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Abstract

The local and global well-posedness for the Cauchy problem for a class of nonlinear Schrödinger equations is studied. The global well-posedness of the problem is proved in the Sobolev space $H^s = H^s(\mathbf{R}^n)$ of fractional order $s > n/2$ under the following assumptions: (1) Concerning the Cauchy data $\phi \in H^s$, $\|\phi; L^2\|$ is relatively small with respect to $\|\phi; \dot{H}^\sigma\|$ for any fixed σ with $n/2 < \sigma \leq s$. (2) Concerning the nonlinearity f, $f(u)$ behaves as a conformal power $u^{1+4/n}$ near zero and has an arbitrary growth rate at infinity.

1 Introduction

This is a sequel to previous papers [20, 21] of the same authors, where the Cauchy problem for the nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u = f(u) \tag{1.1}$$

is considered in space-time $I \times \mathbf{R}^n$, where $I = [-T, T]$ with $T > 0$ for local solutions and $I = \mathbf{R}$ for global solutions. Here u is a complex-valued function of $(t, x) \in I \times \mathbf{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbf{R}^n$, and $f(u)$ is a nonlinear interaction given by a complex-valued function f on $\mathbf{C}$.

There is a large literature on the Cauchy problem for (1.1) and on the asymptotic behavior in time of global solutions (see for instance [2-7, 9-22, 25-27, 29-31] and references therein). The Cauchy problem for (1.1) has been studied in the Sobolev spaces $H^k = H^k(\mathbf{R}^n)$ of integral order $k = 0, 1, 2$ [11, 15, 16, 19, 30, 31], while there arises a new interest in the Cauchy problem in the Sobolev spaces $H^s = H^s(\mathbf{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbf{R}^n)$ of fractional order $s \geq 0$ [3, 7, 17, 20, 21, 25].

To illustrate some of the latest developments of the H^s-theory for (1.1), take for example the single power interaction

$$f(u) = \lambda |u|^{p-1} u, \tag{1.2}$$

where $\lambda \in \mathbf{C}$ and $1 < p < \infty$. A homogeneity argument on (1.1) with (1.2) shows that the power p is critical [resp. subcritical] at the level of H^s if and only if $p = 1 + 4/(n-2s)$ [resp. $p < 1 + 4/(n-2s)$]. Here the critical Sobolev index s associated with the critical power $p = 1 + 4/(n-2s)$ makes sense only when $0 \leq s < n/2$ to keep the critical power finite, while the notion of subcritical power extends to Sobolev spaces H^s for all $s \geq 0$ with the convention that $1 < p < \infty$ when $s \geq n/2$.

Both in the critical and subcritical cases the available results show the existence and uniqueness of global H^s-solutions under the smallness assumption on $\|\phi; \dot{H}^{s_c}\| \equiv \|(-\Delta)^{s_c/2}\phi; L^2\|$ with $s_c = s_c(p) \equiv n/2 - 2/(p-1)$ for the Cauchy data $u(0) = \phi \in H^s$ (see for instance [3, 7, 17, 18, 21] for the critical case and [7, 17, 25] for the subcritical case). The available H^s-theory in fact admits not only the single power such as (1.2) but a wider class of nonlinearities which includes quasihomogeneous nonlinearity. To be more specific, at the cost of smallness assumption on the data, that theory does not require structural assumptions on f related to the conservation laws but do need scaling control of f on the basis of power behavior near zero and at infinity at the same rate p. Concerning the critical Sobolev index s_c associated with p we notice the following elementary facts: (1) $s_c = s$ in the critical case. (2) $s_c < s$ in the subcritical case. (3) p is critical at the level of H^{s_c}. (4) $0 \leq s_c < n/2$. Meanwhile, the H^s-theory requires that p is greater than or equal to $1 + 4/n$, which is the critical power at the level of L^2 as well as the conformal power of the nonlinear Schrödinger equations.

The notion of criticality with homogeneity argument loses its meaning at the level of $H^{n/2}$. In [20] we have nevertheless developed the $H^{n/2}$-theory for (1.1) and proved the existence and uniqueness of global $H^{n/2}$-solutions under the smallness assumption on $\|\phi; H^{n/2}\|$ for the Cauchy data $u(0) = \phi \in H^{n/2}$, where the nonlinearity is supposed to behave as a power $u^{1+4/n}$ near zero and as an exponential function $\exp(\lambda |u|^2)$ with $\lambda > 0$ at infinity. The last growth rate at infinity reminds us of Trudinger's inequality, which replaces the Sobolev embedding $H^s \hookrightarrow L^\infty$ with $s > n/2$ in the borderline case $s = n/2$. In fact, the method in [20] depends on an equivalent sharp Gagliardo-Nirenberg inequality [23, 24] and in view of Trudinger's inequality the last growth rate at infinity seems to be optimal at the level of $H^{n/2}$. Meanwhile, the available $H^{n/2}$-theory distinguishes behavior near zero of admissible nonlinearity from that at infinity and indicates that the right power near zero is $1 + 4/n$, which is the minimal power near zero in the framework of H^s-theory.

The purpose in this paper is to study the H^s-theory with $s > n/2$ and to specify admissible behavior of nonlinearity and minimal requirement on smallness of the Cauchy data to ensure the existence and uniqueness of global H^s-solutions. There are some related works on the global Cauchy problem in the Sobolev space H^k of integral order $k > n/2$ [5, 12, 13, 14, 26], though most of the available works treated the problem from the point of view of regularity and a priori estimates in H^k, starting recursively from the energy space H^1. This requires some structural assumptions on the nonlinearity and, at a technical and essential level, yields restrictions on the admissible space dimensions. We evade the difficulty caused by the classical a priori estimates and, at the cost of smallness of the Cauchy data, look for another possibility for the H^s-theory with $s > n/2$, deserving of a natural extension of the H^s-theory with $s \le n/2$. In this respect, our approach is similar to that of Kato [17], the only exception to the previous theories cited above. The method in this paper differs from that of [17] in that we use homogeneous Besov spaces, where estimates of Leibniz type on the composite function $f(u)$ can be performed efficiently with non integer derivatives and minimal differentiability on f.

In this paper we prove the existence and uniqueness of global H^s-solutions with $s > n/2$ under the following setting:

(1) The Cauchy data $\phi \in H^s$ satisfies the assumption that $\|\phi; L^2\|$ is relatively small with respect to $\|\phi; \dot{H}^\sigma\|$ for any fixed σ with $n/2 < \sigma \le s$.

(2) The nonlinear interaction f behaves as a conformal power $u^{1+4/n}$ near zero and has an arbitrary growth rate at infinity.

To state the main result precisely we introduce the following notation. For any r with $1 \le r \le \infty$, $L^r = L^r(\mathbf{R}^n)$ denotes the Lebesgue space on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $H^s_r = (1-\Delta)^{-s/2}L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $\dot{H}^s_r = (-\Delta)^{-s/2}L^r$ denotes the homogeneous Sobolev space defined in terms of Riesz potentials. For any $s \in \mathbf{R}$ and any r, m with $1 \le r, m \le \infty$, $\dot{B}^s_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj}\|\mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}u; L^r\|\}_{-\infty}^{\infty} \in \ell^m(\mathbf{Z})$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and its inverse, respectively, and φ is a nonnegative function on $\mathbf{R}^n$ with compact support in $\{x \in \mathbf{R}^n\ ;\ 1/2 \le |x| \le 2\}$ such that $\{\varphi(2^{-j}\cdot)\}_{-\infty}^{\infty}$ forms the Littlewood-Paley dyadic decomposition on $\mathbf{R}^n\backslash\{0\}$. The usual Besov space $B^s_{r,m}$ is defined analogously in terms of the dyadic decomposition on $\mathbf{R}^n$. We refer to [1, 9, 28] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity we make abbreviations such as $H^s = H^s_2$, $\dot{H}^s = \dot{H}^s_2$, $B^s_r = B^s_{r,2}$, $\dot{B}^s_r = \dot{B}^s_{r,2}$. For any interval $I \subset \mathbf{R}$ and any Banach space X we denote by $C_b(I; X)$ the space of bounded and strongly continuous functions from I to X and by $L^q(I;X)$ the space of measurable functions u from I to X such that $\|u(\cdot);X\| \in L^q(I)$. We denote by $U(t) = \exp(it\Delta)$ the free propagator, namely the one parameter unitary group which solves the free Schrödinger equation. For any r with $2 \le r \le \infty$ we define $\delta(r) = n/2 - n/r$, namely the optimal rate in time for the boundedness of the free propagator from $L^{r'}$ to L^r, where r' is the exponent dual to r defined by $1/r + 1/r' = 1$. Regarding the space-time integrability properties for the free propagator, it is convenient to call a pair of exponent (q,r) admissible if $0 \le 2/q = \delta(r) < 1$.

The Cauchy problem for the equation (1.1) for the given initial data ϕ will be treated in the form of the integral equation

$$u(t) = \Phi(u)(t) \equiv U(t)\phi - i(G_0 f(u))(t), \tag{NLS}$$

where for any $t_0 \in \mathbf{R} \cup \{\pm\infty\}$ the operator G_{t_0} is defined as

$$(G_{t_0} g)(t) \equiv \int_{t_0}^{t} U(t-\tau) g(\tau) d\tau.$$

The integral equation (NLS) will be studied in the space

$$\dot{X}_{q_0}(I) \equiv C_b(I; H^s) \cap L^{q_0}(I; B^s_{r_0} \cap \dot{B}^0_{r_0})$$

endowed with the metric d given by

$$d(u,v) \equiv \sup_{0 \le 2/q = \delta(r) \le 2/q_0} \|u - v; L^q(I; \dot{B}^0_r)\|,$$

where I is any interval on $\mathbf{R}$, q_0 is some number with $0 \le 1/q_0 \le n/2(n+2)$, and r_0 is the number that makes (q_0, r_0) admissible.

