Stability of a stationary solution for evolving boundaries of symmetric three-phases driven by surface diffusion

Kazuo Ito and Yoshihito Kohsaka

Series #471. September 1999

Stability of a stationary solution for evolving boundaries of symmetric three-phases driven by surface diffusion

Kazuo Ito and Yoshihito Kohsaka
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan

August 26, 1999

Abstract

We prove that the sharp interface model for a three-phase boundary motion by surface diffusion proposed by H. Garcke and A. Novick-Cohen admits a unique global solution provided the initial data fulfills a certain symmetric criterion and is also close to a minimizer of the energy under an area-constraint. This minimizer is also a stationary solution of the present model. Moreover we prove that the global solution converges to the minimizer of the energy as time goes to infinity.

Keywords. free boundary problem, triple junction, global existence, stability, surface diffusion.

AMS(MOS) subject classification. 35K55, 35K65, 35R35, 82C24.

1 Introduction and main results

We study a sharp interface model for a three-phase boundary motion by surface diffusion proposed by H. Garcke and A. Novick-Cohen [11]. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. We consider a situation that three phases of a ternary alloy system in a nonequilibrium state are contained in Ω and they are separated by evolving three interphase boundaries $\Gamma^i(t)$, $i = 1, 2, 3$, where $t \geq 0$ denotes the time variable. Three interfaces are connected by a triple junction $m(t) \in \Omega$ at their one end points and are almost perpendicular to the boundary of Ω, $\partial\Omega$, at their another end points. It is known that a Cahn-Hilliard system with a concentration dependent mobility describes this situation as a phase field model. H. Garcke and A. Novick-Cohen [11] discussed a formal singular limit in this system to derive a sharp interface model for $\Gamma(t) = \bigcup_{i=1}^{3} \Gamma^i(t)$ with a triple junction $m(t)$ in the following: for $i = 1, 2, 3$, and $t > 0$

$$\left.\begin{array}{ll}
(A) & \text{along } \Gamma^i(t);\quad V^i = -l^i\sigma^i\kappa^i_{ss} \text{ :surface diffusion flow equation,} \\
(B) & \text{at } \Gamma^i(t)\cap\partial\Omega; \\
& \Gamma^i(t)\perp\partial\Omega \text{ :contact angle is } \pi/2, \\
& \kappa^i_s = 0 \text{ :no flux condition,} \\
(C) & \text{at } m(t); \\
& \angle(\Gamma^1(t),\Gamma^2(t)) = \theta^3,\quad \angle(\Gamma^2(t),\Gamma^3(t)) = \theta^1,\quad \angle(\Gamma^3(t),\Gamma^1(t)) = \theta^2, \\
& l^1\sigma^1\kappa^1_s = l^2\sigma^2\kappa^2_s = l^3\sigma^3\kappa^3_s \text{ :balance of fluxes,} \\
& \sigma^1\kappa^1 + \sigma^2\kappa^2 + \sigma^3\kappa^3 = 0 \text{ :continuity of the chemical potential}
\end{array}\right\} \quad (1.1)$$

with the initial condition

$$\Gamma(0) = \Gamma_0 = \bigcup_{i=1}^{3}\Gamma^i_0, \quad m(0) = m_0. \tag{1.2}$$

Here, V^i and κ^i stand for the normal velocity and the curvature of $\Gamma^i(t)$ respectively, and s denotes the arc-length parameter of $\Gamma^i(t)$. The sign convention used here is as follows: s runs from $m(t)$, at which $s = 0$, to the point of intersection of $\Gamma^i(t)$ with $\partial\Omega$, at which $s = L[\Gamma^i(t)]$, where $L[\Gamma^i(t)]$ denotes the total length of $\Gamma^i(t)$; $V^i(t,s)$ and $\kappa^i(t,s)$ are computed to the direction of the unit normal $N^i(t,s)$ to $\Gamma^i(t)$ at s, where $(T^i(t,s), N^i(t,s))$ makes the orthogonal coordinate system and $T^i(t,s)$ is the unit tangent to $\Gamma^i(t)$ at s. Moreover l^i, σ^i, and θ^i are positive constants with the constraints $\theta^1 + \theta^2 + \theta^3 = 2\pi$ and

$$\frac{\sigma^1}{\sin\theta^1} = \frac{\sigma^2}{\sin\theta^2} = \frac{\sigma^3}{\sin\theta^3}.$$

The latter condition is called Young's law which is well-known among material scientists. $\angle(\Gamma^i(t),\Gamma^j(t))$ stands for the angle between $\Gamma^i(t)$ and $\Gamma^j(t)$ for $i, j = 1, 2, 3$, $i \neq j$. In [11] Garcke and Novick-Cohen also mathematically studied (1.1)-(1.2) to obtain both a local existence result for $\Gamma_0 \in C^{4+\alpha}$ $(0 < \alpha < 1)$ with a suitable compatibility condition and a uniqueness result in a geometric sense.

Our purpose in this paper is to obtain a unique global solution $\Gamma(t)$ with a triple junction $m(t)$ of (1.1)-(1.2) for $\Gamma_0 \in C^3$ with a suitable compatibility condition in a symmetric framework and also to show its convergence to a stationary solution determined by the initial data as $t \to \infty$. As far as we know this is a first contribution to global results for the problem (1.1)-(1.2).

Let us explain our framework for the problem (1.1)-(1.2) which is devoted to study a symmetric evolution of $\Gamma(t)$ with respect to the x-axis. For this purpose let $\Omega = \{(x,y) \in \mathbf{R}^2; -a < x < 0,\ -b < y < b\}$, where a and b are positive constants large enough. We consider the evolution such that $\Gamma^1(t)$ always stays a segment on the x-axis, and $\Gamma^2(t)$ and $\Gamma^3(t)$ are symmetric with respect to the x-axis and $\Gamma^3(t)$ is in $\{(x,y) \in \Omega; y \geq 0\}$. Let $\theta \in (0, \pi/2)$ and set $\theta^1 = 2\theta$ and $\theta^2 = \theta^3 = \pi - \theta$. For simplicity we put $\sigma^2 = \sigma^3 = 1$ and $l^1 = l^2 = l^3 = 1$. Then Young's law and the condition on the balance of fluxes are simplified to $\sigma^1 = 2\cos\theta$ and $\kappa^1_s = \kappa^2_s = \kappa^3_s = 0$ at $m(t)$, respectively, and by the symmetry the condition on the continuity of the chemical potential is automatically fulfilled. Moreover for brevity we introduce the following terminologies.

Definition 1.1 (i) We say that a family of three curves Γ belongs to $\mathcal{S}_\theta$, if there are $\xi \in (0,a)$ and nonnegative function $u \in C^3[-\xi,0]$ with $u(-\xi) = 0$, $u_x(-\xi) = \tan\theta$, $u_x(0) = 0$, and $(u_{xx}(1+u_x^2)^{-3/2})_x = 0$ at $x = -\xi$ and 0 such that $\Gamma = \Lambda[u,\xi]$ and $m = \mu[\xi]$.

(ii) Let $A > 0$ be a given constant. We say that a family of three curves Γ belongs to $\mathcal{C}_{\theta,A}$ if there are $\xi \in (0,a)$ and nonnegative function $u \in H^2(-\xi,0)$ with $u(-\xi) = 0$, $u_x(-\xi) = \tan\theta$, $u_x(0) = 0$, and $\int_{-\xi}^0 u(x)dx = A$ such that $\Gamma = \Lambda[u,\xi]$ and $m = \mu[\xi]$.

Here we set

$$
\begin{aligned}
&\mu[\xi] := (-\xi, 0)\text{: triple junction,}\\
&\Lambda[u,\xi] := \bigcup_{i=1}^{3} \Lambda^i[u,\xi],\\
&\Lambda^1[u,\xi] := \{(x,0); -a \le x \le -\xi\},\\
&\Lambda^2[u,\xi] := \{(x,-u(x)); -\xi \le x \le 0\},\\
&\Lambda^3[u,\xi] := \{(x,u(x)); -\xi \le x \le 0\}.
\end{aligned}
$$

□

Note that a family of three curves in $\mathcal{S}_\theta$ or in $\mathcal{C}_{\theta,A}$ is symmetric with respect to the x-axis. In view of the structure of the evolution problem (1.1)-(1.2) it can be expected that if $\Gamma_0 \in \mathcal{S}_\theta$, then the solution $\Gamma(t)$ of (1.1)-(1.2) also belongs to $\mathcal{S}_\theta$ for all $t > 0$ as long as it exists. So we go to work the evolution problem (1.1)-(1.2) on $\mathcal{S}_\theta$ and set $\Gamma(t) = \Lambda[u(t,\cdot),\xi(t)]$ with $m(t) = \mu[\xi(t)]$ for $t > 0$ and $\Gamma_0 = \Lambda[u_0,\xi_0]$ with $m_0 = \mu[\xi_0]$. Then $(u(t,x),\xi(t))$ for $t \ge 0$ and $-\xi(t) \le x \le 0$ is the unknown function to be looked for and the equation (1.1) for $t > 0$ is reduced to

$$
\begin{cases}
u_t = -\partial_x\left(\dfrac{1}{(1+u_x^2)^{1/2}}\partial_x\left(\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}\right)\right), & -\xi(t) < x < 0,\\
u_x(t,-\xi(t)) = \tan\theta, \quad u_x(t,0) = 0, & \\
\partial_x\left(\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}\right) = 0, \quad \text{at } x = -\xi(t) \text{ and } 0, & \\
u(t,-\xi(t)) = 0 &
\end{cases}
\tag{1.3}
$$

and the initial condition (1.2) is reduced to

$$
u(0,x) = u_0(x) \quad \text{for } -\xi_0 < x < 0, \quad \xi(0) = \xi_0. \tag{1.4}
$$

Now our task is reduced to solve the problem (1.3)-(1.4).

In the study of the global-in-time solvability of the evolution problem (1.3)-(1.4), the variational analysis for the energy associated with (1.3)-(1.4) plays an essential role. To explain this let $A > 0$ satisfy $\xi_{\theta,A} < a$, where we put

$$
\xi_{\theta,A} = r_{\theta,A}\sin\theta, \tag{1.5}
$$

$$
r_{\theta,A} = \left(\frac{2A}{\theta - \sin\theta\cos\theta}\right)^{1/2}. \tag{1.6}
$$

(This choice of A is only to ensure that $\Gamma_{\theta,A}$ defined below is completely contained in Ω.) For such A, we also set

$$u_{\theta,A}(x) = -r_{\theta,A}\cos\theta + (r_{\theta,A}^2 - x^2)^{1/2}, \quad x \in [-\xi_{\theta,A}, 0] \tag{1.7}$$

and define

$$\Gamma_{\theta,A} = \Lambda[u_{\theta,A}, \xi_{\theta,A}]. \tag{1.8}$$

Then $\Gamma_{\theta,A}$ belongs to $\mathcal{C}_{\theta,A}$ and it consists of one segment $\Gamma^1_{\theta,A} = \Lambda^1[u_{\theta,A}, \xi_{\theta,A}]$ on the x-axis and two circular arcs $\Gamma^i_{\theta,A} = \Lambda^i[u_{\theta,A}, \xi_{\theta,A}]$, $i = 2, 3$, which are symmetric to each other with respect to the x-axis. $\Gamma_{\theta,A}$ is also the unique stationary solution of (1.3) such that the area enclosed by $\Gamma^3_{\theta,A}$, the x-axis, and the y-axis is equal to A. Now we define the energy E on $\mathcal{C}_{\theta,A}$ associated with the problem (1.3)-(1.4) by

$$E[\Gamma] = (a - \xi)\cos\theta + \int_{-\xi}^{0} (1 + u_x^2)^{1/2} dx \quad \text{for } \Gamma = \Lambda[u, \xi] \in \mathcal{C}_{\theta,A}. \tag{1.9}$$

Then it will be shown in Theorem 2.1 that $\Gamma_{\theta,A}$ is the unique minimizer of E for $\Gamma = \Lambda[u, \xi] \in \mathcal{C}_{\theta,A}$. It is possible to say that this minimizing problem is an extended version of several kinds of isoperimetric problems with account of various boundary conditions, which were studied by many researchers by applying abstract methods such as the geometric measure theory. We especially refer the reader to [20]. But in Theorem 2.1 we shall give a direct proof, since our case can be treated by means of piecewise smooth graphs in one dimension. Owing to one-dimensionality, we have no extra difficulties to show the uniqueness and the existence of the minimizer of E on $\mathcal{C}_{\theta,A}$, though many technical difficulties to be overcome arise due to the presence of the triple junction. The result in Theorem 2.1 is a key to investigate the global-in-time solvability of the problem (1.3)-(1.4) for Γ_0 in $C^{2+\alpha}$ with $0 < \alpha \leq 1/2$. In fact the L^2-estimate of the first derivative of the curvature of $\Gamma^3(t)$, which is the key estimate for the global solvability, needs the lower bound of $E[\Gamma(t)]$. The reason why the range of α is restricted to $(0, 1/2]$ is, roughly speaking, due to the estimate of the C^α-seminorm of u_{xx} by the L^2-norm of u_{xxx}. Here we recall that Garcke and Novick-Cohen [11] previously presented the local existence theorem for (1.1)-(1.2) for Γ_0 in $C^{4+\alpha}$ with $0 < \alpha < 1$. So when we try to construct a global solution by extending a local solution, we are forced to improve their result to reconstruct a local solution for Γ_0 having at most $C^{2+\alpha}$-regularity with $0 < \alpha \leq 1/2$. This issue can be settled by applying the optimal regularity result for parabolic problems as comprehensively studied in [15].

Next we should mention that there are two remarkable features of the evolution problem (1.1)-(1.2); one is the area-preserving property and another is the energy-decreasing property [11]. The former means that each area of the 'Δ'-shaped domains enclosed by $\Gamma^1(t) \cup \Gamma^2(t) \cup \partial\Omega$ and by $\Gamma^2(t) \cup \Gamma^3(t) \cup \partial\Omega$ is preserved for all $t \geq 0$ under the motion by (1.1). The latter means that the energy associated with (1.1) for the solution $\Gamma(t)$ of (1.1) is nonincreasing function of t. It will be shown in Section 4 that these two properties as well as the results in Section 2 enable us to obtain a priori estimate of the first derivative of the curvature of $\Gamma^3(t)$ in L^2 when the initial curve Γ_0 and Γ_{θ,A_0} are close to each other in some sense, where A_0 denotes the area enclosed by Γ^3_0, the x-axis, and the y-axis. This procedure originates in X. Chen [3] and C. M. Elliott and H. Garcke [5] where the motions of closed curves are treated. In our case the motions of the triple junction and the end points of $\Gamma(t)$ at $\partial\Omega$ must be also taken into account. When $\theta \in (0,\pi/2)$, we can settle this issue to establish a modified version of their estimates.

Thus one can expect that if Γ_0 is close to Γ_{θ,A_0}, then a global solution $\Gamma(t)$ with a triple junction $m(t)$ of (1.3)-(1.4) exists and converges to Γ_{θ,A_0} as $t \to \infty$. This is in fact the case and we shall show it in our main theorem. To do so it is convenient to introduce the following notations.

Notation. Let $I = [0,1]$. For $0 \le t_0 < t_1 < \infty$ we set $R_{t_0,t_1} = (t_0,t_1] \times I$. For $0 < \alpha < 1$ we define spaces of functions $v(t,\eta)$ for $(t,\eta) \in [t_0,t_1] \times I$ and $\xi(t)$ for $t \in [t_0,t_1]$ as

$$\mathcal{Y}^{2+\alpha}(\overline{R_{t_0,t_1}}) = C^{0,2+\alpha}(\overline{R_{t_0,t_1}}) \cap C^{1/2}([t_0,t_1];C^{\alpha}(I)),$$
$$\mathcal{Y}^{4+\alpha}(\overline{R_{t_0,t_1}}) = \{v \in \mathcal{Y}^{2+\alpha}(\overline{R_{t_0,t_1}}) \cap C^{1,4+\alpha}(R_{t_0,t_1});\, \|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{t_0,t_1}})} < \infty\},$$
$$\mathcal{Z}^1[t_0,t_1] = \{\xi \in C[t_0,t_1] \cap C^1(t_0,t_1];\, \|\xi\|_{\mathcal{Z}^1[t_0,t_1]} < \infty\},$$

where the norms with which the latter two spaces are equipped are defined by

$$\begin{aligned}
\|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{t_0,t_1}})} &= \|v\|_{\mathcal{Y}^{2+\alpha}(\overline{R_{t_0,t_1}})} + \sup_{0<\delta<t_1-t_0} \delta^{1/4}\|v_{\eta\eta\eta}\|_{C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})} \\
&\quad + \sup_{0<\delta<t_1-t_0} \delta^{1/2}\|v_{\eta\eta\eta\eta}\|_{C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})} \\
&\quad + \sup_{0<\delta<t_1-t_0} \delta^{1/2}\|v_t\|_{C^{0,\alpha}(\overline{R_{t_0+\delta,t_1}})}, \\
\|\xi\|_{\mathcal{Z}^1[t_0,t_1]} &= \|\xi\|_{C[t_0,t_1]} + \sup_{0<\delta<t_1-t_0} \delta^{1/2}\|\dot{\xi}\|_{C^1[t_0+\delta,t_1]}.
\end{aligned}$$

Here the spaces such as $C^{j,k+\beta}([t_0,t_1] \times I)$ $(0 \le j \le (k+\beta)/4,\ k = 0,1,\cdots,\ 0 < \beta < 1)$ used here are defined in the usual manner; for their complete descriptions, see [16] or [15]. □

The manner of the establishment of the above spaces is somewhat complicated but it will be seen in Section 3 that they are quite reasonable when they are devoted to the study of the local-in-time solvability for (1.3)-(1.4) with $\Gamma_0 \in \mathcal{S}_\theta$. We also remind the reader that the indices of $\mathcal{Y}$-spaces measure the space-regilarities, whereas the index of $\mathcal{Z}$-space measures the time-regularity.

