Small solutions to nonlinear wave equations in the Sobolev spaces

M. NAKAMURA and T. OZAWA

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M.NAKAMURA[1] and T.OZAWA[2]

1. Graduate School of Information Sciences(GSIS), Tohoku University, Sendai 980-8577, Japan
2. Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

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Abstract

The local and global well-posedness for the Cauchy problem for a class of nonlinear wave equations is studied. The global well-posedness of the problem is proved in the homogeneous Sobolev space $\dot{H}^s = \dot{H}^s(\mathbf{R}^n)$ of fractional order $s > n/2$ under the following assumptions: (1) Concerning the Cauchy data $(\phi, \psi) \in \dot{\mathcal{H}}^s \equiv \dot{H}^s \oplus \dot{H}^{s-1}$, $\|(\phi, \psi); \dot{\mathcal{H}}^{1/2}\|$ is relatively small with respect to $\|(\phi, \psi); \dot{\mathcal{H}}^\sigma\|$ for any fixed σ with $n/2 < \sigma \leq s$. (2) Concerning the nonlinearity f, $f(u)$ behaves as a conformal power $u^{1+4/(n-1)}$ near zero and has an arbitrary growth rate at infinity.

1 Introduction

This is a sequel to previous papers [20, 21, 22] of the same authors, where the Cauchy problem for the nonlinear wave equations of the form

$$\partial_t^2 u - \Delta u = f(u) \tag{1.1}$$

is considered in space-time $I \times \mathbf{R}^n$, where $I = [-T, T]$ with $T > 0$ for local solutions and $I = \mathbf{R}$ for global solutions. Here u is a complex-valued function of $(t, x) \in I \times \mathbf{R}^n$, $\partial_t^2 = \partial^2/\partial t^2$, Δ is the Laplacian in $\mathbf{R}^n$, and $f(u)$ is a nonlinear interaction given by a complex-valued function f on $\mathbf{C}$.

There is a large literature on the Cauchy problem for (1.1) and on the asymptotic behavior in time of global solutions (see for instance [3, 4, 5, 12, 15, 16, 20, 21] and references therein). Recently the Cauchy problem for (1.1) has been studied in the Sobolev spaces

$$H^s = H^s(\mathbf{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbf{R}^n)$$

and in the homogeneous Sobolev spaces

$$\dot{H}^s = \dot{H}^s(\mathbf{R}^n) = (-\Delta)^{-s/2}L^2(\mathbf{R}^n)$$

of fractional order $s \geq 0$ [11, 16, 21, 26, 34].

To illustrate some of the latest developments of the H^s-theory for (1.1), take for example the single power interaction

$$f(u) = \lambda |u|^{p-1}u, \tag{1.2}$$

where $\lambda \in \mathbf{C}$ and $1 < p < \infty$. A homogeneity argument on (1.1) with (1.2), similar to the nonlinear Schrödinger equations [23], shows that the power p must satisfy $p \leq 1+4/(n-2s)$ for the well-posedness of the Cauchy problem for (1.1). Here the critical power $p = 1+4/(n-2s)$ makes sense only when $0 \leq s < n/2$ to keep the critical power finite, while the notion of subcritical power $p < 1+4/(n-2s)$ extends to the homogeneous Sobolev spaces $\dot{H}^s$ for all $s \geq 0$ with the convention that $1 < p < \infty$ when $s \geq n/2$. In the critical case the available results show the existence and uniqueness of global $\dot{H}^s$-solutions with $n \geq 2$ and $s \geq 1/2$ under the smallness assumption on

$$\|(\phi,\psi);\dot{\mathcal{H}}^s\| \equiv (\|\phi;\dot{H}^s\|^2 + \|\psi;\dot{H}^{s-1}\|^2)^{1/2} \tag{1.3}$$

for the Cauchy data

$$(u(0),\partial_t u(0)) = (\phi,\psi) \in \dot{\mathcal{H}}^s \equiv \dot{H}^s \oplus \dot{H}^{s-1}.$$

On one hand, the condition $s \geq 1/2$ is natural in view of the Strichartz estimates in the diagonal case [6, 16, 32] as well as of another representation of the critical power

$$p = 1+4/((n-1)-2(s-1/2)),$$

which is taken into account of natural shift of space dimension as compared to the Schrödinger equation. On the other hand, H.Lindblad and C.D.Sogge have shown that when $s < 1/2$ there exists another critical power p which is smaller than $1+4/(n-2s)$ even for the local well-posedness of (1.1) (see

[14, 16]). In this sense, $s = 1/2$ is also critical exponent for the global well-posedness of (1.1), when the critical power is given by $p = 1 + 4/(n-1)$, which is equal to the conformal power for the nonlinear wave equations. The available H^s-theory in fact admits not only the single power such as (1.2) but a wider class of nonlinearities which includes quasihomogeneous nonlinearity. To be more specific, at the cost of smallness assumption on the data, that theory does not require structural assumptions on f related to the conservation laws but do need scaling control of f on the basis of power behavior near zero and at infinity at the same rate p.

The notion of criticality with homogeneity argument loses its meaning at the level of $H^{n/2}$. In [20] we have nevertheless developed the $H^{n/2}$-theory for (1.1) and proved the existence and uniqueness of global $\dot{H}^{n/2}$-solutions under the smallness assumption on $\|(\phi, \psi); \dot{\mathcal{H}}^{n/2} \cap \dot{\mathcal{H}}^{1/2}\|$ for the Cauchy data

$$(u(0), \partial_t u(0)) = (\phi, \psi) \in \dot{\mathcal{H}}^{n/2} \cap \dot{\mathcal{H}}^{1/2},$$

where the nonlinearity is supposed to behave as a power $u^{1+4/(n-1)}$ near zero and as an exponential function $\exp(\lambda |u|^2)$ with $\lambda > 0$ at infinity. The last growth rate at infinity reminds us of Trudinger's inequality, which replaces the Sobolev embedding $H^s \hookrightarrow L^\infty$ with $s > n/2$ in the borderline case $s = n/2$. In fact, the method in [20] depends on an equivalent sharp Gagliardo-Nirenberg inequality [24, 25] and in view of Trudinger's inequality the last growth rate at infinity seems to be optimal at the level of $H^{n/2}$. Meanwhile, the available $H^{n/2}$-theory distinguishes behavior near zero of admissible nonlinearity from that at infinity and indicates that the right power near zero is $1 + 4/(n-1)$, which is the minimal power near zero in the framework of H^s-theory with $s \geq 1/2$.

The purpose in this paper is to study the H^s-theory with $s > n/2$ and to specify admissible behavior of nonlinearity and minimal requirement on smallness of the Cauchy data to ensure the existence and uniqueness of global $\dot{H}^s$-solutions. There are some related works on the global Cauchy problem in the Sobolev space H^k of integral order $k > n/2$ [2, 7, 27, 28, 29], which treat the problem from the point of view of regularity and a priori estimates in H^k, starting recursively from the energy space $\dot{H}^1$. This requires some structural assumptions on the nonlinearity and, at a technical and essential level, yields restrictions on the admissible space dimensions. We evade the difficulty caused by the classical a priori estimates and, at the cost of smallness of the Cauchy data, look for another possibility for the H^s-theory with $s > n/2$, deserving of a natural extension of the H^s-theory with $s \leq n/2$.

In this paper we prove the existence and uniqueness of global $\dot{H}^s$-solutions with $s > n/2$ under the following setting:

(1) The Cauchy data $(\phi, \psi) \in \dot{\mathcal{H}}^s$ satisfies the assumption that $\|(\phi, \psi); \dot{\mathcal{H}}^{1/2}\|$ is relatively small with respect to $\|(\phi, \psi); \dot{\mathcal{H}}^\sigma\|$ for any fixed σ with $n/2 < \sigma \leq s$.

(2) The nonlinear interaction f behaves as a conformal power $u^{1+4/(n-1)}$ near zero and has an arbitrary growth rate at infinity.

To state the main result precisely we introduce the following notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbf{R}^n)$ denotes the Lebesgue space on $\mathbf{R}^n$. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $H_r^s = (1 - \Delta)^{-s/2} L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbf{R}$ and any r with $1 < r < \infty$, $\dot{H}_r^s = (-\Delta)^{-s/2} L^r$ denotes the homogeneous Sobolev space defined in terms of Riesz potentials. For any $s \in \mathbf{R}$ and any r, m with $1 \leq r, m \leq \infty$, $\dot{B}_{r,m}^s$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj} \|\mathcal{F}^{-1} \varphi(2^{-j}\cdot) \mathcal{F} u; L^r\|\}_{-\infty}^{\infty} \in \ell^m(\mathbf{Z})$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and its inverse, respectively, and φ is a nonnegative function on $\mathbf{R}^n$ with compact support in $\{x \in \mathbf{R}^n \ ; \ 1/2 \leq |x| \leq 2\}$ such that $\{\varphi(2^{-j}\cdot)\}_{-\infty}^{\infty}$ forms the Littlewood-Paley dyadic decomposition on $\mathbf{R}^n \backslash \{0\}$. The usual Besov space $B_{r,m}^s$ is defined analogously in terms of the dyadic decomposition on $\mathbf{R}^n$. We refer to [1, 9, 35] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity we make abbreviations such as $H^s = H_2^s$, $\dot{H}^s = \dot{H}_2^s$, $B_r^s = B_{r,2}^s$, $\dot{B}_r^s = \dot{B}_{r,2}^s$. For any Banach spaces X and Y having a common dense subspace, we put

$$\|a; X \cap Y\| \equiv \max\{\|a; X\|, \|a; Y\|\}$$

for any $a \in X \cap Y$. For any interval $I \subset \mathbf{R}$ and any Banach space X we denote by $C_b(I; X)$ the space of bounded and strongly continuous functions from I to X and by $L^q(I; X)$ the space of measurable functions u from I to X such that $\|u(\cdot); X\| \in L^q(I)$. To describe the free propagator which solves the free wave equations, we define the operators $U(t) \equiv \exp(it\omega)$, $K(t) \equiv \sin(t\omega)/\omega$, $\dot{K}(t) \equiv \cos(t\omega)$, where $\omega \equiv \sqrt{-\Delta}$. For any r with $2 \leq r \leq \infty$ we define $\gamma(r) = (n-1)(1/2 - 1/r)$, namely the optimal rate in time for the norm of $U(t)$ as a bounded operator from $L^{r'}$ to L^r, where r' is the exponent dual to r defined by $1/r + 1/r' = 1$.

