t)\|_{k-9}^2 \leq C\varepsilon^2 \quad \text{for } 0 \leq t < T,$$

provided  $(5.26)_k$  and  $(5.57)_{k-7}$  hold.

2) Suppose that  $(A)_1$  holds. Let  $k \geq 8$ . Then we have

$$(6.51) \quad \|\partial u(t)\|_{k-7}^2 \leq C\varepsilon^2 \quad \text{for } 0 \leq t < T,$$

provided  $(6.2)_k$  and  $(5.57)_{k-5}$  hold.

*Proof.* First we shall prove (6.50). Let either  $(A)_2$  or  $(A)_3$  be satisfied. Then  $\nu = 0$ , hence  $(6.2)_k$  coincides with  $(5.26)_k$ . Let  $k \geq 10$ . Then it follows from Proposition 6.4 and the first part of Corollary 5.15 that

$$(6.52) \quad \|\partial u(t)\|_{k-9}^2 \leq C\varepsilon^2 + C \int_0^t (1+s)^{-2+3\delta} (\varepsilon^2 + \|\partial u\|_{k,s}^2) ds,$$

provided  $(5.26)_{k-2}$  and  $(5.57)_{k-7}$  hold. Moreover we see from Proposition 6.1 and (6.49) that

$$\|\partial u\|_{k,t}^2 \leq C\varepsilon^2 (1+t)^{1/3},$$

provided  $(5.26)_k$  holds, because the right hand side of  $(6.1)_k$  is increasing with respect to  $t$ . Taking  $\delta = 1/18$  in (6.52), we thus obtain

$$\|\partial u(t)\|_{k-9}^2 \leq C\varepsilon^2 \left( 1 + \int_0^\infty (1+s)^{-3/2} ds \right),$$

which yields (6.50).

Next suppose that  $(A)_1$  holds. Then  $0 < \nu < 1/2$ . Hence it follows from (6.36) and the last part of Corollary 5.15 that for  $k \geq 8$

$$\|\partial u(t)\|_{k-7}^2 \leq C\varepsilon^2 + C \int_0^t (1+s)^{-2+3\delta} (\varepsilon^2 + \|\partial u\|_{k,s}^2) ds,$$

provided  $(6.2)_{k-2}$  and  $(5.57)_{k-5}$  hold. Therefore we obtain (6.51) as above, provided  $(6.2)_k$  holds. The proof is complete.

## §7. Proof of Theorem 1.2.

In this section we shall complete the proof of Theorem 1.2. Throughout the present section we assume that conditions (1.4), (1.5), (1.9)<sub>1</sub> and (1.10) are fulfilled. Then (1.9)<sub>2</sub>, (1.9)<sub>3</sub> and (5.18) also hold. Moreover one can assume without loss of generality that (4.7), (4.12) and (4.19) are satisfied, as we have remarked at the end of the proof of Theorem 4.1.

We shall first prove the part 1) of the theorem. Let  $n$  be a fixed positive integer with  $n \geq 17$ , so that

$$(7.1) \quad \left\lceil \frac{n+18}{2} \right\rceil \leq n.$$

Then it follows from Lemma 5.7 with  $\nu = 0$  and  $\delta = 1$  that the Cauchy problem (5.1)-(5.2) has a solution  $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  satisfying (5.26) <sub>$n+17$</sub> , provided  $T = T_{n+3}$  and  $0 < \varepsilon \leq \tilde{\varepsilon}_{n+3}$ . Suppose that one of the hypotheses (A)<sub>2</sub> and (A)<sub>3</sub> is satisfied. Making use of Proposition 6.1 with  $\nu = 0$  and  $k = n + 17$ , we then obtain

$$(7.2) \quad \|\partial u(t)\|_{n+17}^2 \leq C\varepsilon^2(1+t)^{B\Phi(u)(t;0,n+18)} \quad \text{for } 0 \leq t < T,$$

where  $C$  and  $B$  are constants independent of  $T$ ,  $\varepsilon$  and  $u$  if  $0 < \varepsilon \leq 1$ .

