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1. Introduction

Motion of curves driven by anisotropic singular interfacial energy is an important topic in the theory of evolving phase-boundaries. There are nice review articles by M. Gurtin [Gu] and J. Taylor, J. Cahn and C. Handwerker [TCH] for the background of this problem. The governing equation of the motion is considered as a generalization of the curve shortening equation $V = \kappa$ which is well studied (see e.g. [GH], [Gr]). Here V denotes the normal velocity of the closed curve Γ_t depending on time t in the direction of the outward unit normal $\mathbf{n}$ and κ denotes the curvature $-\mathrm{div}\ \mathbf{n}$. For $V = \kappa$ if initial curve Γ_0 is closed and embedded in the plane $\mathbf{R}^2$, the solution $\Gamma_t (t > 0)$ stays smooth and becomes convex in a finite time [Gr]. After that it stays convex and shrinks to a point [GH] in a finite time. No singularities, no self-intersection appear during evolutions. However, if we consider $V = \kappa + 1$, Γ_t may overlap. We need a new notion of solutions to define evolution of Γ_t after pinching as embedded one. For this purpose a level set method has been proposed by S. Osher and A. Sethian [OSe] for a numerical calculation. Based on the theory of viscosity solutions [CIL] its analytic foun-

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dation is established by [CGG1] and [ES1]. It tracks whole evolution even if Γ_t develops singularities (up to fattening). It applies to motion of a hypersurface in higher dimensional spaces. The theory is also adjusted for a surface of arbitrary codimension moved by its mean curvature vector [AS]. For further development of the theory the reader is referred to [G1], [G2] for its analytic aspects and to a book of A. Sethian [Se] for its numerical aspects. Although the level set method still applies to motion by anisotropic interfacial energy if the interfacial energy is enough regular (and convex) [CGG1], [GGo], so far it was not applicable when interfacial energy is not C^1 as pointed out by [TCH].

In this paper we extend the level set method for motion of curves in the plane by singular energy including crystalline energy. Such an extension is not trivial at all since the quantity corresponding to (weighted) curvature is nonlocal. We consider an interface equation of a closed embedded curve Γ_t in the plane $\mathbf{R}^2$ of form

$$V = g(\mathbf{n},\ \Lambda_\gamma(\mathbf{n})) \quad \text{on} \quad \Gamma_t \tag{I}$$

for $t > 0$. The quantity Λ_γ is formally defined by

$$\Lambda_\gamma(\mathbf{n}) = -\mathrm{div}\ \xi(\mathbf{n}), \tag{1.1}$$

where $\xi = \nabla\gamma$ and γ is a given convex, positively homogeneous function of degree 1 in $\mathbf{R}^2$; div denotes the divergence on the curve Γ_t. The quantity γ is called the *interfacial energy* (*density*) and Λ_γ is called an (energetically) *weighted curvature*; ξ is called the *Cahn-Hoffman vector*. (The quantity Λ_γ is invariant under addition of linear functions so we may always assume $\gamma \geq 0$.) We shall assume that g in (I) is continuous on $S^1 \times \mathbf{R}$ and nondecreasing in the second variable, i.e.,

$$g(p, \lambda_1) \leq g(p, \lambda_2) \quad \text{for} \quad \lambda_1 \leq \lambda_2, \quad p \in S^1, \tag{1.2}$$

where $S^1 = \{p \in \mathbf{R}^2,\ |p| = 1\}$. Since $\Lambda_\gamma(\mathbf{n}) = (\tilde{\gamma}'' + \tilde{\gamma})(\theta)\kappa$ for smooth γ with $\tilde{\gamma}(\theta) = \gamma(\cos\theta,\ \sin\theta)$, $\mathbf{n} = (\cos\theta,\ \sin\theta)$ and since the convexity of γ is equivalent to $\tilde{\gamma}'' + \tilde{\gamma} \geq 0$, the problem (I) is at least degenerate parabolic if (1.2) holds and γ is convex. A typical example of the interface equation is

$$V = M(\mathbf{n})(\Lambda_\gamma(\mathbf{n}) + C), \tag{1.3}$$

where M is a positive function (called *mobility*) on S^1 and C is a given constant. If $\gamma(p) = |p|$, $M(\mathbf{n}) \equiv 1$ and $C = 0$, then (1.3) is the curve shortening equation. If $g(p, \lambda) = |\lambda|^{\alpha-1}\lambda$, $\gamma(p) = |p|$, $\alpha > 0$ then (I) is

$$V = |\kappa|^{\alpha-1}\kappa. \tag{1.4}$$

If $\alpha = 1/3$, the equation (1.4) is the affine curvature flow equation (see e.g. [AGLM], [ST], [And], [AST]). It is also important to study the case when γ is not necessarily C^1. A typical example of such γ is piecewise linear convex γ. Such γ is called a *crystalline energy* as proposed by [T1] and [AG]. In this case (1.3) is no longer a PDE since the weighted curvature is nonlocal. We have given a general notion of solutions for (I) in [GG1], [GG3], [GG5] when Γ_t is graph-like, i.e., the graph of a function. In the present paper we would like to adjust their method so that we can handle closed curves. In [GG3], [GG5] we also give backgrounds of the problem so we do not repeat them here.

To construct a global generalized solution $\{\Gamma_t\}_{t\geq 0}$ to (I) with initial data Γ_0, for the moment we assume that $\tilde{\gamma}$ is C^2 and recall as in [CGG1], [GGo] the level set equation corresponding to (I):

$$u_t - |\nabla u|\ g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) = 0 \tag{L}$$

in $\mathbf{R}^2 \times (0, \infty)$ with $\Lambda_\gamma(u) = -\mathrm{div}(\xi(-\nabla u/|\nabla u|))$. Here $u = u(x,t)$ is an auxiliary function defined on $\mathbf{R}^2 \times (0, \infty)$ and $\nabla u = (\partial u/\partial x_1,\ \partial u/\partial x_2) = (\partial_{x_1} u, \partial_{x_2} u)$,

$u_t = \partial u/\partial t = \partial_t u$. Here we use the convention of the orientation so that $\mathbf{n} = -\nabla u/|\nabla u|$. The feature of this equation is that each level set of u moves by (I) at least formally so that the parabolicity of (L) is degenerate in the direction of ∇u. Moreover, (L) has a singularity at least at $\nabla u = 0$. Because of these features the initial-value problem for (L) is not solvable in the class of smooth functions even for smooth initial data so weaker notion of solutions is necessary to solve the problem. In [CGG1] the notion of viscosity solutions [CIL] is adjusted so that it applies to (L). For recent development of the theory of viscosity solutions there are nice review articles published in [BCESS]. We often suppress the word "viscosity". We list here typical results immediately follows from [CGG1] (see also [GGo]).

Classical Results I. *Assume that $\tilde{\gamma}$ is C^2 and that γ is convex. Assume that g is continuous on $S^1 \times \mathbf{R}$, i.e., $g \in C(S^1 \times \mathbf{R})$ and satisfying (1.2) with*

$$(1.5) \qquad \overline{\lim}_{|\lambda|\to\infty} \sup_{p\in S^1} |g(p,\lambda)/\lambda| < \infty.$$

Then following principles are valid.