Now we are in position to state our main result.

Theorem 1.2 *Let $\theta \in (0,\pi/2)$ and let $\Gamma_0 = \Lambda[u_0,\xi_0] \in \mathcal{S}_\theta$. Assume that Γ_0 and Γ_{θ,A_0} are close to each other in the sense that*

$$\|\kappa^3_{0,s}\|^2_{L^2(\Gamma^3_0)} + C_{\theta,A_0}(E[\Gamma_0] - E[\Gamma_{\theta,A_0}]) \text{ is sufficiently small,} \tag{1.10}$$

here $\kappa^3_0(s)$ is the curvature of Γ^3_0 with the argument of its arc-length parameter s and C_{θ,A_0} is a positive constant depending on θ and A_0. Then,

(i) the problem (1.3)-(1.4) admits a unique global solution $\Gamma(t) = \Lambda[u(t,\cdot),\xi(t)]$ with a triple junction $m(t) = \mu[\xi(t)]$ for $t \ge 0$ satisfying

$$(v,\xi) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0,T] \quad \textit{for any } T > 0$$

with $\alpha \in (0,1/2]$, where

$$v(t,\eta) := u(t,-(1-\eta)\xi(t)), \quad (t,\eta) \in [0,\infty) \times I.$$

(ii) Moreover the solution $\Gamma(t)$ converges to Γ_{θ,A_0} uniformly and $E[\Gamma(t)]$ also converges to $E[\Gamma_{\theta,A_0}]$ as $t \to \infty$.

Note that $E[\Gamma_0] - E[\Gamma_{\theta,A_0}]$ is positive, since Γ_{θ,A_0} is the unique minimizer of E on $\mathcal{C}_{\theta,A_0}$ as mentioned above.

Remark 1.3 (i) For the surface diffusion flow equation

$$V = -\kappa_{ss} \quad \text{on } \Gamma(t),\, t > 0, \tag{1.11}$$

(here $\Gamma(t)$ is a single (possibly closed) curve), it is known that a loss of embeddedness [12] ([17] for higher dimension) and a graph-breaking [4] for $\Gamma(t)$ may occur for a class of the initial curves. Hence (1.10) seems necessary.

(ii) For closed curves there are several results related to Theorem 1.2. X. Chen [3] showed for the Hele-Shaw problem that if the initial closed curve Γ_0 is close to a circle, then the global solution $\Gamma(t)$ of the Hele-Shaw problem exists and converges to a (possibly another) circle with the same area enclosed by Γ_0 as $t \to \infty$. Elliott and Garcke [5] also obtained the same results for other area-preserving and curve-shortening motions including (1.11). Escher, Mayer, and Simonett [6, 7], Escher and Simonett [8, 9], extended their results to higher dimensional versions by means of both Amann's sophisticated theory for analytic semigroups and a center manifold analysis. For further informations, see the references cited in [8].

(iii) For second order case there are several results related to the present problem. D. Hilhorst and J. Hulshof [14] showed for the heat equation with similar angle condition to our problem that the solution $\Gamma(t)$ exists in $[0, T]$ for a finite time $T > 0$ and is asymptotically equal to a self-similar shrinking solution which vanishes at T. V. A. Galaktionov, J. Hulshof, and J. L. Vazquez [10] extended the result of [14] to the radially symmetric multi-dimansional case. For three-phase problem with a triple junction, L. Bronsard and F. Reitich [2] dealt with the mean curvature evolution for phase boundaries which are coupled by an angle condition, known as Young's law, at a triple junction. They showed the local-time existence of that problem. □

This paper proceeds as follows. In Section 2 we study a variational analysis for the energy E in (1.9) to give a direct proof that the minimizer of E on $\mathcal{C}_{\theta,A}$ is $\Gamma_{\theta,A}$. Next, by means of an optimal regularity result for parabolic problems (as studied in [15]), in Section 3 we show a local existence result of (1.3)-(1.4) depending on the magnitude of $\Gamma_0 \in C^{2+\alpha}$ $(0 < \alpha < 1)$, Theorem 3.1, which is an improvement of the result obtained in [11] since Garcke and Novick-Cohen [11] treated $C^{4+\alpha}$-initial data. This improvement is necessary to obtain global solutions $\Gamma(t)$ of (1.3)-(1.4) by means of a priori estimates of solutions in $C^{2+\alpha}$ with $0 < \alpha \le 1/2$. For reader's convenience we show an essential part of calculations employed in the proof of Theorem 3.1 in Appendix. By virtue of the results in Sections 2 and 3 and the assumption (1.10) as well as both area-preserving and energy-decreasing propoerties, in Section 4 we obtain a priori estimate of the solution $\Gamma(t)$ in $C^{2+\alpha}$ $(0 < \alpha \le 1/2)$ for $t \ge 0$ and consequently get a unique global solution of (1.3)-(1.4), which proves Theorem 1.2 (i). Finally, in Section 5, we prove Theorem 1.2 (ii) in Theorems 5.5 and 5.6.

Acknowledgement. The authors are grateful to Professor Y. Giga for suggesting them the present problem and continuing encouragements. The first author is partially supported by The Japan Ministry of Education, Science, Sports, and Culture through Grant No. 10740079.

2 Variational analysis for the energy

In this section we carry out a variational analysis for the energy E in (1.9). Our aim here is to show that there is a unique minimizer of E under an area-constraint. The result obtained in this section is useful not only in itself but also in view of a key to investigate the global solvability for the problem (1.3)-(1.4).

As stated in Section 1, we shall give a direct proof without using any sophisticated theory such as the geometric measure theory because the present issue can be settled by describing $\Gamma^1 \cup \Gamma^3$ by means of a piecewise smooth graph in one dimension. Main technical difficulty arises from the triple junction, which makes the setting of the set of curves somewhat complicated. In fact the graph of $\Gamma^1 \cup \Gamma^3$ has less regularity at the triple junction than at any other point.

Let $\theta \in (0, \pi/2)$ and $A > 0$ be given with $\xi_{\theta,A} < a$, where $\xi_{\theta,A}$ is in (1.5). We study the follwing energy-minimizing problem under the area-constraint:

$$\text{minimize } E \text{ on } \mathcal{C}_{\theta,A}.$$

The result is stated as follows.

Theorem 2.1 *(Existence of the unique minimizer). The functional $E : \mathcal{C}_{\theta,A} \to \mathbf{R}$ has a unique minimizer in $\mathcal{C}_{\theta,A}$, which coincides with $\Gamma_{\theta,A}$ in (1.8).*

Proof. (i) (Uniqueness). Assume that $\Gamma_\dagger = \Lambda[u_\dagger, \xi_\dagger] \in \mathcal{C}_{\theta,A}$ is a minimizer of E. Set

$$\mathcal{C}_{\theta,A,\dagger} = \{\Gamma = \Lambda[u, \xi] \in \mathcal{C}_{\theta,A}; \xi = \xi_\dagger\}.$$

Since $\Gamma_\dagger \in \mathcal{C}_{\theta,A,\dagger} \subset \mathcal{C}_{\theta,A}$, we have $E[\Gamma_\dagger] = \inf_{\mathcal{C}_{\theta,A}} E \leq \inf_{\mathcal{C}_{\theta,A,\dagger}} E$, which means that $\Gamma_\dagger$ is also a minimizer of E on $\mathcal{C}_{\theta,A,\dagger}$. From this fact we shall show that $\Gamma_\dagger$ is unique and $\Gamma_\dagger$ coincides with $\Gamma_{\theta,A}$.

In order to show this, let us consider the functionals defined by

$$f_0(w) = (a - \xi_\dagger)\cos\theta + \frac{1}{2}\int_{-\xi_\dagger}^{\xi_\dagger} (1 + w_x^2)^{1/2} dx,$$

$$f_1(w) = \frac{1}{2}\int_{-\xi_\dagger}^{\xi_\dagger} w dx, \quad f_2(w) = w_x(-\xi_\dagger + 0)$$

for $w \in W := \{w \in (H^2 \cap H_0^1)(-\xi_\dagger, \xi_\dagger); w \text{ is even}\}$. Put $w_\dagger = \tilde{u}_\dagger \in W$, where $\tilde{u}_\dagger(x)$ for $x \in [-\xi_\dagger, \xi_\dagger]$ is the even extension of $u_\dagger(x)$ to the interval $[-\xi_\dagger, \xi_\dagger]$. Then, since $\Gamma_\dagger$ is a minimizer of E on $\mathcal{C}_{\theta,A,\dagger}$, $w_\dagger$ gives the minimum of f_0 on W under the constraints $f_1(w) = A$ and $f_2(w) = \tan\theta$. Hence there is a Lagrange multiplier $(\lambda_0, \lambda_1, \lambda_2) \neq (0,0,0)$ such that $\sum_{i=0}^2 \lambda_i f_i'(w_\dagger) = 0$, which yields

$$\lambda_0 \int_{-\xi_\dagger}^{\xi_\dagger} \frac{w_{\dagger,x}}{(1 + w_{\dagger,x}^2)^{1/2}} \zeta_x dx + \lambda_1 \int_{-\xi_\dagger}^{\xi_\dagger} \zeta dx + 2\lambda_2 \zeta_x(-\xi_\dagger + 0) = 0 \quad \text{for } \zeta \in W.$$

Clearly $\lambda_0 \neq 0$. Setting $\lambda = \lambda_1/\lambda_0$, we get

$$\int_{-\xi_\dagger}^{\xi_\dagger} \frac{w_{\dagger,x}}{(1 + w_{\dagger,x}^2)^{1/2}} \zeta_x dx + \lambda \int_{-\xi_\dagger}^{\xi_\dagger} \zeta dx = 0 \quad \text{for any even } \zeta \in C_0^\infty(-\xi_\dagger, \xi_\dagger)$$

or

$$-\left(\frac{w_{\dagger,x}}{(1+w_{\dagger,x}^2)^{1/2}}\right)_x+\lambda=0,\quad x\in(-\xi_\dagger,\xi_\dagger).$$

This implies that the graph of $y=w_\dagger(x)$ is a circular arc with the radius $|\lambda|^{-1}$. Furthermore it must follow from the constraints $f_1(w)=A$ and $f_2(w)=\tan\theta$ that $\xi_\dagger=\xi_{\theta,A}$ and $w_\dagger=\tilde{u}_{\theta,A}$, where $\tilde{u}_{\theta,A}(x)$ for $x\in[-\xi_{\theta,A},\xi_{\theta,A}]$ is the even extension of $u_{\theta,A}(x)$ to the interval $[-\xi_{\theta,A},\xi_{\theta,A}]$. Thus we have proved that the minimizers of $E:\mathcal{C}_{\theta,A}\to\mathbf{R}$ are unique and must coincide with $\Gamma_{\theta,A}=\Lambda[u_{\theta,A},\xi_{\theta,A}]$ in (1.8).

(ii) (Existence). Put $M_{\theta,A}=2\|u_{\theta,A}\|_{H^2(-\xi_{\theta,A},0)}$. We decompose $\mathcal{C}_{\theta,A}$ as

$$\mathcal{C}_{\theta,A}=\bigcup_{M\geq M_{\theta,A}}\mathcal{B}_M, \tag{2.1}$$

where

$$\mathcal{B}_M=\{\Gamma=\Lambda[u,\xi]\in\mathcal{C}_{\theta,A};\|u\|_{H^2(-\xi,0)}\leq M\}\quad\text{for } M\geq M_{\theta,A}. \tag{2.2}$$

The one reason why we introduce the artificial decomposition (2.1) for $\mathcal{C}_{\theta,A}$ is that we can gain several compactnesses guaranteed by the artificial bound M to get a minimizing sequence. Another reason is to ensure the pointwise value of the function u_x at $x=-\xi$.

Now once the fact that

$$E[\Gamma_{\theta,A}]=\inf_{\mathcal{B}_M}E\quad\text{for any } M\geq M_{\theta,A}, \tag{2.3}$$

is verified, then it follows from (2.1) that

$$E[\Gamma_{\theta,A}]=\inf_{\mathcal{C}_{\theta,A}}E,$$

which gives the desired result. So we shall prove (2.3) via four steps.

Step 1. We are going to find a candidate of a minimizer of E on $\mathcal{B}_M$. Since $E\geq 0$ on $\mathcal{B}_M$, by the property of the infimum one can find a sequence $\Gamma_n=\Lambda[u_n,\xi_n]\in\mathcal{B}_M$ such that

$$\lim_{n\to\infty}E[\Gamma_n]=\inf_{\mathcal{B}_M}E. \tag{2.4}$$

In the following we will subtract a number of subsequences of $\{\Gamma_n\}$ one after another; the sequence appeared later, whose explicit notation is unchanged, is whenever a subsequence of the previous one.

Since $\{\xi_n\}$ is bounded, one can find a subsequence of $\{\Gamma_n\}$ (again denoted by $\{\Gamma_n\}$ with $\Gamma_n=\Lambda[u_n,\xi_n]$) such that $\lim_{n\to\infty}\xi_n=\xi_\infty\in[0,a]$. Hereafter, in general, we write the zero-extension of a function w on $(-\xi,0)$ with $0<\xi<a$ to $(-a,0)$ by

$$\bar{w}(x)=\begin{cases} w(x), & x\in(-\xi,0),\\ 0, & x\in(-a,-\xi).\end{cases}$$

Then, since $\{\bar{u}_n\}$ is bounded in $H^1(-a,0)$, one can find a subsequence of $\{\Gamma_n\}$ (again denoted by $\{\Gamma_n\}$ with $\Gamma_n=\Lambda[u_n,\xi_n]$) and a $\Gamma_\infty=\Lambda[u_\infty,\xi_\infty]$ such that

$$\bar{u}_n\to\bar{u}_\infty\quad\text{weakly in } H^1(-a,0), \tag{2.5}$$

$$\bar{u}_n\to\bar{u}_\infty\quad\text{strongly in } L^2(-a,0), \tag{2.6}$$

$$\bar{u}_n(x)\to\bar{u}_\infty(x)\quad\text{a.e. } x\in(-a,0), \tag{2.7}$$

as $n \to \infty$. Moreover it can be proved that there is a positive integer N such that for $n \geq N$, u_n are uniformly extended to $\tilde{u}_n$ on $(-\frac{3}{2}\xi_\infty, 0)$ and satisfy the estimate

$$\|\tilde{u}_n\|_{H^2(-\frac{3}{2}\xi_\infty,0)} \leq \tilde{C}\|u_n\|_{H^2(-\xi_n,0)} \quad \text{for } n \geq N,$$

where $\tilde{C} > 0$ is a constant independent of n. Then, since $\Gamma_n \in \mathcal{B}_M$, we see that $\{\tilde{u}_n\}$ is bounded in $H^2(-\frac{3}{2}\xi_\infty, 0)$. So we can find a subsequence of $\{\tilde{u}_n\}$ (again denoted by $\{\tilde{u}_n\}$) and a $v \in H^2(-\frac{3}{2}\xi_\infty, 0)$ such that

$$\tilde{u}_n \to v \quad \text{weakly in } H^2(-\tfrac{3}{2}\xi_\infty, 0) \text{ as } n \to \infty.$$

Then

$$v = u_\infty \quad \text{on } (-\xi_\infty, 0). \tag{2.8}$$

Indeed, recalling the embedding $H^2(-\frac{3}{2}\xi_\infty, 0) \subset C^{1+1/2}[-\frac{3}{2}\xi_\infty, 0]$, we see that $\{\tilde{u}_n\}$ is bounded in $C^{1+1/2}[-\frac{3}{2}\xi_\infty, 0]$, and by the Ascoli-Arzelà theorem we obtain

$$\tilde{u}_n \to v \quad \text{in } C^1[-\tfrac{3}{2}\xi_\infty, 0] \text{ as } n \to \infty. \tag{2.9}$$

Now (2.7) and (2.9) shows (2.8).

Step 2. We are going to show that $\Gamma_\infty = \Lambda[u_\infty, \xi_\infty]$ obtained as above belongs to $\mathcal{B}_M$. First it it not difficult to check that $\Gamma_\infty \in \mathcal{C}_{\theta,A}$: for example, the equality $u_{\infty,x}(-\xi_\infty + 0) = \tan\theta$ can be verified by virtue of (2.9) and $u_{n,x}(-\xi_n + 0) = \tan\theta$. So it remains to show

$$\|u_\infty\|_{H^2(-\xi_\infty,0)} \leq M. \tag{2.10}$$

Since $\|u_n\|_{H^2(-\xi_n,0)} \leq M$ for $n = 1, 2, \cdots$, we have

$$\liminf_{n\to\infty} \|u_n\|_{H^1(-\xi_n,0)} + \liminf_{n\to\infty} \|u_{n,xx}\|_{L^2(-\xi_n,0)} \leq M.$$

It then follows from (2.5) that

$$\|u_\infty\|_{H^1(-\xi_\infty,0)} \leq \liminf_{n\to\infty} \|u_n\|_{H^1(-\xi_n,0)}.$$

So if the inequality

$$\|u_{\infty,xx}\|_{L^2(-\xi_\infty,0)} \leq \liminf_{n\to\infty} \|u_{n,xx}\|_{L^2(-\xi_n,0)}. \tag{2.11}$$

is verified, then we get (2.10).