Set

$$(7.3) \quad B_n = \max\{1, B, \tilde{C}_n, \tilde{C}_{n+1}, \dots, \tilde{C}_{n+10}\},$$

where  $B$  and  $\tilde{C}_k$  are the constants in (7.2) and (5.57) <sub>$k$</sub>  respectively. Notice that  $B_n$  is independent of  $T$ ,  $\varepsilon$  and  $u$  if  $\varepsilon \leq 1$ . Then we see from Lemma 5.7 that the Cauchy problem (5.1)-(5.2) has a solution  $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$  satisfying

$$(7.4) \quad \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq \frac{1}{3B_n} \quad \text{for } 0 \leq t < T,$$

provided  $T = T_{n+3}$  and  $0 < \varepsilon \leq \min\{1, \tilde{\varepsilon}_{n+3}\}$  with another constant  $\tilde{\varepsilon}_{n+3}$ .

Let  $T$  and  $\varepsilon$  be as above. Since (7.1), (7.3) and (7.4) yield (5.26) <sub>$n+17$</sub>  and (5.57) <sub>$k$</sub>  for  $n \leq k \leq n + 10$ , we see from Proposition 6.4 with  $\delta = 1/18$  and the part 1) of Corollary 5.15 that

$$\|\partial u(t)\|_{n+8}^2 \leq C\varepsilon^2 + C \int_0^t (1+s)^{-2+1/6} \|\partial u\|_{n+17,s}^2 ds \quad \text{for } 0 \leq t < T.$$

Therefore by virtue of (7.2) we obtain

$$(7.5) \quad \|\partial u(t)\|_{n+8}^2 \leq C_n^2 \varepsilon^2 \left( 1 + \int_0^t (1+s)^{-2+1/6+B_n \Phi(u)(s;0,n+18)} ds \right) \\ \text{for } 0 \leq t < T,$$

where  $C_n$  is a constant independent of  $T$ ,  $\varepsilon$  and  $u$ . According to the part 1) of Corollary 5.15, one can also take the constant  $C_n$  so that

$$(7.6) \quad \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq C_n(\varepsilon + \|\partial u\|_{n+8,t}) \\ \text{for } 0 \leq t < T.$$

Moreover (7.4) and (7.5) yield

$$(7.7) \quad \|\partial u(t)\|_{n+8}^2 \leq 3C_n^2 \varepsilon^2 \quad \text{for } 0 \leq t < T,$$

since (7.1) implies that

$$\Phi(u)(t; 0, n+18) \leq \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t}.$$

Therefore from (7.6) we obtain

$$(7.8) \quad \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq C_n(1 + \sqrt{3}C_n)\varepsilon \\ \text{for } 0 \leq t < T,$$

provided  $T = T_{n+3}$  and  $0 < \varepsilon \leq \min\{1, \tilde{\varepsilon}_{n+3}\}$ .

In what follows we shall assume without loss of generality that  $C_n \geq 2$ . Let  $\varepsilon$  be a fixed positive number satisfying

$$(7.9) \quad \varepsilon \leq \min\{1, \tilde{\varepsilon}_{n+3}, (9B_n C_n^2)^{-1}\}.$$

Let  $T(\varepsilon)$  be the least upper bound of positive numbers  $T$  such that the Cauchy problem (5.1)-(5.2) has a solution  $u(t, x) \in C^\infty([0, T) \times \mathbb{R}^3)$  satisfying

$$(7.10) \quad \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq 3C_n^2 \varepsilon \quad \text{for } 0 \leq t < T.$$

Then it follows from (7.8) that  $T(\varepsilon) \geq T_{n+3}$ . Moreover we shall show that

$$(7.11) \quad T(\varepsilon) = \infty.$$

On the contrary suppose that  $T(\varepsilon) < \infty$ . Then we see from the definition of  $T(\varepsilon)$  that the Cauchy problem (5.1)-(5.2) with  $T = T(\varepsilon)$  has a solution  $u(t, x) \in C^\infty([0, T(\varepsilon)) \times \mathbb{R}^3)$  satisfying (7.10). Since (7.9) implies  $3C_n^2\varepsilon \leq 1/(3B_n)$ , we find from the above procedure that (7.8) is valid with  $T = T(\varepsilon)$ , noting that the constants  $B_n$  and  $C_n$  in (7.4), (7.5) and (7.6) are independent of  $T$ . Hence it follows from Majda [20], Theorem 2.2 with  $s = 3$  (or its Corollary 2) that there exists a positive number  $\delta$  such that  $u(t, x)$  is a solution of (5.1)-(5.2) with  $T = T(\varepsilon) + \delta$  satisfying  $u(t, x) \in C^2([0, T(\varepsilon) + \delta) \times \mathbb{R}^3)$ . Moreover we see from Theorem 4.2 that  $u(t, x) \in C^\infty([0, T(\varepsilon) + \delta) \times \mathbb{R}^3)$ . Thus by (7.8) with  $T = T(\varepsilon)$  one can find a positive number  $T_1$  greater than  $T(\varepsilon)$  such that the Cauchy problem (5.1)-(5.2) with  $T = T_1$  has a solution  $u(t, x) \in C^\infty([0, T_1) \times \mathbb{R}^3)$  satisfying