(i) Comparison. *For a given $T \in (0,\infty)$ assume that u and v are sub- and supersolutions of (L) in $\mathbf{R}^2 \times (0,T)$, respectively. Assume that $u_0 \leq v_0$ with $u_0(x) = u^*(x,0)$, $v_0(x) = v_*(x,0)$, $x \in \mathbf{R}^2$ and that u and v are constant outside $B_R \times [0,T)$ for some $R > 0$, where B_R denotes the closed ball of radius R centered at the origin of $\mathbf{R}^2$. Then $u^* \leq v_*$ in $\mathbf{R}^2 \times (0,T)$, where u^* denotes the upper semicontinuous envelope and $v_* = -(-v)^*$.*

(ii) Invariance. *For a given $T \in (0,\infty]$ let u be a subsolution (supersolution) of (L) in $\mathbf{R}^2 \times (0,T)$. Then so is the composite function $\theta \circ u$ provided that $\theta : \mathbf{R} \to \mathbf{R}$ is continuous and nondecreasing.*

(iii) Uniqueness (of level set). *For a given bounded open set D_0 (resp. closed set E_0) in $\mathbf{R}^2$ let $u \in \mathcal{K}_\alpha$ $(\mathbf{R}^2 \times [0,T))$ be a solution of (L) such that*

$$D_0 = \{x \in \mathbf{R}^2;\ u(x,0) > \ell\} \quad (E_0 = \{x \in \mathbf{R}^2;\ u(x,0) \geq \ell\})$$

for some ℓ and $\alpha \in \mathbf{R}$, where $T \in (0,\infty)$. Then the set

$$D = \{(x,t) \in \mathbf{R}^2 \times [0,T);\ u(x,t) > \ell\} (= \{u > \ell\})$$
$$(E = \{(x,t) \in \mathbf{R}^2 \times [0,T);\ u(x,t) \geq \ell\} (= \{u \geq \ell\}))$$

is completely determined by D_0 (resp. E_0) and is independent of the choice of u, α, ℓ. Here $\mathcal{K}_\alpha(\mathbf{R}^2 \times [0,T))$ denotes the space of all continuous function v on $\mathbf{R}^2 \times [0,T)$ such that the support spt$(v-\alpha)$ is contained in some $B_R \times [0,T)$.

(iv) Existence. *For a given $u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ there is a unique solution $u \in \mathcal{K}_\alpha(\mathbf{R}^2 \times [0,T))$ (for any $T > 0$) of (L) in $\mathbf{R}^2 \times (0,T)$ with $u|_{t=0} = u_0$. Here $\mathcal{K}_\alpha(\mathbf{R}^2)$ denotes the space of all continuous functions v on $\mathbf{R}^2$ satisfying spt$(v-\alpha) \subset B_R$ for some $R > 0$.*

This existence and uniqueness of the level set gives a notion of an open (resp. closed) *generalized solution* D (resp. E) of (I) starting from D_0 (resp. E_0), where D_0 (resp. E_0) is an open (a closed) set surrounded by Γ_0. Indeed for given D_0 (resp. E_0) there always exists $u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ such that $D_0 = \{u_0 > \ell\}$, $E_0 = \{u_0 \geq \ell\}$ for some ℓ and $\alpha \in \mathbf{R}$. The set $E \setminus D$ is considered a generalized interface evolution starting from $\Gamma_0 = E_0 \setminus D_0$. In general $\overline{D} \neq E$ even if Γ_0 has no interior [ES1], [G1]. This phenomenon is called fattening. G. Bellettini and M. Paolini [BP] proved that the fattening occurs even if Γ_0 is smooth for $V = k + 1$.

The above classical results have been extended in various settings (see [G1]) and applied to many problems ([BCESS]). For example, C^2 regularity of $\tilde{\gamma}$ is somewhat weakened. In [OhS] and [GSS], $\tilde{\gamma}$ is assumed to be C^1 but the second derivative $\tilde{\gamma}''$ may jump in a finite set of $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ although $\tilde{\gamma}'$ is Lipschitz continuous on $\mathbf{R}/2\pi\mathbf{Z}$. In [IS] and [Go] by modifying the definition of viscosity solutions the assumption (1.5) is removed. Also replacing $\mathcal{K}_\alpha$ by the space of bounded uniformly continuous functions allows to handle unbounded curves as in [IS].

Our goal in this paper is to extend these classical results when $\tilde{\gamma}$ is not necessarily C^1 by introducing a suitable notion of solutions. Indeed, we shall give a notion of solutions to (L) and prove the following Theorem.

Theorem 1.1. *Assume that $\tilde{\gamma}$ is C^2 except a finite set S of $\mathbf{R}/2\pi\mathbf{Z}$ and that $\tilde{\gamma}''$ is bounded on $(\mathbf{R}/2\pi\mathbf{Z}) \setminus S$. Assume that γ is convex and that $\tilde{\gamma} > 0$ on $\mathbf{R}/2\pi\mathbf{Z}$. If $g \in C(S^1 \times \mathbf{R})$ satisfies (1.2), (1.5), then Comparison, Invariance, Uniqueness and Existence principles are valid.*

The assumption (1.5) can be removed by modifying the definition of solution as in [IS]; however, we do not discuss this problem in this paper. The assumption $\tilde{\gamma} > 0$ is imposed to simplify the argument. It is possible to remove this assumption; however, we do not pursue such an extension in the present paper.

It is also important to study the convergence and stability properties of solutions. We first recall classical results.

Classical Results II. *For $\varepsilon \geq 0$ assume that γ_ε is positively homogeneous of degree one in $\mathbf{R}^2$ and convex. Assume that $\tilde{\gamma}_\varepsilon(\theta) = \gamma_\varepsilon(\cos\theta, \sin\theta)$ is C^2. Assume that $g_\varepsilon \in C(S^1 \times \mathbf{R})$ satisfies (1.2) for $\varepsilon \geq 0$. Assume that $\partial_\theta^k \tilde{\gamma}_\varepsilon \to \partial_\theta^k \tilde{\gamma}_0$ uniformly on $\mathbf{R}/2\pi\mathbf{Z}$ for $0 \leq k \leq 2$ as $\varepsilon \to 0$, where $\partial_\theta = \partial/\partial\theta$. Assume that $g_\varepsilon \to g_0$ locally uniformly on $S^1 \times \mathbf{R}$. Assume that*

$$\overline{\lim}_{|\lambda|\to\infty} \sup_{0<\varepsilon<1} \sup_{p\in S^1} |g_\varepsilon(p,\lambda)/\lambda| < \infty. \tag{1.6}$$

Then following properties hold.

(i) Stability. *For $T \in (0,\infty]$ let $u^\varepsilon (\varepsilon > 0)$ be a subsolution of*

$$(\mathrm{L}_\varepsilon) \qquad u_t - |\nabla u| g_\varepsilon(-\nabla u/|\nabla u|,\ \Lambda_{\gamma_\varepsilon}(u)) = 0 \quad in \ \ \Omega \times (0,T),$$

where Ω is an open set in $\mathbf{R}^2$. Then the relaxed upper limit $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$ is a subsolution of (L) with $\gamma = \gamma_0$, $g = g_0$ provided that $\overline{u} < \infty$ on $\overline{\Omega} \times [0,T)$.

(ii) Convergence. *For $T \in (0,\infty)$ let $u^\varepsilon \in \mathcal{K}_{\alpha_\varepsilon}(\mathbf{R}^2 \times [0,T))$ be the solution of (L_ε) with initial data $u_0^\varepsilon \in \mathcal{K}_{\alpha_\varepsilon}(\mathbf{R}^2)$ for $\varepsilon > 0$. Assume that $u_0^\varepsilon \to u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ uniformly as $\varepsilon \to 0$ and spt $(u_0^\varepsilon - \alpha_\varepsilon) \subset B_R$ with R independent of $\varepsilon > 0$. Then $u = \lim_{\varepsilon\to 0} u^\varepsilon$ exists and the convergence is uniform in $\mathbf{R}^2 \times [0,T']$ for any $T' < T$. Moreover, $u \in \mathcal{K}_\alpha(\mathbf{R}^2 \times [0,T))$ is a solution of (L) with the initial data u_0.*

The stability principle is already contained in [CGG1]. Such a version of the stability principle is originally due to G. Barles, B. Perthame and H. Ishii (cf. e.g. [Ba]). The similar statement holds for supersolutions by replacing $\limsup^*$ by $\liminf_*$ with obvious modification. (For the definition of $\limsup^*$ see §7.)