We show (2.11). Put

$$f_n = \begin{cases} u_{n,xx} & \text{for } -\xi_n \leq x \leq 0, \\ 0 & \text{for } -a \leq x \leq -\xi_n, \end{cases}$$

$$f_\infty = \begin{cases} u_{\infty,xx} & \text{for } -\xi_\infty \leq x \leq 0, \\ 0 & \text{for } -a \leq x \leq -\xi_\infty. \end{cases}$$

Then if

$$f_n \to f_\infty \quad \text{weakly in } L^2(-a, 0) \text{ as } n \to \infty \tag{2.12}$$

is verified, then we get $\|f_\infty\|_{L^2(-a,0)} \leq \liminf_{n\to\infty} \|f_n\|_{L^2(-a,0)}$, which shows (2.11). We show (2.12). Let $\varphi \in C^\infty[-a,0]$ be arbitrary. Then

$$\begin{aligned}\int_{-a}^{0} f_n \varphi dx &= \int_{-\xi_n}^{0} u_{n,xx}\varphi dx \\ &= -\tan\theta \cdot \varphi(-\xi_n) - \int_{-\xi_n}^{0} u_{n,x}\varphi_x dx \\ &\to -\tan\theta \cdot \varphi(-\xi_\infty) - \int_{-\xi_\infty}^{0} u_{\infty,x}\varphi_x dx \quad \text{as } n\to\infty,\end{aligned}$$

where we have used (2.5) and the equalities $u_{n,x}(-\xi_n+0) = \tan\theta$ and $u_{n,x}(0) = 0$. An integration by parts in the last term yields

$$\begin{aligned}&\int_{-\xi_\infty}^{0} u_{\infty,x}\varphi_x dx \\ &= -\tan\theta \cdot \varphi(-\xi_\infty) - \int_{-\xi_\infty}^{0} u_{\infty,xx}\varphi dx,\end{aligned}$$

since $u_{\infty,x}(-\xi_\infty+0) = \tan\theta$ and $u_{\infty,x}(0) = 0$. Thus we get

$$\int_{-a}^{0} f_n\varphi dx \to \int_{-a}^{0} f_\infty \varphi dx \quad \text{as } n\to\infty.$$

Since $C^\infty[-a,0]$ is dense in $L^2(-a,0)$, we have proved (2.12) and hence $\Gamma_\infty \in \mathcal{B}_M$ is proved.

Step 3. We shall show that Γ_∞ is a minimizer of $E : \mathcal{B}_M \to \mathbf{R}$, i.e.,

$$E[\Gamma_\infty] = \inf_{\mathcal{B}_M} E. \tag{2.13}$$

The sign $\geq$ is obvious, for $\Gamma_\infty \in \mathcal{B}_M$. To show the opposite sign, we may check that

$$E[\Gamma_\infty] \leq \liminf_{n\to\infty} E[\Gamma_n] \tag{2.14}$$

(recall (2.4)). Observe

$$\begin{aligned}\liminf_{n\to\infty} E[\Gamma_n] \;&\geq\; \liminf_{n\to\infty}(a-\xi_n)\cos\theta + \liminf_{n\to\infty}\int_{-\xi_\infty}^{0}(1+u_{n,x}^2)^{1/2}dx \\ &\quad + \liminf_{n\to\infty}\int_{-\xi_n}^{-\xi_\infty}(1+u_{n,x}^2)^{1/2}dx \\ &=: \; I_1 + I_2 + I_3.\end{aligned}$$

Using $\lim_{n\to\infty}\xi_n = \xi_\infty$ and $\Gamma_n \in \mathcal{B}_M$, one may easily check that

$$I_1 = (a-\xi_\infty)\cos\theta, \quad I_3 = 0.$$

For I_2 we observe that the functional

$$H^1(-a,0) \ni u \mapsto \int_{-\xi_\infty}^{0}(1+u_x^2)^{1/2}dx \in \mathbf{R}$$

is convex and continuous in $H^1(-a,0)$, so that this functional is lower semicontinuous with respect to the weak topology in $H^1(-a,0)$. Thus we see that $I_2 \geq \int_{-\xi_\infty}^0 (1+u_{\infty,x}^2)^{1/2}dx$ and (2.14) is verified.

Step 4. Now a similar discussion employed in (i) leads to the fact that $\xi_\infty = \xi_{\theta,A}$ and $u_\infty = u_{\theta,A}$ and thus we have proved (2.3). The proof is complete. □

A similar argument can be valid for the length of $\Gamma^3 = \Lambda^3[u,\xi]$,

$$L[\Gamma^3] = \int_{-\xi}^0 (1+u_x^2)^{1/2}dx.$$

Thus we get the following result.

Corollary 2.2 *The functional* $L : \Gamma^3 \mapsto L[\Gamma^3]$ *with* $\Gamma = \bigcup_{i=1}^3 \Gamma^i \in \mathcal{C}_{\theta,A}$ *has a unique minimizer* $\Gamma_{\theta,A}$ *defined by (1.8).*

3 Local existence

In this section, we study a local existence result for (1.3)-(1.4) for the initial data $\Gamma_0 \in C^{2+\alpha}$ with $0<\alpha<1$. We once again point out that H. Garcke and A. Novick-Cohen [11] previously obtained a local existence result for (1.1)-(1.2) when the initial data Γ_0 belongs to $C^{4+\alpha}$ with $0<\alpha<1$. Our result is an improvement of that of [11] in the sense that the existence time of the local solution depends upon the magnitude of Γ_0 in $C^{2+\alpha}$ with $0<\alpha<1$. This improvement is useful in investigating the global solvability for (1.3)-(1.4), which will be clarified in Section 4. In order to obtain our local existence results for the less regular data, we shall use the optimal regularity theory for parabolic problems as in [15] instead of the theory by V. A. Solonnikov [18] used in [11].

We shall first obtain the ξ's equation. To do so we assume that (u,ξ) is a solution of the problem (1.3) without the equation $u(t,-\xi(t))=0$ and (1.4) under the condition $u_0(-\xi_0)=0$. Then the equation $u(t,-\xi(t))=0$ and the one

$$\dot{\xi}(t) = -\ C_1(\theta)u_{xxxx}(t,-\xi(t)) + C_2(\theta)u_{xx}^3(t,-\xi(t)) \tag{3.1}$$

are equivalent, where $C_1(\theta) := 1/[(1+\tan^2\theta)^2\tan\theta]$, $C_2(\theta) := 3(1+5\tan^2\theta)/[(1+\tan^2\theta)^4\tan\theta]$. Here and hereafter, for simplicity of the notation we often use $\dot{\xi}$ instead of $d\xi/dt$. In order to see this equivalence, we first assume that the equation $u(t,-\xi(t)) = 0$ is satisfied. We differentiate it with respect to t and use the boundary conditions $u_x(t,-\xi(t)) = \tan\theta$, and $\partial_x(u_{xx}/(1+u_x^2)^{3/2}) = 0$ at $x=-\xi(t)$. Then we get (3.1). Conversely, if (3.1) is satisfied, then we integrate it with respect to t. By virtue of the condition $u_0(-\xi_0)=0$, we then get the equation $u(t,-\xi(t))=0$.

From now on, we shall study the problem (1.3) without the equation $u(t,-\xi(t))=0$, (1.4) and (3.1) with $\xi(0)=\xi_0$ for the initial data $(u_0(x),\xi_0)$ satisfying $u_0(-\xi_0)=0$, $u_{0x}(-\xi_0)=\tan\theta$ and $u_{0x}(0)=0$.

In order to normalize the coordinate x, we perform the change of variables for each $t\geq 0$ as follows:

$$\eta = 1+\frac{x}{\xi(t)}, \quad v(t,\eta) = u(t,-(1-\eta)\xi(t)).$$

Then, the problem (1.3) without the equation $u(t, -\xi(t)) = 0$, (1.4) and (3.1) become the form for $(t, \eta) \in (0, T) \times (0, 1)$:

$$v_t = f(\eta, v_\eta, v_{\eta\eta}, v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, \xi, \dot{\xi}), \tag{3.2}$$

$$v_\eta(t, 0) = \xi(t) \tan\theta, \tag{3.3}$$

$$v_\eta(t, 1) = 0, \tag{3.4}$$

$$v_{\eta\eta\eta}(t, 0) = \frac{3\tan\theta}{1+\tan^2\theta} \cdot \frac{v_{\eta\eta}^2(t, 0)}{\xi(t)}, \tag{3.5}$$

$$v_{\eta\eta\eta}(t, 1) = 0, \tag{3.6}$$

$$v(0, \eta) = v_0(\eta) := u_0(-(1-\eta)\xi_0), \tag{3.7}$$

$$\dot{\xi}(t) = -\, C_1(\theta) \frac{v_{\eta\eta\eta\eta}(t, 0)}{\xi^4(t)} + C_2(\theta) \frac{v_{\eta\eta}^3(t, 0)}{\xi^6(t)}, \tag{3.8}$$

$$\xi(0) = \xi_0, \tag{3.9}$$

where

$$\begin{aligned} & f(\eta, v_\eta, v_{\eta\eta}, v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, \xi, \dot{\xi}) \\ &= -\frac{1}{(\xi^2+v_\eta^2)^2} v_{\eta\eta\eta\eta} + \frac{10 v_\eta v_{\eta\eta}}{(\xi^2+v_\eta^2)^3} v_{\eta\eta\eta} + \frac{3(\xi^2 - 5v_\eta^2) v_{\eta\eta}^3}{(\xi^2+v_\eta^2)^4} - \frac{(1-\eta)\dot{\xi} v_\eta}{\xi}. \end{aligned}$$

Thus our task is reduced to solve the problem (3.2)-(3.9) locally in time.

Now we are in position to state our local existence theorem.

Theorem 3.1 *(Local existence).* *Let $\alpha \in (0, 1)$. Let us assume that $v_0 \in C^{2+\alpha}(I)$, and that v_0 satisfies $v_0(0) = 0, v_{0\eta}(0) = \xi_0 \tan\theta, v_{0\eta}(1) = 0$. Let us also assume that $\xi_0 > 0$. Then, there exists a $T_1 = T_1(\xi_0, 1/\|v_0\|_{C^{2+\alpha}(I)}) > 0$ such that the problem (3.2)-(3.9) has a unique solution $(v, \xi) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T_1}}) \times \mathcal{Z}^1[0, T_1]$.*

Before proving this theorem, we need some preliminaries. We shall first linearize (3.2)-(3.9) about the initial data. For later convenience, we introduce a parameter $\tau \geq 0$, which is regarded as initial time. If a pair of functions $(\bar{v}, \bar{\xi})$, which has a suitable regularity, is given, then, for C^4-function $U : [0, 1] \ni \eta \to U(\eta) \in \mathbf{R}$, we define a differential operator $\mathcal{A}_\tau$ by

$$\begin{aligned} \mathcal{A}_\tau U \;:=\; & -\frac{1}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau, \eta))^2} \partial_\eta^4 U + 10 \frac{\bar{v}_\eta(\tau, \eta) \bar{v}_{\eta\eta}(\tau, \eta)}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau, \eta))^3} \partial_\eta^3 U \\ & + C_1(\theta)(1-\eta) \frac{\bar{v}_\eta(\tau, \eta)}{\bar{\xi}^5(\tau)} \partial_\eta^4 U(0). \end{aligned}$$

In addition, we set

$$\begin{aligned} F_\tau(t, \eta) \;:=\; & -\{\frac{1}{(\bar{\xi}^2(t) + \bar{v}_\eta^2(t, \eta))^2} - \frac{1}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau, \eta))^2}\} \bar{v}_{\eta\eta\eta\eta}(t, \eta) \\ & + 10 \{\frac{\bar{v}_\eta(t, \eta) \bar{v}_{\eta\eta}(t, \eta)}{(\bar{\xi}^2(t) + \bar{v}_\eta^2(t, \eta))^3} - \frac{\bar{v}_\eta(\tau, \eta) \bar{v}_{\eta\eta}(\tau, \eta)}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau, \eta))^3}\} \bar{v}_{\eta\eta\eta}(t, \eta) \\ & + C_1(\theta)(1-\eta)(\frac{\bar{v}_\eta(t, \eta)}{\bar{\xi}^5(t)} - \frac{\bar{v}_\eta(\tau, \eta)}{\bar{\xi}^5(\tau)}) \bar{v}_{\eta\eta\eta\eta}(t, 0) \\ & + \frac{3(\bar{\xi}^2(t) - 5\bar{v}_\eta^2(t, \eta)) \bar{v}_{\eta\eta}^3(t, \eta)}{(\bar{\xi}^2(t) + \bar{v}_\eta^2(t, \eta))^4} - C_2(\theta) \frac{(1-\eta) \bar{v}_\eta(t, \eta)}{\bar{\xi}^7(t)} \bar{v}_{\eta\eta}^3(t, 0). \end{aligned}$$

In the proof of Theorem 3.1, we use $\tau = 0$. Let $T > 0$. Then, for given $(\bar{v}, \bar{\xi}) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0,T]$ with $(\bar{v}(0,\cdot), \bar{\xi}(0)) = (v_0(\cdot), \xi_0)$, we consider the linearized problem of (3.2)-(3.7) as follows:

$$\begin{cases} v_t = \mathcal{A}_0 v + F_0(t,\eta) \quad \text{in } R_{0,T}, \\ v_\eta(t,0) = \bar{\xi}(t) \tan\theta, \\ v_\eta(t,1) = 0, \\ v_{\eta\eta\eta}(t,0) = \dfrac{3\tan\theta}{1+\tan^2\theta} \cdot \dfrac{\bar{v}_{\eta\eta}^2(t,0)}{\bar{\xi}(t)}, \\ v_{\eta\eta\eta}(t,1) = 0, \\ v(0,\eta) = v_0(\eta). \end{cases} \tag{3.10}$$

Moreover, for v satisfying (3.10), we consider the equation of the form

$$\begin{cases} \dot{\xi}(t) = -\, C_1(\theta) \dfrac{v_{\eta\eta\eta\eta}(t,0)}{\bar{\xi}^4(t)} + C_2(\theta) \dfrac{v_{\eta\eta}^3(t,0)}{\bar{\xi}^6(t)}, \\ \xi(0) = \xi_0. \end{cases} \tag{3.11}$$

Remark 3.2 When we linearize (3.2)-(3.7), we should take care that the initial data Γ_0 is given in $C^{2+\alpha}$, and that the singularity of $\dot{\xi}$ at $t = 0$ is expected to be equal to that of $v_{\eta\eta\eta\eta}$. So we first plug (3.8) into (3.2) and use f as

$$\begin{aligned} & f(\eta, v_\eta, v_{\eta\eta}, v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, \xi, v_{\eta\eta\eta\eta}(\cdot,0), v_{\eta\eta}(\cdot,0)) \\ &= -\frac{1}{(\xi^2+v_\eta^2)^2} v_{\eta\eta\eta\eta} + \frac{10 v_\eta v_{\eta\eta}}{(\xi^2+v_\eta^2)^3} v_{\eta\eta\eta} + \frac{3(\xi^2 - 5v_\eta^2) v_{\eta\eta}^3}{(\xi^2+v_\eta^2)^4} \\ &\quad + C_1(\theta) \frac{(1-\eta) v_\eta}{\xi^5} v_{\eta\eta\eta\eta}(\cdot,0) - C_2(\theta) \frac{(1-\eta) v_\eta}{\xi^7} v_{\eta\eta}^3(\cdot,0). \end{aligned} \tag{3.12}$$

Then we linearize f in (3.12) with respect to $v_{\eta\eta\eta}, v_{\eta\eta\eta\eta}, v_{\eta\eta\eta\eta}(\cdot,0)$ around (v_0, ξ_0) to get the linearized equation $v_t = \mathcal{A}_0 v + F_0(t,\eta)$ in $R_{0,T}$. We should note that, owing to the third term of $\mathcal{A}_0 v$, which is a contribution of the triple junction, the operator $\mathcal{A}_0$ is different from usual elliptic operators. For details, see the first half of Appendix.

For the unique existence results of the problem (3.10) and the problem (3.11), we have the following lemma.

Lemma 3.3 *Let $\alpha \in (0,1)$ and let us assume that $(\bar{v}, \bar{\xi}) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0,T]$ with $(\bar{v}(0,\cdot), \bar{\xi}(0)) = (v_0(\cdot), \xi_0)$, which satisfies the assumption of Theorem 3.1. Then,*
(i) there exists a unique solution $v \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})$ of (3.10),
(ii) for v obtained in (i), there exists a unique solution $\xi \in \mathcal{Z}^1[0,T]$ of (3.11).