$$\langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq \frac{5}{2}C_n^2\varepsilon \quad \text{for } 0 \leq t < T_1,$$

because  $C_n(1 + \sqrt{3}C_n) < (5/2)C_n^2$  and the left hand side of (7.8) is continuous in  $t$ . This contradicts the definition of  $T(\varepsilon)$ . Therefore we obtain (7.11).

By  $\varepsilon_0$  we denote the right hand side of (7.9). Let  $0 < \varepsilon \leq \varepsilon_0$ . Then we have proved that the Cauchy problem (1.1)-(1.2) has a solution  $u(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^3)$  satisfying

$$(7.12) \quad \langle u \rangle_{0,n,t} + [\partial u]_{1,n,t} \leq 3C_n^2\varepsilon \quad \text{for } t \geq 0.$$

Since  $3C_n^2\varepsilon \leq 1/(3B_n)$ , we see that (7.4) holds with any positive number  $T$ . Set  $k = n+1$  and  $C = 3(1 + 1/C_1)C_n^2$ . Then (7.12) implies (1.16). Moreover from (7.7) we obtain (1.17) with another constant  $C$ , making use of (5.4) and (6.3)<sub>1</sub>. Thus we have found a solution  $u(t, x)$  of (1.1)-(1.2) having the properties stated in the part 1) of the theorem.

Finally we shall show that such a solution  $u(t, x)$  is unique, provided  $|u(t, x)| + |\partial u(t, x)| \leq \delta$  for  $(t, x) \in [0, \infty) \times \mathbb{R}^3$ , where  $\delta$  is the constant in (4.6). The procedure is similar to John [10], Appendix 1. We write the modified equation (4.20) as

$$(7.13) \quad \tilde{a}_0(w)\partial_t w = \sum_{j=1}^3 \tilde{a}_j(w)\partial_j w + \tilde{b}(w)$$

and take the  $\delta$  so small as (4.19) holds, where

$$w = \begin{bmatrix} u \\ \partial u \end{bmatrix}, \quad \tilde{a}_0 = \begin{bmatrix} I_p & 0 \\ 0 & aa_0 \end{bmatrix},$$

$$\tilde{a}_j = \begin{bmatrix} 0 & 0 \\ 0 & aa_j \end{bmatrix} \quad (j = 1, 2, 3) \quad \text{and} \quad \tilde{b} = \begin{bmatrix} \partial_t u \\ ab \end{bmatrix}.$$

Then we have

$$(7.14) \quad \tilde{a}_0(w) \geq \tilde{C}I_{5p} \quad \text{for} \quad w \in \mathbb{R}^{5p},$$

where  $\tilde{C}$  is the same constant as in (4.19).

Let  $T$  be an arbitrarily fixed positive number. Let  $w^k(t, x) \in C^1([0, T] \times \mathbb{R}^3)$ ,  $k = 1, 2$ , be two solutions of the equation (7.13) in  $(0, T) \times \mathbb{R}^3$  such that  $w^1(0, x) = w^2(0, x)$  for  $x \in \mathbb{R}^3$  and that  $w^1(t, x) = w^2(t, x) = 0$  for  $|x| \geq L$  and  $0 \leq t \leq T$  with some positive constant  $L$ . Set  $w(t, x) = w^1(t, x) - w^2(t, x)$ . Then we have

$$\tilde{a}_0(w^1)\partial_t w = \sum_{j=1}^3 \tilde{a}_j(w^1)\partial_j w + F(w^1, w^2),$$

where

$$F(w^1, w^2) = \{\tilde{a}_0(w^2) - \tilde{a}_0(w^1)\}\partial_t w^2$$

$$- \sum_{j=1}^3 \{\tilde{a}_j(w^2) - \tilde{a}_j(w^1)\}\partial_j w^2 + \tilde{b}(w^1) - \tilde{b}(w^2).$$