The convergence property is found in [Ca] where u_0^ε is assumed to be independent of ε. To prove the convergence property we construct a barrier near $t = 0$ which prevents the initial layer for the sequence u_ε and apply the stability and the comparison principle. We shall explain the detail in §8 when we prove our version of convergence results stated below. This method, which is by now standard [Ba], does not require any estimates on derivatives of u^ε.

Theorem 1.2. *Assume that $\tilde{\gamma}_\varepsilon$ is C^2 except a finite set S_ε of $\mathbf{R}/2\pi\mathbf{Z}$ and that $\tilde{\gamma}_\varepsilon''$ is bounded on $\mathbf{R}/2\pi\mathbf{Z} \setminus S_\varepsilon$ for $\varepsilon \geq 0$. Assume that γ_ε is convex for $\varepsilon \geq 0$. Assume that $g_\varepsilon \in C(S^1 \times \mathbf{R})$ satisfies (1.2) for $\varepsilon \geq 0$ and (1.6). If $\tilde{\gamma}_\varepsilon \to \tilde{\gamma}_0 > 0$ is uniform in $\mathbf{R}/2\pi\mathbf{Z}$ and $g_\varepsilon \to g_0$ is locally uniform on $S^1 \times \mathbf{R}$ as $\varepsilon \to 0$, then Stability and Convergence principles are valid.*

Note that our assumption on the convergence of γ_ε does not require the uniform convergence of $\tilde{\gamma}_\varepsilon''$. Also the singular set S_ε is allowed to depend on ε.

Remark 1.3. All results in Theorem 1.1 and Theorem 1.2 are still valid even if we impose spatially periodic boundary condition. To state such results we have to replace $\mathbf{R}^2$ by $\mathbf{T}^2 = (\mathbf{R}/\omega_1\mathbf{Z}) \times (\mathbf{R}/\omega_2\mathbf{Z})$ when it is related to domain of definition of solutions. The proof is essentially the same so we do not write it explicitly.

Since our singular $\tilde{\gamma}_0$ can be always approximated by a smooth $\tilde{\gamma}_\varepsilon$ keeping the convexity of γ_ε, our theorem says that the solution of the initial value problem of (L) with singular γ can be obtained as limit of the solution of the approximate problem (L_ε). This justifies our notion of solutions which we shall give in §2.

If γ is crystalline, S. Angenent and M. E. Gurtin [AG] and J. Taylor [T1] (see also [T2], [T3]) proposed to restrict a class of piecewise linear curves Γ_t called admissible and derive a system of ODEs corresponding to (1.3). Such a solution is called an admissible crystalline flow for (1.3). For general γ it is proposed to approximate a crystalline energy γ_ε and use the ODE system as numerical algorithm to approximate original problem. For example, for the curve shortening equation $V = \kappa$ the convergence of the crystalline algorithm has been proved by [FG] and [GirK1] when Γ_t is graph-like. In fact, P. Girão and R. Kohn [GirK1] (see also [GirK2]) derived rate of convergence among other results. Later their results has been extended to closed convex curves by [Gir]. The idea has been extended to (1.4) by [UY]. General convergence results in [GG5] yield the convergence of crystalline algorithm for very general equations when Γ_t is graph-like by extending the notion of viscosity solutions. In [GG7] the convergence of derivatives is also discussed as well as crystalline algorithm for a general interface equation (I). When Γ_t is a closed curve the convergence of the crystalline algorithm has been established [ISo] for $V = \kappa$ by adapting stability property of solutions. Theorem

1.2 yields the convergence of level set (or a generalized solution)

$$E^\varepsilon := \{u^\varepsilon \geq \ell\} \to E := \{u \geq \ell\} \quad \text{in } \mathbf{R}^2 \times [0, T'],\ T' < T$$

in the Hausdorff distance topology provided that there is no fattening for E i.e., $E = \overline{D}$ where $D = \{u > \ell\}$. In particular, the extinction time of E^ε converges to that of E. Theorem 1.2 also yields the convergence of the cross-section $E^\varepsilon(t) \to E(t)$ uniformly in $t \in [0, T']$ if $E^\varepsilon(0) \to E(0)$ (in the Hausdorff distance topology) provided that $E(t)$ is continuous in t and $E(t) = \overline{D(t)}$ in $\mathbf{R}^2$, where $D(t) = \{x \in \mathbf{R}^2; (x,t) \in D\}$. Since it is not difficult to prove that an admissible crystalline flow is regarded as our level set flow (cf. [GG2]), our convergence results for E^ε is useful to prove the convergence of crystalline algorithm for various problems which we shall discuss in our forthcoming paper [GG8].

It is an important problem whether an admissible crystalline flow for (1.3) with crystalline energy γ is a reasonable interpretation of solution not only as approximation. The work of [FG] and [EGS] proved that an admissible crystalline flow is a solution of variational inequality corresponding to (1.3) when $C = 0$ and Γ_0 is graph-like. For (1.3) with $C = 0$ (zero driving term) F. Almgren and J. Taylor [AT] investigated a semi-discretized implicit scheme introduced by [ATW]. This approximate solution converges as time grid tends to zero [ATW] by taking a subsequence and the limit weak solution is called a flat curvature flow. In [AT] weakly admissible crystalline flow (called weakly admissible crystal in [GGu], [GG5]) is the unique flat curvature flow with the same initial data provided that two adjacet facets do not vanish simultaneously when the interfacial energy is symmetric, i.e., $\gamma(p) = \gamma(-p)$. Our convergence theory also justify weakly admissible crystalline flow as a limit of smoother problem. In our theory the interfacial energy need not be symmetric.

In [GG1], [GG3], [GG5] we have introduced a new notion of viscosity type solutions for (I) when Γ_t is graph-like and establishes Comparison, Existence, Stability and Convergence principles for graph-like solutions. We extend the notion to level set solutions so that Invariance property holds. The main difficulty lies in the proof of Comparison principle. Since it is already two dimensional, compared with one dimensional it would be complicated if we attack the problem directly. To circumvent this inconvenience we reduce the problem to graph-like functions by establishing "a slicing lemma". In its simplest form it asserts that

$$u^{\#}(t,x_1) = \sup\{x_2;\ u(x_1,x_2,t) \geq \ell\}$$

is a subsolution of the graph-version of (I) provided that u is a subsolution of (L) and that $u^{\#} < \infty$. (Such a statement itself holds for higher dimensional problem when $\gamma \in C^2(\mathbf{R}^n \setminus \{0\})$.) By this lemma one can reduce the proof of the comparison principle to that of graph-like solutions although comparison principle for graph-like solutions does not apply directly. We note that even for graph-like solutions the standard maximum principle [CIL] for semicontinuous functions does not apply since the curvature is nonlocal as pointed out in [GG3]. However, necessary steps have been already prepared in [GG3].

The slicing lemma is also useful to prove the stability principle. It actually reduces the problem for the stability [GG5] of graph-like solutions. We also note that it is possible to prove the stability principle directly without appealing the slicing lemma (cf. [MHG]). Constructing barriers near $t = 0$, we can prove the convergence principle by the stability. Existence of solutions of (L) with singular interfacial energy follows from the convergence and the existence of solutions [CGG1] for (L_ε) with smooth $\tilde{\gamma}_\varepsilon$ by approximating $\gamma = \gamma_0$ by γ_ε and $g = g_0 = g_\varepsilon$.