To describe the nonlinear interaction f with an arbitrary growth at infinity as well as with a vanishing behavior as a power p at zero, we introduce the following assumption $(N)_{s,p}$ with $0 \le s < \infty$ and $1 \le p < \infty$.

$(N)_{s,p}$ $f \in C^{[s]}(\mathbf{C};\mathbf{C})$ and there exists a nonnegative, nondecreasing function M on $\mathbf{R}_+$ such that for all k with $0 \le k \le [s]$ $f^{(k)}$ satisfies the estimate

$$|f^{(k)}(z)| \le |z|^{(p-k)_+} M(|z|),$$

$$|f^{([s])}(z_1) - f^{([s])}(z_2)|$$
$$\le \begin{cases} |z_1 - z_2|^{p-[s]} M(|z_1| \vee |z_2|) & \text{if } s < p < [s]+1 \\ |z_1 - z_2| (|z_1| \vee |z_2|)^{(p-[s]-1)_+} M(|z_1| \vee |z_2|) & \text{otherwise} \end{cases}$$

for all $z, z_1, z_2 \in \mathbf{C}$.

Here $f^{(k)}$ denotes any of the k-th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(k)}|$ denotes the maximum of the moduli of those derivatives. For $a, b \in \mathbf{R}$ we denote by $a \vee b$ and $a \wedge b$ the maximum and minimum of a and b, respectively, by a_+ the maximum of a and 0, and by $[a]$ the greatest integer that is less than or equal to a.

With the notation above we now state the main result in this paper. For any $s, \sigma, \gamma_0, \gamma_\sigma, \gamma_s > 0$ we define

$$\dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s) \equiv \{\phi \in H^s \ ; \ \|\phi; L^2\| \le \gamma_0, \ \|\phi; \dot{H}^\sigma\| \le \gamma_\sigma, \ \|\phi; \dot{H}^s\| \le \gamma_s\}.$$

Theorem 1. 1 *Let $n \ge 1$, $s > n/2$, $1 \le p < \infty$, and let $f \in (N)_{s,p}$. Let σ be any number with $n/2 < \sigma \le s$. Then there exists q_0 such that $0 \le 1/q_0 \le n/2(n+2)$ with the following property.*

(1) For any $\gamma_0, \gamma_\sigma > 0$ there exists $T > 0$ such that for any $\gamma_s > 0$ and any $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ (NLS) *has a unique local solution u in $\dot{X}_{q_0}([-T, T])$.*

(2) Let $(-T_*, T^*)$, $T_*, T^* > 0$, be the maximal interval of existence of the solution in (1). If $1 < p < 1 + 4/n$, then for any fixed $\gamma_\sigma > 0$, T_* and T^* are estimated from below as

$$T_* \wedge T^* \ge C\gamma_0^{-4(p-1)/(4-n(p-1))}, \tag{1.3}$$

where the positive constant C is independent of $\gamma_0, \gamma_s > 0$ and $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$. Especially T_* and T^* can be taken sufficiently large by taking γ_0 sufficiently small.

(3) If $T^* < \infty$ [resp. $T_* < \infty$], then $\|u(t); H^\sigma\|$ blows up at $t = T^*$ [resp. $t = -T_*$].

(4) If $p \ge 1 + 4/n$, then for any $\gamma_\sigma > 0$ there exists $\gamma_0 > 0$ with the following property.

(4a) For any $\gamma_s > 0$ and any data $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ at $t_0 = 0$, (NLS) *has a unique global solution in $\dot{X}_{q_0}(\mathbf{R})$. Moreover there exists a unique pair $(\phi_+, \phi_-) \in H^s \oplus H^s$ such that*

$$\|u(t) - U(t)\phi_\pm; H^s\| \to 0 \quad as \quad t \to \pm\infty. \tag{1.4}$$

(4b) For any $\gamma_s > 0$ and any data $\phi_- \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ at $t_0 = -\infty$, there exists a unique global solution u of (NLS) *in $\dot{X}_{q_0}(\mathbf{R})$ and a unique state $\phi_+ \in H^s$ such that (1.4) holds. Moreover the scattering operator $S : \phi_- \mapsto \phi_+$ is well-defined on $\dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ to H^s and is continuous in the following sense: If $\{\phi_-^j\}_{j=1}^\infty \subset \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ satisfies*

$$\|\phi_- - \phi_-^j; L^2\| \to 0 \tag{1.5}$$

as $j \to \infty$, then it follows that

$$\|\phi_+ - \phi_+^j; H^\nu\| \to 0 \tag{1.6}$$

as $j \to \infty$ for any ν with $0 \le \nu < s$, where $\phi_+^j = S(\phi_-^j)$.

(5) The solutions given by (1) and (4) have the continuous dependence on the initial data. Namely, for any initial data $\phi, \{\phi_j\}_{j=1}^\infty$ in $\dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, let $u, \{u_j\}_{j=1}^\infty$ be the corresponding solutions of (NLS) *given by (1) or (4). If $\{\phi_j\}_{j=1}^\infty$ satisfies*

$$\|\phi - \phi_j; L^2\| \to 0$$

as $j \to \infty$, then

$$d(u, u_j) \to 0 \quad \textit{and} \quad \|u - u_j; L^\infty(I; H^\nu)\| \to 0$$

as $j \to \infty$ for any ν with $0 \le \nu < s$, where $I = [-T, T]$ for (1) and $I = \mathbf{R}$ for (4).

Remark 1. The assumptions in (4) of the theorem cover for instance the nonlinearities given by complex polynomials in u and $\bar{u}$ with the lowest degree of $5, 3$, and 2 for $n = 1, 2 \le n \le 3$, and $n \ge 4$, respectively. More generally, the nonlinearities given by those polynomials multiplied by any smooth functions fall within the scope of the theorem. As regards the power behavior near zero that is compatible with minimal regularity of f, the assumption $(N)_{s,p}$ requires that $s < p$ if s is not an integer. This improves the previous assumption $[s] + 1 < p$ in [17].

Remark 2. As in the $H^{n/2}$-theory developed in [20], the H^s-theory with $s > n/2$ above distinguish behavior near zero of admissible nonlinearity from that at infinity and indicates that the right power for the former is again $1 + 4/n$, which is critical at the level of L^2 and is the minimal power near zero in the framework of the available H^s-theory with $s < n/2$. In contrast to the $H^{n/2}$-theory, however, arbitrary growth rate at infinity is admissible for the nonlinearity of the H^s-theory with $s > n/2$.

The method of proof of the theorem depends on the Strichartz estimates and on the Leibniz type estimate on the composite function $f \circ u$, both of which are described in terms of homogeneous Besov spaces. The former is the same as in [3, 7, 20, 21] and the latter partly generalizes the results in [3, 7, 8, 21]. Specifically, we prove the following proposition. In this paper we follow the following conventions: Different positive constants might be denoted by the same letter C. Different nonnegative, nondecreasing functions on $\mathbf{R}_+$ might be denoted by the same letter M.

Proposition 1.1 *Let $s > 0$ and $p \geq 1$. Let ℓ, r, m satisfy $1 \leq \ell < \infty$, $2 \leq m \leq \infty$, $2 \leq r < \infty$, $1/\ell = (p-1)/m + 1/r$. Let f satisfy $(N)_{s,p}$. Then*

$$\|f(u); L^\ell\| \leq M(\|u; L^\infty\|)\|u; L^m\|^{p-1}\|u; L^r\|, \tag{1.7}$$

$$\|f(u); \dot{B}^s_\ell\|$$
$$\leq \begin{cases} M(\|u; L^\infty\|)\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad 0 < s < 2, \\ M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_m\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad s \geq 2, \ m \neq \infty, \\ M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; L^\infty \cap \dot{B}^0_\infty\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad s \geq 2, \ m = \infty, \end{cases} \tag{1.8}$$

where $\|u; L^\infty \cap \dot{B}^0_\infty\| = \|u; L^\infty\| \vee \|u; \dot{B}^0_\infty\|$.

We prove the proposition in Section 2. The proof uses equivalent norms on homogeneous Besov spaces in terms of modulus of continuity with the first and second differences and the convexity properties and embedding theorems in homogeneous Besov spaces. We prove the main theorem in Section 3. We solve the integral equation (NLS) by a partial contraction argument in the complete metric space $\dot{X}_{q0}(I, R_0, R_\sigma, R_s)$ with $R_\mu > 0$, $\mu = 0, \sigma, s$.

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2 Proof of the proposition

In this section we prove the proposition. The proof is carried out along the line of the argument in [7, 21] with a number of necessary modifications due to the extra factor M and the exceptional case $s \geq p$ in the assumption $(N)_{s,p}$. For completeness we start all over again and give a full proof appropriate for the present setting, though the major part looks as a repetition of the previous argument.