The Cauchy problem for the equation (1.1) with given data (ϕ, ψ) will be treated in the form of the integral equation

$$u(t) = \Phi(u)(t) \equiv \dot{K}(t)\phi + K(t)\psi + \textstyle\int_0^t K(t-\tau)f(u(\tau))d\tau. \qquad \text{(NLW)}$$

The integral equation (NLW) will be studied in the space

$$\dot{X}_{s_0}(I) \equiv C_b(I; \dot{H}^s \cap \dot{H}^{s_0}) \cap L^{q_0}(I; \dot{B}^{s-s_0}_{r_0} \cap \dot{B}^0_{r_0})$$

for some s_0 with $0 \le s_0 \le 1/2$, endowed with the metric d given by

$$d(u,v) \equiv \|u - v; L^\infty(I; \dot{H}^{s_0}) \cap L^{q_0}(I; \dot{B}^0_{r_0})\|, \tag{1.4}$$

where I is any interval on $\mathbf{R}$, and q_0, r_0 are defined by

$$1/q_0 \equiv (n-1)s_0/(n+1), \quad 1/r_0 \equiv 1/2 - 2/(n-1)q_0. \tag{1.5}$$

To describe the nonlinear interaction f with an arbitrary growth at infinity as well as with a vanishing behavior as a power p at zero, we introduce the following assumption $(N)_{s,p}$ with $0 \le s < \infty$ and $1 \le p < \infty$.

$(N)_{s,p}$ $f \in C^{[s]}(\mathbf{C}; \mathbf{C})$ and there exists a nonnegative, nondecreasing function M on $\mathbf{R}_+$ such that for all k with $0 \le k \le [s]$ $f^{(k)}$ satisfies the estimate

$$|f^{(k)}(z)| \le |z|^{(p-k)_+} M(|z|),$$

$$|f^{([s])}(z_1) - f^{([s])}(z_2)| \le \begin{cases} |z_1 - z_2|^{p-[s]} M(|z_1| \vee |z_2|) & \text{if } s < p < [s]+1, \\ |z_1 - z_2|(|z_1| \vee |z_2|)^{(p-[s]-1)_+} M(|z_1| \vee |z_2|) & \text{otherwise} \end{cases}$$

for all $z, z_1, z_2 \in \mathbf{C}$.

Here $f^{(k)}$ denotes any of the k-th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(k)}|$ denotes the maximum of the moduli of those derivatives. For $a, b \in \mathbf{R}$ we denote by $a \vee b$ and $a \wedge b$ the maximum and minimum of a and b, respectively, by a_+ the maximum of a and 0, and by $[a]$ the greatest integer that is less than or equal to a.

With the notation above we now state the main result in this paper. For any $s_0 \ge 0$ and any $s, \sigma, \gamma_{s_0}, \gamma_\sigma, \gamma_s > 0$ we define the set of the data by

$$\begin{aligned} &\dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s) \\ \equiv\ &\{(\phi, \psi) \in \cap_{\mu = s_0, \sigma, s} \dot{\mathcal{H}}^\mu \ ; \ \|(\phi,\psi); \dot{\mathcal{H}}^\mu\| \le \gamma_\mu \ \text{ for } \ \mu = s_0, \sigma, s\}. \end{aligned}$$

Theorem 1. 1 *Let $n \geq 2$, $s > n/2$, $1+2/n \leq p < \infty$. Let σ be any number with $n/2 < \sigma \leq s$. Then there exists s_0 with $0 \leq s_0 \leq 1/2$ such that for any f satisfying $(N)_{s-s_0,p}$, the following property holds.*

(1) For any $\gamma_{s_0}, \gamma_\sigma > 0$ there exists $T > 0$ such that for any $\gamma_s > 0$ and any $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$ (NLW) *has a unique local solution u in $\dot{X}_{s_0}([-T, T])$.*

(2) Let $(-T_*, T^*)$, $T_*, T^* > 0$, be the maximal interval of existence of the solution in (1). If $T^* < \infty$ [resp. $T_* < \infty$], then

$$\|(u(t), \partial_t u(t)); \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^\sigma\|$$

blows up at $t = T^*$ [resp. $t = -T_*$].

(3) If $p < 1 + 4/(n-1)$, then for any fixed $\gamma_\sigma > 0$, T_* and T^* are estimated from below as

$$T_* \wedge T^* \geq C\gamma_{s_0}^{-2(p-1)/(4-(p-1)(n-2s_0))}, \tag{1.6}$$

where s_0 satisfies $4-(p-1)(n-2s_0) > 0$, the positive constant C is independent of $\gamma_{s_0}, \gamma_s > 0$ and $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$. Especially T_* and T^* can be taken sufficiently large by taking γ_{s_0} sufficiently small.

(4) If $p \geq 1 + 4/(n-1)$, then, taking s_0 as $s_0 = 1/2$, it follows that for any $\gamma_\sigma > 0$ there exists $\gamma_{s_0} > 0$ with the following property.

(4a) For any $\gamma_s > 0$ and any initial data $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$, (NLW) *has a unique global solution in $\dot{X}_{s_0}(\mathbf{R})$. Moreover there exists unique two pairs (ϕ_+, ψ_+) and (ϕ_-, ψ_-) in $\dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$ such that*

$$\|(u(t) - v_\pm(t), \partial_t(u(t) - v_\pm(t))); \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s\| \to 0 \quad as \quad t \to \pm\infty, \tag{1.7}$$

where $v_\pm(t) \equiv \dot{K}(t)\phi_\pm + K(t)\psi_\pm$.

(4b) For any $\gamma_s > 0$ and any data $(\phi_-, \psi_-) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$, there exists a unique global solution u of (NLW) *in $\dot{X}_{s_0}(\mathbf{R})$ and a unique state $(\phi_+, \psi_+) \in \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$ such that (1.7) holds. Moreover the scattering operator $S : (\phi_-, \psi_-) \mapsto (\phi_+, \psi_+)$ is well-defined on $\dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$ to $\dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$ and is continuous in the following sense. If $\{(\phi_-^j, \psi_-^j)\}_{j=1}^\infty \subset \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$ satisfies*

$$\|(\phi_- - \phi_-^j, \psi_- - \psi_-^j); \dot{\mathcal{H}}^{s_0}\| \to 0 \tag{1.8}$$

as $j \to \infty$, then

$$\|(\phi_+ - \phi_+^j, \psi_+ - \psi_+^j); \dot{\mathcal{H}}^\nu\| \to 0 \tag{1.9}$$

as $j \to \infty$ for any ν with $s_0 \leq \nu < s$, where $(\phi_+^j, \psi_+^j) = S((\phi_-^j, \psi_-^j))$.

(5) The solutions given by (1) and (4) have the continuous dependence on the initial data. Namely, for any initial data $(\phi, \psi), \{(\phi_j, \psi_j)\}_{j=1}^{\infty}$ in $\dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$, let $u, \{u_j\}_{j=1}^{\infty}$ be the corresponding solutions of (NLW) *given by (1) or (4). If $\{(\phi_j, \psi_j)\}_{j=1}^{\infty}$ satisfies*

$$\|(\phi - \phi_j, \psi - \psi_j); \dot{\mathcal{H}}^{s_0}\| \to 0$$

as $j \to \infty$, then

$$d(u, u_j) \to 0 \quad and \quad \|(u - u_j, \partial_t(u - u_j)); L^\infty(I; \dot{\mathcal{H}}^\nu)\| \to 0$$

as $j \to \infty$ for any ν with $s_0 \le \nu < s$, where $I = [-T, T]$ for (1) and $I = \mathbf{R}$ for (4).

Remark 1. The assumptions in (4) of the theorem cover for instance the nonlinearities given by complex polynomials in u and $\bar{u}$ with the lowest degree of $5, 3$, and 2 for $n = 2, 3 \le n \le 4$, and $n \ge 5$, respectively. More generally, the nonlinearities given by those polynomials multiplied by any smooth functions fall within the scope of the theorem. As regards the power behavior near zero that is compatible with minimal regularity of f, the assumption $(N)_{s-s_0,p}$ requires that $s - s_0 < p$ if $s - s_0$ is not an integer.

Remark 2. As in the $H^{n/2}$-theory developed in [20], the H^s-theory with $s > n/2$ above distinguish behavior near zero of admissible nonlinearity from that at infinity and indicates that the right power for the former is again $1 + 4/(n-1)$, which is critical at the level of $\dot{H}^{1/2}$ and is the minimal power near zero in the framework of the available H^s-theory with $1/2 \le s < n/2$. In contrast to the $H^{n/2}$-theory, however, arbitrary growth rate at infinity is admissible for the nonlinearity of the H^s-theory with $s > n/2$.

Remark 3. In Theorem 1. 1 , the assumption on the minimal growth rate at the origin of the nonlinearity such as $1 + 2/n \le p$ arises from the gap of the regularity of the Strichartz estimates in the homogeneous Sobolev space. Let us consider the nonlinear interaction (1.2). The number $p = 1 + 2/n$ arises as follows. By Lemma 2. 4 below, we have

$$\| \int_0^t K(t-s) f(u(s)) ds; L^\infty([-T, T]; L^2)\| \le C \| |u|^{p-1} u; L^1([-T, T]; \dot{H}^{-1})\|, \tag{1.10}$$

where the constant C is independent of u. To apply the contraction argument, the RHS in (1.10) needs to be estimated in the form of a power of

$\|u; L^\infty([-T,T]; L^2)\|$. By the embedding $L^{2n/(n+2)} \hookrightarrow \dot{H}^{-1}$, which is the dual to $\dot{H}^1 \hookrightarrow L^{2n/(n-2)}$, the RHS in (1.10) is estimated by

$$CT\|u; L^\infty([-T,T]; L^{2np/(n+2)})\|^p,$$

so that a contraction argument goes through provided that $p = 1 + 2/n$.