The bibliography of [GG3] and [GG5] includes many references to recent work on the motion by nonlocal curvature. We take this opportunity to note some other, related article not cited there and not mentioned elsewhere in this introduction. Modified Stefan problem with crystalline interfacial energy is studied by P. Rybka [R1, 2, 3] for both quasi-static approximation and the original version. There the moving boundary is always assumed to be admissible crystalline flow. Since the driving force term C of (1.3) is not spatially homogeneous, facet may break as suggested in [GG4]. Thus, the above assumption may be sometimes too strong unless moving boundary is small so the nonlocal curvature is big. Also there is a work by A. Stancu [St1, 2] discussing the way of shrinking for (1.3) with $C = 0$ for admissible convex crystalline flow. The reader is referred to a review [G3] for the behaviour of solutions of (1.3) including when $\tilde{\gamma}$ is smooth and $\tilde{\gamma}'' + \tilde{\gamma} > 0$. Very recently, the result in [St2] has been extended by S. Yazaki [Ya1] for $V = M(\mathbf{n})|\Lambda_\gamma(\mathbf{n})|^{\alpha-1}\Lambda_\gamma(\mathbf{n})$, $\alpha > 0$, when γ is crystalline; see also [Ya2, 3] for related works. There are also three dimensional analog for crystalline flow [GGuM]. However, we realize that in this case flat portion may break as pointed out by G. Bellettini, M. Novaga and M. Paolini [BNP] and J. Yunger [Y]. Based on our convergence results, R. Schätzle and the first author [GS] proposed to consider an anisotropic Allen-Cahn equation with γ_ε approximating singular energy. Using our convergence results they prove the internal layer of solutions converges to generalized solution of (1.3) (up to fattening). This result extends the convergence result for crystalline energy in [BGN] in the sense that the convergence is global-in-time and that M may not be propotional to γ. We also note that explicit solutions for an anisotropic Allen-Cahn equations with crystalline γ has been proposed in [TC].

This paper is organized as follows. In §2 we introduce the notion of solution to (L). In §3 we establish invariance principle. In §4 we recall several properties of graph-like solutions in [GG3], [GG5] and related properties. In §5 we compare level set solutions with graph-like solutions. Especially we establish the slicing lemma. In §6 we prove the comparison principle. At this stage we complete the proof of Theorem 1 except (iv) in §6. In §7 we prove the stability principle by using the slicing lemma. In §8 we prove the convergence principle. As an application we prove the existence principle and the convergence of a level set of (L_ε).

Main results in this paper have been announced in [GG6].

2. Definition of solutions

We define solutions of the level set equation of the interface equation

$$V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n})). \tag{I}$$

2.1. Class of interfacial energy density. Let $\mathcal{I}$ denote the set of all γ satisfying

(i) $\gamma : \mathbf{R}^2 \to [0, \infty)$ is convex, positively homogeneous of degree one and $\gamma > 0$ on $S^1 = \{p \in \mathbf{R}^2;\ |p| = 1\}$;

(ii) γ is C^2 on $S^1 \setminus \mathcal{N}$ and all second derivatives of γ is bounded on $S^1 \setminus \mathcal{N}$ for some finite set $\mathcal{N}$ on S^1.

The set $\mathcal{N}$ is called the *singular set* of $\gamma \in \mathcal{I}$.

If we define the Frank diagram

$$\text{Frank } \gamma = \{p \in \mathbf{R}^2;\ \gamma(p) \le 1\},$$

for positively homogeneous function γ of degree one, the above conditions on γ is expressed in a geometric way. For example the convexity of γ is equivalent to that of Frank γ. The property $\gamma > 0$ on S^1 is equivalent to the boundedness of Frank γ. The condition (ii) is equivalent to say that ∂ (Frank γ) is C^2 except finitely many points called vertices and its curvature is bounded outside vertices.

2.2. Class of evolution law. Let $\mathcal{G}_0$ denote the set of all continuous functions $g : S^1 \times \mathbf{R} \to (-\infty, \infty)$ satisfying

(i) $\lambda \mapsto g(p,\lambda)$ is nondecreasing for all $p \in S^1$,

(ii) $\limsup_{|\lambda|\to\infty} \sup_{p\in S^1} |g(p,\lambda)/\lambda| < \infty$.

2.3. Class of test functions. Let U be an open set in $\mathbf{R}^2$. A C^2-curve c in U is *faceted* in U at $x_0 = (x_{01}, x_{02}) \in c$ if c is represented as the graph of a function in some neighborhood I of x_{01} or x_{02} which is faceted in I in the sense of [GG3]. In other words there exists a segment (nontrivial closed connected maximal interval) $\overline{z_0 z_1}$ on c containing x_0 with the properly that near $z_0 \in U$ (and $z_1 \in U$) the set $c \setminus \overline{z_0 z_1}$ is contained in one of open half planes (depending on z_0 and z_1) separated by the line containing $\overline{z_0 z_1}$. For $\gamma \in \mathcal{I}$ let $\mathcal{N}$ be the singular set of γ. We say that an oriented C^2-curve c in U (with normal vector field $\mathbf{n}$ called an orientation of c) is a C^2_γ-curve if c is faceted at each $x_0 \in c$ provided that $\mathbf{n}(x_0)$ belongs to $\mathcal{N}$. The segment containing $x_0 \in c$ is called a *facet* with orientation $\mathbf{n}(x_0)$. We say that $h \in C^2(U)$ belongs to $C^2_\gamma(U)$ if

(i) each ℓ-level set of h, i.e., $\{x \in U;\ h(x) = \ell\}$ is C^2_γ-curve in $U' = U \setminus \{x \in U;\ \nabla h(x) = 0\}$;

(ii) a facet with orientation $\mathbf{n} \in \mathcal{N}$ on the ℓ-level set of h in U' is continuous in ℓ in the topology of the Hausdorff distance.

For an interval J let $A_\gamma(U \times J)$ denote the space of a function φ on $U \times J$ of form

$$\varphi(x,t) = h(x) + k(t),\ \ k \in C^1(J),\ \ h \in C^2_\gamma(U).$$

2.4. Nonlocal weighted curvature. It is convenient to represent $\gamma(\mathbf{m})$ by the argument of $\mathbf{m}$. For $\mathbf{m} = (\cos\theta,\ \sin\theta)$ let $\tilde{\gamma}(\theta)$ denote $\gamma(\mathbf{m}(\theta))$. For $\mathbf{m} \in \mathcal{N}$ we set

$$\Delta(\mathbf{m}) = \tilde{\gamma}'(\theta+0) - \tilde{\gamma}'(\theta-0),$$

which is the jump of $\tilde{\gamma}'$ accross θ.

Let c be a C^2_γ-curve with orientation $\mathbf{n}$ in U. We define nonlocal curvature Λ_γ at $x_0 \in c$ by

$$\Lambda_\gamma(x_0;\ c) = (\tilde{\gamma}''(\theta) + \tilde{\gamma}(\theta))\kappa(x_0;\ c) \qquad \text{if } \mathbf{n}(x_0) = (\cos\theta,\ \sin\theta) \notin \mathcal{N},$$
$$\Lambda_\gamma(x_0;\ c) = \chi(x_0;\ c)\Delta(\mathbf{n}(x_0))/L(x_0) \quad \text{if } \mathbf{n}(x_0) \in \mathcal{N},$$

where $\kappa(x; c)$ is the curvature of c at x_0 in the direction of $\mathbf{n}(x_0)$. Here $L = L(x_0)$ is the length of the facet containing x_0 and $\chi = 1$ (resp. -1) if c is contained in a half space

$$H = \{p \in \mathbf{R}^2; (p - x_0) \cdot \mathbf{n}(x_0) \geq 0\} \quad (\text{resp. } \mathbf{R}^2 \setminus \operatorname{int} H)$$

is some neighborhood of the facet containing x_0; otherwise χ is assigned to be equal to zero. The number χ is called the *transition number*. The quantity $\Lambda_\gamma(x_0; c)$ is independent of the choice of U.