Hence, similarly to the proof of Proposition 6.1, we find from (4.7), (4.18) and (7.14) that

$$\int_{\mathbb{R}^3} |w(t, x)|^2 dx \leq C \int_0^t ds \int_{\mathbb{R}^3} |w(s, x)|^2 dx \quad \text{for} \quad 0 \leq t \leq T,$$

where  $C$  is a constant depending only on  $\tilde{C}$ ,  $\partial_w \tilde{a}_j(w)$ ,  $j = 0, 1, 2, 3$ ,  $\partial_w \tilde{b}(w)$  for  $w \in \mathbb{R}^{5p}$  and  $\partial w^k(t, x)$  for  $0 \leq t \leq T$ ,  $|x| \leq L$ ,  $k = 1, 2$ . Therefore by Gronwall's inequality we obtain  $w(t, x) = 0$  for  $(t, x) \in [0, T] \times \mathbb{R}^3$ . Thus we prove the part 1).

Next we shall prove the part 2). The plan is analogous to the proof of the part 1). So we shall point out only the differences. Let  $n$  be a fixed positive integer with  $n \geq 13$ , so that

$$(7.1)' \quad \left\lfloor \frac{n+14}{2} \right\rfloor \leq n.$$

Let  $\nu$  be an arbitrary positive number with  $\nu < 1/2$ . Then it follows from Lemma 5.7 that the Cauchy problem (5.1)-(5.2) with  $T = T_{n+3}$  and  $0 < \varepsilon \leq \tilde{\varepsilon}_{n+3}$  has a solution  $u(t, x) \in C^\infty([0, T) \times \mathbb{R}^3)$  satisfying (6.2) $_{n+13}$ . Moreover we have

$$(7.2)' \quad \|\partial u(t)\|_{n+13}^2 \leq C\varepsilon^2(1+t)^{B\Phi(u)(t;\nu,n+14)} \quad \text{for } 0 \leq t < T,$$

making use of Proposition 6.1.

Set

$$(7.3)' \quad B_n = \max\{1, B, \tilde{C}_n, \tilde{C}_{n+1}, \dots, \tilde{C}_{n+8}\},$$

where  $B$  and  $\tilde{C}_k$  are the constants in (7.2)' and (5.57) $_k$  respectively. Then (7.4) is replaced by

$$(7.4)' \quad \langle u \rangle_{\nu,n,t} + [\partial u]_{1,n,t} \leq \frac{1}{3B_n} \quad \text{for } 0 \leq t < T.$$

Suppose the hypothesis (A) $_1$  is satisfied. Then we see from Proposition 6.4 and the part 2) of Corollary 5.15 that

$$\|\partial u(t)\|_{n+6}^2 \leq C\varepsilon^2 + C \int_0^t (1+s)^{-2+1/6} \|\partial u\|_{n+13,s}^2 ds \quad \text{for } 0 \leq t < T,$$

since (7.1)', (7.3)' and (7.4)' imply (6.2) $_{n+13}$  and (5.57) $_k$  for  $n \leq k \leq n+8$ . Therefore (7.2)' yields

$$(7.5)' \quad \|\partial u(t)\|_{n+6}^2 \leq C_n^2 \varepsilon^2 \left( 1 + \int_0^t (1+s)^{-2+1/6+B_n\Phi(u)(s;\nu,n+14)} ds \right) \\ \text{for } 0 \leq t \leq T.$$

By virtue of the part 2) of Corollary 5.15, one can also assume that

$$(7.6)' \quad \langle u \rangle_{\nu,n,t} + [\partial u]_{1+\nu,n,t} \leq C_n(\varepsilon + \|\partial u\|_{n+6,t}) \quad \text{for } 0 \leq t < T.$$