To consider the local solvability of (NLW) for low vanishing rate p such as $1 \le p < 1 + 2/n$, which is excepted to complement Theorem 1. 1 , we consider (NLW) in the function space $C_b(I; H^s)$ endowed with the metric d_0 defined by

$$d_0(u,v) \equiv \|u - v; L^\infty(I; L^2)\|,$$

where I is an interval on $\mathbf{R}$.

For any $\mu \in \mathbf{R}$ we denote by $\mathcal{H}^\mu$ the Hilbert space $H^\mu \oplus H^{\mu-1}$ endowed with the norm

$$\|(\phi,\psi); \mathcal{H}^\mu\| = (\|\phi; H^\mu\|^2 + \|\psi; H^{\mu-1}\|^2)^{1/2}.$$

For any given $\sigma, s \ge 0$ and $\gamma_\mu > 0, \mu = 0, \sigma, s$, we denote by $A^s(\gamma_0, \gamma_\sigma, \gamma_s)$ the set of the data defined by

$$A^s(\gamma_0,\gamma_\sigma,\gamma_s) \equiv \{(\phi,\psi) \in \mathcal{H}^s \mid \|(\phi,\psi); \mathcal{H}^\mu\| \le \gamma_\mu,\ \ \mu = 0,\sigma,s\}.$$

Theorem 1. 2 *Let $n \ge 1$, $s > n/2$, $1 \le p < \infty$. Let σ be any number with $n/2 < \sigma \le s$. Then there exists s_0 with $0 \le s_0 \le 1$ such that for any f satisfying $(N)_{(s-s_0)_+,p}$, the following property holds.*

(1) For any $\gamma_0, \gamma_\sigma > 0$ there exists $T > 0$ such that for any $\gamma_s > 0$ and any $(\phi,\psi) \in A^s(\gamma_0,\gamma_\sigma,\gamma_s)$ (NLW) *has a unique local solution in $C_b([-T,T]; H^s)$.*

(2) Let $(-T_*, T^*)$, $T_*, T^* > 0$, be the maximal interval of existence of the solution in (1). If $T^* < \infty$ [resp. $T_* < \infty$], then $\|(u(t), \partial_t u(t)); \mathcal{H}^\sigma\|$ blows up at $t = T^*$ [resp. $t = -T_*$].

(3) If $p \ne 1$, then for any $\gamma_\sigma > 0$ and any data $(\phi,\psi) \in \mathcal{H}^s$ with $\|(\phi,\psi); \mathcal{H}^\sigma\| \le \gamma_\sigma$, T^* and T_* can be taken sufficiently large by taking $\|(\phi,\psi); \mathcal{H}^0\|$ sufficiently small.

(4) The solutions given by (1) have the continuous dependence on the initial data. Namely, for any initial data (ϕ,ψ), $\{(\phi_j,\psi_j)\}_{j=1}^\infty$ in $A^s(\gamma_0,\gamma_\sigma,\gamma_s)$, let u, $\{u_j\}_{j=1}^\infty$ be the corresponding solutions given by (1). If $\{(\phi_j,\psi_j)\}_{j=1}^\infty$ satisfies

$$\|(\phi - \phi_j, \psi - \psi_j); \mathcal{H}^0\| \to 0$$

as $j \to \infty$, *then*

$$d_0(u, u_j) \to 0$$

and

$$\|(u - u_j, \partial_t(u - u_j)); L^\infty([-T, T]; \mathcal{H}^\nu)\| \to 0 \tag{1.11}$$

as $j \to \infty$ *for any* ν *with* $0 \le \nu < s$.

The method of proofs of the theorems depends on the Strichartz estimates and on the Leibniz type estimate on the composite function $f \circ u$, both of which are described in terms of homogeneous Besov spaces. The former is the same as in [6, 16, 33] and the latter partly generalizes the results in [23], which we have shown in [19]. In this paper we follow the following conventions: Different positive constants might be denoted by the same letter C. Different nonnegative, nondecreasing functions on $\mathbf{R}_+$ might be denoted by the same letter M.

2 Proofs of the theorems

In this section we prove the theorems. We use the following estimates of Leibniz type, and embeddings to control the behavior of the nonlinear term at infinity in the framework of homogeneous Besov spaces. For any interval $I \subset \mathbf{R}$ possibly unbounded, $|I|$ denotes the length of I and $\bar{I}$ denotes its closure in $\mathbf{R} \cup \{\pm\infty\}$ equipped with the natural topology.

Lemma 2. 1 (Proposition 1.1 in [19]) *Let* $s > 0$ *and* $p \ge 1$. *Let* ℓ, r, m *satisfy* $1 \le \ell < \infty$, $2 \le m \le \infty$, $2 \le r < \infty$, $1/\ell = (p-1)/m + 1/r$. *Let* f *satisfy* $(N)_{s,p}$. *Then*

$$\|f(u); L^\ell\| \le M(\|u; L^\infty\|)\|u; L^m\|^{p-1}\|u; L^r\|, \tag{2.1}$$

$$\|f(u); \dot{B}^s_\ell\| \le \begin{cases} M(\|u; L^\infty\|)\|u; L^m\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad 0 < s < 2, \\ M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_m\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad s \ge 2, \ m \ne \infty, \\ M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; L^\infty \cap \dot{B}^0_\infty\|^{p-1}\|u; \dot{B}^s_r\| & \text{if} \quad s \ge 2, \ m = \infty. \end{cases} \tag{2.2}$$

Lemma 2. 2 (Lemma 3.2 in [19]) *Let* $n \ge 1$, $-\infty < \mu_0 < n/2 < \mu < \infty$. *Then*

$$\dot{H}^{\mu_0} \cap \dot{H}^\mu \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow L^\infty \cap \dot{B}^0_{\infty,2}.$$

Moreover,

$$\begin{aligned}\|u; L^\infty\| \vee \|u; \dot{B}^0_{\infty,2}\| &\le C\|u; \dot{B}^0_{\infty,1}\| \\ &\le C\|u; \dot{H}^{\mu_0}\|^{1-\theta}\|u; \dot{H}^{\mu}\|^{\theta},\end{aligned}$$

where $\theta = (n/2 - \mu_0)/(\mu - \mu_0)$.

To solve (NLW) by a contraction argument we define the function space as follows. For any interval $I \subset \mathbf{R}$, $s > 0$, $\sigma > 0$, $s_0 \ge 0$ and $R_\mu > 0$, $\mu = s_0, \sigma, s$, let $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$ be a function space given by

$$\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s) \equiv \{u \ ; \ \|u; \dot{X}^\mu_{s_0}(I)\| \le R_\mu, \ \mu = s_0, \sigma, s\}$$

endowed with the metric d defined by (1.4), where

$$\dot{X}^\mu_{s_0} = L^\infty(I; \dot{H}^\mu) \cap L^{q_0}(I; \dot{B}^{\mu - s_0}_{r_0})$$

with norm

$$\|u; \dot{X}^\mu_{s_0}(I)\| \equiv \|u; L^\infty(I; \dot{H}^\mu)\| \vee \|u; L^{q_0}(I; \dot{B}^{\mu - s_0}_{r_0})\|$$

for any $\mu \in \mathbf{R}$.

Lemma 2. 3 *Let $0 \le s_0 \le 1/2$. Let $s_0 < \sigma \le s$. Then for any interval $I \subset \mathbf{R}$, $R_\mu > 0$, $\mu = s_0, \sigma, s$, $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$ is a complete metric space.*

Proof of Lemma 2. 3 . Let $\{u_j\}_{j=1}^\infty$ be a Cauchy sequence in $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$. Then

$$\sup_{j \ge 1} \|u_j; L^\infty(I; \dot{H}^\mu)\| \le R_\mu$$

for $\mu = s_0, \sigma, s$. By the definition of d, there exists u in $L^\infty(I; \dot{H}^{s_0})$ such that u_j strongly converges to u in $L^\infty(I; \dot{H}^{s_0})$. Especially it follows that $\|u; L^\infty(I; \dot{H}^{s_0})\| \le R_{s_0}$. Since $\dot{H}^\mu$ is reflexive for any $\mu \in \mathbf{R}$, for almost every $t \in I$ there exists a subsequence $\{u_{j,1}(t)\}_{j=1}^\infty \subset \{u_j(t)\}_{j=1}^\infty$ and $u^\mu(t) \in \dot{H}^\mu$, $\mu = \sigma, s$, such that $u_{j,1}(t)$ weakly converges to $u^\mu(t)$ in $\dot{H}^\mu$ and $\|u^\mu(t); \dot{H}^\mu\| \le R_\mu$ for $\mu = \sigma, s$. Since we have

$$\mathcal{F}^{-1} C_0^\infty(\mathbf{R}^n \setminus \{0\}) \subset \dot{H}^{-\mu} = (\dot{H}^\mu)^*$$

for any $\mu \in \mathbf{R}$, for any $\chi \in \mathcal{F}^{-1} C_0^\infty(\mathbf{R}^n \setminus \{0\})$ the pairing $\langle u_{j,1}(t), \chi\rangle$ converges to $\langle u^\mu(t), \chi\rangle$ for $\mu = \sigma, s$, and to $\langle u(t), \chi\rangle$ as $j \to \infty$. Therefore we have $u(t) = u^\mu(t)$ in $\mathcal{S}'/\mathcal{P}$, so that $\|u(t); \dot{H}^\mu\| \le R_\mu$ for $\mu = \sigma, s$, where $\mathcal{P}$ denotes the set of polynomials. Since the last inequality is preserved when t runs almost everywhere on I, we obtain $\|u; L^\infty(I; \dot{H}^\mu)\| \le R_\mu$ for $\mu = \sigma, s$.