We use the following equivalent norm on the homogeneous Besov space $\dot{B}^s_{\ell,q}$

$$\|v; \dot{B}^s_{\ell,q}\| \simeq \sum_{|\alpha|=N} \left(\int_0^\infty t^{-1-\sigma q} \sup_{|y|\leq t} \|\tau_y \partial^\alpha v + \tau_{-y}\partial^\alpha v - 2\partial^\alpha v; L^\ell\|^q dt\right)^{1/q} \tag{2.1}$$

which holds for $s = N + \sigma$ with a nonnegative integer N and $0 < \sigma < 2$, where τ_y is the translation by $y \in \mathbf{R}^n$, as well as the norm

$$\|v; \dot{B}^s_{\ell,q}\| \simeq \sum_{|\alpha|=N} \left(\int_0^\infty t^{-1-\sigma q} \sup_{|y|\leq t} \|\tau_y \partial^\alpha v - \partial^\alpha v; L^\ell\|^q dt\right)^{1/q} \tag{2.2}$$

which holds for $s = N + \sigma$ with $0 < \sigma < 1$. The inequality (1.7) follows from the Hölder inequality. For the first line in (1.8) with $0 < s < 1$ we use (2.2) with $N = 0$ and $0 < \sigma = s < 1$, the last inequality in $(N)_{s,p}$ with $(p - [s] - 1)_+ = p - 1$, and the Hölder inequality. From now on we prove the proposition with $s \geq 1$ for real-valued functions for simplicity since the proof for complex-valued functions is analogous if we regard f as a function of two variables u and $\bar{u}$. We first consider the case $1 \leq s < 2$. For that purpose we use (2.1) with $N = 0$ and $1 \leq \sigma = s < 2$. For simplicity we put

$$v = \tau_y u, \quad w = \tau_{-y} u.$$

Then the second difference of $f(u)$ is written as

$$\begin{aligned}
& \tau_y(f(u)) + \tau_{-y}(f(u)) - 2f(u) \\
= \ & f(v) + f(w) - 2f(u) \\
= \ & f'(u)(v + w - 2u) + \int_0^1 (f'(\theta v + (1-\theta)u) - f'(u))d\theta(v - u) \\
& + \int_0^1 (f'(\theta w + (1-\theta)u) - f'(u))d\theta(w - u). \qquad (2.3)
\end{aligned}$$

Here we distinguish three cases: (a) $s < 2 \leq p$. (b)$s < p < 2$. (c) $p \leq s < 2$.

• The case (a): $s < 2 \leq p$.
The term on the RHS of (2.3) is estimated by

$$\begin{aligned}
M(\|u; L^\infty\|)(|u|^{p-1}|v + w - 2u| + (|v| \vee |u|)^{p-2}|v - u|^2 \\
+(|w| \vee |u|)^{p-2}|w - u|^2). \qquad (2.4)
\end{aligned}$$

We substitute (2.4) into (2.1) and use the Hölder inequality to reproduce Besov norms on the basis of (2.1) and (2.2) to obtain

$$\begin{aligned}
& \|f(u); \dot{B}^s_\ell\| \\
\leq \ & M(\|u; L^\infty\|)(\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\| + \|u; L^m\|^{p-2}\|u; \dot{B}^{s/2}_{2q,4}\|^2), \quad (2.5)
\end{aligned}$$

where $1/q = 1/\ell - (p-2)/m$. Regarding the last norm on the RHS of (2.5), we use the interpolation inequality [8]

$$\|u; \dot{B}^{s/2}_{2q,4}\| \leq \|u; \dot{B}^s_r\|^{1/2}\|u; \dot{B}^0_{m,\infty}\|^{1/2}$$

with $1/q = 1/r + 1/m$ and the embedding $L^m \hookrightarrow \dot{B}^0_{m,\infty}$ to obtain the first line in (1.8).

• The case (b): $s < p < 2$.
The term on the RHS of (2.3) is estimated by

$$M(\|u; L^\infty\|)(|u|^{p-1}|v + w - 2u| + |v - u|^p + |w - u|^p). \tag{2.6}$$

We substitute (2.6) into (2.1) and use the Hölder inequality to reproduce Besov norms on the basis of (2.1) and (2.2) to obtain

$$\begin{aligned} &\|f(u); \dot{B}^s_\ell\| \\ \leq\ & M(\|u; L^\infty\|)(\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\| + \|u; \dot{B}^{s/p}_{\ell p, 2p}\|^p). \end{aligned} \tag{2.7}$$

Regarding the last norm on the RHS of (2.7), we use the interpolation inequality [8]

$$\|u; \dot{B}^{s/p}_{\ell q, 2p}\| \leq \|u; \dot{B}^s_r\|^{1/p}\|u; \dot{B}^0_{m,\infty}\|^{1-1/p}$$

with $s/p = \theta s$, $1/\ell p = \theta/r + (1-\theta)/m$, $\theta = 1/p$, and the embedding $L^m \hookrightarrow \dot{B}^0_{m,\infty}$ to obtain the first line in (1.8).

• The case (c): $p \leq s < 2$.
The term on the RHS of (2.3) is estimated by

$$M(\|u; L^\infty\|)(|u|^{p-1}|v + w - 2u| + |v - u|^2 + |w - u|^2). \tag{2.8}$$

We substitute (2.8) into (2.1) and use the Hölder inequality to reproduce Besov norms on the basis of (2.1) and (2.2) to obtain

$$\begin{aligned} &\|f(u); \dot{B}^s_\ell\| \\ \leq\ & M(\|u; L^\infty\|)(\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\| + \|u; \dot{B}^{s/2}_{2\ell,4}\|^2). \end{aligned} \tag{2.9}$$

Regarding the last norm on the RHS of (2.9), we use the interpolation inequality [8]

$$\begin{aligned} \|u; \dot{B}^{s/2}_{2\ell,4}\| &\leq \|u; \dot{B}^s_r\|^{1/2}\|u; \dot{B}^0_{m/(p-1),\infty}\|^{1/2} \\ &\leq \|u; \dot{B}^s_r\|^{1/2}\|u; \dot{B}^0_{m,\infty}\|^{(p-1)/2}\|u; \dot{B}^0_{\infty,\infty}\|^{(2-p)/2} \end{aligned}$$

with the embedding $L^m \hookrightarrow \dot{B}^0_{m,\infty}$ $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$. Then the RHS of (2.9) is estimated by

$$M(\|u; L^\infty\|)(1 + \|u; L^\infty\|^{2-p})\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\|$$

and therefore the first line in (1.8) follows by the preceding argument and by replacing $M(\rho)$ by $M(\rho)(1+\rho)^{2-p}$, though the latter is still denoted by $M(\rho)$ by the convention made before.

We next consider the case $s \geq 2$. We decompose $s = N + \sigma$ with $1 \leq \sigma < 2$ and an integer $N \geq 1$ and we use (2.1). For that purpose we differentiate $f \circ u$ with respect to the spatial variables of total order $|\alpha| = N$ to obtain

$$\partial^\alpha(f(u)) = \sum_{k=1}^{N} \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha \\ |\beta_j|\geq 1}} \frac{\alpha!}{k! \prod_{j=1}^{k} \beta_j!} f^{(k)}(u) \prod_{j=1}^{k} \partial^{\beta_j} u.$$

We consider the second differences of the terms of the form

$$f^{(k)}(u)\partial^{\beta_1}u \cdots \partial^{\beta_k}u \tag{2.10}$$

with $1 \leq k \leq N$, $\beta_1 + \cdots + \beta_k = \alpha$, $|\alpha| = N$. For simplicity we put

$$\begin{aligned} &F(u) = f^{(k)}(u), \quad v = \tau_y u, \quad w = \tau_{-y} u, \\ &u_j = \partial^{\beta_j} u, \quad v_j = \partial^{\beta_j} v, \quad w_j = \partial^{\beta_j} w. \end{aligned}$$

Then the second difference of (2.10) is written as

$$\begin{aligned}
& \tau_y(f^{(k)}(u) \prod_j \partial^{\beta_j} u) + \tau_{-y}(f^{(k)}(u) \prod_j \partial^{\beta_j} u) - 2 f^{(k)}(u) \prod_j \partial^{\beta_j} u \\
= \;& F(v) \prod_j v_j + F(w) \prod_j w_j - 2F(u) \prod_j u_j \\
= \;& F(u)(\prod_j v_j + \prod_j w_j - 2\prod_j u_j) + (F(v) - F(u))(\prod_j v_j - \prod_j u_j) \\
& +(F(w) - F(u))(\prod_j w_j - \prod_j u_j) + (F(v) + F(w) - 2F(u)) \prod_j u_j \\
= \;& F(u) \sum_j (v_j + w_j - 2u_j) \prod_{a<j} u_a \prod_{b>j} v_b \\
& -F(u) \sum_j \sum_{i>j} (w_j - u_j)(v_i - u_i) \prod_{\substack{a<i \\ a\neq j}} u_a \prod_{b>i} v_b \\
& +F(u) \sum_j \sum_{i>j} (w_j - u_j)(w_i - u_i) \prod_{\substack{a<i \\ a\neq j}} u_a \prod_{b>i} w_b \\
& +(F(v) - F(u)) \sum_j (v_j - u_j) \prod_{a<j} u_a \prod_{b>j} v_b \\
& +(F(w) - F(u)) \sum_j (w_j - u_j) \prod_{a<j} u_a \prod_{b>j} w_b \\
& +F'(u)(v + w - 2u) \prod_j u_j
\end{aligned}$$

$$+\int_0^1 (F'(\theta v+(1-\theta)u)-F'(u))d\theta(v-u)\prod_j u_j$$

$$+\int_0^1 (F'(\theta w+(1-\theta)u)-F'(u))d\theta(w-u)\prod_j u_j \tag{2.11}$$

According to the decomposition above, we denote by I, II, III , $\cdots$, VIII the first, second, $\cdots$, eighth terms on the RHS of the last equality of (2.11), respectively. To estimate I in L^ℓ by the Hölder inequality we define the exponents m_a with $1 \le a \le k$ by

$$1/m_a = \begin{cases} \mu_a/r + (1-\mu_a)/m & \text{if } \ p \ge k, \\ \mu_a/r + (1-\mu_a)(p-1)/m(k-1) & \text{if } \ p < k, \end{cases}$$

where

$$\mu_a = \begin{cases} |\beta_a|/s & \text{if } \ a \ne j, \\ (|\beta_a|+\sigma)/s & \text{if } \ a = j. \end{cases}$$

We now distinguish two cases: (a) $p \ge k$. (b) $p < k$.

• The case (a): $p \ge k$.