This lemma is essentially obtained by the first half of Appendix. Indeed, by virtue of that, we can apply the optimal regularity theory of analytic semigroups to the linearized problem (3.10), and (i) is proved (see [15]). We omit its detailed proof. Once (i) is verified, then (ii) is obvious.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We explain an outline of the proof. We shall carry out detailed calculation in Appendix.

In order to obtain local existence result for the full problem (3.2)-(3.9), we shall use a fixed point argument. So we let

$$\begin{aligned}\mathcal{D} \ := \ & \{(v,\xi) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0,T] \ ; \\ & (v(0,\cdot),\xi(0)) = (v_0(\cdot),\xi_0), \ \|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\xi\|_{\mathcal{Z}^1[0,T]} \le K\},\end{aligned}$$

for positive bounded parameters K, T satisfying $KT^{1/2} \le \xi_0/4$. Here we note that, if $\xi \in \mathcal{D}$, then ξ satisfies

$$\xi_0/2 \le \xi(t) \le 3\xi_0/2 \quad \text{for } t \in [0,T] \tag{3.13}$$

by virtue of $KT^{1/2} \le \xi_0/4$. Moreover we define the map Φ as

$$\Phi : \mathcal{D} \ni (\bar{v},\bar{\xi}) \mapsto (v,\xi) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0,T],$$

where (v,ξ) is the unique solution of (3.10) and (3.11) established in Lemma 3.3. We shall show that, for a suitable constant K and small enough T depending on ξ_0 and $\|v_0\|_{C^{2+\alpha}(I)}$, the map Φ is contracting in $\mathcal{D}$. Once this is verified, then the map Φ has a unique fixed point in $\mathcal{D}$. Of course, this fixed point is a unique solution of the full problem (3.2)-(3.9) in $\mathcal{D}$.

We shall first prove that Φ maps $\mathcal{D}$ into itself when we choose suitable K and T. By means of Appendix, if v satisfies the linearized problem (3.10), we get the estimate below;

$$\|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} \le M_0 + NT^{\nu}, \tag{3.14}$$

where M_0 is a constant depending on $\xi_0, \|v_0\|_{C^{2+\alpha}(I)}, \theta$, and N is a constant depending on $\xi_0, \|v_0\|_{C^{2+\alpha}(I)}, \theta, \alpha, \gamma$ $(0 < \alpha < \gamma < 1), K$ (throughout this section any constant depending on the preceding quantities will be denoted by N whose value may be different in each occasion), and $\nu = \min\{\alpha/4, (\gamma-\alpha)/4\} \in (0, 1/4)$. In particular, we emphasize the dependences of M_0 on $\xi_0, \|v_0\|_{C^{2+\alpha}(I)}$ and of N on K. By virtue of Appendix and the basic theory of elliptic equations, we can eventually check that M_0 is an increasing function of $1/\xi_0, \|v_0\|_{C^{2+\alpha}(I)}$, and that N is an increasing function of K. Then, according to (3.11) and (3.14),

$$\|\xi\|_{\mathcal{Z}^1[0,T]} \le \xi_0 + C_1(\theta)\left(\frac{2}{\xi_0}\right)^4 M_0 + NT^{\nu}.$$

Hence, choosing

$$K = 2\{\xi_0 + (1 + 16C_1(\theta)\xi_0^{-4})M_0\}, \tag{3.15}$$

and

$$T_0 = \min\{\left(\frac{K}{2N}\right)^{1/\nu}, \left(\frac{\xi_0}{4K}\right)^2\}, \tag{3.16}$$

we obtain that

$$\|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\xi\|_{\mathcal{Z}^1[0,T]} \le K \quad \text{for } T \le T_0. \tag{3.17}$$

That is, Φ maps $\mathcal{D}$ into itself.

Next we prove that the map Φ is a contraction on $\mathcal{D}$ for a suitable choice of T. Let $(\bar{v}_1,\bar{\xi}_1), (\bar{v}_2,\bar{\xi}_2) \in \mathcal{D}$ with $T \le T_0$, and put $(v_1,\xi_1) = \Phi(\bar{v}_1,\bar{\xi}_1), (v_2,\xi_2) = \Phi(\bar{v}_2,\bar{\xi}_2)$.

Moreover, let $V = v_1 - v_2$, $\Xi = \xi_1 - \xi_2$, $\bar{V} = \bar{v}_1 - \bar{v}_2$, $\bar{\Xi} = \bar{\xi}_1 - \bar{\xi}_2$. Then the function V satisfies

$$\begin{cases} V_t = \mathcal{A}_0 V + \tilde{F}(t,\eta) \quad \text{in } R_{0,T}, \\ V_\eta(t,0) = \bar{\Xi}(t)\tan\theta, \\ V_\eta(t,1) = 0, \\ V_{\eta\eta\eta}(t,0) = \dfrac{3\tan\theta}{1+\tan^2\theta}(b_1(t)\bar{V}_{\eta\eta}(t,0) + b_2(t)\bar{\Xi}(t)), \\ V_{\eta\eta\eta}(t,1) = 0, \\ V(0,\eta) = 0, \end{cases} \tag{3.18}$$

and, for V satisfying (3.18), the function Ξ satisfies

$$\begin{cases} \dfrac{d}{dt}\Xi(t) = -\dfrac{C_1(\theta)}{\bar{\xi}^4(t)}V_{\eta\eta\eta\eta}(t,0) + b_3(t)V_{\eta\eta}(t,0) \\ \qquad\qquad + [b_4(t)v_{2\eta\eta\eta\eta}(t,0) + b_5(t)]\bar{\Xi}(t), \\ \Xi(0) = 0, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} \tilde{F}(t,\eta) &= (F_0 \text{ evaluated at } \bar{v}_1 \text{ and } \bar{\xi}_1) - (F_0 \text{ evaluated at } \bar{v}_2 \text{ and } \bar{\xi}_2) \\ &= -\{\frac{1}{(\bar{\xi}_1^4(t)+\bar{v}_{1\eta}^2)^2} - \frac{1}{(\xi_0^2+v_{0\eta}^2)^2}\}\bar{V}_{\eta\eta\eta\eta} \\ &\quad + g_1(\bar{\xi}_i,\bar{v}_{i\eta})\bar{v}_{2\eta\eta\eta\eta}\bar{V}_\eta + g_2(\bar{\xi}_i,\bar{v}_{i\eta})\bar{v}_{2\eta\eta\eta\eta}\bar{\Xi} + \cdots, \end{aligned}$$

and $b_1(t),\cdots,b_5(t)$ are functions of $\bar{\xi}_i(t), \bar{v}_{i\eta\eta}(t,0)$ $(i=1,2)$ without significant singularities. Here, applying Appendix again, we obtain

$$\|V\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} \le NT^\nu(\|\bar{V}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1[0,T]}). \tag{3.20}$$

Then, by means of (3.19) and (3.20),

$$\|\Xi\|_{\mathcal{Z}^1[0,T]} \le NT^\nu(\|\bar{V}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1[0,T]}).$$

Thus we derive

$$\|V\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\Xi\|_{\mathcal{Z}^1[0,T]} \le NT^\nu(\|\bar{V}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} + \|\bar{\Xi}\|_{\mathcal{Z}^1[0,T]}). \tag{3.21}$$

Consequently Φ is a contraction on $\mathcal{D}$ for $T \le T_1$, where

$$T_1 = \min\{\left(\frac{1}{2N}\right)^{1/\nu}, T_0\}. \tag{3.22}$$

This completes the proof of Theorem 3.1. □

Remark 3.4 According to (3.15), (3.16) and (3.22), T_1 is an increasing function of its arguments. So, if we choose $(v(t_0,\cdot),\xi(t_0))$ (where $t_0 > 0$) as the initial data, T_1 is an increasing function of $\xi(t_0), 1/\|v(t_0,\cdot)\|_{C^{2+\alpha}(I)}$. Thus, as long as there exist constants ν_1,ν_2 such that

$$0 < \nu_1 \le \xi(t_0), \quad \|v(t_0,\cdot)\|_{C^{2+\alpha}(I)} \le \nu_2 < \infty,$$

T_1 does not shrink to 0. In fact,

$$T_1(\xi(t_0),\ 1/\|v(t_0,\cdot)\|_{C^{2+\alpha}(I)}) \ge T_1(\nu_1, 1/\nu_2) > 0.$$

This property is used in Section 4.

In the same way as we obtain the inequality (3.21), we can derive a uniqueness result. Since the proof can be done identically as in the derivation of (3.21), we only show its statement without the proof.

Proposition 3.5 *(Uniqueness).* *Let τ_0 (≥ 0) be a given constant, and let $(v_{\tau_0}, \xi_{\tau_0})$ be a given data satisfying $\|v_{\tau_0}\|_{C^{2+\alpha}(I)} < \infty$ and $\xi_{\tau_0} > 0$ for $\alpha \in (0,1)$. Then, solutions of the problem (3.2)-(3.6) and (3.8) on the time interval $[\tau_0, \tau_0 + T]$ with $T > 0$ for the initial condition $(v(\tau_0, \cdot), \xi(\tau_0)) = (v_{\tau_0}(\cdot), \xi_{\tau_0})$ are unique in $\mathcal{Y}^{4+\alpha}(\overline{R_{\tau_0,\tau_0+T}}) \times \mathcal{Z}^1[\tau_0, \tau_0 + T]$.*

In the following we show that the solution obtained in Theorem 3.1 has a further regularity for $t \in (0, T_1]$.

Theorem 3.6 *(Regularity).* *Let $\alpha \in (0,1)$ and let (v, ξ) be the solution obtained in Theorem 3.1. Then, (v, ξ) also satisfies*

$$(v, \xi) \in C^{1,6+\alpha}((0, T_1] \times I) \times C^{1+(2+\alpha)/4}(0, T_1].$$

Proof. Let $\delta \in (0, T_1)$ be arbitrarily fixed. Then we study the problem (3.2)-(3.6) and (3.8) on the time interval $[\delta, \delta + T]$ with $T > 0$ for the initial data $v(\delta, \cdot) \in C^{4+\alpha}(I)$ and $\xi(\delta) > 0$, where (v, ξ) is the solution obtained in Theorem 3.1. Now we set $R_{\delta+\varepsilon,\delta+T} = (\delta + \varepsilon, \delta + T] \times I$ for $\varepsilon \in (0, T)$, and define the spaces as

$$\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}}) := \{w \in C^{1,4+\alpha}(\overline{R_{\delta,\delta+T}}) \cap C^{1,6+\alpha}(R_{\delta,\delta+T}) \ ; \ \|w\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} < \infty\},$$
$$\mathcal{Z}^{(6+\alpha)/4}[\delta, \delta + T] := \{\zeta \in C^1[\delta, \delta + T] \cap C^{1+(2+\alpha)/4}(\delta, \delta + T] \ ; \ \|\zeta\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]} < \infty\},$$

where the norms are defined by

$$\begin{aligned}
\|w\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} &:= \|w\|_{C^{1,4+\alpha}(\overline{R_{\delta,\delta+T}})} + \sup_{0<\varepsilon<T} \varepsilon^{1/4} \|\partial_\eta^5 w\|_{C^{0,\alpha}(\overline{R_{\delta+\varepsilon,\delta+T}})} \\
&\quad + \sup_{0<\varepsilon<T} \varepsilon^{1/2} \|\partial_\eta^6 w\|_{C^{0,\alpha}(\overline{R_{\delta+\varepsilon,\delta+T}})} \\
&\quad + \sup_{0<\varepsilon<T} \varepsilon^{1/2} \|w_t\|_{C^{0,2+\alpha}(\overline{R_{\delta+\varepsilon,\delta+T}})} \\
\|\zeta\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]} &:= \|\zeta\|_{C^1[\delta,\delta+T]} + \sup_{0<\varepsilon<T} \varepsilon^{1/2} [\dot{\zeta}]_{C^{(2+\alpha)/4}[\delta+\varepsilon,\delta+T]}.
\end{aligned}$$

We shall obtain the further regularity result via three steps.

Step 1. We shall first linearize the problem (3.2)-(3.6) and (3.8) about $(v(\delta, \cdot), \xi(\delta))$. For given $(\bar{w}, \bar{\zeta}) \in \mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}}) \times \mathcal{Z}^{(6+\alpha)/4}[\delta, \delta + T]$ with $(\bar{w}(\delta, \cdot), \bar{\zeta}(\delta)) = (v(\delta, \cdot), \xi(\delta))$, we consider the linearized problem as follows:

$$\begin{cases}
w_t = \mathcal{A}_\delta w + F_\delta(t, \eta) \quad \text{in } R_{\delta,T}, \\
w_\eta(t, 0) = \bar{\zeta}(t) \tan\theta, \\
w_\eta(t, 1) = 0, \\
w_{\eta\eta\eta}(t, 0) = \dfrac{3\tan\theta}{1 + \tan^2\theta} \cdot \dfrac{\bar{w}_{\eta\eta}^2(t, 0)}{\bar{\zeta}(t)}, \\
w_{\eta\eta\eta}(t, 1) = 0, \\
w(\delta, \eta) = v(\delta, \eta).
\end{cases} \tag{3.23}$$

Moreover, for w satisfying (3.23), we consider the problem

$$\begin{cases} \dot{\zeta}(t) = -\, C_1(\theta)\dfrac{w_{\eta\eta\eta\eta}(t,0)}{\zeta^4(t)} + C_2(\theta)\dfrac{w_{\eta\eta}^3(t,0)}{\zeta^6(t)}, \\ \zeta(\delta) = \xi(\delta). \end{cases} \tag{3.24}$$

Then, using the same argument as we obtain Lemma 3.3, we can show that there exists a unique solution $(w, \zeta) \in \mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}}) \times \mathcal{Z}^{(6+\alpha)/4}[\delta, \delta+T]$ of the linearized problem (3.23) and the problem (3.24).

Step 2. In order to use a fixed point argument, we let

$$\begin{aligned} \mathcal{E} \ := \ & \{(w,\zeta) \in \mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}}) \times \mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T] \ ; \\ & (w(\delta,\cdot),\zeta(\delta)) = (v(\delta,\cdot),\xi(\delta)), \ \|w\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} + \|\zeta\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]} \le \tilde{K}\}, \end{aligned}$$

for positive bounded parameters $\tilde{K}, T$ satisfying $\tilde{K}T \le \xi(\delta)/2$. Here we note that, if $\xi \in \mathcal{E}$, then ξ satisfies $\xi(\delta)/2 \le \xi(t) \le 3\xi(\delta)/2$ for any $t \in [\delta, \delta+T]$ by virtue of $\tilde{K}T \le \xi(\delta)/2$. Moreover we define the map Φ_δ as

$$\Phi_\delta : \mathcal{E} \ni (\bar{w}, \bar{\zeta}) \mapsto (w, \zeta) \in \mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}}) \times \mathcal{Z}^{(6+\alpha)/4}[\delta, \delta+T],$$

where w and ζ are the unique solutions of the problems (3.23) and (3.24) respectively. We claim that, for a suitable constant $\tilde{K}$ and small enough T, the map Φ_δ is contracting on $\mathcal{E}$. Once this is verified, then a unique fixed point of Φ_δ exists in $\mathcal{E}$. Of course, this fixed point is the desired solution of the problem (3.2)-(3.6) and (3.8) with the initial condition $(w(\delta,\cdot), \zeta(\delta)) = (v(\delta,\cdot), \xi(\delta))$.

We verify this claim. Let us choose

$$\tilde{K} = 2(\xi(\delta) + M_\delta), \tag{3.25}$$

$$T_2 = \min\{\left(\frac{\tilde{K}}{2\tilde{N}}\right)^{1/\nu}, \frac{\xi(\delta)}{2\tilde{K}}, \left(\frac{1}{2\tilde{N}}\right)^{1/\nu}\}, \tag{3.26}$$

where M_δ is a constant depending on $\xi(\delta), \|v(\delta,\cdot)\|_{C^{4+\alpha}(I)}, \theta$, and $\tilde{N}$ is a constant depending on $\xi(\delta), \|v(\delta,\cdot)\|_{C^{4+\alpha}(I)}, \theta, \alpha, \gamma, \tilde{K}$. By the same argument about constants M_0, N of the inequality (3.14), we can eventually check that M_δ is an increasing function of $1/\xi(\delta), \|v(\delta,\cdot)\|_{C^{4+\alpha}(I)}$, and that $\tilde{N}$ is an increasing function of $\tilde{K}$. Then, in the same way as we obtain the inequality (3.17), we are led to

$$\|w\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} + \|\zeta\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]} \le \tilde{K} \quad \text{for } T \le T_2.$$

That is, Φ_δ maps $\mathcal{E}$ into itself. The next task is to see that Φ_δ is contracting in $\mathcal{E}$. To do so, let $(\bar{w}_i, \bar{\zeta}_i) \in \mathcal{E}$ $(i = 1, 2)$ with $T \le T_2$, and put $(w_i, \zeta_i) = \Phi_\delta(\bar{w}_i, \bar{\zeta}_i)$ $(i = 1, 2)$. Then, in the same way as we obtain the inequality (3.21), we derive, for $T \le T_2$,

$$\begin{aligned} & \|w_1 - w_2\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} + \|\zeta_1 - \zeta_2\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]} \\ & \le \frac{1}{2}(\|\bar{w}_1 - \bar{w}_2\|_{\mathcal{Y}^{6+\alpha}(\overline{R_{\delta,\delta+T}})} + \|\bar{\zeta}_1 - \bar{\zeta}_2\|_{\mathcal{Z}^{(6+\alpha)/4}[\delta,\delta+T]}), \end{aligned}$$

which shows that Φ_δ is a contraction on $\mathcal{E}$. Thus the above claim is verified.