When $s_0 = 0$, the proof of the lemma is completed by the above argument. We consider the case $s_0 > 0$ in the following. By the definition of d, there exists v in $L^{q_0}(I; \dot{B}^0_{r_0})$ such that u_j strongly converges to v in $L^{q_0}(I; \dot{B}^0_{r_0})$ as $j \to \infty$. Especially we have $\|v; L^{q_0}(I; \dot{B}^0_{r_0})\| \le R_{s_0}$. Since $L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})$ is reflexive for any $\mu \in \mathbf{R}$ and

$$\sup_{j\ge1} \|u_j; L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})\| \le R_\mu$$

for $\mu = \sigma, s$, there exists a subsequence $\{u_{j,2}\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ and $v^\mu \in L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})$ for $\mu = \sigma, s$ such that $u_{j,2}$ weakly converges to v^μ in $L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})$ and $\|v^\mu; L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})\| \le R_\mu$ for $\mu = \sigma, s$. Now since $\{u_j\}$ satisfies

$$\sup_{j\ge1} \|u_j; L^{q_0}(I; \dot{B}^{s-s_0}_{r_0})\| \le R_s,$$

therefore by the embeddings $\dot{B}^0_r \hookrightarrow L^r$ for any $2 \le r < \infty$, $L^r \cap \dot{B}^\mu_r \hookrightarrow B^\mu_r$ for any $\mu > 0$, $1 \le r \le \infty$, we have

$$\sup_{j\ge1} \|u_j; L^{q_0}(I; B^{s-s_0}_{r_0})\| \le CR_{s_0} \vee R_s.$$

Since $L^{q_0}(I; B^{s-s_0}_{r_0})$ is reflexive, there exists a subsequence $\{u_{j,3}\}_{j=1}^\infty \subset \{u_{j,2}\}_{j=1}^\infty$ and $w \in L^{q_0}(I; B^{s-s_0}_{r_0})$ such that $u_{j,3}$ weakly converges to w in $L^{q_0}(I; B^{s-s_0}_{r_0})$ as $j \to \infty$. By the embedding $B^{s-s_0}_{r_0} \hookrightarrow B^{\sigma-s_0}_{r_0}$, $u_{j,3}$ also weakly converges to w in $L^{q_0}(I; B^{\sigma-s_0}_{r_0})$. Therefore by the embeddings $B^{\mu-s_0}_{r_0} \hookrightarrow \dot{B}^{\mu-s_0}_{r_0}$ for $\mu = \sigma, s$, we have $w = v^\mu$ in $L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})$ for $\mu = \sigma, s$. Since u_j also converges to v in $L^{q_0}(I; L^{r_0})$ by the embedding $\dot{B}^0_{r_0} \hookrightarrow L^{r_0}$, $2 \le r_0 < \infty$, therefore by the embedding $B^{s-s_0}_{r_0} \hookrightarrow L^{r_0}$, we have $w = v$ in $L^{q_0}(I; L^{r_0})$. So that we have $v(t) = v^\mu(t)$ in $\mathcal{S}'/\mathcal{P}$ for almost every $t \in I$, and $\|v; L^{q_0}(I; \dot{B}^{\mu-s_0}_{r_0})\| \le R_\mu$ for $\mu = \sigma, s$.

Since u_j strongly converges to u in $L^\infty(I; \dot{H}^{s_0})$ and to v in $L^{q_0}(I; \dot{B}^0_{r_0})$ as $j \to \infty$, there exists a subsequence $\{u_{j,4}\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ such that for almost every $t \in I$ $u_{j,4}(t)$ strongly converges to $u(t)$ in $\dot{H}^{s_0}$ and to $v(t)$ in $\dot{B}^0_{r_0}$ as $j \to \infty$. Since

$$\mathcal{F}^{-1}C_0^\infty(\mathbf{R}^n \setminus \{0\}) \subset (\dot{H}^{s_0})^* \cap (\dot{B}^0_{r_0})^*,$$

we have $\langle u(t), \chi\rangle = \langle v(t), \chi\rangle$ for any $\chi \in \mathcal{F}^{-1}C_0^\infty(\mathbf{R}^n \setminus \{0\})$, so that $u(t) = v(t)$ in $\mathcal{S}'/\mathcal{P}$ for almost every $t \in I$. Therefore we have $d(u_j, u) \to 0$ as

$j \to \infty$ and $\|u; \dot{X}^{\mu}_{s_0}(I)\| \le R_\mu$ for $\mu = s_0, \sigma, s$. Consequently we obtain the lemma. $\square$

Let n, s, p, σ be as in Theorem 1. 1 . Let p_0 satisfy

$$1 + 2/n \le p_0 \le (1 + 4/(n-1)) \wedge p.$$

Let s_0 be a number defined by

$$s_0 \equiv \begin{cases} \frac{(n+1)(n(p_0-1)-2)}{2(2n(p_0-1)+n-1)} & \text{if } p_0 < 1 + \frac{3n+1}{n(n-1)}, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then we have $0 \le s_0 \le 1/2$. For simplicity, we put $p_* \equiv 1+(3n+1)/n(n-1)$ in the following. Let q_0, r_0 be numbers defined by (1.5). Then by a direct computation, we have

$$0 \le 1/q_0 \le (n-1)/2(n+1) \le 1/r_0 \le 1/2, \tag{2.3}$$

$$s_0 = n(1/2 - 1/r_0) - 1/q_0. \tag{2.4}$$

We define

$$\kappa \equiv 2 - (p_0 - 1)(n/2 - s_0). \tag{2.5}$$

We note that $\kappa \ge 0$ and that $\kappa = 0$ holds if and only if $p_0 = 1 + 4/(n-1)$. Indeed, if $p_0 < p_*$, then by the definition of s_0, $\kappa > 0$ is equivalent to

$$n(n-1)(p_0-1)^2 + (n^2 - 7n + 2)(p_0 - 1) - 4(n-1) < 0.$$

Since the above quadratic polynomial with respect to $p_0 - 1$ is negative for $p_0 = 1$ and $p_0 = p_*$, therefore we have $\kappa > 0$ for $p_0 < p_*$. If $p_0 \ge p_*$, then $\kappa \ge 0$ is equivalent to

$$p_0 \le 1 + 4/(n-1).$$

Therefore we have $\kappa \ge 0$ for $p_0 \ge p_*$, and especially, we have $\kappa = 0$ if and only if $p_0 = 1 + 4/(n-1)$.

Let $\tilde{r}$ and $\tilde{q}$ be defined by

$$1/\tilde{r} \equiv p_0/r_0, \qquad 1/\tilde{q} \equiv \kappa + p_0/q_0. \tag{2.6}$$

Then by the definition of $\tilde{r}$ and $\tilde{q}$ and by (2.4) we have

$$n(1/2 - 1/r_0) - 1/q_0 = 2 + n(1/2 - 1/\tilde{r}) - 1/\tilde{q}, \tag{2.7}$$

$$(n+3)/2(n+1) \le 1/\tilde{q} \le 1, \quad (n+1)/2n \le 1/\tilde{r} \le (n+2)/2n. \tag{2.8}$$

Indeed, if $p_0 < p_*$, then by the definition of κ and q_0 we have $\tilde{q} = 1$. If $p_0 \geq p_*$, then $\tilde{q}$ is rewritten as

$$1/\tilde{q} = (5n+3)/2(n+1) - (p_0 - 1)n(n-1)/2(n+1).$$

Therefore we have $1/\tilde{q} = (n+3)/2(n+1)$ for $p_0 = 1 + 4/(n-1)$ and $\tilde{q} = 1$ for $p_0 = p_*$. We rewrite (2.7) as

$$1/\tilde{r} = (n+4)/2n - (n+1)/n(n-1)q_0 - 1/n\tilde{q}$$

to obtain

$$1/\tilde{r} = \begin{cases} (n+2)/2n - (n+1)/n(n-1)q_0 & \text{for } p < p_*, \\ (n+3)/2n - 1/n\tilde{q} & \text{for } p \geq p_*, \end{cases}$$

since $\tilde{q} = 1$ for $p_0 < p_*$, $1/q_0 = (n-1)/2(n+1)$ for $p_0 \geq p_*$. So that by (2.3) and the inequalities on $\tilde{q}$ in (2.8), we have the inequalities on $\tilde{r}$ in (2.8). By this argument, especially we also have

$$1/\tilde{r} + 2/(n-1)\tilde{q} \geq (n+3)/2(n-1). \tag{2.9}$$

Now let $1/r^* \equiv (p_0 - 1)/(p-1)r_0$. Then we have

$$1/\tilde{r} = (p-1)/r^* + 1/r_0. \tag{2.10}$$

Since $r_0, \tilde{r}, r^*$ satisfy $0 < 1/r_0, 1/r^* \leq 1/2$ and $1/2 < 1/\tilde{r} \leq 1$ by (2.3) and (2.8), applying the Lemma 2. 1 to the composite function $f(u)$, for any μ with $0 < \mu \leq s$ we have

$$\|f(u); \dot{B}^{\mu}_{\tilde{r}}\| \leq M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_{r^*}\|^{p-1}\|u; \dot{B}^{\mu}_{r_0}\|. \tag{2.11}$$

By the convex inequality

$$\|u; \dot{B}^0_{r^*}\| \leq \|u; \dot{B}^0_\infty\|^{1-\theta}\|u; \dot{B}^0_{r_0}\|^{\theta},$$

where $\theta \equiv (p_0 - 1)/(p-1)$, the RHS of (2.11) is estimated by

$$M(\|u; L^\infty \cap \dot{B}^0_\infty\|)\|u; \dot{B}^0_{r_0}\|^{p_0 - 1}\|u; \dot{B}^{\mu}_{r_0}\|,$$

where we denote by the same letter M the nondecreasing function on $[0, \infty)$. Therefore by the definition of $\tilde{q}$ in (2.6) and the fact $\kappa \geq 0$, applying the Hölder inequality in the time variable, we have

$$\begin{aligned} &\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu}_{\tilde{r}})\| \\ \leq\ & M(\|u; L^\infty(I; L^\infty \cap \dot{B}^0_\infty)\|)|I|^{\kappa}\|u; L^{q_0}(I; \dot{B}^0_{r_0})\|^{p_0-1}\|u; L^{q_0}(I; \dot{B}^{\mu}_{r_0})\| \end{aligned}$$

for any interval $I \subset \mathbf{R}$. Similarly, by (2.1) and the Hölder inequality in space and time variables we also have

$$\begin{aligned} & \|f(u); L^{\tilde{q}}(I; L^{\tilde{r}})\| \\ \leq\ & M(\|u; L^{\infty}(I; L^{\infty})\|)|I|^{\kappa}\|u; L^{q_0}(I; L^{r_0})\|^{p_0-1}\|u; L^{q_0}(I; L^{r_0})\|. \end{aligned}$$