We estimate I in L^ℓ as

$$\begin{aligned} &\|\mathrm{I}; \mathrm{L}^\ell\| \\ \le\ & \sum_j \|F(u); L^{m/(p-k)}\| \|v_j + w_j - 2u_j; L^{m_j}\| \prod_{a<j} \|u_a; L^{m_a}\| \prod_{b>j} \|v_b; L^{m_b}\| \\ \le\ & M(\|u; L^\infty\|)\|u; L^m\|^{p-k} \sum_j \|v_j + w_j - 2u_j; L^{m_j}\| \prod_{a\ne j} \|u; \dot{H}^{|\beta_a|}_{m_a}\|, \end{aligned} \tag{2.12}$$

where we have used the Hölder inequality with $\sum_a 1/m_a + (p-k)/m = 1/\ell$. On the RHS of the last inequality of (2.12), the dependence on $y \in \mathbf{R}^n$ is given by the L^{m_j}-norms, from which the Besov norms are reproduced as

$$\begin{aligned} &\left(\int_0^\infty t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{I}; \mathrm{L}^\ell\|^2 \mathrm{dt}\right)^{1/2} \\ \le\ & M(\|u; L^\infty\|)\|u; L^m\|^{p-k} \sum_j \|u; \dot{B}^{|\beta_j|+\sigma}_{m_j}\| \prod_{a\ne j} \|u; \dot{H}^{|\beta_a|}_{m_a}\|. \end{aligned} \tag{2.13}$$

We estimate the Besov and Sobolev norms on the RHS of (2.13) by interpolation inequalities and embedding theorems. By the definition of m_j, μ_j, we have

$$\|u; \dot{B}^{|\beta_j|+\sigma}_{m_j}\| \le \|u; \dot{B}^s_r\|^{\mu_j} \|u; \dot{B}^0_m\|^{1-\mu_j}. \tag{2.14}$$

By a similar computation and the embedding $\dot{B}^s_p \hookrightarrow \dot{H}^s_p$ with $2 \le p < \infty$, we have

$$\|u; \dot{H}^{|\beta_a|}_{m_a}\| \le C\|u; \dot{B}^{|\beta_a|}_{m_a}\| \le C\|u; \dot{B}^s_r\|^{\mu_a} \|u; \dot{B}^0_m\|^{1-\mu_a}, \tag{2.15}$$

$$\|u; L^m\| \le C\|u; \dot B^0_m\| \quad \text{if } m \neq \infty. \tag{2.16}$$

By (2.14), (2.15), and (2.16), the RHS of (2.13) is estimated by

$$\begin{cases} M(\|u; L^\infty\|)\|u; \dot B^0_m\|^{p-1}\|u; \dot B^s_r\| & \text{if } m \neq \infty, \\ M(\|u; L^\infty\|)\|u; L^\infty \cap \dot B^0_\infty\|^{p-1}\|u; \dot B^s_\ell\| & \text{if } m = \infty. \end{cases} \tag{2.17}$$

• The case: (b) $p < k$.
We estimate I in L^ℓ as

$$\begin{aligned} &\|\mathrm{I}; \mathrm{L}^\ell\| \\ \le\ & \sum_j \|F(u); L^\infty\| \|v_j + w_j - 2u_j; L^{m_j}\| \prod_{a \neq j} \|u; \dot H^{|\beta_a|}_{m_a}\|, \end{aligned} \tag{2.18}$$

where we have used the Hölder inequality with $\sum_a 1/m_a = 1/\ell$. As in (2.13) we obtain

$$\begin{aligned} &\left(\int_0^\infty t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{I}; \mathrm{L}^\ell\|^2 \mathrm{d}t\right)^{1/2} \\ \le\ & M(\|u; L^\infty\|) \sum_j \|u; \dot B^{|\beta_j|+\sigma}_{m_j}\| \prod_{a \neq j} \|u; \dot H^{|\beta_a|}_{m_a}\|. \end{aligned} \tag{2.19}$$

We estimate the Besov and Sobolev norms on the RHS of (2.19) by interpolation inequalities and embedding theorems. By the definition of m_j, μ_j, we have

$$\begin{aligned} &\|u; \dot B^{|\beta_j|+\sigma}_{m_j}\| \le \|u; \dot B^s_r\|^{\mu_j} \|u; \dot B^0_{(k-1)m/(p-1)}\|^{1-\mu_j} \\ \le\ & \|u; \dot B^s_r\|^{\mu_j} \|u; \dot B^0_m\|^{(1-\mu_j)(p-1)/(k-1)} \|u; \dot B^0_\infty\|^{(1-\mu_j)(k-p)/(k-1)}. \end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned} &\|u; \dot H^{|\beta_a|}_{m_a}\| \le C\|u; \dot B^{|\beta_a|}_{m_a}\| \\ \le\ & C\|u; \dot B^s_r\|^{\mu_a} \|u; \dot B^0_m\|^{(1-\mu_a)(p-1)/(k-1)} \|u; \dot B^0_\infty\|^{(1-\mu_a)(k-p)/(k-1)}. \end{aligned} \tag{2.21}$$

By (2.20) and (2.21), the RHS of (2.19) is estimated by

$$M(\|u; L^\infty\|)\|u; \dot B^0_\infty\|^{k-p}\|u; \dot B^0_m\|^{p-1}\|u; \dot B^s_r\|. \tag{2.22}$$

This proves that the contribution of I is estimated in the form of the LHS of (2.13) and (2.19) by the RHS of the second and third lines in (1.8) with $M(\rho)$ replaced by $M(\rho)(1+\rho)^{(N-p)_+}$. A similar argument works on the contribution of VI and implies that the integral on the LHS of (2.13) and (2.19) with I replaced by VI is estimated by the RHS of the second and third lines in (1.8) with $M(\rho)$ replaced by $M(\rho)(1+\rho)^{(N+1-p)_+}$.

We next consider the contribution of II. We define the exponents m_a with $1 \le a \le k$ by

$$1/m_a = \begin{cases} \mu_a/r + (1-\mu_a)/m & \text{if } \ p \ge k, \\ \mu_a/r + (1-\mu_a)(p-1)/m(k-1) & \text{if } \ p < k, \end{cases}$$

where

$$\mu_a = \begin{cases} |\beta_a|/s & \text{if } \ a \ne i, j, \\ (|\beta_a| + \sigma/2)/s & \text{if } \ a = i, j. \end{cases}$$

By the Hölder inequality we estimate II in L^ℓ in the same way as in the preceding argument and we substitute the resulting estimate into the integral on the LHS of (2.13) with I replaced by II, where the dependence on y is given only by two factors $\|w_j - u_j; L^{m_j}\|$ and $\|v_i - u_i; L^{m_i}\|$. Then we use the Schwarz inequality for the integral with respect to t and we reconstruct Besov norms in the form of (2.2) to obtain

$$\begin{aligned} & \Big(\int_0^\infty t^{-1-2\sigma} \sup_{|y|\le t} \|\mathrm{II}; L^\ell\|^2 dt\Big)^{1/2} \\ \le\ & M(\|u; L^\infty\|)\|u; L^m\|^{(p-k)_+} \\ & \sum_j \sum_{i>j} \|u; \dot{B}^{|\beta_j|+\sigma/2}_{m_j,4}\| \|u; \dot{B}^{|\beta_i|+\sigma/2}_{m_i,4}\| \prod_{a\ne i,j} \|u; \dot{H}^{|\beta_a|}_{m_a}\|. \end{aligned} \tag{2.23}$$

We use the interpolation inequalities

$$\begin{aligned} & \|u; \dot{B}^{|\beta_a|+\sigma/2}_{m_a}\| \\ \le & \begin{cases} \|u; \dot{B}^s_r\|^{\mu_a} \|u; \dot{B}^0_m\|^{1-\mu_a} & \text{if } \ p \ge k, \\ \|u; \dot{B}^s_r\|^{\mu_a} \|u; \dot{B}^0_m\|^{(1-\mu_a)(p-1)/(k-1)} \|u; \dot{B}^0_\infty\|^{(1-\mu_a)(k-p)/(k-1)} & \\ & \text{if } \ p < k, \end{cases} \end{aligned}$$

for $a = i, j$, the embedding $\dot{B}^s_m \hookrightarrow \dot{B}^s_{m,4}$, and (2.15) and (2.21) for $a \ne i, j$ on the RHS of (2.23) to conclude that the LHS of (2.23) is estimated by the RHS of the second and third lines in (1.8) with $M(\rho)$ replaced by $M(\rho)(1+\rho)^{(N-p)_+}$. The same estimate holds for the LHS of (2.23) with II replaced by III . A similar argument shows that the LHS of (2.23) with II replaced by IV and V is estimated by the RHS of the second and third lines in (1.8) with $M(\rho)$ replaced by $M(\rho)(1+\rho)^{(N+1-p)_+}$. Moreover, the LHS of (2.23) with II replaced by VII and VIII is shown to be estimated by the RHS of the second and third lines in (1.8) with $M(\rho)$ replaced by $M(\rho)(1+\rho)^{(N+2-p)_+}$ except when $k = N$ and $s = N + \sigma < p < N+2 = [s]+1$. The last case is treated exactly in the same way as in [21] by taking the additional factor $M(\|u; L^\infty\|)$ into account. □

3 Proof of the theorem

In this section we prove the theorem. We use the following estimates of Strichartz type, and embeddings to control the behavior of the nonlinear term at infinity in the framework of homogeneous Besov spaces. For any interval $I \subset \mathbf{R}$ possibly unbounded, $|I|$ denotes the length of I and $\bar{I}$ denotes its closure in $\mathbf{R} \cup \{\pm\infty\}$ equipped with the natural topology.