Consequently, for $T \in (0, T_2]$, there exists a unique solution $(w, \zeta) \in \mathcal{E}$ of the problem (3.2)-(3.6) and (3.8) with the initial condition $(w(\delta, \cdot), \zeta(\delta)) = (v(\delta, \cdot), \xi(\delta))$. Then, by virtue of Proposition 3.5, we see that this solution (w, ζ) coincides with (v, ξ) obtained in Theorem 3.1. Thus, (v, ξ) has the desired regularity in $[\delta, \delta + T_2]$ for any $\delta \in (0, T_1)$.

Step 3. As far as $\delta + T_2 < T_1$ for $\delta \in (0, T_1)$, we repeat the same procedure as in Step 2. Then we need an estimate concerning the infimum of T_2. In order to see this, we use (3.13) and (3.17) to get, for T_1 obtained in Theorem 3.1,

$$\xi(t) \geq \frac{\xi_0}{2} > 0, \quad \|v(t, \cdot)\|_{C^{4+\alpha}(I)} \leq K\delta^{-1/2} \quad \text{for } t \in [\delta, T_1], \tag{3.27}$$

where K is the constant established by (3.15). So, by means of (3.26), we obtain

$$T_2\left(\xi(t),\ 1/\|v(t, \cdot)\|_{C^{4+\alpha}(I)}\right) \geq T_2\left(\xi_0/2,\ \delta^{1/2}/K\right) > 0 \quad \text{for } t \in [\delta, T_1]. \tag{3.28}$$

That is, as long as $t \in [\delta, T_1]$, T_2 evaluated at $\xi(t)$ and $1/\|v(t, \cdot)\|_{C^{4+\alpha}(I)}$ does not shrink to 0. Consequently, (v, ξ) has the desired regularity in $(0, T_1]$ since δ is arbitrary in $(0, T_1)$. This completes the proof of Theorem 3.6. □

Remark 3.7 Repeating the same way as we obtain Theorem 3.6, we see that the solution obtained in Theorem 3.1 also belongs to $C^{1,2m+2+\alpha}((0, T_1] \times I) \times C^{(2m+2+\alpha)/4}(0, T_1]$ for any positive integer m and $\alpha \in (0, 1)$.

4 Global existence

The purpose of this section is to obtain global solutions of (3.2)-(3.9) and consequently of (1.3)-(1.4) when the initial data Γ_0 and Γ_{θ, A_0} are sufficiently close to each other in some sense. The precedure to prove this is as follows. First we show that if the above condition is fulfilled, then the derivative of the curvature κ of $\Gamma^3(t)$, κ_s, is always small (in this and the remaining sections, for simplicity, we omit the upper indices 3 of κ^3 and V^3). In this step we need C^6-regularity for solutions of (3.2)-(3.9). Of course this requirement is fulfilled by virtue of Theorem 3.6. Next, using this result, we derive a priori estimates of the solutions $(v(t, \eta), \xi(t))$ of (3.2)-(3.9) in a suitable norm and we then obtain the desired global existence result.

In order to derive a priori estimate we employ a similar method to those of X. Chen [3] and C. M. Elliott and H. Garcke [5] but they only studied the motion of closed curves. In our case the point is how to deal with the motion of the triple junction. We shall settle this issue to establish a modified version of the estimates obtained in [3] and [5].

First we mention the energy-decreasing and the area-preserving properties found by Garcke and Novick-Cohen [11]. Let $\Gamma(t)$ be the solution of (1.3) with the initial data $\Gamma_0 \in \mathcal{S}_\theta$, which has an enough regularity such as at least $\Gamma(t) \in C^3$ up to $t = 0$. Let $A(t)$ be the area enclosed by $\Gamma^3(t)$, the x-axis, and the y-axis, and also let A_0 be the area enclosed by Γ_0^3, the x-axis, and the y-axis. In the following we set for simplicity

$$L(t) = L[\Gamma^3(t)]. \tag{4.1}$$

Then we have:

Lemma 4.1 [11] *There hold for* $t \geq 0$

$$\frac{dE[\Gamma(t)]}{dt} = -\int_0^{L(t)} \kappa_s(t,s)^2 ds, \tag{4.2}$$

$$A(t) = A_0. \tag{4.3}$$

Next we show a priori estimates for $E[\Gamma(t)]$ and $L[\Gamma^3(t)]$.

Proposition 4.2 *There hold*

$$E[\Gamma_{\theta,A}] \leq E[\Gamma(t)] \leq E[\Gamma_0], \tag{4.4}$$
$$L[\Gamma^3_{\theta,A}] \leq L(t) \leq E[\Gamma_0] \tag{4.5}$$

for $t \geq 0$.

Proof. We show (4.4). The first inequality is due to Theorem 2.1 and the second inequality is due to (4.2), which shows (4.4).

Next we show (4.5). The first inequality is due to Corollary 2.2 and the second inequality follows from the inequality $L(t) \leq E[\Gamma(t)]$ and (4.4). □

In the following let $\Gamma(t)$ be the solution of (1.3)-(1.4) with the regularity in Theorem 3.6. We investigate the boundary values of the derivatives of the curvature κ.

Lemma 4.3 *There hold for* $t > 0$

(i) $\kappa_{ss}(t,0) = -\dfrac{d\xi}{dt}\sin\theta$,

(ii) $\kappa_{sss}(t,0) = -\kappa(t,0)\dfrac{d\xi}{dt}\cos\theta$,

(iii) $\kappa_{sss}(t, L(t)) = 0$.

Proof. (i): Differentiating the equality $u(t, -\xi(t)) = 0$ with respect to t and using the condition $u_x(t, -\xi(t)) = \tan\theta$, we get

$$u_t(t, -\xi(t)) = \frac{d\xi}{dt}\tan\theta. \tag{4.6}$$

On the other hand, from the equation $V = -\kappa_{ss}$ we have

$$\kappa_{ss}(t,s) = -\frac{u_t}{(1+u_x^2)^{1/2}}(t,x). \tag{4.7}$$

Here the relation between s and x is

$$s = \int_{-\xi(t)}^{x} (1+u_x^2)^{1/2}(t,x')dx'. \tag{4.8}$$

Now, evaluating (4.7) at $s = 0$ and using the equality $u_x(t, -\xi(t)) = \tan\theta$ and (4.6), we obtain (i).

(ii): Differentiating (4.7) with respect to s and using $\partial/\partial s = (1+u_x^2)^{-1/2}\partial/\partial x$, we have

$$\kappa_{sss}(t,s) = \left(-\frac{u_{tx}}{1+u_x^2} + \frac{u_t u_x u_{xx}}{(1+u_x^2)^2}\right)(t,x). \tag{4.9}$$

On the other hand, differentiating the equality $u_x(t,-\xi(t)) = \tan\theta$ with respect to t, we get

$$u_{tx}(t,-\xi(t)) = u_{xx}(t,-\xi(t))\frac{d\xi}{dt}. \tag{4.10}$$

Evaluating (4.9) at $s=0$ and using (4.10) and the condition $u_x(t,-\xi(t)) = \tan\theta$, we get (ii).

(iii): It follows from the condition $u_x(t,0)=0$ that $u_{tx}(t,0)=0$. With this in mind we evaluate (4.9) at $s=L(t)$ to get (iii). □

We are going to obtain an a priori estimate for κ_s in $L^2(\Gamma^3(t))$ by taking care of the boundary values obtained in Lemma 4.3 and employing the method as in [3] and [5]. For this purpose we use a new parametrization of $\Gamma^3(t)$ as

$$\Gamma^3(t) = \{\hat{F}(t,p) \in \mathbf{R}^2; p \in [p_0(t), p_1(t)]\}$$

satisfying

$$\begin{aligned}
&\varphi : [p_0(t), p_1(t)] \ni p \mapsto x \in [-\xi(t), 0] : \text{monotone increasing},\\
&\hat{F}(t,p_0(t)) = (-\xi(t),0), \quad \hat{F}(t,p_1(t)) = (0,u(t,0)),\\
&\hat{F}_t \cdot \hat{F}_p = 0 \quad \text{for } t \geq 0 \text{ and } p \in [p_0(t), p_1(t)],
\end{aligned}$$

where $\cdot$ denotes the standard inner product in $\mathbf{R}^2$. We write the curvature $\kappa(t,s)$ and the normal velocity $V(t,s)$ in this new coordinate as

$$\hat{\kappa}(t,p) = \kappa(t,s), \quad \hat{V}(t,p) = V(t,s).$$

Here p and s are linked via φ and (4.8). Then $\hat{\kappa}(t,p)$ satisfies the identity

$$\hat{\kappa}_t = \Delta\hat{V} + (\hat{\kappa})^2\hat{V},$$

where

$$\Delta\hat{V} := \frac{1}{\hat{g}}(\hat{V}_{pp} - \frac{1}{2}\frac{\hat{g}_p}{\hat{g}}\hat{V}_p), \quad \hat{g}(t,p) := \hat{F}_p(t,p)\cdot\hat{F}_p(t,p)$$

(see [13]). Since the equation $V = -\kappa_{ss}$ is equivalent to the one $\hat{V} = -\Delta\hat{\kappa}$, we get the equation

$$\hat{\kappa}_t = -\Delta^2\hat{\kappa} - \hat{\kappa}^2\Delta\hat{\kappa} \quad \text{for } t \geq 0 \text{ and } p \in [p_0(t), p_1(t)]. \tag{4.11}$$

We multiply (4.11) by $-2\Delta\hat{\kappa}$ and integrate it on $\Gamma^3(t)$. Then a straightforward computation gives

$$\begin{aligned}
-\int_{p_0(t)}^{p_1(t)} 2\Delta\hat{\kappa}\cdot\hat{\kappa}_t\sqrt{\hat{g}}dp &= \frac{d}{dt}\int_0^{L(t)}\kappa_s^2 ds + \int_0^{L(t)}\kappa\kappa_s^2\kappa_{ss}ds,\\
\int_{p_0(t)}^{p_1(t)} 2\Delta\hat{\kappa}(\Delta^2\hat{\kappa} + \hat{\kappa}^2\Delta\hat{\kappa})\sqrt{\hat{g}}dp &= \int_0^{L(t)} 2\kappa_{ss}(\kappa_{ssss} + \kappa^2\kappa_{ss})ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{d}{dt}\|\kappa_s(t)\|_2^2 &= -\int_0^{L(t)}\kappa\kappa_s^2\kappa_{ss}ds + 2\int_0^{L(t)}\kappa_{ss}\kappa_{ssss}ds\\
&\quad +2\int_0^{L(t)}\kappa^2\kappa_{ss}^2 ds\\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Here and hereafter we set for simplicity

$$\|\cdot\|_2 = \|\cdot\|_{L^2(\Gamma^3(t))}, \quad \|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Gamma^3(t))}.$$

We estimate J_i, $i = 1, 2, 3$.

Lemma 4.4 *There hold for* $t > 0$

$$|J_1| \leq (L(t)^3\|\kappa_s(t)\|_2^2 + \theta L(t)^{3/2}\|\kappa_s(t)\|_2)\|\kappa_{sss}(t)\|_2^2, \tag{4.12}$$

$$J_2 \leq 2(\frac{L(t)^{3/2}}{\tan\theta}\|\kappa_s(t)\|_2 + \frac{\theta}{\tan\theta} - 1)\|\kappa_{sss}(t)\|_2^2, \tag{4.13}$$

$$\begin{aligned} J_3 \leq\ & 2(L(t)^3\|\kappa_s(t)\|_2^2 + 2\theta L(t)^{3/2}\|\kappa_s(t)\|_2 + \delta_1)\|\kappa_{sss}(t)\|_2^2 \\ & + C(\delta_1)\theta^4 L(t)^{-4}\|\kappa_s(t)\|_2^2, \end{aligned} \tag{4.14}$$

where $\delta_1 > 0$ *is arbitrary small constant and* $C(\delta_1)$ *is a constant depending on* δ_1.

Proof. We use the following inequalities, which are easily derived by the fact that $\kappa_s = 0$ at $s = 0$ and $L(t)$:

$$\|(\kappa - \kappa_{av})(t)\|_\infty \leq L(t)^{1/2}\|\kappa_s(t)\|_2, \tag{4.15}$$

$$\|\partial_s^j\kappa(t)\|_2 \leq L(t)\|\partial_s^{j+1}\kappa(t)\|_2, \tag{4.16}$$

$$\|\partial_s^j\kappa(t)\|_\infty \leq L(t)^{1/2}\|\partial_s^{j+1}\kappa(t)\|_2 \tag{4.17}$$

for $j = 1, 2$, here $\kappa_{av}(t)$ is the averaged curvature of $\kappa(t, s)$ defined by

$$\kappa_{av}(t) = \frac{1}{L(t)}\int_0^{L(t)} \kappa(t, s)ds. \tag{4.18}$$

We also note the fact

$$\kappa_{av}(t) = -\frac{\theta}{L(t)}. \tag{4.19}$$

Indeed, if we let $\omega(t, s)$ be the angle of the unit normal of $\Gamma^3(t)$ from the x-axis at s, then $\kappa(t, s) = \partial\omega/\partial s(t, s)$. It then follows from (4.18) that

$$\kappa_{av}(t) = \frac{1}{L(t)}\int_0^{L(t)} \frac{\partial\omega}{\partial s}ds = \frac{1}{L(t)}(\omega(t, L(t)) - \omega(t, 0)).$$

Note that the boundary conditions $u_x(t, -\xi(t)) = \tan\theta$ and $u_x(t, 0) = 0$ mean $\omega(t, 0) = \pi/2 + \theta$ and $\omega(t, L(t)) = \pi/2$, respectively. Thus we obtain (4.19).

To show (4.12), we use (4.19) to get

$$\begin{aligned} |J_1| &\leq \int_0^{L(t)} |\kappa - \kappa_{av}||\kappa_s|^2|\kappa_{ss}|ds + \int_0^{L(t)} |\kappa_{av}||\kappa_s|^2|\kappa_{ss}|ds \\ &\leq (\|\kappa - \kappa_{av}\|_\infty\|\kappa_s\|_2^2\|\kappa_{ss}\|_\infty + \frac{\theta}{L}\|\kappa_s\|_2^2\|\kappa_{ss}\|_\infty)(t). \end{aligned}$$

Now (4.15)-(4.17) yield (4.12).

We show (4.13). An integration by parts and Lemma 4.3 imply

$$J_2 = -2\sin\theta\cos\theta\kappa(t, 0)(\frac{d\xi}{dt})^2 - 2\|\kappa_{sss}(t)\|_2^2. \tag{4.20}$$

It also follows from Lemma 4.3 (i) and (4.17) with $j = 2$ that

$$(\frac{d\xi}{dt})^2 = \frac{1}{\sin^2\theta}\kappa_{ss}(t,0)^2 \leq \frac{L(t)}{\sin^2\theta}\|\kappa_{sss}(t)\|_2^2.$$

Then the above inequality together with (4.15) and (4.19) yields

$$\begin{aligned}
&|\sin\theta\cos\theta\kappa(t,0)(\frac{d\xi}{dt})^2| \\
&\leq \frac{L(t)}{\tan\theta}(|\kappa(t,0)-\kappa_{av}(t)| + |\kappa_{av}(t)|)\|\kappa_{sss}(t)\|_2^2 \\
&\leq \frac{1}{\tan\theta}(L(t)^{3/2}\|\kappa_s(t)\|_2 + \theta)\|\kappa_{sss}(t)\|_2^2.
\end{aligned}$$

We plug this into (4.20) to get (4.13).