Therefore by Lemma 2. 2 , we have

$$\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu-s_0}_{\tilde{r}})\| \leq M(R_{s_0} \vee R_{\sigma})|I|^{\kappa} R_{s_0}^{p_0-1} R_{\mu} \tag{2.12}$$

for any $\mu = s_0, \sigma, s$, and any $u \in \dot{X}_{s_0}(I, R_{s_0}, R_{\sigma}, R_s)$, where we have used the embeddings $L^{\tilde{r}} \hookrightarrow \dot{B}^0_{\tilde{r}}$, $\dot{B}^0_{r_0} \hookrightarrow L^{r_0}$ for the case $s_0 = 0$. Now, for any $t_0 \in \bar{I}$ and any data $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_{\sigma}, \gamma_s)$, let Φ_{t_0} be an operator defined as

$$\Phi_{t_0}(u)(t) \equiv \dot{K}(t)\phi + K(t)\psi + \int_{t_0}^{t} K(t-s) f(u(s)) ds.$$

Lemma 2. 4 ([6]) *Let $\mu, \rho, \tilde{\rho} \in \mathbf{R}$. Let $2 \leq q, r \leq \infty$, $1 \leq \tilde{q}, \tilde{r} \leq 2$ satisfy the following conditions:*

$$\begin{aligned} & 1/r + 2/(n-1)q \leq 1/2, \quad (1/q, 1/r) \neq (1/2, (n-3)/2(n-1)), \\ & 1/\tilde{r} + 2/(n-1)\tilde{q} \geq (n+3)/2(n-1), \ (1/\tilde{q}, 1/\tilde{r}) \neq (1/2, (n+1)/2(n-1)), \\ & \mu = \rho + n(1/2 - 1/r) - 1/q = 2 + \tilde{\rho} + n(1/2 - 1/\tilde{r}) - 1/\tilde{q}. \end{aligned}$$

Then for any interval $I \subset \mathbf{R}$ and any $t_0 \in \bar{I}$ $U(\cdot)$ satisfies the estimates

$$\begin{aligned} & \|U(\cdot)\phi; L^q(\mathbf{R}; \dot{B}^{\rho}_r)\| \leq C\|\phi; \dot{H}^{\mu}\| \\ & \|\int_{t_0}^{t} \frac{U(t-s)}{\omega} h(s) ds; L^q(I; \dot{B}^{\rho}_r)\| \leq C\|h; L^{\tilde{q}}(I; \dot{B}^{\tilde{\rho}}_{\tilde{r}})\|, \end{aligned}$$

where the constant C is independent of t_0 and I.

By (1.5), (2.3), (2.4), (2.7), (2.8) and (2.9) we apply Lemma 2. 4 , and then by (2.12) we have

$$\begin{aligned} \|\Phi_{t_0}(u); \dot{X}^{\mu}_{s_0}(I)\| & \leq C\|(\phi, \psi); \dot{\mathcal{H}}^{\mu}\| + C\|f(u); L^{\tilde{q}}(I; \dot{B}^{\mu-s_0}_{\tilde{r}})\| \\ & \leq C\gamma_{\mu} + M(R_{s_0} \vee R_{\sigma})|I|^{\kappa} R_{s_0}^{p_0-1} R_{\mu} \end{aligned} \tag{2.13}$$

for $\mu = s_0, \sigma, s$, any $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_{\sigma}, \gamma_s)$ and any $u \in \dot{X}_{s_0}(I, R_{s_0}, R_{\sigma}, R_s)$.

On the other hand, by the same argument as above, we also have

$$d(\Phi_{t_0}(u), \Phi_{t_0}(v)) \leq M(R_{s_0} \vee R_{\sigma})|I|^{\kappa} R_{s_0}^{p_0-1} d(u, v) \tag{2.14}$$

for any $u, v \in \cup_{R>0}\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R)$. Indeed, by Lemma 2. 4 we have

$$d(\Phi_{t_0}(u), \Phi_{t_0}(v)) \leq C\|f(u) - f(v); L^{\tilde{q}}(I; L^{\tilde{r}})\|,$$

where we have used the embedding $L^{\tilde{r}} \hookrightarrow \dot{B}^0_{\tilde{r}}$. By the Hölder inequality in space and time variables with (2.6) we have

$$\begin{aligned} &\|f(u) - f(v); L^{\tilde{q}}(I; L^{\tilde{r}})\| \\ \leq\ & M(\|u; L^\infty(I; L^\infty)\| \vee \|v; L^\infty(I; L^\infty)\|)|I|^\kappa \\ & (\|u; L^{q_0}(I; L^{r_0})\| \vee \|v; L^{q_0}(I; L^{r_0})\|)^{p_0-1}\|u - v; L^{q_0}(I; L^{r_0})\|. \quad (2.15) \end{aligned}$$

Therefore by Lemma 2. 2 and the embedding $\dot{B}^0_{r_0} \hookrightarrow L^{r_0}$ we obtain (2.14).

By the above argument, if $\gamma_\mu > 0$, $R_\mu > 0$, $\mu = s_0, \sigma, s$, and $I \subset \mathbf{R}$ satisfy

$$C\gamma_\mu + M(R_{s_0} \vee R_\sigma)|I|^\kappa R_{s_0}^{p_0-1} R_\mu \leq R_\mu, \quad (2.16)$$

$$M(R_{s_0} \vee R_\sigma)|I|^\kappa R_{s_0}^{p_0-1} \leq 1/2 \quad (2.17)$$

for $\mu = s_0, \sigma, s$, then Φ_{t_0} is a contraction map on $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$. Since $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$ is a complete metric space by Lemma 2. 3 , Φ_{t_0} has a unique fixed point in $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$ and the solution u of (NLW) with the data $u(0) = \phi$, $\partial_t u(0) = \psi$ is given by the fixed point of Φ_0. Let $R_\mu \equiv 2C\gamma_\mu$, $\mu = s_0, \sigma, s$, in (2.16) and (2.17), then (2.16) and (2.17) are rewritten as

$$M(\gamma_{s_0} \vee \gamma_\sigma)|I|^\kappa \gamma_{s_0}^{p_0-1} \leq 1 \quad (2.18)$$

for some nondecreasing function M.

(1) Let $t_0 = 0$ and $I = [-T, T]$, $T > 0$, in the above argument. Since $\kappa > 0$ for $p_0 < 1 + 4/(n-1)$, for any $\gamma_\mu > 0$, $\mu = s_0, \sigma$, there exists $T > 0$ which satisfies (2.18). Then for any $\gamma_s > 0$ and any $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$, (NLW) has a unique solution u in $\dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$.

We see that the solution u satisfies $u \in C(I; \dot{H}^{s_0} \cap \dot{H}^s)$ by using Lemma 2. 4 with (2.12) and the unitarity of the operator $U(t)$.

The solution u is also a unique solution of (NLW) in $\dot{X}_{s_0}(I)$ by the following lemma. Indeed, if u and v are the solutions with the same initial data (ϕ, ψ), then by that lemma, there exists $\varepsilon > 0$ such that

$$(u(t), \partial_t u(t)) = (v(t), \partial_t v(t)) \quad \text{in} \quad \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s \quad (2.19)$$

for any $t \in [-\varepsilon, \varepsilon] \cap I$. By repetition of the argument with $t_1 = \pm j\varepsilon$, $j \geq 1$, we have (2.19) for any $t \in I$.

Lemma 2. 5 *Let u, v be the solutions of* (NLW) *in* $\dot{X}_{s_0}(I)$. *Then there exists $\varepsilon > 0$ with the following property. If*

$$(u(t), \partial_t u(t)) = (v(t), \partial_t v(t)) \qquad in \quad \dot{\mathcal{H}}^s \cap \dot{\mathcal{H}}^{s_0} \tag{2.20}$$

holds for some $t = t_1 \in I$, then (2.20) holds for any $t \in [t_1 - \varepsilon, t_1 + \varepsilon] \cap I$.

Proof of Lemma 2. 5 . Let $R_\mu > 0$, $\mu = s_0, \sigma, s$, be numbers such that $u, v \in \dot{X}_{s_0}(I, R_{s_0}, R_\sigma, R_s)$. Let u and v satisfy (2.20) for some $t = t_1 \in I$. Then u and v satisfy

$$w(t) = \Phi(w)(t) = \dot{K}(t - t_1)u(t_1) + K(t - t_1)\partial_t u(t_1) + \int_{t_1}^{t} K(t - s) f(w(s)) ds$$

for $w = u, v$. Therefore by the same argument on (2.14) we have

$$d_{t_1,\varepsilon}(\Phi(u), \Phi(v)) \leq M(R_{s_0} \vee R_\sigma)(2\varepsilon)^\kappa R_{s_0}^{p_0 - 1} d_{t_1,\varepsilon}(u, v),$$

where $d_{t_1,\varepsilon}$ denotes the metric d restricted on $I_{t_1,\varepsilon} \equiv [t_1 - \varepsilon, t_1 + \varepsilon] \cap I$, and M is independent of t_0 and ε. Therefore taking $\varepsilon > 0$ sufficiently small, we have $d_{t_1,\varepsilon}(u, v) = 0$, where ε is independent of $t_1 \in I$. Especially $u(t) = v(t)$ in $\dot{H}^{s_0}$ a.e $t \in I_{t_1,\varepsilon}$. By the continuity of u and v on I with values in $\dot{H}^{s_0}$, we have $u(t) = v(t)$ in $\dot{H}^{s_0}$ for any $t \in I_{t_1,\varepsilon}$. This also implies $u(t) = v(t)$ in $\dot{H}^s$ and $\partial_t u(t) = \partial_t v(t)$ in $\dot{H}^{s_0 - 1} \cap \dot{H}^{s-1}$ for any $t \in I_{t_1,\varepsilon}$. □

(2) We consider the case $T^* < \infty$. The proof for $T_* < \infty$ follows quite similarly. Now let

$$\gamma_* \equiv \sup_{0 \leq t < T^*} \|(u(t), \partial_t u(t)); \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s\| < \infty.$$

Let $\varepsilon > 0$ be a number which satisfies

$$M(\gamma_*)\varepsilon^\kappa \gamma_*^{p_0 - 1} \leq 1,$$

where M is the same function appearing in (2.18). Let $t_0 > 0$ be a number such that $T^* \in (t_0, t_0 + \varepsilon)$. Then by the same argument as in the proof of (1), (NLW) has a unique solution $\tilde{u}_+$ in $\dot{X}_{s_0}([t_0, t_0 + \varepsilon])$ with $\tilde{u}_+(t_0) = u(t_0)$ and $\partial_t \tilde{u}_+(t_0) = \partial_t u(t_0)$. Let $\tilde{u}(t) = u(t)$ for $t \in [0, t_0)$, $\tilde{u}(t) = \tilde{u}_+(t)$ for $t \in [t_0, t_0 + \varepsilon]$. Then $\tilde{u}$ is a solution of (NLW) in $\dot{X}_{s_0}([0, t_0 + \varepsilon])$ with $\tilde{u}(0) = u(0)$ and $\partial_t \tilde{u}(0) = \partial_t u(0)$. This contradicts the definition of T^*.