Lemma 3. 1 [2, 3, 7, 16, 32] *The free propagator U satisfies the following estimates.*
(1) For any $s \in \mathbf{R}$ and any admissible pair (q, r)

$$\|U(\cdot)\phi; L^q(\mathbf{R}; \dot{B}^s_r)\| \le C\|\phi; \dot{H}^s\|.$$

(2) For any $s \in \mathbf{R}$, any admissible pairs (q_1, r_1) and (q_2, r_2), for any interval $I \subset \mathbf{R}$, and for any $t_0 \in \bar{I}$, the operator G_{t_0} satisfies the estimate

$$\|G_{t_0}v; L^{q_1}(I; \dot{B}^s_{r_1})\| \le C\|v; L^{q_2'}(I; \dot{B}^s_{r_2'})\|,$$

where the constant C is independent of I and t_0.

Lemma 3. 2 *Let $n \ge 1$, $-\infty < \mu_0 < n/2 < \mu < \infty$. Then*

$$\dot{H}^{\mu_0} \cap \dot{H}^{\mu} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow L^\infty \cap \dot{B}^0_{\infty,2}.$$

Moreover,

$$\begin{aligned} \|u; L^\infty\| \vee \|u; \dot{B}^0_{\infty,2}\| &\le C\|u; \dot{B}^0_{\infty,1}\| \\ &\le C\|u; \dot{H}^\mu\|^\theta \|u; \dot{H}^{\mu_0}\|^{1-\theta}, \end{aligned}$$

where $\theta = (n/2 - \mu_0)/(\mu - \mu_0)$.

Proof of Lemma 3. 2 . Let φ be the function appearing in the definition of the Besov space. Since $\sum_{k=j-1}^{j+1} \varphi(2^{-k}\cdot)\varphi(2^{-j}\cdot) = \varphi(2^{-j}\cdot)$ for any $j \in \mathbf{Z}$, we have

$$\|u; \dot{B}^0_{\infty,1}\| \le \sum_{j\in\mathbf{Z}} \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}\varphi(2^{-k}\cdot)\varphi(2^{-j}\cdot)\mathcal{F}u; L^\infty\|.$$

By the Young inequality and the equality

$$\|\mathcal{F}^{-1}(\varphi(2^{-k}\cdot)); L^2\| = 2^{nk/2}\|\varphi; L^2\|,$$

we have

$$\|u; \dot{B}^0_{\infty,1}\| \le C\sum_{j\in\mathbf{Z}} 2^{nj/2}\|\mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}u; L^2\|.$$

For any $\ell \in \mathbf{Z}$, factorizing $2^{nj/2}$ as $2^{(n/2-\mu)j}2^{\mu j}$ for $j \geq \ell$, $2^{(n/2-\mu_0)j}2^{\mu_0 j}$ for $j < \ell$, we have

$$\begin{aligned} & \|u; \dot{B}^0_{\infty,1}\| \\ \leq \; & C\sum_{j\geq\ell} 2^{(n/2-\mu)j}\|u; \dot{B}^{\mu}_{2,\infty}\| + C\sum_{j<\ell} 2^{(n/2-\mu_0)j}\|u; \dot{B}^{\mu_0}_{2,\infty}\|, \end{aligned} \tag{3.1}$$

where we have used the definition $\|u; \dot{B}^{\rho}_{2,\infty}\| = \sup_j 2^{\rho j}\|\mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}u; L^2\|$. Since the RHS of (3.1) is estimated by

$$(C2^{(n/2-\mu_0)\ell}a^{-\theta} + C2^{(n/2-\mu)\ell}a^{1-\theta})\|u; \dot{B}^{\mu_0}_{2,\infty}\|^{1-\theta}\|u; \dot{B}^{\mu}_{2,\infty}\|^{\theta},$$

with $a \equiv \|u; \dot{B}^{\mu}_{2,\infty}\|/\|u; \dot{B}^{\mu_0}_{2,\infty}\|$ and $\theta \equiv (n/2-\mu_0)/(\mu-\mu_0)$, we put

$$\ell = \left[\frac{1}{\mu-\mu_0}\log_2 a\right],$$

to obtain $2^{(n/2-\mu_0)\ell}a^{-\theta} \leq 1$ and $2^{(n/2-\mu)\ell}a^{1-\theta} \leq 2^{\mu-n/2}$. Therefore we have

$$\|u; \dot{B}^0_{\infty,1}\| \leq C\|u; \dot{B}^{\mu_0}_{2,\infty}\|^{1-\theta}\|u; \dot{B}^{\mu}_{2,\infty}\|^{\theta}.$$

By the embedding $\dot{H}^{\rho} = \dot{B}^{\rho}_{2,2} \hookrightarrow \dot{B}^{\rho}_{2,\infty}$ for any $\rho \in \mathbf{R}$, we obtain the first embedding in the lemma.

Concerning the second embedding, $\dot{B}^0_{\infty,1} \hookrightarrow \dot{B}^0_{\infty,2}$ is obvious (see [1, Theorem 6.3.1]). As for $\dot{B}^0_{\infty,1} \hookrightarrow L^{\infty}$, since $\sum_{j\in\mathbf{Z}}\varphi(2^{-j}x) = 1$ for $x \in \mathbf{R}^n \setminus \{0\}$, we have

$$\begin{aligned} \|u; L^{\infty}\| &\leq \sum_{j\in\mathbf{Z}} \|\mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}u; L^{\infty}\| \\ &= \|u; \dot{B}^0_{\infty,1}\|, \end{aligned}$$

as required. □

To solve (NLS) by a contraction argument we define the function space as follows. For any interval $I \subset \mathbf{R}$, $s > 0$, $\sigma > 0$, $R_\mu > 0$, $\mu = 0, \sigma, s$, and $1 \leq q_0 \leq \infty$, let $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$ be a function space given by

$$\dot{X}_{q_0}(I, R_0, R_\sigma, R_s) \equiv \{u \; ; \; \|u; \dot{X}^{\mu}_{q_0}(I)\| \leq R_\mu, \; \mu = 0, \sigma, s\}$$

endowed with the metric d defined by $d(u,v) = \|u - v; \dot{X}^0_{q_0}(I)\|$, where

$$\|u; \dot{X}^{\mu}_{q_0}(I)\| \equiv \sup_{0\leq 2/q=\delta(r)\leq 2/q_0} \|u; L^q(I; \dot{B}^{\mu}_r)\|$$

for any $\mu \in \mathbf{R}$.

Lemma 3. 3 *Let $0 \le 1/q_0 \le n/2(n+2)$. Let $0 < \sigma \le s$. Then for any interval $I \subset \mathbf{R}$, $R_\mu > 0$, $\mu = 0, \sigma, s$, $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$ is a complete metric space.*

Proof of Lemma 3. 3 . Let $\{u_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$. Then $\sup_{j\ge1} \|u_j; L^\infty(I; \dot{H}^\mu)\| \le R_\mu$ for any $\mu = 0, \sigma, s$. By the definition of d, there exists u in $L^\infty(I; \dot{H}^0)$ such that u_j strongly converges to u in $L^\infty(I; \dot{H}^0)$. Especially it follows that $\|u; L^\infty(I; \dot{H}^0)\| \le R_0$. Since $\dot{H}^\mu$ is reflexive for any $\mu \in \mathbf{R}$, for almost every $t \in I$ there exists a subsequence $\{u_{j,1}(t)\}_{j=1}^{\infty} \subset \{u_j(t)\}_{j=1}^{\infty}$ and $u^\mu(t) \in \dot{H}^\mu$, $\mu = \sigma, s$, such that $u_{j,1}(t)$ weakly converges to $u^\mu(t)$ in $\dot{H}^\mu$ and $\|u^\mu(t); \dot{H}^\mu\| \le R_\mu$ for $\mu = \sigma, s$. Since we have

$$\mathcal{F}^{-1} C_0^\infty(\mathbf{R}^n \setminus \{0\}) \subset \dot{H}^{-\mu} = (\dot{H}^\mu)^*$$

for any $\mu \in \mathbf{R}$, for any $\chi \in \mathcal{F}^{-1} C_0^\infty(\mathbf{R}^n \setminus \{0\})$ the pairing $\langle u_{j,1}(t), \chi\rangle$ converges to $\langle u^\mu(t), \chi\rangle$ for $\mu = \sigma, s$, and to $\langle u(t), \chi\rangle$ as $j \to \infty$. Therefore we have $u(t) = u^\mu(t)$ in $\mathcal{S}'/\mathcal{P}$, so that $\|u(t); \dot{H}^\mu\| \le R_\mu$ for $\mu = \sigma, s$, where $\mathcal{P}$ denotes the set of polynomials. Since the last inequality is preserved when t runs almost everywhere on I, we obtain $\|u; L^\infty(I; \dot{H}^\mu)\| \le R_\mu$ for $\mu = \sigma, s$.