We show (4.14). We split J_3 as

$$\begin{aligned}
J_3 &= 2\int_0^{L(t)} |\kappa-\kappa_{av}|^2\kappa_{ss}^2 ds + 4\kappa_{av}\int_0^{L(t)}(\kappa-\kappa_{av})\kappa_{ss}^2 ds \\
&\quad +2\kappa_{av}^2\int_0^{L(t)}\kappa_{ss}^2 ds \\
&=: J_{3,1}+J_{3,2}+J_{3,3}.
\end{aligned}$$

Applying (4.15), (4.16) with $j = 2$, and (4.19) to $J_{3,1}$ and $J_{3,2}$, we have

$$\begin{aligned}
J_{3,1} &\leq 2L(t)^3\|\kappa_s(t)\|_2^2\|\kappa_{sss}(t)\|_2^2, \\
J_{3,2} &\leq 4\theta L(t)^{3/2}\|\kappa_s(t)\|_2\|\kappa_{sss}(t)\|_2^2.
\end{aligned}$$

An integration by parts in $J_{3,3}$ employing the condition that $\kappa_s = 0$ at $s = 0$ and $L(t)$ yields

$$\begin{aligned}
J_{3,3} &= -2\kappa_{av}(t)^2\int_0^{L(t)}\kappa_s\kappa_{sss}ds \leq 2\kappa_{av}(t)^2\|\kappa_s(t)\|_2\|\kappa_{sss}(t)\|_2 \\
&\leq \frac{1}{2\delta_1}\left(\frac{\theta}{L(t)}\right)^4\|\kappa_s(t)\|_2^2 + 2\delta_1\|\kappa_{sss}(t)\|_2^2.
\end{aligned}$$

Thus (4.14) is proved. □

It is observed that $1-\theta/\tan\theta > 0$ when $\theta \in (0,\pi/2)$. Hence we can choose $\delta_1 > 0$ so small so that $1-\theta/\tan\theta-\delta_1 > 0$. We fix such a δ_1 and put

$$H(l,\lambda) = 2(1-\frac{\theta}{\tan\theta}-\delta_1) - \{3l^3\lambda^2 + (5\theta+\frac{2}{\tan\theta})l^{3/2}\lambda\} \quad \text{for } l,\ \lambda > 0.$$

Then from Lemma 4.4 we get

$$\begin{aligned}
&\frac{d}{dt}\|\kappa_s(t)\|_2^2 + H(L(t),\|\kappa_s(t)\|_2)\|\kappa_{sss}(t)\|_2^2 \\
&\leq C(\delta_1)\theta^4 L(t)^{-4}\|\kappa_s(t)\|_2^2.
\end{aligned}$$

Integrating this on $[0,t]$ and using Proposition 4.2 and (4.2), we arrive at

$$\begin{aligned}&\|\kappa_s(t)\|_2^2+\int_0^t H(E[\Gamma_0],\|\kappa_s(\tau)\|_2)\|\kappa_{sss}(\tau)\|_2^2d\tau\\&\le\|\kappa_{0,s}\|_2^2+C_{\theta,A_0}(E[\Gamma_0]-E[\Gamma_{\theta,A_0}]).\end{aligned}\tag{4.21}$$

Here we put

$$C_{\theta,A_0}=C(\delta_1)\theta^4L[\Gamma^3_{\theta,A_0}]^{-4}.\tag{4.22}$$

Now we are ready to show a priori estimate of κ_s when Γ_0 and Γ_{θ,A_0} are sufficiently close to each other. To make the statement precise, we put

$$\rho_0^2:=\|\kappa_{0,s}\|_2^2+C_{\theta,A_0}(E[\Gamma_0]-E[\Gamma_{\theta,A_0}]).\tag{4.23}$$

Then we say that Γ_0 and Γ_{θ,A_0} are close to each other if $H(E[\Gamma_0],\rho_0)>0$. The following proposition establishes a modified version of a priori estimate presented in [3] and [5].

Proposition 4.5 *The following two statements hold.*
(i) If $E[\Gamma_0]=E[\Gamma_{\theta,A_0}]$, then $\Gamma(t)=\Gamma_{\theta,A_0}$ for $t\ge 0$.
(ii) If $E[\Gamma_0]>E[\Gamma_{\theta,A_0}]$ and $H(E[\Gamma_0],\rho_0)>0$, then

$$\|\kappa_s(t)\|_2^2+H(E[\Gamma_0],\rho_0)\int_0^t\|\kappa_{sss}(\tau)\|_2^2d\tau\le\rho_0^2\quad \textit{for } t\ge 0.\tag{4.24}$$

Remark 4.6 Proposition 4.5 together with (4.15) and (4.5) shows that if Γ_0 and Γ_{θ,A_0} are close to each other, then $\kappa(t,s)$ is close to $\kappa_{av}(t)$ for $t\ge 0$ and $s\in[0,L(t)]$.

Proof of Proposition 4.5. (i): If $E[\Gamma_0]=E[\Gamma_{\theta,A_0}]$, then it follows from (4.4) that $E[\Gamma(t)]=E[\Gamma_0]=E[\Gamma_{\theta,A_0}]$ for $t\ge 0$. By (4.2), (4.15), and (4.19) we see that $\kappa(t,s)=\kappa_{av}(t)=-\theta/L(t)$ for $t\ge 0$ and $s\in[0,L(t)]$, which shows that $\Gamma^3(t)$ is a circular arc with the radius $L(t)/\theta$. Moreover, recalling the condition $u_x(t,0)=0$, we see that the center of $\Gamma^3(t)$ is on the y-axis. Now the condition $u_x(t,-\xi(t))=\tan\theta$ together with the above fact implies $\xi(t)=L(t)\sin\theta/\theta$. Then

$$\begin{aligned}E[\Gamma(t)]&=(a-\xi(t))\cos\theta+L(t)\\&=a\cos\theta+(1-\frac{\sin\theta\cos\theta}{\theta})L(t).\end{aligned}\tag{4.25}$$

Thus $L(t)$ is a constant and $\Gamma(t)$ must be stationary and coincide with Γ_{θ,A_0} for $t\ge 0$, otherwise a contradiction to (4.3) occurs.

(ii): If $E[\Gamma_0]>E[\Gamma_{\theta,A_0}]$, the proof can proceed as in [5] by virtue of the boundary condition $\kappa_s=0$ at $s=0$ and $L(t)$, so we omit the details. □

By virtue of Proposition 4.5 we can obtain a number of a priori estimates. The contributions arised from the motion of the triple junction emerge as various functions of θ. Throughout the remaining part of this paper we always assume the following conditions on initial data Γ_0:

$$\begin{aligned}&E[\Gamma_0]>E[\Gamma_{\theta,A_0}],&&H(E[\Gamma_0],\rho_0)>0,\\&\theta+E[\Gamma_0]^{3/2}\rho_0<\pi/2,&&L[\Gamma^3_{\theta,A_0}]\sin\theta/\theta-\tfrac{1}{2}E[\Gamma_0]^{5/2}\rho_0>0.\end{aligned}\tag{4.26}$$

The reason why the third and fourth conditions are imposed will be clarified in Remarks 4.8 and 4.12.

We begin to show a priori estimate of $\omega(t,s)$, the angle of the unit normal of $\Gamma^3(t)$ from the x-axis at s.

Proposition 4.7 *There holds*

$$\frac{\pi}{2} - E[\Gamma_0]^{3/2}\rho_0 \le \omega(t,s) \le \frac{\pi}{2} + \theta + E[\Gamma_0]^{3/2}\rho_0 \tag{4.27}$$

for $t \ge 0$ *and* $s \in [0, L(t)]$.

Remark 4.8 Owing to the third assumption in (4.26), the right hand side of (4.27) is less than π, which means that $\Gamma^3(t)$ is prevented from a graph-breaking.

Proof of Proposition 4.7. We only show the estimtate from below. The estimate from above can be checked almost identically.

Since $\partial\omega/\partial s = \kappa$, we have

$$\begin{aligned} \omega(t,s) &= \frac{\pi}{2} + \theta + \int_0^s \kappa(t,\sigma)d\sigma \\ &= \frac{\pi}{2} + \theta + \kappa_{av}(t)s + \int_0^s (\kappa(t,\sigma) - \kappa_{av}(t))d\sigma. \end{aligned}$$

We use (4.19) in the third term and (4.15) in the last term. Then,

$$\omega(t,s) \ge \frac{\pi}{2} + \theta(1 - \frac{s}{L(t)}) - L(t)^{3/2}\|\kappa_s(t)\|_2.$$

Now (4.5) and (4.24) yield the desired result. □

Lemma 4.9 *There holds*

$$-(E[\Gamma_0]^{1/2}\rho_0 + \frac{\theta}{L[\Gamma^3_{\theta,A_0}]}) \le \kappa(t,s) \le E[\Gamma_0]^{1/2}\rho_0 - \frac{\theta}{E[\Gamma_0]} \tag{4.28}$$

for $t \ge 0$ *and* $s \in [0, L(t)]$.

Proof. We write

$$\kappa(t,s) = (\kappa(t,s) - \kappa_{av}(t)) + \kappa_{av}(t).$$

Now a similar argument employed in the proof of Proposition 4.7 gives (4.28). □

Now we shall derive a priori estimate for $\xi(t)$. For this purpose, we need some observations. Let $(X,Y)(t,s)$ be the parametrization of $\Gamma^3(t)$ with respect to its arc-length parameter s. Then the unit tangent of $\Gamma^3(t)$ at s is given by $(\sin\omega(t,s), -\cos\omega(t,s))$. Thus we have

$$(X,Y)(t,s) = (-\xi(t) + \int_0^s \sin\omega(t,\sigma)d\sigma, -\int_0^s \cos\omega(t,\sigma)d\sigma). \tag{4.29}$$

Let $\Gamma^3_{av}(t)$ be the circular arc in $\{(x,y); x \le 0,\ y \ge 0\}$ with the radius $|\kappa_{av}(t)|^{-1} = L(t)/\theta$ and with the center on the y-axis and with $\angle(\Gamma^3_{av}(t), x\text{-axis})|_{x=-\xi_{av}(t)} = \theta$, where $-\xi_{av}(t)$ is the x-coordinate of the point at which $\Gamma^3_{av}(t)$ intersects to the x-axis. Then it is easily seen that the total length of $\Gamma^3_{av}(t)$ is also equal to $L(t)$. Let $(X_{av}, Y_{av})(t,s) \in \mathbf{R}^2$ be the parametrization of $\Gamma^3_{av}(t)$ with its arc-length parameter $s \in [0, L(t)]$. Let $\omega_{av}(t,s)$ be the angle of the unit normal of $\Gamma^3_{av}(t)$ from the x-axis at s. Then as in (4.29) we have

$$(X_{av}, Y_{av})(t,s) = (-\xi_{av}(t) + \int_0^s \sin\omega_{av}(t,\sigma)d\sigma, -\int_0^s \cos\omega_{av}(t,\sigma)d\sigma). \tag{4.30}$$

The following lemma enables us to relate $\Gamma^3(t)$ with $\Gamma^3_{av}(t)$.

Lemma 4.10 *There hold for $t \geq 0$ and $s \in [0, L(t)]$*

$$|X(t,s) - X_{av}(t,s)| \leq \frac{1}{2}L(t)^{5/2}\|\kappa_s(t)\|_2, \tag{4.31}$$

$$|Y(t,s) - Y_{av}(t,s)| \leq \frac{1}{2}L(t)^{5/2}\|\kappa_s(t)\|_2. \tag{4.32}$$

Proof. We employ the method as in [3]. Note the facts that

$$\omega_{av}(t,0) = \frac{\pi}{2} + \theta, \quad \xi_{av}(t) = L(t)\frac{\sin\theta}{\theta}, \tag{4.33}$$

which are easily seen by an elementary geometric observation. Since

$$X(t, L(t)) = X_{av}(t, L(t)) = 0,$$

we have

$$\xi(t) = \int_0^{L(t)} \sin\omega(t,\sigma)d\sigma, \quad \xi_{av}(t) = \int_0^{L(t)} \sin\omega_{av}(t,\sigma)d\sigma. \tag{4.34}$$

On the other hand, we integrate the identities $\partial\omega/\partial s = \kappa$ and $\partial\omega_{av}/\partial s = \kappa_{av}$ with respect to s. By virtue of the first of (4.33) and (4.15), we have

$$\begin{aligned} |\omega(t,s) - \omega_{av}(t,s)| &\leq \int_0^s |\kappa(t,\sigma) - \kappa_{av}(t)|d\sigma \\ &\leq L(t)^{1/2}\|\kappa_s\|_2 s. \end{aligned} \tag{4.35}$$

Then, by (4.34) and (4.35),

$$\begin{aligned} |X(t,s) - X_{av}(t,s)| &= |\int_{L(t)}^s \sin\omega(t,\sigma)d\sigma - \int_{L(t)}^s \sin\omega_{av}(t,\sigma)d\sigma| \\ &\leq \int_s^{L(t)} |\omega(t,\sigma) - \omega_{av}(t,\sigma)|d\sigma \\ &\leq L(t)^{1/2}\|\kappa_s(t)\|_2 \int_s^{L(t)} \sigma d\sigma \\ &\leq \frac{1}{2}L(t)^{5/2}\|\kappa_s(t)\|_2. \end{aligned}$$

Thus we have proved (4.31). Similar argument as in (4.31) gives (4.32). □

The following proposition establishes a priori bounds for the triple junction.

Proposition 4.11 *There holds*

$$L[\Gamma^3_{\theta,A_0}]\frac{\sin\theta}{\theta} - \frac{1}{2}E[\Gamma_0]^{5/2}\rho_0 \leq \xi(t) \leq E[\Gamma_0]\frac{\sin\theta}{\theta} + \frac{1}{2}E[\Gamma_0]^{5/2}\rho_0 \tag{4.36}$$

for $t \geq 0$.

Remark 4.12 Owing to the fourth assumption in (4.26), it is assured that $\xi(t) > 0$ for $t \geq 0$.

Proof of Proposition 4.11. It follows from (4.33) and (4.5) that

$$L[\Gamma^3_{\theta,A_0}]\frac{\sin\theta}{\theta} \leq \xi_{av}(t) \leq E[\Gamma_0]\frac{\sin\theta}{\theta}.$$

Evaluating (4.31) at $s = 0$ and using (4.5), (4.24), and the above inequality, we get (4.36). □

Now we can show a priori estimate of v.

Proposition 4.13 *Let $\alpha \in (0, 1/2]$. Then we have*

$$\|v(t)\|_{C^{2+\alpha}(I)} \leq B_0 \quad \textit{for } t \geq 0. \tag{4.37}$$

Here $B_0 > 0$ is a constant depending only on Γ_0 through ρ_0, A_0, and $E[\Gamma_0]$.

Proof. First we differentiate $v(t, \eta) = Y(t, s)$ with respect to η to get

$$v_\eta = Y_s \cdot (\xi^2 + v_\eta^2)^{1/2}. \tag{4.38}$$

On the other hand, by (4.29), $Y_s = -\cos\omega$. Thus

$$v_\eta^2 = \xi^2\frac{\cos^2\omega}{\sin^2\omega}.$$

It then follows from (4.27) and (4.36) that there is a constant $B_{0,1}$ depending on Γ_0 such that

$$\|v_\eta(t)\|_{C(I)} \leq B_{0,1} \quad \text{for } t \geq 0.$$

Moreover, since the condition $v(t, 0) = 0$ yields the estimate $|v(t, \eta)| \leq \|v_\eta(t)\|_{C(I)}$, we obtain

$$\|v(t)\|_{C^1(I)} \leq B_{0,1} \quad \text{for } t \geq 0. \tag{4.39}$$

Next we differentiate (4.38) with respect to η. With the equality $Y_{ss} = \kappa\sin\omega$ in mind we obtain

$$v_{\eta\eta} = \frac{\kappa\sin\omega \cdot (\xi^2 + v_\eta^2)}{1 + v_\eta\cos\omega \cdot (\xi^2 + v_\eta^2)^{-1/2}}.$$

Now (4.28), (4.36), and (4.39) imply that there is a constant $B_{0,2}$ depending on Γ_0 such that

$$\|v_{\eta\eta}(t)\|_{C(I)} \leq B_{0,2} \quad \text{for } t \geq 0. \tag{4.40}$$

Finally we show that there is a constant $B_{0,3}$ depending on Γ_0 such that

$$[v_{\eta\eta}(t, \cdot)]_{C^\alpha[0,1]} \leq B_{0,3} \quad \text{for } t \geq 0 \text{ with } \alpha \in (0, 1/2]. \tag{4.41}$$

Indeed, differentiating (4.38) twice with respect to η and using $Y_{sss} = \kappa_s\sin\omega + \kappa^2\cos\omega$, we have

$$\begin{aligned}
&(1 + \frac{v_\eta\cos\omega}{(\xi^2 + v_\eta^2)^{1/2}})v_{\eta\eta\eta} \\
&= (\kappa_s\sin\omega + \kappa^2\cos\omega)(\xi^2 + v_\eta^2)^{3/2} + 3\kappa v_\eta v_{\eta\eta}\sin\omega \\
&\quad - \frac{\xi^2 v_{\eta\eta}^2\cos\omega}{(\xi^2 + v_\eta^2)^{3/2}}.
\end{aligned}$$

We take the $L^2(I)$-norm in the both sides and use (4.28), (4.36), (4.39), and (4.40). Consequently we can find a constant $B_{0,3}$ depending on Γ_0 such that

$$\|v_{\eta\eta\eta}(t)\|_{L^2(I)} \leq B_{0,3} \quad \text{for } t \geq 0.$$

Now (4.41) immediately follows from the above inequality when $\alpha \in (0, 1/2]$. □

Now by virtue of Propositions 4.11 and 4.13 one can always solve (3.2)-(3.9) on the time intervals $[0, T_*]$, $[T_*, 2T_*]$, $[2T_*, 3T_*]$, $\cdots$, for a $T_* > 0$ which is determined only on Γ_0. Thus we arrive at the following global existence result.

Theorem 4.14 *(Global existence). Let $\alpha \in (0, 1/2]$. Assume (4.26). Then the equation (3.2)-(3.9) admits a unique global solution $(v, \xi) \in \mathcal{Y}^{4+\alpha}(\overline{R_{0,T}}) \times \mathcal{Z}^1[0, T]$ for any $T > 0$. Consequently, the problem (1.1)-(1.2) admits a unique global solution in $C^{1+\alpha/4, 4+\alpha}$ for $t > 0$ provided the initial data belongs to $\mathcal{S}_\theta$ and is also close to Γ_{θ, A_0}.*

Remark 4.15 The global solutions obtained in Theorem 4.14 also possess the additional regularities guaranteed in Theorem 3.6 and Remark 3.7 with $\alpha \in (0, 1/2]$.