If $\gamma \equiv 1$, $\Lambda_\gamma(x_0; c) = -1$ when c is a unit circle with outward orientation. If Frank γ is smooth and its inward curvature is positive, then it is not difficult to see that $\Lambda_\gamma(x_0; c) = -1$ for the boundary curve c of the Wulff shape

$$\text{Wulff } \gamma = \bigcap_{\substack{|p|=1 \\ p \in \mathbf{R}^2}} \{x \in \mathbf{R}^2;\ x \cdot p \leq \gamma(p)\}.$$

In other words Wulff γ plays a role of disk for usual curvature. Our Λ_γ is defined so that $\Lambda_\gamma(x_0; c)$ for $c = \partial$ (Wulff γ) (with outward orientation) equals -1 independent of $x_0 \in c$ provided that c is C^2_γ.

For a function $h \in C^2_\gamma(U)$ and $x_0 \in U \setminus \{x \in U;\ \nabla h(x) = 0\}$, we set $\Lambda_\gamma(h)(x_0) = \Lambda_\gamma(x_0; c)$, where $\Lambda_\gamma(x_0; c)$ is the nonlocal curvature at x_0 of C^2_γ-curve $c : h(x) = h(x_0)$ with orientation $\mathbf{n}(x_0) = -\nabla h(x_0)/|\nabla h(x_0)|$.

2.5. A level set equation. An equation for a function u on $\mathbf{R}^2 \times (0, T)$ is called a level set equation of (I) if each level set of a solution u moves by (I) at least formally at the place where $\nabla u \neq 0$. It is uniquely determined and it is formally written as

$$\text{(L)} \qquad u_t - |\nabla u| g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) = 0$$

with the orienation $\mathbf{n} = -\nabla u/|\nabla u|$ of level set of u.

Definition 2.1(Solution of a level set equation). Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. Let Ω be an open set in $\mathbf{R}^2$. A real-valued function u on $Q = \Omega \times (0,T)$ is a *subsolution* of (L) in Q if the upper semicontinuous envelope $u^* < \infty$ in $\overline{\Omega} \times [0,T)$ and

$$\text{(i)} \qquad \varphi_t(\hat{x},\hat{t}) - |\nabla\varphi(\hat{x},\hat{t})| g(\mathbf{m}, \Lambda_\gamma(\varphi(\cdot,\hat{t}))(\hat{x})) \leq 0 \quad \text{when} \quad \nabla\varphi(\hat{x},\hat{t}) \neq 0$$

with $\mathbf{m} = -\nabla\varphi(\hat{x},\hat{t})/|\nabla\varphi(\hat{x},\hat{t})|$, and

$$\text{(ii)} \qquad \varphi_t(\hat{x},\hat{t}) \leq 0 \quad \text{when} \quad \nabla\varphi(\hat{x},\hat{t}) = 0 \quad \text{and} \quad \nabla\nabla\varphi(\hat{x},\hat{t}) = 0,$$

whenever $(\varphi, (\hat{x},\hat{t})) \in C(Q) \times Q$ satisfies

$$\text{(2.1)} \qquad \max_Q(u^* - \varphi) = (u^* - \varphi)(\hat{x},\hat{t})$$

and the restriction of φ in some open neighborhood $U \times (t_1,t_2) (\subset Q)$ of $(\hat{x},\hat{t})$ belongs to $A_\gamma(U \times (t_1,t_2))$. As usual, *supersolution* is defined by replacing $u^* < \infty$ by $u_* > -\infty$, max by min and inequalities (i), (ii) by the reverse ones. If u is both sub- and supersolution, it is called a (*viscosity*) *solution*. Here u_* is the lower semicontinuous envelope defined by $u_* = -(-u)^*$. The upper semicontinuous envelope u^* is the smallest upper semicontinuous function on $\overline{Q}$ greater than u in Q. In the above definition $\varphi \in A_\gamma$ $(U \times (t_1,t_2))$ is called a *test function* of u in Q at $(\hat{x},\hat{t})$. One may replace Q in (2.1) by $U \times (t_1,t_2)$ and $\varphi \in A_\gamma(U \times (t_1,t_2))$ need not be defined outside $U \times (t_1,t_2)$.

As usual in the definition of viscosity solutions [CIL] to see that u is a subsolution it suffices to check (i), (ii) for any $\varphi \in A_\gamma(D)$ that satisfies

$$\max_D(u^* - \varphi) = (u^* - \varphi)(\hat{x}, \hat{t}) > \sup_{\partial D}(u^* - \varphi)$$

for some neighborhood $D = U' \times (t_1', t_2')(\subset Q)$ of $(\hat{x}, \hat{t})$.

Remark 2.2. When Frank γ is C^2 and (1.5) holds, there is a definition of solutions in [CGG1]. The above definition is not the same as in [CGG1]. It turns out that under (1.5) it is equivalent to that in [CGG1] when Frank γ is C^2 as noted in [G2]. It is also possible to define reasonable notion of solution without (1.5) as in [IS].

Remark 2.3. If u is a subsolution of (L) in Q then, by definition, u is a subsolution of (L) in $Q' = \Omega' \times (t_0, t_1)$ for any open set of Ω' in Ω and $0 \le t_0 < t_1 \le T$. However, since the problem is not purely local, u may not a subsolution of (L) in Q even if u is a subsolution of (L) in some neighborhood of each point $(\hat{x}, \hat{t}) \in Q$.

Remark 2.4. If u itself belongs to $A_\gamma(U \times (t_1, t_2))$, then u is a subsolution in $U \times (t_1, t_2)$ if and only if (i), (ii) holds for $\varphi = u$ at each point $(\hat{x}, \hat{t})$ of $U \times (t_1, t_2)$.

Remark 2.5. We may take Q as any relatively open set in $\mathbf{R}^2 \times (0, T]$ in Definition 2.1. Of course, one should replace $U \times (t_1, t_2)$ by $U \times (t_1, t_2) \cap (\mathbf{R}^2 \times (0, T]) \subset Q$ and assume $u^* < \infty$ on the parabolic closure of Q, in particular on $Q \cap \{t = T\}$.

Lemma 2.6 (Localization). *Assume that (1.5) holds. Let Ω be an open set in $\mathbf{R}^2$. Let u be a subsolution (resp. supersolution) of (L) in $\Omega \times (0, T)$. Then for each $T' \in (0, T)$, u is a subsolution (supersolution) of (L) in $\Omega \times (0, T']$.*

The proof is standard and essentially the same as in [CGG2, §8] so we do not repeat it here.

3. Invariance

One of the key properties of the level set equation is that the motion of the each level set is independent of other level sets. To show this property it is a key to observe the next lemma which asserts that the equation is invariant under the coordinate change of the dependent variable.