When $q_0 = \infty$, the proof of the lemma is completed by the above argument. We consider the case $q_0 \ne \infty$ in the following. Let r_0 be a number such that (q_0, r_0) is an admissible pair. By the definition of d, there exists v in $L^{q_0}(I; \dot{B}^0_{r_0})$ such that u_j strongly converges to v in $L^{q_0}(I; \dot{B}^0_{r_0})$ as $j \to \infty$. Especially we have $\|v; L^{q_0}(I; \dot{B}^0_{r_0})\| \le R_0$. Since $L^{q_0}(I; \dot{B}^\mu_{r_0})$ is reflexive for any $\mu \in \mathbf{R}$ and $\sup_{j\ge1} \|u_j; L^{q_0}(I; \dot{B}^\mu_{r_0})\| \le R_\mu$ for $\mu = \sigma, s$, there exists a subsequence $\{u_{j,2}\}_{j=1}^{\infty} \subset \{u_j\}_{j=1}^{\infty}$ and $v^\mu \in L^{q_0}(I; \dot{B}^\mu_{r_0})$ for $\mu = \sigma, s$ such that $u_{j,2}$ weakly converges to v^μ in $L^{q_0}(I; \dot{B}^\mu_{r_0})$ and $\|v^\mu; L^{q_0}(I; \dot{B}^\mu_{r_0})\| \le R_\mu$ for $\mu = \sigma, s$. Now since $\{u_j\}$ satisfies $\sup_{j\ge1} \|u_j; L^{q_0}(I; \dot{B}^s_{r_0})\| \le R_s$, by the embeddings $\dot{B}^0_r \hookrightarrow L^r$ for any $2 \le r < \infty$, $L^r \cap \dot{B}^\mu_r \hookrightarrow B^\mu_r$ for any $\mu > 0$, $1 \le r \le \infty$, we have $\sup_{j\ge1} \|u_j; L^{q_0}(I; B^s_{r_0})\| \le CR_0 \vee R_s$. Since $L^{q_0}(I; B^s_{r_0})$ is reflexive, there exists a subsequence $\{u_{j,3}\}_{j=1}^{\infty} \subset \{u_{j,2}\}_{j=1}^{\infty}$ and $w \in L^{q_0}(I; B^s_{r_0})$ such that $u_{j,3}$ weakly converges to w in $L^{q_0}(I; B^s_{r_0})$ as $j \to \infty$. By the embedding $B^s_{r_0} \hookrightarrow B^\sigma_{r_0}$, $u_{j,3}$ also weakly converges to w in $L^{q_0}(I; B^\sigma_{r_0})$. Therefore by the embeddings $B^\mu_{r_0} \hookrightarrow \dot{B}^\mu_{r_0}$ for $\mu = \sigma, s$, we have $w = v^\mu$ in $L^{q_0}(I; \dot{B}^\mu_{r_0})$ for $\mu = \sigma, s$. Since u_j also converges to v in $L^{q_0}(I; L^{r_0})$ by the embedding $\dot{B}^0_{r_0} \hookrightarrow L^{r_0}$, $2 \le r_0 < \infty$, by the embedding $B^s_{r_0} \hookrightarrow L^{r_0}$, we have $w = v$ in $L^{q_0}(I; L^{r_0})$. So that we have $v(t) = v^\mu(t)$ in $\mathcal{S}'/\mathcal{P}$ for almost every $t \in I$, and $\|v; L^{q_0}(I; \dot{B}^\mu_{r_0})\| \le R_\mu$ for $\mu = \sigma, s$.

Since u_j strongly converges to u in $L^\infty(I; \dot{H}^0)$ and to v in $L^{q_0}(I; \dot{B}^0_{r_0})$ as $j \to \infty$, there exists a subsequence $\{u_{j,4}\}_{j=1}^{\infty} \subset \{u_j\}_{j=1}^{\infty}$ such that for almost every $t \in I$, $u_{j,4}(t)$ strongly converges to $u(t)$ in $\dot{H}^0$ and to $v(t)$

in $\dot{B}^0_{r_0}$ as $j \to \infty$. Since $\mathcal{F}^{-1}C_0^\infty(\mathbf{R}^n \setminus \{0\}) \subset (\dot{H}^0)^* \cap (\dot{B}^0_{r_0})^*$, we have $\langle u(t), \chi \rangle = \langle v(t), \chi \rangle$ for any $\chi \in \mathcal{F}^{-1}C_0^\infty(\mathbf{R}^n \setminus \{0\})$, so that $u(t) = v(t)$ in $\mathcal{S}'/\mathcal{P}$ for almost every $t \in I$.

For any $\mu = 0, \sigma, s$, and any admissible pair (q, r) with $1/q \leq 1/q_0$, by the convex inequality

$$\|w; L^q(I; \dot{B}^\mu_r)\| \leq C\|w; L^\infty(I; \dot{H}^\mu)\|^{1-q_0/q}\|w; L^{q_0}(I; \dot{B}^\mu_{r_0})\|^{q_0/q},$$

we have $d(u_j, u) \to 0$ as $j \to \infty$ and $\|u; \dot{X}^\mu_{q_0}(I)\| \leq R_\mu$ for $\mu = 0, \sigma, s$. Consequently we obtain the lemma. □

Let n, s, p, f, σ be as in Theorem 1. 1 . Let p_0 satisfy $1 \leq p_0 \leq (1+4/n) \wedge p$ and $p_0 \neq 1$ if $p \neq 1$. Let q_0 be the number given by

$$\frac{1}{q_0} = \frac{n}{4}\left(1 - \frac{n+4}{(n+2)p_0}\right)_+ .$$

Then q_0 satisfies $0 \leq 1/q_0 \leq n/2(n+2)$. Let $\tilde{q}$ be the number given by

$$\frac{1}{\tilde{q}} \equiv \frac{n+4}{4} - p_0\left(\frac{n}{4} - \frac{1}{q_0}\right).$$

Then $\tilde{q}$ satisfies $(n+4)/2(n+2) \leq 1/\tilde{q} \leq 1$. Let r_0, $\tilde{r}$ be numbers given by

$$1/r_0 \equiv 1/2 - 2/nq_0, \quad 1/\tilde{r} \equiv (n+4)/2n - 2/n\tilde{q},$$

and let $1/r^* \equiv (p_0 - 1)/(p-1)r_0$. Then by a direct computation, we see that (q_0, r_0), $(\tilde{q}', \tilde{r}')$ are admissible pairs, and that $r_0, \tilde{r}, r^*$ satisfy $2 \leq r_0, r^* < \infty$ and

$$1/\tilde{r} = (p-1)/r^* + 1/r_0. \tag{3.2}$$

Applying Proposition 1. 1 to the composite function $f(u)$, for any μ with $0 < \mu \leq s$ we have

$$\|f(u); \dot{B}^\mu_{\tilde{r}}\| \leq M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_{r^*}\|^{p-1}\|u; \dot{B}^\mu_{r_0}\|. \tag{3.3}$$

By the convex inequality

$$\|u; \dot{B}^0_{r^*}\| \leq \|u; \dot{B}^0_\infty\|^{1-\theta}\|u; \dot{B}^0_{r_0}\|^\theta,$$

where $\theta \equiv (p_0 - 1)/(p-1)$, the RHS of (3.3) is estimated by

$$M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_{r_0}\|^{p_0-1}\|u; \dot{B}^\mu_{r_0}\|,$$

where we denote by the same letter M the nondecreasing function on $[0,\infty)$. Let κ be a number given by

$$\kappa \equiv 1 - n(p_0 - 1)/4. \tag{3.4}$$

Then κ satisfies

$$1/\tilde{q} = \kappa + p_0/q_0, \quad \kappa \geq 0. \tag{3.5}$$

Indeed, the first equation follows from the definition of $\tilde{q}$ immediately and the second inequality follows from $p_0 \leq 1 + 4/n$. Especially we have $\kappa = 0$ if and only if $p_0 = 1 + 4/n$.

Therefore by the Hölder inequality in the time variable with (3.5), we have

$$\begin{aligned}
&\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu}_{\tilde{r}})\| \\
\leq\ & M(\|u; L^{\infty}(I; L^{\infty} \cap \dot{B}^{0}_{\infty}\|)|I|^{\kappa}\|u; L^{q_0}(I; \dot{B}^{0}_{r_0})\|^{p_0-1}\|u; L^{q_0}(I; \dot{B}^{\mu}_{r_0})\|
\end{aligned}$$

for any interval $I \subset \mathbf{R}$. Similarly, by (1.7) and the Hölder inequality in space and time variables, we also have

$$\begin{aligned}
&\|f(u); L^{\tilde{q}}(I; L^{\tilde{r}})\| \\
\leq\ & M(\|u; L^{\infty}(I; L^{\infty})\|)|I|^{\kappa}\|u; L^{q_0}(I; L^{r_0})\|^{p_0-1}\|u; L^{q_0}(I; L^{r_0})\|.
\end{aligned}$$

Therefore by Lemma 3. 2 , we have

$$\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu}_{\tilde{r}})\| \leq M(R_0 \vee R_\sigma)|I|^{\kappa}R_0^{p_0-1}R_\mu \tag{3.6}$$

for any $\mu = 0, \sigma, s$, and any $u \in \dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$, where we have used the embeddings $L^{\tilde{r}} \hookrightarrow \dot{B}^{0}_{\tilde{r}}$, $\dot{B}^{0}_{r_0} \hookrightarrow L^{r_0}$ for the case $\mu = 0$. Now, for any $t_0 \in \bar{I}$ and any data $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, let Φ_{t_0} be an operator defined as

$$\Phi_{t_0}(u)(t) \equiv U(t)\phi - iG_{t_0}(f(u))(t).$$

Since (q_0, r_0), $(\tilde{q}', \tilde{r}')$ are admissible pairs, by Lemma 3. 1 and (3.6), we have

$$\begin{aligned}
\|\Phi_{t_0}(u); \dot{X}^{\mu}_{q_0}(I)\| &\leq C\|\phi; \dot{H}^{\mu}\| + C\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu}_{\tilde{r}})\| \\
&\leq C\gamma_\mu + M(R_0 \vee R_\sigma)|I|^{\kappa}R_0^{p_0-1}R_\mu
\end{aligned} \tag{3.7}$$

for any $\mu = 0, \sigma, s$, any $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ and any $u \in \dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$.