5 Convergence to the minimizer of the energy

This section gives the proofs that the global solution of (1.3)-(1.4) obtained in Theorem 4.14 converges to the minimizer Γ_{θ, A_0} of the energy E and the energy of the solution converges to $E[\Gamma_{\theta, A_0}]$ as $t \to \infty$.

The fundamental tool to obtain this result is the following fact.

Lemma 5.1 *Assume that $\Gamma(t) = \bigcup_{i=1}^3 \Gamma^i(t)$ is a (at least) C^6-global solution of (1.3)-(1.4) with a C^3-initial data $\Gamma_0 \in \mathcal{S}_\theta$. Let $\kappa(t, s)$ be the curvature of $\Gamma^3(t)$. Then,*

$$\|\kappa_s(t)\|_2 \to 0 \quad \text{as } t \to \infty.$$

This lemma can be proved as in [5, Theorem 6.4] by employing the same argument to derive (4.21), so we omit its proof.

Let $\Gamma(t)$ be the global solution of (1.3)-(1.4) for the initial data $\Gamma_0 \in \mathcal{S}_\theta$ constructed in Theorem 4.14. Put

$$\Gamma_{av}(t) = \bigcup_{i=1}^{3} \Gamma_{av}^i(t), \tag{5.1}$$

where

$$\begin{aligned} \Gamma_{av}^1(t) &= \{(x, 0); x \in [-a, -\xi_{av}(t)]\}, \\ \Gamma_{av}^2(t) &= \{(X_{av}, -Y_{av})(t, s); s \in [0, L(t)]\}, \\ \Gamma_{av}^3(t) &= \{(X_{av}, Y_{av})(t, s); s \in [0, L(t)]\} \end{aligned}$$

for $t \geq 0$ and $(X_{av}, Y_{av})(t, s)$ is given in (4.30).

Lemma 5.2 *Put $E_\star := \lim_{t\to\infty} E[\Gamma(t)]$ and $L_\star := \theta(E_\star - a\cos\theta)/(\theta - \sin\theta\cos\theta)$. Then,*

$$E[\Gamma_{av}(t)] \to E_\star, \tag{5.2}$$

$$L(t) \to L_\star \tag{5.3}$$

as $t \to \infty$.

Proof. Since $E[\Gamma_{av}(t)] = (a - \xi_{av}(t))\cos\theta + L(t)$, we have

$$E[\Gamma_{av}(t)] - E[\Gamma(t)] = (\xi(t) - \xi_{av}(t))\cos\theta.$$

Then (4.31) at $s = 0$ implies

$$|\xi(t) - \xi_{av}(t)| \leq \frac{1}{2}L(t)^{5/2}\|\kappa_s(t)\|_2 \leq \frac{1}{2}E[\Gamma_0]^{5/2}\|\kappa_s(t)\|_2.$$

Moreover, from Lemma 5.1 it follows that the right hand side converges to 0 as $t \to \infty$. Thus we get (5.2).

In order to check (5.3), we observe, as in (4.25),

$$E[\Gamma_{av}(t)] = a\cos\theta + (1 - \frac{\sin\theta\cos\theta}{\theta})L(t).$$

Using (5.2) and passing to the limit as $t \to \infty$ in the above equality, we get (5.3). □

The next lemma shows that the large-time profile of $\Gamma(t)$ is approximated by $\Gamma_{av}(t)$.

Lemma 5.3

$$\lim_{t\to\infty} \sup_{s\in[0,L(t)]} (|X - X_{av}| + |Y - Y_{av}|)(t, s) = 0.$$

Proof. It follows from (4.31) and (4.32) that

$$\sup_{s\in[0,L(t)]} (|X - X_{av}| + |Y - Y_{av}|)(t, s) \leq L(t)^{5/2}\|\kappa_s(t)\|_2.$$

Now (5.3) and Lemma 5.1 give the desired result. □

We put

$$\xi_\star := L_\star\frac{\sin\theta}{\theta}, \quad \kappa_\star := -\frac{\theta}{L_\star}. \tag{5.4}$$

Define a circular arc in $\{(x, y); x \leq 0,\ y \geq 0\}$ by

$$(X_\star, Y_\star)(s) = (-\xi_\star + \int_0^s \sin(\frac{\pi}{2} + \theta + \kappa_\star\sigma)d\sigma, -\int_0^s \cos(\frac{\pi}{2} + \theta + \kappa_\star\sigma)d\sigma)$$

for $s \in [0, L_\star]$ and set

$$\Gamma_\star = \bigcup_{i=1}^{3}\Gamma_\star^i \in \mathcal{S}_\theta,$$

where

$$\begin{aligned}\Gamma_\star^1 &= \{(x, 0); x \in [-a, -\xi_\star]\},\\ \Gamma_\star^2 &= \{(X_\star, -Y_\star)(s); s \in [0, L_\star]\},\\ \Gamma_\star^3 &= \{(X_\star, Y_\star)(s); s \in [0, L_\star]\}.\end{aligned}$$

The next lemma shows that the large-time profile of $\Gamma_{av}(t)$ is described by $\Gamma_\star$.

Lemma 5.4

$$\Gamma_{av}(t) \to \Gamma_\star \quad \textit{as } t \to \infty.$$

Proof. Passing to the limit as $t \to \infty$ in the second equation in (4.33) and (4.19), we have

$$\lim_{t\to\infty} \xi_{av}(t) = \xi_\star, \quad \lim_{t\to\infty} \kappa_{av}(t) = \kappa_\star.$$

Then, by the first equation in (4.33),

$$\begin{aligned} \omega_{av}(t,s) &= \omega_{av}(t,0) + \int_0^s \frac{\partial \omega_{av}}{\partial \sigma}(t,\sigma)d\sigma \\ &= \frac{\pi}{2} + \theta + \kappa_{av}(t)s \to \frac{\pi}{2} + \theta + \kappa_\star s \quad \text{as } t \to \infty \end{aligned} \tag{5.5}$$

for $s \in [0, L_\star]$. Now, passing to the limit as $t \to \infty$ in (4.30), we obtain the desired result.
□

Thus we arrive at the desired result on the convergence of the solution as $t \to \infty$.

Theorem 5.5

$$\Gamma(t) \to \Gamma_{\theta,A_0} \quad \textit{as } t \to \infty.$$

Proof. By virtue of Lemmas 5.3 and 5.4 we get

$$\Gamma(t) \to \Gamma_\star \quad \text{as } t \to \infty. \tag{5.6}$$

Let $A_\star$ be the area enclosed by $\Gamma^3_\star$, the x-axis, and the y-axis. If $\Gamma_\star \neq \Gamma_{\theta,A_0}$, then $A_\star$ is not equal to A_0. On the other hand (4.3) and (5.6) imply that $A_\star$ must coincide with A_0. This yields a contradiction and we then conclude that $\Gamma_\star$ must coincide with Γ_{θ,A_0}. This completes the proof. □

We can also show the convergences of both the length and the energy of the solution as $t \to \infty$. The key point is to make use of the area-preserving property.

Theorem 5.6 *We have*

$$\lim_{t\to\infty} L[\Gamma^3(t)] = L[\Gamma^3_{\theta,A_0}], \tag{5.7}$$

$$\lim_{t\to\infty} E[\Gamma(t)] = E[\Gamma_{\theta,A_0}]. \tag{5.8}$$

Proof. We first show (5.7). We use the following fact.

Fact. Let $D \subset \mathbf{R}^2$ be a simply connected bounded domain with the piecewise smooth boundary ∂D with the total length L. Suppose that ∂D is parameterized by its arc-length parameter s running clockwise as $\partial D = \{F(s) \in \mathbf{R}^2; s \in [0, L]\}$. Let $\nu(s)$ be the outward unit normal field on ∂D. Then,

$$\text{the area of } D = \frac{1}{2}\int_0^L F(s)\cdot\nu(s)ds. \tag{5.9}$$

We apply the formula (5.9) to the domain enclosed by $\Gamma^3(t)$, the y-axis, and the x-axis to get

$$\begin{aligned} A(t) &= \frac{1}{2}\int_0^{L(t)} (X,Y)(t,s)\cdot(\cos\omega, \sin\omega)(t,s)ds \\ &\quad + \frac{1}{2}\int_{u(t,0)}^0 (0,y)\cdot(1,0)dy + \frac{1}{2}\int_0^{-\xi(t)} (x,0)\cdot(0,-1)dx. \end{aligned}$$

The last two terms of the right hand side vanish. We use (4.3) in the left hand side and use (4.29) with (4.34) in the first term of the right hand side. Moreover, after using Fubini's theorem, we have

$$A_0 = -\int_0^{L(t)} \left(\int_0^s \cos\omega(t,\sigma)d\sigma \right) \sin\omega(t,s)ds. \tag{5.10}$$

On the other hand, using (4.35), (4.5), and Lemma 5.1, we get

$$|\omega(t,s) - \omega_{av}(t,s)| \le E[\Gamma_{\theta,A_0}]^{3/2} \|\kappa_s(t)\|_2 \to 0 \quad \text{as } t \to \infty$$

for $s \in [0, L_\star]$. Recalling (5.5) and (5.4), we have

$$\omega_{av}(t,s) \to \frac{\pi}{2} + \theta - \frac{\theta}{L_\star}s \quad \text{as } t \to \infty$$

for $s \in [0, L_\star]$. Consequently, we have

$$\omega(t,s) \to \frac{\pi}{2} + \theta - \frac{\theta}{L_\star}s \quad \text{as } t \to \infty$$

for $s \in [0, L_\star]$. Now we let $t \to \infty$ in (5.10) and calculate the integral to obtain

$$L_\star = r_{\theta,A_0}\theta, \tag{5.11}$$

where r_{θ,A_0} is defined in (1.5). This shows (5.7).

Now (5.8) is a direct consequence of (5.11) and the definition of $L_\star$ in the statement of Lemma 5.2. This completes the proof. □

A Appendix

In this section we shall prove the inequalities (3.14) and (3.20). To prove them, we use the optimal regularity theory of analytic semigroups as comprehensively studied in [15].

Here we summarize a basic abstract framework used below. Let X be a Banach space with norm $\|\cdot\|$ and let $A : D(A) \subset X \to X$ be a linear operator satisfying the following condition:

$$\begin{cases} \text{there exist constants } \omega \in \mathbf{R},\ \mu \in (\pi/2, \pi),\ M > 0, \text{ which depend on } A, \\ \text{such that} \\ \quad (i) \quad \rho(A) \supset R_{\mu,\omega} = \{\lambda \in \mathbf{C};\ \lambda \ne \omega, |\arg(\lambda - \omega)| < \mu\}, \\ \quad (ii) \quad \|(\lambda - A)^{-1}\|_{L(X)} \le \dfrac{M}{|\lambda - \omega|} \quad \text{for } \lambda \in R_{\mu,\omega}, \\ \text{where } \rho(A) \text{ is a resolvent set of } A. \end{cases} \tag{A.1}$$

If A satisfies the condition (A.1), A is said to be sectorial in X. Then A generates an analytic semigroup e^{tA} in X for $t \ge 0$. In addition, a family of intermediate spaces between $D(A)$ and X can be defined by

$$D_A(\beta, \infty) = \{\phi \in X\ ;\ [\phi]_{D_A(\beta,\infty)} = \sup_{0<t\le 1} \|t^{1-\beta} A e^{tA}\phi\| < \infty\}, \quad 0 < \beta < 1.$$

They are Banach spaces under the norm

$$\|\phi\|_{D_A(\beta,\infty)} = \|\phi\| + [\phi]_{D_A(\beta,\infty)}.$$

Moreover the following estimate is known. For $k \in \mathbf{N}$, $\beta_1, \beta_2 \in (0,1)$, there exists a constant $C = C(k, \beta_1, \beta_2, A)$ such that

$$\|t^{k-\beta_1+\beta_2} A^k e^{tA}\|_{L(D_A(\beta_1,\infty), D_A(\beta_2,\infty))} \leq C \quad \text{for } 0 < t \leq 1. \tag{A.2}$$

The statement holds also for $k = 0$, provided $\beta_1 \leq \beta_2$.

We shall return to our problem. In the remaining part of this section, we use C_α, C_θ, etc. to denote various constants. Here their subindices are written to emphasize what they depend on. Let $\tau \geq 0$. We define two differential operators $\mathcal{A}_\tau^{(1)}$ and $\mathcal{A}_\tau^{(2)}$, for C^4-function $U \ : \ [0,1] \ni \eta \mapsto U(\eta) \in \mathbf{R}$ and given $(\bar{v}, \bar{\xi}) \in \mathcal{Y}^{4+\alpha}(\overline{R_{\tau,\tau+T}}) \times \mathcal{Z}^1[\tau, \tau+T]$, by

$$\mathcal{A}_\tau^{(1)} U := -\frac{1}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau,\eta))^2} \partial_\eta^4 U + 10 \frac{\bar{v}_\eta(\tau,\eta)\bar{v}_{\eta\eta}(\tau,\eta)}{(\bar{\xi}^2(\tau) + \bar{v}_\eta^2(\tau,\eta))^3} \partial_\eta^3 U,$$

$$\mathcal{A}_\tau^{(2)} U := C_1(\theta)(1-\eta) \frac{\bar{v}_\eta(\tau,\eta)}{\bar{\xi}^5(\tau)} \partial_\eta^4 U(0).$$

If we define $X := C[0,1]$ and

$$D(A_\tau^{(1)}) := \{U \in C^4[0,1] \ ; \ \partial_\eta U(0) = \partial_\eta U(1) = \partial_\eta^3 U(0) = \partial_\eta^3 U(1) = 0\},$$

then $A_\tau^{(1)} : \ X \supset D(A_\tau^{(1)}) \ni U \mapsto \mathcal{A}_\tau^{(1)} U \in X$ is the realization of $\mathcal{A}_\tau^{(1)}$ in X and satisfies the condition (A.1) where constants ω, μ, M depend on $\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$ (see [19]). Therefore, $A_\tau^{(1)}$ is sectorial in X. In particular, we note that these constants increase with $1/\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$. Moreover, we obtain the following equalities:

$$D_{A_\tau^{(1)}}(\beta,\infty) = \begin{cases} C^{4\beta}[0,1], & \text{if } 0 < \beta < 1/4, \\ C_{b_1}^{4\beta}[0,1], & \text{if } 1/4 < \beta < 3/4, \\ C_{b_2}^{4\beta}[0,1], & \text{if } 3/4 < \beta < 1. \end{cases} \tag{A.3}$$

with equivalence of norms (see e.g. [1], [16]), where

$$C_{b_1}^{4\beta}[0,1] := \{U \in C^{4\beta}[0,1] \ ; \ \partial_\eta U(0) = \partial_\eta U(1) = 0\},$$

$$C_{b_2}^{4\beta}[0,1] := \{U \in C^{4\beta}[0,1] \ ; \ \partial_\eta U(0) = \partial_\eta U(1) = \partial_\eta^3 U(0) = \partial_\eta^3 U(1) = 0\}.$$

It can be checked that constants, which assure equivalence of norms, increase with $1/\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$. From now on we represent constants concerned with equivalence of norms as C_{eq}. Here $\beta = 1/4$ and $\beta = 3/4$ are sensitive cases. But these cases are unnecessary in our paper, so we don't make mention of them.

Now let $A_\tau^{(2)} : \ X \supset D(A_\tau^{(1)}) \ni U \mapsto \mathcal{A}_\tau^{(2)} U \in X$ be the realization of $\mathcal{A}_\tau^{(2)}$ in X. Then we are led to the following lemma.

Lemma A.1 *Let $A_\tau := A_\tau^{(1)} + A_\tau^{(2)}$. Then,*
(i) A_τ is a sectorial operator in X.
(ii) there exists a constant C, which increases with $1/\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$, such that

$$|\lambda| \|(\lambda I - A_\tau)^{-1}\|_{L(X)} \le C \quad \textit{for } \lambda \in R_{A_\tau^{(1)}}$$

where $R_{A_\tau^{(1)}} = \{\lambda \in \mathbf{C} \; ; \; \lambda \neq \omega(A_\tau^{(1)}), \; |\arg(\lambda - \omega(A_\tau^{(1)}))| < \mu(A_\tau^{(1)})\}$.

Proof. First, we prove that A_τ is a sectorial operator in X. As stated above, it is known that $A_\tau^{(1)}$ is a sectorial operator in X (see [19]). Moreover, according to [15, Section 2.4], if $A_\tau^{(2)}$ is a bounded linear operator from $D(A_\tau^{(1)})$ to $D_{A_\tau^{(1)}}(\frac{\alpha}{4}, \infty)$ $(=: X_\alpha)$ for $\alpha \in (0,1)$, then A_τ is a sectorial operator in X. Thus we shall prove

$$\|A_\tau^{(2)} U\|_{X_\alpha} \le C \|U\|_{D(A_\tau^{(1)})} \quad \text{for } U \in D(A_\tau^{(1)}), \tag{A.4}$$

where C is a constant independent of U. In fact, one can prove

$$\|\partial_\eta^4 U\|_\infty \le C_\tau \|U\|_{D(A_\tau^{(1)})},$$

where C_τ is a constant depending on $\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$. Then, by means of (A.3) and using this inequality, we get

$$\begin{aligned}
\|A_\tau^{(2)} U\|_{X_\alpha} &\le C_{eq,\tau} \, \|A_\tau^{(2)} U\|_{C^\alpha[0,1]} \\
&= C_{eq,\tau} \, \|C_1(\theta)(1-\eta)\frac{\bar{v}_\eta(\tau,\cdot)}{\bar{\xi}^5(\tau)} \partial_\eta^4 U(0)\|_{C^\alpha[0,1]} \\
&\le C_{\tau,\theta} \, \|\frac{\bar{v}_\eta(\tau,\cdot)}{\bar{\xi}^5(\tau)}\|_{C^\alpha[0,1]} \cdot |\partial_\eta^4 U(0)| \\
&\le C'_{\tau,\theta} \, \|U\|_{D(A_\tau^{(1)})}.
\end{aligned} \tag{A.5}$$

In particular, we note that a constant $C_{eq,\tau}$ increases with $1/\bar{\xi}(\tau)$, $\|\bar{v}(\tau,\cdot)\|_{C^{2+\alpha}[0,1]}$. Thus the inequality (A.4) is proved.