(3) Let $t_0 = 0$ and $I = [-T, T]$, $T > 0$. If $p < 1 + 4/(n-1)$, then we take p_0 as $p_0 = p$. Since κ and p satisfy $\kappa \neq 0$ and $p \neq 1$, for any fixed γ_σ we take T in (2.18) as any number with

$$T \leq C\gamma_{s_0}^{-2(p-1)/(4-(p-1)(n-2s_0))}, \tag{2.21}$$

where the constant C is independent of $\gamma_s > 0$ and $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$. So that we conclude (1.6) by (2.21).

(4) (4a) In the argument before the proof of (1), if p satisfies $p \geq 1 + 4/(n-1)$, then we take p_0 for $p_0 = 1+4/(n-1)$ and therefore $\kappa = 0$ in (2.18) by (2.5). Since M in (2.18) is independent of I, therefore for any given $\gamma_\sigma > 0$ there exists small γ_0 such that (2.18) holds with $|I|^\kappa$ replaced by 1. With these $\gamma_\mu > 0$, $\mu = s_0, \sigma$, fixed, for any $\gamma_s > 0$ and $(\phi, \psi) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$ we see that Φ is a contraction map on $\dot{X}_{s_0}(\mathbf{R}, R_{s_0}, R_\sigma, R_s)$ for some $R_\mu > 0$, $\mu = s_0, \sigma, s$. The fixed point u is unique in $\dot{X}_{s_0}(\mathbf{R})$. Indeed, for any $t_0 \in \mathbf{R}$ and any data at $t = t_0$, by the same argument as in the proof of (1) we conclude that local solutions are unique in $\dot{X}_{s_0}([t_0 - T, t_0 + T])$ for some $T > 0$, where T is determined only by the $\dot{\mathcal{H}}^\mu$-norm, $\mu = s_0, \sigma$, of the data. Since the norms $\|(u(t), \partial_t u(t)); \dot{\mathcal{H}}^\mu\|$, $\mu = s_0, \sigma$, are bounded, we conclude the uniqueness of solutions in $\dot{X}_{s_0}(\mathbf{R})$.

Let (ϕ_+, ψ_+) and (ϕ_-, ψ_-) be defined by

$$\phi_\pm = \phi + \int_0^{\pm\infty} K(-s)f(u(s))ds, \quad \psi_\pm = \psi + \int_0^{\pm\infty} \dot{K}(-s)f(u(s))ds. \tag{2.22}$$

Then $(\phi_\pm, \psi_\pm) \in \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$. Indeed, by Lemma 2. 4 , for any $\mu \in \mathbf{R}$ we have

$$\begin{aligned} \|\phi_\pm; \dot{H}^\mu\| &\leq \|\phi; \dot{H}^\mu\| + \|\textstyle\int_t^{\pm\infty} K(t-s)f(u(s))ds; L^\infty(\mathbf{R}; \dot{H}^\mu)\| \\ &\leq \|\phi; \dot{H}^\mu\| + C\|f(u); L^{\tilde{q}}(\mathbf{R}; \dot{B}_{\tilde{r}}^{\mu - s_0})\|, \end{aligned} \tag{2.23}$$

where we have used the continuity of $K *_t f(u)$ with respect to the time variable. Since the last term of the last inequality in (2.23) is estimated as (2.12) with $|I|^\kappa$ replaced by 1, $\|\phi_\pm; \dot{H}^\mu\|$ is finite for $\mu = s_0, s$. It follows that $\psi_\pm \in \dot{H}^{s_0 - 1} \cap \dot{H}^{s-1}$ analogously.

With these $\phi_\pm$ and $\psi_\pm$, (1.7) holds. Indeed, since $v_\pm$ are rewritten as

$$v_\pm(t) = \dot{K}(t)\phi + K(t)\psi + \int_0^{\pm\infty} K(t-s)f(u(s))ds,$$

by the same argument in (2.23), we have

$$\begin{aligned} \|u(t) - v_+(t); \dot{H}^\mu\| &\leq \|\textstyle\int_\tau^\infty K(\tau - s)f(u(s))ds; L_\tau^\infty([t, \infty); \dot{H}^\mu)\| \\ &\leq C\|f(u); L^{\tilde{q}}([t, \infty); \dot{B}_{\tilde{r}}^{\mu - s_0})\|. \end{aligned} \tag{2.24}$$

Since the RHS of the last inequality is estimated by (2.12) with $|I|^{\kappa}$ replaced by 1, and is bounded uniformly in t for $\mu = s_0, s$, we conclude that $u(t) - v_+(t)$ converges to zero in $\dot{H}^{s_0} \cap \dot{H}^s$ as $t \to \infty$. By an analogous argument for $\partial_t u(t) - \partial_t v_+(t)$, $u(t) - v_-(t)$ and $\partial_t u(t) - \partial_t v_-(t)$, we conclude (1.7).

To show the uniqueness of the pairs $(\phi_\pm, \psi_\pm)$ in (1.7), it suffices to show that

$$\lim_{t\to\infty} \|(\dot{K}(t)\phi_0 + K(t)\psi_0, \partial_t(\dot{K}(t)\phi_0 + K(t)\psi_0); \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s\| = 0 \qquad (2.25)$$

implies $\phi_0 = \psi_0 = 0$. If (2.25) holds, then we have

$$\lim_{t\to\infty} \|U(t)(\phi_0 - i\omega^{-1}\psi_0); \dot{H}^{\mu}\| \vee \|U(-t)(\phi_0 + i\omega^{-1}\psi_0); \dot{H}^{\mu}\| = 0$$

for any $\mu = s_0, s$, where $\omega \equiv \sqrt{-\Delta}$. Therefore by the unitarity of $U(t)$, we conclude $\phi_0 = \psi_0 = 0$.

(4b) For any $\gamma_s > 0$ and any data $(\phi_-, \psi_-) \in \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$, by the argument before the proof of (1), $\Phi_{-\infty}$ has a unique fixed point u in

$$\dot{X}_{s_0}(\mathbf{R}, R_{s_0}, R_\sigma, R_s),$$

where $R_\mu > 0$, $\mu = s_0, \sigma, s$, are chosen to satisfy $R_\mu = 2C\gamma_\mu$, $\mu = s_0, \sigma, s$, with the same constant C appearing in (2.16). Let (ϕ, ψ) and (ϕ_+, ψ_+) be defined by the equations (2.22). Then, as in the argument on (2.23), we have $(\phi, \psi), (\phi_+, \psi_+) \in \dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$. Now u is rewritten as

$$u(t) = \dot{K}(t)\phi + K(t)\psi + \int_0^t K(t-s)f(u(s))ds,$$

which implies exactly that u is a solution of (NLW) in $\dot{X}_{s_0}(\mathbf{R}, R_{s_0}, R_\sigma, R_s)$ with the data $u(0) = \phi$, $\partial_t u(0) = \psi$. u is also a unique solution of (NLW) in $\dot{X}_{s_0}(\mathbf{R})$ which satisfies

$$\|(u(t) - v_-(t), \partial_t(u(t) - v_-(t))); \dot{H}^{s_0} \cap \dot{H}^s\| \to 0 \qquad (2.26)$$

as $t \to -\infty$, where $v_-(t) \equiv \dot{K}(t)\phi_- + K(t)\psi_-$. Indeed, let v be a solution of (NLW) in $\dot{X}_{s_0}(\mathbf{R})$ which satisfies (2.26) with u replaced by v. Then v satisfies $v = \Phi_{-\infty}(v)$ and for sufficiently small $t_0 < 0$,

$$\|v; \dot{X}_{s_0}((-\infty, t_0])\| \leq 2C\gamma_\mu$$

for any $\mu = s_0, \sigma, s$, since

$$\Big| \|v(t); \dot{H}^{\mu}\| - \|v_-(t); \dot{H}^{\mu}\| \Big|$$

converges to zero as $t \to -\infty$ and $\|v_-(t); \dot{H}^\mu\| \le \gamma_\mu$ by the conservation of the energy of the free propagator, namely

$$\|(v_-(t), \partial_t v_-(t)); \dot{\mathcal{H}}^\mu\| = \|(\phi_-, \psi_-); \dot{\mathcal{H}}^\mu\| \tag{2.27}$$

for any $t \in \mathbf{R}$. Here (2.27) follows from

$$\begin{aligned}
& \|v_-(t); \dot{H}^\mu\|^2 \\
= \ & \frac{1}{2}\{\|(\phi_-, \psi_-); \dot{\mathcal{H}}^\mu\|^2 \\
& \qquad + \mathrm{Re}(\omega^\mu(\phi_- - i\omega^{-1}\psi_-), \omega^\mu \exp(-2it\omega)(\phi_- + i\omega^{-1}\psi_-))\}, \\
& \|\partial_t v_-(t); \dot{H}^{\mu-1}\|^2 \\
= \ & \frac{1}{2}\{\|(\phi_-, \psi_-); \dot{\mathcal{H}}^\mu\|^2 \\
& \qquad - \mathrm{Re}(\omega^\mu(\phi_- - i\omega^{-1}\psi_-), \omega^\mu \exp(-2it\omega)(\phi_- + i\omega^{-1}\psi_-))\}.
\end{aligned}$$