Lemma 3.1 (Invariance). *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. Assume that (1.5) holds. Let u be a subsolution (resp. supersolution) of (L) in $Q = \Omega \times (0, T)$, where Ω is an open set in $\mathbf{R}^2$. Let $\theta : \mathbf{R} \to \mathbf{R}$ be a continuous and nondecreasing function. Then, the composite function $\theta \circ u$ is also a subsolution (supersolution) of (L) on Q.*

Remark 3.2. For subsolution (resp. supersolution) the continuity of θ can be weakened. We only need to assume that θ is upper (lower) semicontinuous with values in $\mathbf{R} \cup \{-\infty\}$ ($\mathbf{R} \cup \{+\infty\}$). We also remark that Q can be replaced by any relatively open set in $\mathbf{R}^2 \times (0, T]$. In particular, one can take $Q = \mathbf{R}^2 \times (0, T]$.

Proposition 3.3. Lemma 3.1 holds for $\theta \in C^2(\mathbf{R})$ with $\theta' > 0$.

Proof. We give a proof for subsolution u, since the proof for supersolution is symmetric. We may assume that u is upper semicontinuous. Suppose that $\varphi \in A_\gamma(U \times (t_1, t_2))$ is a test function of $v := \theta \circ u$ in Q at $(\hat{x}, \hat{t})$ so that φ is of form $\varphi(x, t) = h(x) + k(t)$, $h \in C^2_\gamma(U)$, $k \in C^1(t_1, t_2)$. We may assume that $(v - \varphi)(\hat{x}, \hat{t}) = 0$ and that U is bounded without loss of generality. We may also assume that φ is bounded by taking $U \times (t_1, t_2)$ smaller. Since $D := \{(x, t) \in U \times (t_1, t_2); \varphi(x, t) \in \mathrm{Im}\, \theta\}$ is open and $\varphi(\hat{x}, \hat{t}) = v(\hat{x}, \hat{t}) = \theta \circ u(\hat{x}, \hat{t}) \in \mathrm{Im}\, \theta$, there is a neighborhood $\tilde{D} := \tilde{U} \times (\tilde{t_1}, \tilde{t_2}) \subset D$ of $(\hat{x}, \hat{t})$ such that the restriction of φ in $\tilde{D}$ belongs to $A_\gamma(\tilde{D})$. By the choice of $\tilde{D}$ we have

$$\max_{\tilde{D}} (u - \psi) = (u - \psi)(\hat{x}, \hat{t}) = 0,$$

with $\psi := \eta \circ \varphi$ on $\tilde{D}$, where $\eta = \theta^{-1}$.

Unfortunately, ψ may not be a separable function. So we shall modify ψ to construct a test function $\tilde{\psi}(x,t) = \tilde{h}(x) + \tilde{k}(t) \in A_\gamma(\tilde{D})$ of $(\hat{x}, \hat{t})$ satisfying (i)-(iii):

(i) $\tilde{k} \in C^1(\tilde{t_1}, \tilde{t_2})$, $\tilde{h} = A \circ h \in C^2_\gamma(\tilde{U})$ with some $A \in C^2(\mathbf{R})$ satisfying $A' > 0$ on the image Im $h|_{\tilde{U}}$;

(ii) $\psi \leq \tilde{\psi}$ in $\tilde{D}$ and the equality holds at $(x, \hat{t})$ satisfying $\tilde{h}(x) = \tilde{h}(\hat{x})$;

(iii) $\tilde{\psi}_t(\hat{x}, \hat{t}) = \psi_t(\hat{x}, \hat{t})$.

Indeed, since U and φ are bounded, there is a modulus ω (i.e., a nondecreasing function $\omega \in C[0, \infty)$ satisfying $\omega(0) = 0$) and

$$\psi(x,t) \leq \psi(x,\hat{t}) + \psi_t(x,\hat{t})(t - \hat{t}) + \omega(|t - \hat{t}|)|t - \hat{t}| \quad \text{for} \quad (x,t) \in \tilde{D}.$$

If we set $\rho(t) = \int_t^{2t} \omega(\sigma) d\sigma$, evidently,

$$\omega(|t - \hat{t}|)(t - \hat{t}) \leq \rho(t - \hat{t})$$

and $\rho \in C^1(\mathbf{R})$. Moreover, there is $M_1 > 0$ such that $|\eta''(h(x) + k(\hat{t}))| < M_1$ for all $x \in \tilde{U}$, which implies

$$\begin{aligned}
&|\{\psi_t(x,\hat{t}) - \psi_t(\hat{x},\hat{t})\}(t - \hat{t})| \\
\leq &|M_1(h(x) - h(\hat{x}))k'(\hat{t})(t - \hat{t})| \\
\leq &\delta(h(x) - h(\hat{x}))^2 + M_2(t - \hat{t})^2 \quad \text{for} \quad (x,t) \in \tilde{D}
\end{aligned}$$

with $M_2 := M_1^2 \cdot (k'(\hat{t}))^2/4\delta$, where $\delta > 0$ is to be determined later. Thus we have

$$\psi(x,t) \leq \tilde{\psi}(x,t) = \tilde{h}(x) + \tilde{k}(t) \quad \text{for} \quad (x,t) \in \tilde{D} \tag{3.1}$$

with

$$\begin{aligned}
&\tilde{h}(x) = \psi(x,\hat{t}) + \delta(h(x) - h(\hat{x}))^2, \\
&\tilde{k}(t) = \psi_t(\hat{x},\hat{t})(t - \hat{t}) + M_2(t - \hat{t})^2 + \rho(t - \hat{t}) \\
&\text{i.e.} \quad \tilde{h} = A \circ h \quad \text{with} \quad A(z) = \eta(z + g(\hat{t})) + \delta(z - h(\hat{x}))^2.
\end{aligned}$$

Since h is bounded on U, we take $\delta > 0$ small so that $A' > 0$ on Im $h|_{\tilde{U}}$. The equality in (3.1) holds at $(x, \hat{t})$ with $\tilde{h}(x) = \tilde{h}(\hat{x})$, which implies

$$\begin{aligned} 0 = (u - \psi)(\hat{x}, \hat{t}) &= \max_{\tilde{D}}(u - \psi) \\ &\geq \max_{\tilde{D}}(u - \tilde{\psi}) \geq (u - \tilde{\psi})(\hat{x}, \hat{t}) = 0, \end{aligned}$$

so that $\tilde{\psi} \in A_\gamma(\tilde{D})$ is a desired test function of u at $(\hat{x}, \hat{t})$. Similar argument is found in Lemma 4.5 in a different context.

We easily have $\nabla\tilde{\psi}(\hat{x}, \hat{t}) = \eta'(\varphi(\hat{x}, \hat{t}))\nabla\varphi\ (\hat{x}, \hat{t}) =$ (resp. $\neq$) 0 if $\nabla\varphi(\hat{x}, \hat{t}) =$ (resp. $\neq$) 0. We also have $\nabla\nabla\tilde{\psi}(\hat{x}, \hat{t}) = 0$ if $\nabla\varphi(\hat{x}, \hat{t}) = 0$ and $\nabla\nabla\varphi(\hat{x}, \hat{t}) = 0$.

Now we show that v is a subsolution of (L).