Next, we prove the second half of Lemma A.1. For $\lambda \in R_{A_\tau^{(1)}}$ and $g \in X$, we consider the resolvent equation

$$\lambda u - A_\tau^{(1)} u - A_\tau^{(2)} u = g.$$

According to [15, Section 2.4] again, we get

$$|\lambda| \|u\| \le \left[2 \left(M(A_\tau^{(1)}) \frac{|\lambda| + 1}{|\lambda - \omega(A_\tau^{(1)})|} + 1 \right) \left(1 + \|A_\tau^{(2)}\|_{L(D(A_\tau^{(1)}), \; X_\alpha)} \right) + 1 \right] \|g\|$$

Using this inequality and (A.5), we can verify the second half of Lemma A.1. Thus we complete the proof of Lemma A.1. □

We first prove the inequality (3.14). Let v satisfy (3.10). In order to reduce the inhomogeneous problem (3.10) to a homogeneous problem at the boundaries, we introduce an auxiliary function ψ defined by

$$\psi(t,\eta) := \left(\eta \bar{\xi}(t) \tan\theta + \frac{\eta^3}{3!} \cdot \frac{3 \tan\theta}{1 + \tan^2\theta} \cdot \frac{\bar{v}_{\eta\eta}^2(t,0)}{\bar{\xi}(t)} \right) h(\eta)$$

where $h \in C^\infty[0,1]$ satisfying $h'(\eta) < 0$ for $\eta \in (1/4, 3/4)$, $h(\eta) \equiv 1$ for $\eta \in [0, 1/4]$, $h(\eta) \equiv 0$ for $\eta \in [3/4, 1]$. Then it is seen that $v - \psi$ is homogeneous at the boundaries. From this fact and Lemma A.1.(i), we can represent $v - \psi$ as the variation of constants formula by means of the analytic semigroup e^{tA_0}. After a simple computation, this formula finally takes the form, for $0 \le t \le T$,

$$v(t,\cdot) = v^{(1)}(t,\cdot) + v^{(2)}(t,\cdot) + v^{(3)}(t,\cdot)$$

where

$$\begin{aligned} v^{(1)}(t,\cdot) &= e^{tA_0}(v_0 - \psi(0,\cdot)), \\ v^{(2)}(t,\cdot) &= \int_0^t e^{(t-\sigma)A_0}[F_0(\sigma,\cdot) + \mathcal{A}_0\psi(\sigma,\cdot)]d\sigma, \\ v^{(3)}(t,\cdot) &= -A_0 \int_0^t e^{(t-\sigma)A_0}[\psi(\sigma,\cdot) - \psi(0,\cdot)]d\sigma + \psi(0,\cdot). \end{aligned}$$

Applying the theory of analytic semigroups (see [15]), we get

$$\begin{aligned} \|v^{(1)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} &\le C_0 \ \|v_0 - \psi(0,\cdot)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)} \\ \|v^{(2)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} &\le C_{0,\alpha} \sup_{0<\delta<T} \delta^{1/2} \sup_{t\in[\delta,T]} \|F_0(t,\cdot) + \mathcal{A}_0\psi(t,\cdot)\|_{D_{A_0}(\frac{\alpha}{4},\infty)}. \end{aligned}$$

In particular, we can verify that a constant C_0 increases with $1/\xi_0$, $\|v_0\|_{C^{2+\alpha}[0,1]}$. Hereafter any constant depending on these quantities will be denoted by C_0.

In order to obtain the estimate of $\|v^{(3)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})}$, we set

$$z(t) = \int_0^t e^{(t-\sigma)A_0}[\psi(\sigma,\cdot) - \psi(0,\cdot)]d\sigma. \tag{A.6}$$

Then z satisfies

$$v^{(3)}(t,\cdot) = -A_0 z(t) + \psi(0,\cdot) = -\frac{d}{dt}z(t) + \psi(t,\cdot). \tag{A.7}$$

For the function z, we have the following estimates.

Lemma A.2 *Let z be a function represented by (A.6). Then, there exist constants $\gamma \in (\alpha, 1)$ and $N = N(\xi_0, \|v_0\|_{C^{2+\alpha}[0,1]}, \alpha, \gamma, K, \theta)$ such that*

$$\begin{aligned} &(i) \quad \|\frac{d}{dt}z(t)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)} \le NT^{(\gamma-\alpha)/4}, \\ &(ii) \quad \sup_{0<\delta<T} \delta^{1/4} \sup_{t\in[\delta,T]} \|\frac{d}{dt}z(t)\|_{D_{A_0}(\frac{3+\alpha}{4},\infty)} \le NT^{(\gamma-\alpha)/4}, \\ &(iii) \quad \sup_{0<\delta<T} \delta^{1/2} \sup_{t\in[\delta,T]} \|A_0\frac{d}{dt}z(t)\|_{D_{A_0}(\frac{\alpha}{4},\infty)} \le NT^{(\gamma-\alpha)/4}, \\ &(iv) \quad \|A_0 z\|_{C^{1/2}([0,T];D_{A_0}(\frac{\alpha}{4},\infty))} \le NT^{(\gamma-\alpha)/4}. \end{aligned}$$

Proof. We shall prove the inequality (i). By means of simple calculation, we see

$$\begin{aligned} \frac{d}{dt}z(t) &= \int_0^t A_0 e^{(t-\sigma)A_0}[\psi(\sigma,\cdot) - \psi(t,\cdot)]d\sigma + e^{tA_0}[\psi(t,\cdot) - \psi(0,\cdot)] \\ &=: J_1(t) + J_2(t). \end{aligned}$$

Moreover, for $\sigma, t \in [0,T]$, we get

$$\begin{aligned}\psi(\sigma,\eta)-\psi(t,\eta) &= \eta h(\eta)(\bar{\xi}(\sigma)-\bar{\xi}(t))\tan\theta \\ &\quad+\frac{3\tan\theta}{(1+\tan^2\theta)\cdot 3!}\cdot\eta^3 h(\eta)\left(\frac{\bar{v}^2_{\eta\eta}(\sigma,0)}{\bar{\xi}(\sigma)}-\frac{\bar{v}^2_{\eta\eta}(t,0)}{\bar{\xi}(t)}\right).\end{aligned}$$

First, we shall consider the estimate of $\|J_1(t)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)}$. Since $\eta \mapsto \eta h(\eta)$ is in $D_{A_0}(\frac{\gamma}{4},\infty)$ for any $\gamma \in (0,1)$, we choose γ in the interval $(\alpha, 1)$. Then, using the inequalities (A.2) and $|\bar{\xi}(t)-\bar{\xi}(\sigma)| \le 2K(t-\sigma)^{1/2}$ $(\sigma, t \in [0,T],\ \sigma \neq t)$, we obtain

$$\begin{aligned}&\int_0^t \|A_0 e^{(t-\sigma)A_0}[\eta h(\cdot)(\bar{\xi}(\sigma)-\bar{\xi}(t))]\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)}d\sigma \\ &\le C_0\int_0^t (t-\sigma)^{-((2+\alpha)/4-\gamma/4)-1}\|\eta h\|_{D_{A_0}(\frac{\gamma}{4},\infty)}|\bar{\xi}(\sigma)-\bar{\xi}(t)|d\sigma \\ &\le C_{0,K}\int_0^t (t-\sigma)^{(\gamma-\alpha)/4-1}d\sigma \\ &\le C_{0,K,\alpha,\gamma}\ T^{(\gamma-\alpha)/4}\end{aligned}$$

On the other hand, since $\eta \mapsto \eta^3 h(\eta)$ is in $D_{A_0}(\frac{2+\gamma}{4},\infty)$ for any $\gamma \in (0,1)$, we are led to

$$\begin{aligned}&\int_0^t \|A_0 e^{(t-\sigma)A_0}[\eta^3 h(\cdot)\left(\frac{\bar{v}^2_{\eta\eta}(\sigma,0)}{\bar{\xi}(\sigma)}-\frac{\bar{v}^2_{\eta\eta}(t,0)}{\bar{\xi}(t)}\right)]\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)}d\sigma \\ &\le C_0\int_0^t (t-\sigma)^{-((2+\alpha)/4-(2+\gamma)/4)-1}\|\eta^3 h\|_{D_{A_0}(\frac{2+\gamma}{4},\infty)}\left|\frac{\bar{v}^2_{\eta\eta}(\sigma,0)}{\bar{\xi}(\sigma)}-\frac{\bar{v}^2_{\eta\eta}(t,0)}{\bar{\xi}(t)}\right|d\sigma \\ &\le C_{0,K}\int_0^t (t-\sigma)^{(\gamma-\alpha)/4-1}(|\bar{v}_{\eta\eta}(\sigma,0)-\bar{v}_{\eta\eta}(t,0)|+|\bar{\xi}(\sigma)-\bar{\xi}(t)|)d\sigma \\ &\le C_{0,K}\int_0^t (t-\sigma)^{\gamma/4-1}d\sigma\cdot([\bar{v}_{\eta\eta}(\cdot,0)]_{C^{\alpha/4}[0,T]}+2KT^{1/2-\alpha/4}) \\ &\le C_{0,K,\gamma}\ T^{\gamma/4}\end{aligned}$$

Thus we obtain

$$\|J_1(t)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)} \le C_{0,K,\alpha,\gamma,\theta}\ T^{(\gamma-\alpha)/4}$$

In the same way we also get the estimate of $\|J_2(t)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)}$. Hence we obtain the inequality (i).

We can also prove the inequalities (ii), (iii), (iv) in a similar way as above by using technique in [15, Chapter 4] and leave the details of their proof to the interested reader. □

Hence, by means of these estimates, (A.3) and (A.7), we obtain

$$\|v^{(3)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} \le \|\psi(0,\cdot)\|_{C^{2+\alpha}[0,1]} + C_{eq,0,\alpha,\gamma,K,\theta}\ T^{(\gamma-\alpha)/4}.$$

Consequently, we are led to

$$\begin{aligned}\|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} &\le \|v^{(1)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})}+\|v^{(2)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})}+\|v^{(3)}\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} \\ &\le C_0\|v_0-\psi(0,\cdot)\|_{D_{A_0}(\frac{2+\alpha}{4},\infty)} \\ &\quad + C_{0,\alpha}\sup_{0<\delta<T}\delta^{1/2}\sup_{t\in[\delta,T]}\|F_0(t,\cdot)+\mathcal{A}_0\psi(t,\cdot)\|_{D_{A_0}(\frac{\alpha}{4},\infty)} \\ &\quad + \|\psi(0,\cdot)\|_{C^{2+\alpha}[0,1]}+C_{eq,0,\alpha,\gamma,K,\theta}\ T^{(\gamma-\alpha)/4}.\end{aligned}$$

In addition, since $D_{A_0}(\frac{2+\alpha}{4},\infty) = C^{2+\alpha}_{b_1}[0,1]$ and $D_{A_0}(\frac{\alpha}{4},\infty) = C^{\alpha}[0,1]$ with equivalence of norms, we get

$$\begin{aligned} \|v\|_{\mathcal{Y}^{4+\alpha}(\overline{R_{0,T}})} &\le C_{eq,0}\|v_0 - \psi(0,\cdot)\|_{C^{2+\alpha}[0,1]} \\ &\quad + C_{eq,0,\alpha} \sup_{0<\delta<T} \delta^{1/2} \sup_{t\in[\delta,T]} \|F_0(t,\cdot) + \mathcal{A}_0\psi(t,\cdot)\|_{C^{\alpha}[0,1]} \\ &\quad + \|\psi(0,\cdot)\|_{C^{2+\alpha}[0,1]} + C_{eq,0,\alpha,\gamma,K,\theta}\, T^{(\gamma-\alpha)/4} \\ &\le M_0 + C_{eq,0,\alpha,K,\theta}\, T^{\alpha/4} + C_{eq,0,\alpha,\gamma,K,\theta}\, T^{(\gamma-\alpha)/4} \end{aligned}$$

where M_0 is a constant depending on ξ_0, $\|v_0\|_{C^{2+\alpha}[0,1]}$, θ. In particular, we note that M_0 increases with $1/\xi_0$, $\|v_0\|_{C^{2+\alpha}[0,1]}$. This completes the proof of the inequality (3.14).

Next, we prove the inequality (3.20). Let V satisfy (3.18). In order to obtain a homogeneous problem at the boundaries, we introduce a function Ψ defined by

$$\Psi(t,\eta) = \left[\eta\bar{\Xi}(t)\tan\theta + \frac{\eta^3}{3!}\cdot\frac{3\tan\theta}{1+\tan^2\theta}\cdot\left(b_1(t)\bar{V}_{\eta\eta}(t,0) + b_2(t)\bar{\Xi}(t)\right)\right] h(\eta).$$

Then, since A_0 is sectorial, we can represent V as

$$V(t,\cdot) = \int_0^t e^{(t-\sigma)A_0}[\tilde{F}(\sigma,\cdot) + \mathcal{A}_0\psi(\sigma,\cdot)]d\sigma - A_0\int_0^t e^{(t-\sigma)A_0}[\Psi(\sigma,\cdot) - \Psi(0,\cdot)]d\sigma.$$

In the same way as we have proved the inequality (3.14), we can also prove the inequality (3.20). So we omit the details.

References

[1] P. Acquistapace and B. Terreni, Hölder classes with boundary conditions as interporation spaces, Math. Z., **195** (1987), 451–471.

[2] L. Bronsard and F. Reitich, On three-phase boundary motion and the singular limit of a vector- valued Ginzburg-Landau equation, Arch. Rat. Mech. Anal., **124** (1993), 355–379.

[3] X. Chen, The Hele-Shaw problem and area-preserving curve-shortening motions, Archive Rational Mech. Anal., **123** (1993), 117–151.

[4] C. M. Elliott and S. Maeier-Paape, Losing a graph with surface diffusion, preprint.

[5] C. M. Elliott and H. Garcke, Existence results for diffusive surface motion laws, Advances in Mathematical Sciences and Applications, **7** (1997), 467–490.

[6] J. Escher, U. F. Mayer, and G. Simonett, The surface diffusion flow for immersed hypersurfraces, SIAM J. Math. Anal., **29** (1998), 1419–1433.

[7] J. Escher, U. F. Mayer, and G. Simonett, On the surface diffusion flow, NSE-6 Conference, Amann et al. Eds., 1998.

[8] J. Escher and G. Simonett, Moving surfaces and abstract parabolic evolution equations, Topics in Nonlinear Analysis. The Hebert Amann Anniversary Volume, J. Escher and G. Simonett, Eds., PNLDE **35**, Birkhäuser (1999), 183–212.

[9] J. Escher and G. Simonett, The volume preserving mean curvature flow near spheres, Proc. Amer. Math. Soc., **126** (1998), 2789–2796.

[10] V. A. Galaktionov, J. Hulshof, and J. L. Vazquez, Extinction and focusing behaviour of spherical and annular flames described by a free boundary problem, J. Math. Pures Appl., **76** (1997), 563–608.

[11] H. Garcke and A. Novick-Cohen, A singular limit for a system of degenerate Cahn-Hilliard equations, to appear in Adv. Diff. Equations.

[12] Y. Giga and K. Ito, On pinching of curves moved by surface diffusion, Comm. Appl. Anal., **2** (1998), 393–405.

[13] M. E. Gurtin, Thermodynamics of evolving phase boundaries in the plane, Clarendon Press, Oxford (1993).

[14] D. Hilhorst and J. Hulshof, A free boundary focusing problem, Proc. Amer. Math. Soc., **121** (1994), 1193–1202.

[15] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, 1995.

[16] A. Lunardi, E. Sinestrari, and W. Von Wahl, A semigroup approach to the time dependent parabolic initial-boundary value problem, Diff. Int. Eqns. **5** (1992), 1275–1306.

[17] U. F. Mayer and G. Simonett, Self-intersections for the surface diffusion and the volume-preserving mean curvature flow, submitted.

[18] V. A. Solonnikov, Boundary value problems in physics, Proceedings of the Steklov Institute of Mathematics, Vol. LXXXIII. (1965).

[19] H. B. Stewart, Generation of analytic semigroups by strongly elliptic operators under general boundary conditions, Trans. Amer. Math. Soc., **259** (1980), 299–310.

[20] P. Sternberg and W. P. Ziemer, Local minimisers of a three-phase partition problem with triple junctions, Proc. Roy. Soc. Edinburgh., **124A** (1994), 1059–1073.