Therefore v is also a fixed point of $\Phi_{-\infty}$ in

$$\dot{X}_{s_0}((-\infty, t_0], R_{s_0}, R_\sigma, R_s).$$

Since $\Phi_{-\infty}$ has a unique fixed point in $\dot{X}_{s_0}((-\infty, t_0], R_{s_0}, R_\sigma, R_s)$, we have $u(t) = v(t)$ for any $t \in (-\infty, t_0]$. Especially we have $u(t_0) = v(t_0)$. Since for any $t_0 \in \mathbf{R}$ and any given data at $t = t_0$, local solutions of (NLW) are unique in $\dot{X}_{s_0}([t_0 - T, t_0 + T])$ for some $T > 0$, where T is determined only by the $\dot{\mathcal{H}}^\mu$-norm, $\mu = s_0, \sigma$, of the data, and the norms of $(u(t), \partial_t u(t))$ and $(v(t), \partial_t v(t))$ in $\dot{\mathcal{H}}^\mu$, $\mu = s_0, \sigma$, are bounded with respect to the time variable, we conclude $u = v$ in $\dot{X}_{s_0}(\mathbf{R})$.

Similarly to the argument on (2.24) we have (1.7) for these u, (ϕ_+, ψ_+), (ϕ_-, ψ_-), and we also have the uniqueness of (ϕ_+, ψ_+) for (ϕ_-, ψ_-) with (1.7).

By the above argument, for any $\gamma_s > 0$, we are able to define the scattering operator S on $\dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$ to $\dot{\mathcal{H}}^{s_0} \cap \dot{\mathcal{H}}^s$ as $S((\phi_-, \psi_-)) = (\phi_+, \psi_+)$ by way of a unique global solution u of (NLW) in $\dot{X}_{s_0}(\mathbf{R})$ with (1.7). Moreover, the correspondence is given by (2.22) or equivalently by

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} K(-s) f(u(s)) ds, \quad \psi_+ = \psi_- + \int_{-\infty}^{\infty} \dot{K}(-s) f(u(s)) ds. \tag{2.28}$$

We show the continuity of the operator S. For any

$$\{(\phi_-^j, \psi_-^j)\}_{j=1}^\infty \subset \dot{A}^s(\gamma_{s_0}, \gamma_\sigma, \gamma_s)$$

which satisfies (1.8), let $\{u^j\}_{j=1}^{\infty}$ be the fixed points in $\dot{X}_{s_0}(\mathbf{R}, R_{s_0}, R_\sigma, R_s)$ of the operators $\{\Phi^j_{-\infty}\}_{j=1}^{\infty}$ given by

$$\Phi^j_{-\infty}(v)(t) \equiv \dot{K}(t)\phi^j_- + K(t)\psi^j_- + \int_{-\infty}^{t} K(t-s)f(v(s))ds$$

for $j \geq 1$. Then by Lemma 2. 4 we have, as in (2.14),

$$\begin{aligned} & d(\Phi^j_{-\infty}(u), \Phi^j_{-\infty}(u^j)) \\ \leq\ & C\|(\phi_- - \phi^j_-, \psi_- - \psi^j_-); \dot{\mathcal{H}}^{s_0}\| + M(R_{s_0} \vee R_\sigma)R_{s_0}^{p_0-1}d(u, u^j) \end{aligned}$$

for any $j \geq 1$. Since (2.17) holds with $|I|^\kappa$ replaced with 1, we conclude that $d(u, u^j) \to 0$ as $j \to \infty$. On the other hand, by (2.28), we have

$$\begin{aligned} & \|\phi_+ - \phi^j_+; \dot{H}^{s_0}\| \\ \leq\ & \|\phi_- - \phi^j_-; \dot{H}^{s_0}\| \\ & +\|\int_{-\infty}^{\infty} K(t-s)(f(u(s)) - f(u^j(s)))ds; L^\infty(\mathbf{R}; \dot{H}^{s_0})\| \end{aligned}$$

for any $j \geq 1$. By Lemma 2. 4 , the same argument used on (2.15) shows that the last term on the RHS of the last inequality is estimated by

$$M(R_{s_0} \vee R_\sigma)R_{s_0}^{p_0-1}d(u, u^j).$$

Therefore we have $\|\phi_+ - \phi^j_+; \dot{H}^{s_0}\| \to 0$ as $j \to \infty$. By the convex inequality

$$\begin{aligned} & \|\phi_+ - \phi^j_+; \dot{H}^{\nu}\| \\ \leq\ & \|\phi_+ - \phi^j_+; \dot{H}^{s_0}\|^{(s-\nu)/(s-s_0)}\|\phi_+ - \phi^j_+; \dot{H}^{s}\|^{(\nu-s_0)/(s-s_0)}, \end{aligned}$$

we have $\|\phi_+ - \phi^j_+; \dot{H}^{\nu}\| \to 0$ as $j \to \infty$ for any ν with $s_0 \leq \nu < s$. Analogously we also have $\|\psi_+ - \psi^j_+; \dot{H}^{\nu-1}\| \to 0$ as $j \to \infty$ for any ν with $s_0 \leq \nu < s$. Therefore we obtain (1.9).

(5) Let p_0, s_0 be as in the proof of (1) or (4). Let $t_0 = 0$ and let $I = [-T, T]$ for (1), $I = \mathbf{R}$ for (4). Since $\{u_j\}_{j=1}^{\infty}$ satisfies

$$u_j(t) = \dot{K}(t)\phi_j + K(t)\psi_j + \int_0^t K(t-s)f(u_j(s))ds$$

for any $j \geq 1$, by Lemma 2. 4 and the embedding $L^{\tilde{r}} \hookrightarrow \dot{B}^0_{\tilde{r}}$ for $1 < \tilde{r} \leq 2$, we have

$$d(u, u_j) \leq C\|(\phi - \phi_j, \psi - \psi_j); \dot{\mathcal{H}}^{s_0}\| + C\|f(u) - f(u_j); L^{\tilde{q}}(I; L^{\tilde{r}})\| \quad (2.29)$$

for any $j \geq 1$, where the constant C is independent of I, (ϕ, ψ), u, $\{(\phi_j, \psi_j)\}_{j=1}^{\infty}$ and $\{u_j\}_{j=1}^{\infty}$. As we have shown in the proof of (1) and (4), for any given initial data in $\dot{A}^s(\gamma_{s0}, \gamma_\sigma, \gamma_s)$, (NLW) has a unique solution in $\dot{X}_{s0}(\mathbf{R}, R_{s0}, R_\sigma, R_s)$, where $R_\mu > 0$, $\mu = s_0, \sigma, s$, are chosen to satisfy $R_\mu = 2C\gamma_\mu$, $\mu = s_0, \sigma, s$, and C is the same constant appearing in (2.16), and the solution is also unique in $\dot{X}_{s0}(I)$. Therefore u, $\{u_j\}_{j=1}^{\infty}$ are in $\dot{X}_{s0}(\mathbf{R}, R_{s0}, R_\sigma, R_s)$. So that by the same argument on (2.15), we have

$$C\|f(u) - f(u_j); L^{\tilde{q}}(I; L^{\tilde{r}})\| \leq M(R_{s0} \vee R_\sigma)|I|^\kappa R_{s0}^{p_0-1} d(u, u_j),$$

where $|I|^\kappa$ is disregarded for (4). Since (2.17) holds for these R_μ, $\mu = s_0, \sigma, s$, we conclude that $d(u, u_j) \to 0$ as $j \to \infty$. Especially we obtain

$$\|u - u_j; L^\infty(I; \dot{H}^{s0})\| \to 0 \quad \text{as} \quad j \to \infty.$$

By the convex inequality such as

$$\begin{aligned} & \|u(t) - u_j(t); \dot{H}^\nu\| \\ \leq \; & \|u(t) - u_j(t); \dot{H}^{s0}\|^{(s-\nu)/(s-s0)} \|u(t) - u_j(t); \dot{H}^s\|^{(\nu-s0)/(s-s0)}, \end{aligned}$$

the bound $\|w; L^\infty(I; \dot{H}^s)\| \leq R_s$ for $w = u, u_j$, $j \geq 1$, we conclude that

$$\|u - u_j; L^\infty(I; \dot{H}^\nu)\| \to 0$$

as $j \to \infty$ for any $s_0 \leq \nu < s$. Concerning

$$\|\partial_t(u - u_j); L^\infty(I; \dot{H}^{\nu-1})\| \to 0$$

as $j \to \infty$ for any ν with $s_0 \leq \nu < s$, the proof is analogously and omitted. Consequently we obtain the required results. □

Proof of Theorem 1. 2. To solve (NLW) by a contraction argument we define the function space as follows. For any interval $I \subset \mathbf{R}, s > 0, \sigma > 0, R_\mu > 0, \mu = 0, \sigma, s$, let $X(I, R_0, R_\sigma, R_s)$ be a function space defined by

$$X(I, R_0, R_\sigma, R_s) \equiv \{u; \|u \; ; \; L^\infty(I, H^\mu)\| \leq R_\mu, \mu = 0, \sigma, s\}$$

endowed with metric d_0.

We first consider the linear estimate on the operator K in the Sobolev space.