On the other hand, by the same argument as above, we also have

$$d(\Phi_{t_0}(u), \Phi_{t_0}(v)) \leq M(R_0 \vee R_\sigma)|I|^{\kappa}R_0^{p_0-1}d(u,v) \tag{3.8}$$

for any

$$u, v \in \cup_{R>0}\dot{X}_{q_0}(I, R_0, R_\sigma, R).$$

Indeed, by Lemma 3. 1 , we have

$$d(\Phi_{t_0}(u), \Phi_{t_0}(v)) \leq C\|f(u) - f(v); L^{\tilde{q}}(I; L^{\tilde{r}})\|,$$

where we have used the embedding $L^{\tilde{r}} \hookrightarrow \dot{B}^0_{\tilde{r}}$. By the Hölder inequality in space and time variables with $1/\tilde{r} = p_0/r_0$ and (3.5), we have

$$\begin{aligned} & \|f(u) - f(v); L^{\tilde{q}}(I; L^{\tilde{r}})\| \\ \leq\ & M(\|u; L^\infty(I; L^\infty)\| \vee \|v; L^\infty(I; L^\infty)\|)|I|^\kappa \\ & (\|u; L^{q_0}(I; L^{r_0})\| \vee \|v; L^{q_0}(I; L^{r_0})\|)^{p_0-1}\|u - v; L^{q_0}(I; L^{r_0})\|. \quad (3.9) \end{aligned}$$

Therefore by Lemma 3. 2 and the embedding $\dot{B}^0_{r_0} \hookrightarrow L^{r_0}$, we obtain (3.8).

By the above argument, if $\gamma_\mu > 0$, $R_\mu > 0$, $\mu = 0, \sigma, s$, and $I \subset \mathbf{R}$ satisfy

$$C\gamma_\mu + M(R_0 \vee R_\sigma)|I|^\kappa R_0^{p_0-1} R_\mu \leq R_\mu, \quad (3.10)$$

$$M(R_0 \vee R_\sigma)|I|^\kappa R_0^{p_0-1} \leq 1/2 \quad (3.11)$$

for any $\mu = 0, \sigma, s$, then Φ_{t_0} is a contraction map on $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$. Since $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$ is a complete metric space by Lemma 3. 3 , Φ_{t_0} has a unique fixed point in $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$, and the solution u of (NLS) with the data $u(0) = \phi$ is given by the fixed point of Φ_0. Let $R_\mu \equiv 2C\gamma_\mu$, $\mu = 0, \sigma, s$ in (3.10) and (3.11), then (3.10) and (3.11) are rewritten as

$$M(\gamma_0 \vee \gamma_\sigma)|I|^\kappa \gamma_0^{p_0-1} \leq 1 \quad (3.12)$$

for some nondecreasing function M.

(1) Let $t_0 = 0$ and $I = [-T, T]$, $T > 0$, in the above argument. Since $\kappa > 0$ for $p_0 < 1 + 4/n$, for any $\gamma_\mu > 0$, $\mu = 0, \sigma$, there exists small $T > 0$ which satisfies (3.12). Then for any $\gamma_s > 0$ and any $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, (NLS) has a unique solution u in $\dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$.

We see that the solution u satisfies $u \in C(I; H^s)$ by using Lemma 3. 1 with (3.6) and the fact that the free propagator forms a unitary group in H^s.

The solution u is also a unique solution of (NLS) in $\dot{X}_{q_0}(I)$ by the following lemma. Indeed, if u and v are the solutions with the same initial data, then by that lemma, there exists $\varepsilon > 0$ such that $u(t) = v(t)$ in H^s for any $t \in [-\varepsilon, \varepsilon] \cap I$. By repetition of the argument with $t_1 = \pm j\varepsilon$, $j \geq 1$, we have $u(t) = v(t)$ for any $t \in I$.

Lemma 3. 4 *Let u, v be the solutions of* (NLS) *in $\dot{X}_{q_0}(I)$. Then there exists $\varepsilon > 0$ with the following property. If $u(t_1) = v(t_1)$ in H^s for some $t_1 \in I$, then $u(t) = v(t)$ in H^s for any $t \in [t_1 - \varepsilon, t_1 + \varepsilon] \cap I$.*

Proof of Lemma 3.4. Let $R_\mu > 0$, $\mu = 0, \sigma, s$, be numbers such that $u, v \in \dot{X}_{q_0}(I, R_0, R_\sigma, R_s)$. Since u and v satisfy $u(t_1) = v(t_1)$ and

$$w(t) = \Phi(w)(t) = U(t - t_1)w(t_1) + \int_{t_1}^{t} U(t-s)f(w(s))ds$$

for $w = u, v$, by the same argument on (3.8), we have

$$d_{t_1,\varepsilon}(\Phi(u), \Phi(v)) \leq M(R_0 \vee R_\sigma)(2\varepsilon)^\kappa R_0^{p_0-1} d_{t_1,\varepsilon}(u, v),$$

where $d_{t_1,\varepsilon}$ denotes the metric d restricted on $I_{t_1,\varepsilon} \equiv [t_1 - \varepsilon, t_1 + \varepsilon] \cap I$, and M is independent of t_0 and ε. Therefore taking $\varepsilon > 0$ sufficiently small, we have $d_{t_1,\varepsilon}(u, v) = 0$, where ε is independent of $t_1 \in I$. Especially $u(t) = v(t)$ in $\dot{H}^0 = L^2$ a.e $t \in I_{t_1,\varepsilon}$. By the continuity of $u(t)$ and $v(t)$ in L^2 for $t \in I$, we have $u(t) = v(t)$ in L^2 for any $t \in I_{t_1,\varepsilon}$. Therefore we have $u(t) = v(t)$ in H^s for any $t \in I_{t_1,\varepsilon}$. □

(2) Let $t_0 = 0$ and $I = [-T, T]$, $T > 0$. If $1 < p < 1 + 4/n$, then we take p_0 as $p_0 = p$. Since κ and p satisfy $\kappa \neq 0$ and $p \neq 1$, for any fixed γ_σ we take T in (3.12) as any number with

$$T \leq C\gamma_0^{-4(p-1)/(4-n(p-1))}, \tag{3.13}$$

where the constant C is independent of $\gamma_s > 0$ and $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$. So that we conclude (1.3) by (3.13).

(3) We consider the case $T^* < \infty$. The proof for $T_* < \infty$ follows quite similarly. Now let $\gamma \equiv \sup_{0 \leq t < T^*} \|u(t); H^\sigma\| < \infty$. Let $\varepsilon > 0$ be a number which satisfies

$$M(\gamma)\varepsilon^\kappa \gamma^{p_0-1} \leq 1,$$

where M is the same function appearing in (3.12). Let $t_0 > 0$ be a number such that $T^* \in (t_0, t_0 + \varepsilon)$. Then by the same argument as in the proof of (1), (NLS) has a unique solution $\tilde{u}_+$ in $\dot{X}_{q_0}([t_0, t_0 + \varepsilon])$ with $\tilde{u}_+(t_0) = u(t_0)$. Let $\tilde{u}(t) = u(t)$ for $t \in [0, t_0)$, $\tilde{u}(t) = \tilde{u}_+(t)$ for $t \in [t_0, t_0 + \varepsilon]$. Then $\tilde{u}$ is a solution of (NLS) in $\dot{X}_{q_0}([0, t_0 + \varepsilon])$ with $\tilde{u}(0) = u(0)$. This contradicts the definition of T^*.

(4) (4a) In the argument before the proof of (1), if p satisfies $p \geq 1 + 4/n$, then we take p_0 for $p_0 = 1 + 4/n$ and therefore $\kappa = 0$ in (3.12) by (3.4). Since M in (3.12) is independent of I, for any given $\gamma_\sigma > 0$ there exists γ_0 such that (3.12) holds with $|I|^\kappa$ replaced by 1. With these $\gamma_\mu > 0$, $\mu = 0, \sigma$, fixed, for any $\gamma_s > 0$ and $\phi \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, we see that Φ is a contraction map on $\dot{X}_{q_0}(\mathbf{R}, R_0, R_\sigma, R_s)$ for some $R_\mu > 0$, $\mu = 0, \sigma, s$. The fixed point u is unique in $\dot{X}_{q_0}(\mathbf{R})$. Indeed, for any $t_0 \in \mathbf{R}$ and for any data at $t = t_0$, by

the same argument as in the proof of (1) we conclude that local solutions are unique in $\dot{X}_{q_0}([t_0 - T, t_0 + T])$ for some $T > 0$, where T is determined only by the $\dot{H}^\mu$-norm, $\mu = 0, \sigma$, of the data. Since the norms $\|u(t); \dot{H}^\mu\|$, $\mu = 0, \sigma$, are bounded, we conclude the uniqueness of solutions in $\dot{X}_{q_0}(\mathbf{R})$.

Let ϕ_+ and ϕ_- be given by

$$\phi_\pm = \phi + \int_0^{\pm\infty} U(-s) f(u(s)) ds. \tag{3.14}$$

Then $\phi_+, \phi_- \in H^s$. Indeed, by Lemma 3. 1 , for any $\mu \in \mathbf{R}$ we have

$$\begin{aligned} \|\phi_\pm; \dot{H}^\mu\| &\leq \|\phi; \dot{H}^\mu\| + \| \textstyle\int_t^{\pm\infty} U(t-s) f(u(s)) ds; L^\infty(\mathbf{R}; \dot{H}^\mu)\| \\ &\leq \|\phi; \dot{H}^\mu\| + C\|f(u); L^{\tilde{q}}(\mathbf{R}; \dot{B}^\mu_{\tilde{r}})\|, \end{aligned} \tag{3.15}$$

where we have used the continuity of $U *_t f(u)$ with respect to the time variable. Since the last term of the last inequality is estimated as (3.6) with $|I|^\kappa$ replaced by 1, $\|\phi_\pm; \dot{H}^\mu\|$ is finite for $\mu = 0, s$. By the embedding $\dot{H}^0 \cap \dot{H}^s \hookrightarrow H^s$, we conclude that $\phi_+, \phi_- \in H^s$.