When $\nabla\varphi(\hat{x}, \hat{t}) \neq 0$, we have

$$\begin{aligned} &-\frac{\nabla\tilde{\psi}(\hat{x}, \hat{t})}{|\nabla\tilde{\psi}(\hat{x}, \hat{t})|} = -\frac{\nabla\varphi(\hat{x}, \hat{t})}{|\nabla\varphi(\hat{x}, \hat{t})|} =: \hat{\mathbf{m}}, \\ &\{x \in \tilde{U};\ \tilde{h}(x) = \tilde{h}(\hat{x})\} = \{x \in \tilde{U};\ h(x) = h(\hat{x})\}, \\ &\Lambda_\gamma(\tilde{\psi}(\cdot, \hat{t}))(\hat{x}) = \Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x}), \end{aligned}$$

which implies

$$\begin{aligned} &\tilde{\psi}_t(\hat{x}, \hat{t}) - |\nabla\tilde{\psi}(\hat{x}, \hat{t})| g(\hat{\mathbf{m}},\ \Lambda_\gamma(\tilde{\psi}(\cdot, \hat{t}))(\hat{x})) \\ =&\eta'(\varphi(\hat{x}, \hat{t}))\{\varphi_t(\hat{x}, \hat{t}) - |\nabla\varphi(\hat{x}, \hat{t})| g(\hat{\mathbf{m}},\ \Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x}))\}. \end{aligned}$$

Since $\varphi \in A_\gamma(\tilde{D})$ is a test function of u at $(\hat{x}, \hat{t})$, the right hand side of the previous equality is nonpositive.

If $\nabla\varphi(\hat{x}, \hat{t}) = 0$ and $\nabla\nabla\varphi(\hat{x}, \hat{t}) = 0$, we see that

$$\tilde{\psi}_t(\hat{x}, \hat{t}) = \eta'(\varphi(\hat{x}, \hat{t}))\varphi_t(\hat{x}, \hat{t}) \leq 0$$

by the definition of subsolution for u. ■

To derive Lemma 3.1 from Proposition 3.3 we shall approximate θ by C^2 functions having positive derivative.

Proposition 3.4. *Let θ be as in Lemma 3.1. Then there is a sequence $\{\theta_j\}_{j=1}^{\infty} \subset C^2(\mathbf{R})$ such that $\theta_j' > 0$ for $j \geq 1$ and $\lim_{j\to\infty} |\theta - \theta_j| = 0$.*

We give a proof for the reader's convenience, although it is elementary.

Proof. We may assume that $\theta = \theta(z)$ is strictly increasing by adding δ tanh z with small $\delta > 0$. For $j = 1, 2, \cdots$, there is a unique $z_{j,m} \in \mathbf{R}$ such that $\theta(z_{j,m}) = m/j + \theta(0)$ for all $m \in \mathbf{Z}$ satisfying $\inf_{\mathbf{R}} \theta < m/j + \theta(0) < \sup_{\mathbf{R}} \theta$. Let $\rho_j \in C(\mathbf{R})$ be the piecewise linear function with corners $(z_{j,m}, \theta(z_{j,m}))$, so that $\sup_{\mathbf{R}} |\theta - \rho_j| < 1/j$. By mollifying corners of ρ_j, we can choose $\theta_j \in C^2(\mathbf{R})$ such that $\theta_j' > 0$ and $\theta_j(z) = \rho_j(z)$ for $z \in \mathbf{R}$ satisfying $\theta(0) + (m-1)/j + 1/(4j) < z < \theta(0) + m/j - 1/(4j)$ and $\inf_{\mathbf{R}} \theta < z < \sup_{\mathbf{R}} \theta$. Since $\sup_{\mathbf{R}} |\rho_j - \theta_j| < 1/(2j)$, this θ_j is a desired function. ■

Lemma 3.5. *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. For $j = 1, 2, \cdots$, let v_j be a subsolution (resp. supersolution) of (L) in $Q := \Omega \times (0, T)$, where Ω is an open set in $\mathbf{R}^2$. Then $v := \limsup\limits_{j\to\infty}{}^* v_j$ (resp. $v := \liminf\limits_{j\to\infty}{}_* v_j$) is a subsolution (supersolution) of (L) in Q provided that $v < \infty$ (resp. $v > -\infty$) in Q.*
(For the definition of $\limsup^*$ and $\liminf_*$, see §7.)

Remark 3.6. This lemma is a special case of the stability principle (Theorem 7.1). However, our proof for Theorem 7.1 depends on the invariance lemma (Lemma 3.1) except the case 2. For this reason we have to prove Lemma 3.5 independent of the proof of Theorem 7.1 except its case 2.

Proof. We prove only for subsolution since the proof for supersolution is symmetric. We may assume that v_j is upper semicontinuous. Suppose that $\varphi \in A_\gamma(D)$ is a test function of v in D at $(\hat{x},\hat{t})$ with $D = U \times (t_1, t_2)$, so that $\varphi(x,t) = h(x) + k(t)$, $h \in C^2_\gamma(U)$, $k \in C^1(t_1,t_2)$. We may also assume that $(v-\varphi)(\hat{x},\hat{t}) = 0$. Let $\hat{\mathbf{m}}$ denote $-\nabla\varphi(\hat{x},\hat{t})/|\nabla\varphi(\hat{x},\hat{t})|$ when $\nabla\varphi(\hat{x},\hat{t}) \neq 0$.

Case A $(\nabla\varphi(\hat{x},\hat{t}) \neq 0$ and $\hat{\mathbf{m}} \in \mathcal{N})$. The proof is standard as in [Ba] but we give it for completeness. Since $\mathcal{N}$ is a discrete set, there are a small positive number b and an open neighborhood $\tilde{D} := \tilde{U} \times (\tilde{t_1}, \tilde{t_2}) \subset D$ of $(\hat{x},\hat{t})$ such that $\nabla\tilde{\varphi} \neq 0$ and $-\nabla\tilde{\varphi}/|\nabla\tilde{\varphi}| \notin \mathcal{N}$ in $\tilde{D}$, where

$$\tilde{\varphi}(x,t) = \varphi(x,t) + a(t-\hat{t})^2 + b(x-\hat{x})^4$$

with $a > 0$. This implies that $\tilde{\varphi} \in A_\gamma(\tilde{D})$ is a test function of v in $\tilde{D}$, and that $(\hat{x},\hat{t})$ attains the strict maximum of $v - \tilde{\varphi}$ over $\tilde{D}$. For $j = 1, 2, \cdots$, let $(x_j, t_j) \in \tilde{D}$ satisfy

$$\max\nolimits_{\tilde{D}}(v_j - \tilde{\varphi}) = (v_j - \tilde{\varphi})(x_j, t_j), \tag{3.2}$$

which implies that $\tilde{\varphi} \in A_\gamma(\tilde{D})$ is a test function of subsolution v_j in $\tilde{D}$ at (x_j, t_j). By convergence of maximum points (e.g. [GG5, Lemma 2.16], [Ba, lemma 4.2, lemma 2.2]), we see that (x_j, t_j) converges to $(\hat{x},\hat{t})$ as j tends to 0. Thus we may assume that $\nabla\tilde{\varphi}(x_j,t_j) \neq 0$ so that

$$0 \geq \tilde{\varphi}_t(x_j,t_j) - |\nabla\tilde{\varphi}(x_j,t_j)| g(\mathbf{m}_j,\ \Lambda_\gamma(\tilde{\varphi}(\cdot,t_j))(x_j)) \tag{3.3}$$

with $\mathbf{m}_j = -\nabla\tilde{\varphi}(x_j,t_j)/|\nabla\tilde{\varphi}(x_j,t_j)|$. By sending j to the infinity in (3.3) we obtain

$$\begin{aligned} 0 &\geq \tilde{\varphi}_t(\hat{x},\hat{t}) - |\nabla\tilde{\varphi}(\hat{x},\hat{t})| g(\hat{\mathbf{m}},\ \Lambda_\gamma(\tilde{\varphi}(\cdot,\hat{t}))(\hat{x})) \\ &= \varphi_t(\hat{x},\hat{t}) - |\nabla\varphi(\hat{x},\hat{t})| g(\hat{\mathbf{m}},\ \Lambda_\gamma(\tilde{\varphi}(\cdot,\hat{t}))(\hat{x})). \end{aligned}$$

Case B ($\nabla\varphi(\hat{x},\hat{t}) \neq 0$ and $\hat{\mathrm{m}} \in \mathcal{N}$). For a function $r \in C^2_\gamma(Z)$ and $x_0 \in Z\setminus\{x \in Z;\ \nabla r(x) = 0\}$, let $L(r)(x_0)$ denote the length of the facet (i.e. the maximal closed segment) containing x_0 of C^2_γ-curve $c : r(x) = r(x_0)$ with orientation $\mathbf{n}(x_0) = -\nabla r(x_0)/|\nabla r(x_0)| \in \mathcal{N}$, and let $R(r)(x_0) \subset Z$ denote the facet containing x_0 of c with orientation $\mathbf{n}(x_0)$.