Lemma 2. 6 *For any $\mu \in \mathbf{R}$, any interval $I = [-T, T]$, $T > 0$, and any ℓ with*

$$1/2 \le 1/\ell \le (n+2)/2n \wedge 1,$$

it follows that

$$\|\dot{K}(t)\phi + K(t)\psi; L^\infty(I; H^\mu)\| \le C(1+T)\|(\phi,\psi); \mathcal{H}^\mu\|,$$

$$\begin{aligned} & \| \int_0^t K(t-s)h(s)ds; L^\infty(I; H^\mu)\| \\ \le \quad & C(1+T)T\|h; L^\infty(I; B_\ell^{(\mu-1+n(1/\ell-1/2))+})\| \end{aligned} \tag{2.30}$$

for any $(\phi,\psi) \in \mathcal{H}^\mu$, $h \in L^\infty(I; B_\ell^{(\mu-1+n(1/\ell-1/2))+})$, where the constant C is independent of (ϕ,ψ), h and T.

Proof of Lemma 2. 6 . By the following elementary inequality

$$|\sin(t|\xi|)|/|\xi| \le C(1+|t|)/\langle\xi\rangle$$

with constant C independent of t and ξ, we have

$$\|K(t)\psi; L^2\| \le C(1+T)\|\psi; H^{-1}\|.$$

As a simple superposition we have

$$\| \int_0^t K(t-s)h(s)ds; L^\infty(I; L^2)\| \le C(1+T)T\|h; L^\infty(I; H^{-1})\|$$

By the embeddings $B_\ell^{-1+n(1/\ell-1/2)} \hookrightarrow H^{-1}$ for any ℓ with

$$1/2 \le 1/\ell \le (n+2)/2n \wedge 1,$$

and $B_r^{\mu+} \hookrightarrow B_r^\mu$ for any $\mu \in \mathbf{R}$ and $1 \le r \le \infty$, we obtain the lemma for $\mu = 0$. Since $(1-\Delta)^{\mu/2}$ is an isomorphism on B_r^s to $B_r^{s-\mu}$ for any $\mu, s \in \mathbf{R}$ and $1 \le r \le \infty$, we obtain the lemma. □

Let $1 \le p < \infty$. Let p_0 be any number with

$$1 \le p_0 \le \min\{1+2/n, 2, p\}, \quad p_0 \ne 1 \ \text{ if } \ p \ne 1.$$

Let $s_0 \equiv 1 - n(p_0-1)/2$ and let f satisfy $(N)_{(s-s_0)+,p}$. Let $1/\tilde{r} \equiv p_0/2$. Let $1/r^* \equiv (p_0-1)/2(p-1)$ for $p \ne 1$, $1/r^* \equiv 1/2$ for $p = 1$. Since $\tilde{r}, r^*$ satisfy

$$1/\tilde{r} = (p-1)/r^* + 1/2,$$

for any μ with $0 < \mu \leq s$ we have by Lemma 2. 1

$$\|f(u); \dot{B}^{\mu}_{\tilde{r}}\| \leq M(\|u; L^{\infty} \cap \dot{B}^{0}_{\infty}\|)\|u; \dot{B}^{0}_{r^*}\|^{p-1}\|u; \dot{B}^{\mu}_{2}\|,$$

We continue the estimate on the RHS of the last inequality on the basis of the convex inequality

$$\|u; \dot{B}^{0}_{r^*}\| \leq \|u; \dot{B}^{0}_{\infty}\|^{1-\theta}\|u; \dot{B}^{0}_{2}\|^{\theta}$$

with $\theta \equiv (p_0 - 1)/(p - 1)$ to obtain the estimate of the form

$$\|f(u); \dot{B}^{\mu}_{\tilde{r}}\| \leq M(\|u; L^{\infty} \cap \dot{B}^{0}_{\infty}\|)\|u; \dot{B}^{0}_{2}\|^{p_0-1}\|u; \dot{B}^{\mu}_{2}\|.$$

Therefore for any μ with $0 < \mu \leq s$ we have

$$\|f(u); L^{\infty}(I; \dot{B}^{\mu}_{\tilde{r}})\| \leq M(R_{\sigma})R_0^{p_0-1}\|u; L^{\infty}(I; \dot{B}^{\mu}_{2})\| \tag{2.31}$$

for any $u \in \cup_{R_s>0}X(I, R_0, R_{\sigma}, R_s) \cap L^{\infty}(I; \dot{B}^{\mu}_{2})$, where we have used the embedding $H^{\sigma} \hookrightarrow L^{\infty} \cap \dot{B}^{0}_{\infty}$ for any $\sigma > n/2$. Similarly we also have

$$\|f(u); L^{\infty}(I; L^{\tilde{r}})\| \leq M(R_{\sigma})R_0^{p_0-1}\|u; L^{\infty}(I; L^2)\| \tag{2.32}$$

for any $u \in \cup_{R_s>0}X(I, R_0, R_{\sigma}, R_s)$. By (2.31) and (2.32), we obtain

$$\|f(u); L^{\infty}(I; B^{\mu}_{\tilde{r}})\| \leq M(R_{\sigma})R_0^{p_0-1}\|u; L^{\infty}(I; H^{\mu})\| \tag{2.33}$$

for any μ with $0 \leq \mu \leq s$ and any $u \in \cup_{R_s>0}X(I, R_0, R_{\sigma}, R_s) \cap L^{\infty}(I; H^{\mu})$, where we have used $B^{\mu}_{r} = L^{r} \cap \dot{B}^{\mu}_{r}$ for $\mu > 0, 1 \leq r \leq \infty$, and $L^{\tilde{r}} \hookrightarrow B^{0}_{\tilde{r}}$ for $1 < \tilde{r} \leq 2$. Since $s_0, \tilde{r}$ satisfy $s_0 = 1 - n(1/\tilde{r} - 1/2)$, by Lemma 2. 6 and (2.33) for any μ with $0 \leq \mu \leq s$ we have

$$\begin{aligned}
&\|\Phi(u); L^{\infty}(I; H^{\mu})\| \\
\leq\ & C(1+T)\|(\phi, \psi); \mathcal{H}^{\mu}\| + C(1+T)T\|f(u); L^{\infty}(I; B^{(\mu-s_0)+}_{\tilde{r}})\| \\
\leq\ & C(1+T)\gamma_{\mu} + (1+T)TM(R_{\sigma})R_0^{p_0-1}\|u; L^{\infty}(I; H^{\mu})\|
\end{aligned}$$

for any $u \in \cup_{R_s>0}X(I, R_0, R_{\sigma}, R_s) \cap L^{\infty}(I; H^{\mu})$, where we have used the embedding $B^{\mu}_{2} \hookrightarrow B^{(\mu-s_0)+}_{2}$ for $\mu \geq 0$ at the last inequality. Similarly we also have

$$d_0(\Phi(u), \Phi(v)) \leq (1+T)TM(R_{\sigma})R_0^{p_0-1}d_0(u, v)$$

for any $u, v \in \cup_{R_s>0}X(I, R_0, R_{\sigma}, R_s)$. So that if $\gamma_{\mu}, R_{\mu}, \mu = 0, \sigma, s$, satisfy

$$C(1+T)\gamma_{\mu} + (1+T)TM(R_{\sigma})R_0^{p_0-1}R_{\mu} \leq R_{\mu} \tag{2.34}$$

$$(1+T)TM(R_{\sigma})R_0^{p_0-1} \leq 1/2, \tag{2.35}$$

then Φ is a contraction map on $X(I, R_0, R_\sigma, R_s)$ and the unique solution u of (NLW) is given by the fixed point of Φ. Now taking R_μ, $\mu = 0, \sigma, s$, for $R_\mu \equiv 2C(1+T)\gamma_\mu$, $\mu = 0, \sigma, s$, (2.34) and (2.35) are rewritten as

$$(1+T)^{p_0} T M((1+T)\gamma_\sigma)\gamma_0^{p_0 - 1} \leq 1 \tag{2.36}$$

for some nondecreasing function M.

(1) For any $\gamma_0, \gamma_\sigma > 0$ there exists $T > 0$ which satisfies (2.36). With this T fixed, for any $\gamma_s > 0$ and any $(\phi, \psi) \in A^s(\gamma_0, \gamma_\sigma, \gamma_s)$ (NLW) has a unique solution u in $X(I, R_0, R_\sigma, R_s)$ for some $R_\mu > 0$, $\mu = 0, \sigma, s$, by the above argument. $u \in C(I; H^s)$ follows from Lemma 2. 6 , (2.33) and the Lebesgue convergence theorem. The uniqueness of solutions in $C_b(I; H^s)$ follows from the fact that for any $t_0 \in \mathbf{R}$ and any data at $t = t_0$ the existence time of solutions are estimated only in terms of the H^σ-norm of the data such as (2.36).

(2) If $T^* < \infty$ and $\|(u(t), \partial_t u(t)); \mathcal{H}^s\|$ is bounded on $[0, T^*)$, then by an argument similar to that in the proof of (2) of Theorem 1. 1 , we are able to show that the solution extends beyond T^*, which contradicts the definition of T^*. As for T_*, the proof is analogous and omitted.

(3) If $p \neq 1$, then $p_0 \neq 1$. Therefore for any fixed $\gamma_\sigma > 0$, T in (2.36) can be taken sufficiently large by taking γ_0 sufficiently small. So that for any data $(\phi, \psi) \in \mathcal{H}^s$ with $\|(\phi, \psi); \mathcal{H}^\sigma\| \leq \gamma_\sigma$, T_* and T^* can be taken sufficiently large by taking $\|(\phi, \psi); \mathcal{H}^0\|$ sufficiently small.

(4) The proof of (4) is similar to that of (5) in Theorem 1. 1 . Namely we have

$$\begin{aligned} d_0(u, u_j) &\leq C(1+T)\|(\phi - \phi_j, \psi - \psi_j); \mathcal{H}^0\| \\ &\quad + C(1+T)T\|f(u) - f(u_j); L^\infty(I; L^{\tilde{r}})\|, \end{aligned}$$

and the last term is estimated by

$$(1+T)TM(R_\sigma)R_0^{p_0 - 1} d_0(u, u_j)$$

for some $R_0, R_\sigma > 0$ which satisfy (2.35). Therefore we have $d_0(u, u_j) \to 0$ as $j \to \infty$, from which (1.11) follows on the basis of the convex inequality

$$\|w; H^\nu\| \leq \|w; L^2\|^{(s-\nu)/s} \|w; H^s\|^{\nu/s}$$

for any ν with $0 \leq \nu < s$ and any $w \in H^s$. □

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