With these $\phi_\pm$, (1.4) holds. Indeed, since $U(t)\phi_\pm$ are rewritten as

$$U(t)\phi_\pm = U(t)\phi + \int_0^{\pm\infty} U(t-s) f(u(s)) ds,$$

by the same argument in (3.15), we have

$$\begin{aligned} \|u(t) - U(t)\phi_+; \dot{H}^\mu\| &\leq \| \textstyle\int_\tau^\infty U(\tau - s) f(u(s)) ds; L^\infty_\tau([t, \infty); \dot{H}^\mu)\| \\ &\leq C\|f(u); L^{\tilde{q}}([t, \infty); \dot{B}^\mu_{\tilde{r}})\|. \end{aligned} \tag{3.16}$$

Since the RHS of the last inequality is estimated by (3.6) with $|I|^\kappa$ replaced by 1 and is bounded uniformly in t for any $\mu = 0, s$, we conclude that $u(t) - U(t)\phi_+$ converges to zero in $\dot{H}^0 \cap \dot{H}^s$ as $t \to \infty$. By the embedding $\dot{H}^0 \cap \dot{H}^s \hookrightarrow H^s$, we conclude (1.4) for ϕ_+. By an analogous argument on $u(t) - U(t)\phi_-$, we conclude (1.4).

The uniqueness of the pair (ϕ_+, ϕ_-) satisfying (1.4) follows from the unitarity of the free propagator.

(4b) For any $\gamma_s > 0$ and any data $\phi_- \in \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, by the argument before the proof of (1), $\Phi_{-\infty}$ has a unique fixed point u in $\dot{X}_{q_0}(\mathbf{R}, R_0, R_\sigma, R_s)$, where $R_\mu > 0$, $\mu = 0, \sigma, s$, are chosen to satisfy $R_\mu = 2C\gamma_\mu$, $\mu = 0, \sigma, s$, with the same constant C appearing in (3.10). Let ϕ and ϕ_+ be given by the equations (3.14). Then, as in the argument on (3.15), we have $\phi, \phi_+ \in H^s$. Now u is rewritten as

$$u(t) = U(t)\phi + \int_0^t U(t-s) f(u(s)) ds,$$

which implies exactly that u is a solution of (NLS) in $\dot{X}_{q0}(\mathbf{R}, R_0, R_\sigma, R_s)$ with the data $u(0) = \phi$. We claim that u is also a unique solution of (NLS) in $\dot{X}_{q0}(\mathbf{R})$ which satisfies

$$\|u(t) - U(t)\phi_-; H^s\| \to 0 \tag{3.17}$$

as $t \to -\infty$. Indeed, let v be a solution of (NLS) in $\dot{X}_{q0}(\mathbf{R})$ which satisfies (3.17) with u replaced by v. Then v satisfies $v = \Phi_{-\infty}(v)$ and for sufficiently small $t_0 < 0$,

$$\|v; \dot{X}^\mu_{q0}((-\infty, t_0])\| \leq 2C\gamma_\mu$$

for any $\mu = 0, \sigma, s$, since $\|v(t); \dot{H}^\mu\|$ converges to $\|\phi_-; \dot{H}^\mu\|$ as $t \to -\infty$ for any $\mu = 0, \sigma, s$. Therefore v is also a fixed point of $\Phi_{-\infty}$ in

$$\dot{X}_{q0}((-\infty, t_0], R_0, R_\sigma, R_s).$$

Since $\Phi_{-\infty}$ has a unique fixed point in $\dot{X}_{q0}((-\infty, t_0], R_0, R_\sigma, R_s)$, we have $u(t) = v(t)$ for any $t \in (-\infty, t_0]$. Especially we have $u(t_0) = v(t_0)$. Since for any $t_0 \in \mathbf{R}$ and any given data at $t = t_0$, local solutions of (NLS) are unique in $\dot{X}_{q0}([t_0 - T, t_0 + T])$ for some $T > 0$, where T is determined only by the $\dot{H}^\mu$-norm, $\mu = 0, \sigma$, of the data, and $\|u(t); \dot{H}^\mu\|$, $\|v(t); \dot{H}^\mu\|$, $\mu = 0, \sigma$, are uniformly bounded with respect to the time variable, we conclude $u = v$ in $\dot{X}_{q0}(\mathbf{R})$.

Similarly to the argument on (3.16) we have (1.4) for these $u, \phi_\pm$, and we also have the uniqueness of ϕ_+ for ϕ_- with (1.4).

By the above argument, for any $\gamma_s > 0$, we are able to define the scattering operator S on $\dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ to H^s as $S(\phi_-) = \phi_+$ by way of a unique global solution u of (NLS) in $\dot{X}_{q0}(\mathbf{R})$ with (1.4). Moreover, the correspondence is given by (3.14) or equivalently by

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} U(-s) f(u(s)) ds. \tag{3.18}$$

We show the continuity of the operator S. For any $\{\phi_-^j\}_{j=1}^\infty \subset \dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$ which satisfies (1.5), let $\{u^j\}_{j=1}^\infty$ be the fixed points in

$$\dot{X}_{q0}(\mathbf{R}, R_0, R_\sigma, R_s)$$

of the operators $\{\Phi_{-\infty}^j\}_{j=1}^\infty$ given by

$$\Phi_{-\infty}^j(v)(t) \equiv U(t)\phi_-^j + \int_{-\infty}^{t} U(t-s) f(v(s)) ds$$

for $j \geq 1$. Then by Lemma 3. 1 , we have, as in (3.8)

$$d(\Phi_{-\infty}^j(u), \Phi_{-\infty}^j(u^j)) \leq C\|\phi_- - \phi_-^j; \dot{H}^0\| + M(R_0 \vee R_\sigma) R_0^{p_0 - 1} d(u, u^j)$$

for any $j \geq 1$. Since (3.11) holds with $|I|^\kappa$ replaced with 1, we conclude that $d(u, u^j) \to 0$ as $j \to \infty$. On the other hand, by (3.18), we have

$$\begin{aligned}\|\phi_+ - \phi_+^j; L^2\| &\leq \|\phi_- - \phi_-^j; L^2\| \\ &+ \|\int_{-\infty}^{\infty} U(t-s)(f(u(s)) - f(u^j(s)))ds; L^\infty(\mathbf{R}; L^2)\|\end{aligned}$$

for any $j \geq 1$. By Lemma 3. 1 , the argument used on (3.9) shows that the last term on the RHS of the last inequality is estimated by

$$M(R_0 \vee R_\sigma) R_0^{p_0-1} d(u, u^j).$$

Therefore we have $\|\phi_+ - \phi_+^j; L^2\| \to 0$ as $j \to \infty$. By the convex inequality

$$\|\phi_+ - \phi_+^j; \dot{H}^\nu\| \leq \|\phi_+ - \phi_+^j; L^2\|^{1-\nu/s} \|\phi_+ - \phi_+^j; \dot{H}^s\|^{\nu/s}$$

for any ν with $0 \leq \nu < s$, we obtain (1.6).

(5) Let p_0, q_0 be as in the proof of (1) or (4). Let $t_0 = 0$ and let $I = [-T, T]$ for (1), $I = \mathbf{R}$ for (4). Since $\{u_j\}_{j=1}^\infty$ satisfies

$$u_j(t) = U(t)\phi_j + \int_0^t U(t-s) f(u_j(s)) ds$$

for any $j \geq 1$, by Lemma 3. 1 and the embedding $L^{\tilde{r}} \hookrightarrow \dot{B}^0_{\tilde{r}}$ for $1 < \tilde{r} \leq 2$, we have

$$d(u, u_j) \leq C\|\phi - \phi_j; L^2\| + C\|f(u) - f(u_j); L^{\tilde{q}}(I; L^{\tilde{r}})\| \tag{3.19}$$

for any $j \geq 1$, where the constant C is independent of I, ϕ, u, $\{\phi_j\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$. As we have shown in the proof of (1) and (4), for any given initial data in $\dot{A}^s(\gamma_0, \gamma_\sigma, \gamma_s)$, (NLS) has a unique solution in $\dot{X}_{q_0}(\mathbf{R}, R_0, R_\sigma, R_s)$, where $R_\mu > 0$, $\mu = 0, \sigma, s$, are chosen to satisfy $R_\mu = 2C\gamma_\mu$, $\mu = 0, \sigma, s$, and C is the same constant appearing in (3.10), and the solution is also unique in $\dot{X}_{q_0}(I)$. Therefore u, $\{u_j\}_{j=1}^\infty$ are in $\dot{X}_{q_0}(\mathbf{R}, R_0, R_\sigma, R_s)$. So that by the same argument on (3.9), we have

$$C\|f(u) - f(u_j); L^{\tilde{q}}(I; L^{\tilde{r}})\| \leq M(R_0 \vee R_\sigma)|I|^\kappa R_0^{p_0-1} d(u, u_j),$$

where $|I|^\kappa$ is disregarded for (4). Since (3.11) holds for these R_μ, $\mu = 0, \sigma, s$, we conclude that $d(u, u_j) \to 0$ as $j \to \infty$. Especially we obtain

$$\|u - u_j; L^\infty(I; L^2)\| \to 0$$

as $j \to \infty$. By the convex inequality

$$\|u(t) - u_j(t); \dot{H}^\nu\| \leq \|u(t) - u_j(t); L^2\|^{1-\nu/s} \|u(t) - u_j(t); \dot{H}^s\|^{\nu/s},$$

the bound

$$\sup_j \|u_j; L^\infty(I;\dot{H}^s)\| \vee \|u; L^\infty(I;\dot{H}^s)\| \leq R_s,$$

and the embedding $L^2 \cap \dot{H}^\nu \hookrightarrow H^\nu$, $\nu \geq 0$, we conclude that

$$\|u - u_j; L^\infty(I; H^\nu)\| \to 0$$

as $j \to \infty$ for any ν with $0 \leq \nu < s$. □

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