Since (7.4)' and (7.5)' yield

$$(7.7)' \quad \|\partial u(t)\|_{n+6}^2 \leq 3C_n^2 \varepsilon^2 \quad \text{for } 0 \leq t < T,$$

we therefore obtain

$$(7.8)' \quad \langle u \rangle_{\nu, n, t} + [\partial u]_{1+\nu, n, t} \leq C_n(1 + \sqrt{3}C_n)\varepsilon \quad \text{for } 0 \leq t < T.$$

For a positive number  $\varepsilon$  satisfying (7.9) we obtain  $T(\varepsilon)$  as before, where (7.10) is replaced by

$$(7.10)' \quad \langle u \rangle_{\nu, n, t} + [\partial u]_{1+\nu, n, t} \leq 3C_n^2\varepsilon \quad \text{for } 0 \leq t < T.$$

Then we obtain (7.11) analogously to the proof of the part 1). Thus we prove as before the part 2). This completes the proof of Theorem 1.2.

### Appendix.

In this appendix we prove that condition (1.9)<sub>1</sub> implies (1.9)<sub>2</sub>. For notational convenience we assume that  $m = 1$  and  $c_1 = 1$ , and write  $p_i = p$ ,  $N^i = N$ ,  $N^{i0} = N^0$  and  $u^i = u = (u_1, u_2, \dots, u_p)$ . Besides, we abbreviate (1.4)<sub>1</sub> as

$$F(u, u', u'') = \sum_{a, b=0}^3 A^{ab}(u, u') \partial_a \partial_b u + B(u, u').$$

Then (1.9)<sub>1</sub> and (1.9)<sub>2</sub> can be written as, respectively,

$$(1) \quad \tilde{N}(\tilde{u}, v, w; X) = 0$$

for any  $\tilde{u}, v, w \in \mathbb{R}^p$  and  $X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$  satisfying  $X_0^2 = X_1^2 + X_2^2 + X_3^2$ , and

$$(2) \quad N(u, \partial u, \partial^2 u) = N^0(\partial u, \partial^2 u) + \sum_{j=1}^p u_j (\partial A^{00} / \partial u_j)(0, 0) \square u.$$

Moreover, taking  $v = w = 0$  in (1), we see easily that  $N(u, \partial u, \partial^2 u)$  is linear in  $u$ , hence one can write

$$(3) \quad \begin{aligned} & N(u, \partial u, \partial^2 u) - N^0(\partial u, \partial^2 u) \\ &= \sum_{j=1}^p u_j \left( \sum_{a, b=0}^3 C_j^{ab} \partial_a \partial_b u + \sum_{a=0}^3 D_j^a \partial_a u \right), \end{aligned}$$

where  $C_j^{ab}$  and  $D_j^a$  are constant  $p \times p$  matrices such that  $C_j^{ab} = (\partial A^{ab}/\partial u_j)(0, 0)$  and  $C_j^{ab} = C_j^{ba}$ . Therefore it suffices to prove the following three relations for each  $j = 1, 2, \dots, p$ :

$$(4) \quad D_j^a = 0 \quad \text{for} \quad a = 0, 1, 2, 3,$$

$$(5) \quad C_j^{ab} = 0 \quad \text{for} \quad a, b = 0, 1, 2, 3 \quad \text{with} \quad a \neq b$$

and

$$(6) \quad C_j^{aa} = -C_j^{00} \quad \text{for} \quad a = 1, 2, 3.$$

It follows from (1) and (3) that

$$\sum_{a,b=0}^3 X_a X_b C_j^{ab} w + \sum_{a=0}^3 X_a D_j^a v = 0$$

holds for any  $v, w \in \mathbb{R}^p$  and  $X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$  satisfying  $X_0^2 = X_1^2 + X_2^2 + X_3^2$ .

Taking  $X_0 = \pm(X_1^2 + X_2^2 + X_3^2)^{1/2}$ , we have

$$2 \sum_{a=1}^3 X_a C_j^{a0} w + D_j^0 v = 0$$

and

$$\sum_{a,b=1}^3 X_a X_b (\delta_{ab} C_j^{00} + C_j^{ab}) w + \sum_{a=1}^3 X_a D_j^a v = 0$$

for any  $v, w \in \mathbb{R}^p$  and any  $(X_1, X_2, X_3) \in \mathbb{R}^3$ . From the last identity with  $v = 0$  we get (6). Hence (4) and (5) follow. Thus we obtain (2) from (3).  $\square$

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