First of all, let $\theta \in C^2(\mathbf{R})$ satisfy that $\theta' > 0$ in $\mathbf{R}^2$, $\theta'(\hat{z}) = 1$, $\theta(\hat{z}) = \hat{z}$ and $\theta(z) > z$ for $z \neq \hat{z}$, where $\hat{z} = h(\hat{x})$. Let $f = \theta \circ h$ and let $\psi(x,t)$ denote $f(x) + k(t) + a(t-\hat{t})^2$ in D with some $a > 0$, so that $\psi = \varphi$ in $\{(x,\hat{t}) \in D;\ h(x) = h(\hat{x})\}$ and $\psi > \varphi$ otherwise. We note that $R(f)(\hat{x}) = R(h)(\hat{x})$, $\Lambda_\gamma(f)(\hat{x}) = \Lambda_\gamma(h)(\hat{x})$, $L(f)(\hat{x}) = L(h)(\hat{x})$ since ℓ_0-level set of h equals $\theta(\ell_0)$-level set of f for each $\ell_0 \in \mathbf{R}$. Moreover, we have $\nabla f(\hat{x}) = \nabla h(\hat{x})$ since $\theta'(\hat{z}) = 1$. We shall construct a "nice" test function $\tilde{\varphi}(\geq \psi)$ of v with the property that (x_0, t_0) attains maximum of $v - \tilde{\varphi}$ over suitable neighborhood $\tilde{D} \subset D$ of $R(h)(\hat{x}) \times \{\hat{t}\}$ if and only if $x_0 \in Y := \{x \in R(h)(\hat{x});\ h(x) = h(\hat{x})\}$ and $t_0 = \hat{t}$. Similar modification is often used in [GG3, §8.3–§8.4], [GG5, Lemma 3.1] for a function of one variable. We do not give a detailed construction of $\tilde{\varphi}$ but note that it depends on the sign of $\Lambda_\gamma(\varphi(\cdot,\hat{t}))(\hat{x})$.

Case B-1 ($\Lambda_\gamma(\varphi(\cdot,\hat{t}))(\hat{x}) < 0$). For each $\delta \in (0,1)$ there is an open neighborhood $\tilde{U} \subset U$ of $\hat{R} := R(f)(\hat{x})$ and $\tilde{f} \in C^2_\gamma(\tilde{U})$ such that $\Lambda_\gamma(\tilde{f})(\hat{x}) < 0$, $\tilde{f} \geq f$ in $\tilde{U}$, $\tilde{f} = f$ in $\hat{R}$, $\tilde{f} > f$ in $\{x \in \tilde{U};\ f(x) = f(\hat{x})\} \setminus \hat{R}$, $\nabla\tilde{f}(\hat{x}) = \nabla f(\hat{x})$ $(= \nabla h(\hat{x}))$, $L(\tilde{f})(\hat{x}) = L(f)(\hat{x}) + \delta$ and $\hat{R} \subset R(\tilde{f})(\hat{x})$ and that each end point of segment $\hat{R}$ does not coincide with that of $R(\tilde{f})(\hat{x})$. Setting $\tilde{\varphi}(x,t) = \tilde{f}(x) + k(t) + a(t-\hat{t})^2$ with some $a > 0$, we see that $\tilde{\varphi} \in A_\gamma(\tilde{D})$ is a test function of v in $\tilde{D}$ at $(\hat{x},\hat{t})$, where $\tilde{D} = \tilde{U} \times (t_1, t_2)$.

Suppose that $(x_j, t_j) \in \tilde{D}$, $j = 1, 2, \cdots$ satisfies (3.2), then by [GG5, Lemma 2.16] (x_j, t_j) converges to $(\hat{y}, \hat{t})$ with some $\hat{y} \in Y$ by taking a subsequence. For large j we have

$$0 \geq \tilde{\varphi}_t(x_j, t_j) - |\nabla\tilde{\varphi}(x_j, t_j)| g(\hat{\mathrm{m}},\ \Lambda_\gamma(\tilde{\varphi}(\cdot, t_j))(x_j)) \tag{3.4}$$

since $\hat{y}$ is not an end point of segment $R(\tilde{f})(\hat{x})$ and since $-\nabla\tilde{\varphi}(x_j, t_j)/|\nabla\tilde{\varphi}(x_j, t_j)|$ equals $\hat{\mathrm{m}}$ by the definition of $C^2_\gamma(\tilde{U})$ and the discreteness of $\mathcal{N}$. Sending j to the infinity we have

$$\begin{aligned} 0 &\geq \tilde{\varphi}_t(\hat{y}, \hat{t}) - |\nabla\tilde{\varphi}(\hat{y}, \hat{t})| g(\hat{\mathrm{m}},\ \Lambda_\gamma(\tilde{\varphi}(\cdot, \hat{t}))(\hat{y})) \\ &= \varphi_t(\hat{x}, \hat{t}) - |\nabla\varphi(\hat{x}, \hat{t})| g(\hat{\mathrm{m}}, -\Delta(\hat{\mathrm{m}})/(L(\varphi(\cdot, \hat{t}))(\hat{x}) + \delta)), \end{aligned}$$

since $\nabla\tilde{f}(\hat{y}) = \nabla\tilde{f}(\hat{x})$ by the definition of $C^2_\gamma(\tilde{U})$ and $\nabla\tilde{f}(\hat{x}) = \nabla h(\hat{x}) = \nabla\varphi(\hat{x}, \hat{t})$. The last inequality holds for each $\delta \in (0, 1)$. Thus we obtain that

$$0 \geq \varphi_t(\hat{x}, \hat{t}) - |\nabla\varphi(\hat{x}, \hat{t})| g(\hat{\mathrm{m}},\ \Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x})).$$

Case B-2 ($\Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x}) > 0$). There is an open neighborhood $\tilde{U} \subset U$ of $\hat{R} := R(f)(\hat{x})$ and $\tilde{f} \in C^2_\gamma(\tilde{U})$ such that $\Lambda_\gamma(\tilde{f})(\hat{x}) > 0$, $\tilde{f} \geq f$ in $\tilde{U}$, $\tilde{f} = f$ in $\hat{R}$, $\tilde{f} > f$ in $\{x \in \tilde{U};\ f(x) = f(\hat{x})\} \setminus \hat{R}$, $\nabla\tilde{f}(\hat{x}) = \nabla f(x)$, $R(\tilde{f})(\hat{x}) = \hat{R}$. We set $\tilde{\varphi}(x, t) = \tilde{f}(x) + k(t) + a(t - \hat{t})^2$ with some $a > 0$ for $(x, t) \in \tilde{D} := \tilde{U} \times (t_1, t_2)$, so that $\tilde{\varphi} \in A_\gamma(\tilde{D})$ is a test function of v in Q at $(\hat{x}, \hat{t})$.

Suppose that $(x_j, t_j) \in \tilde{D}$, $j = 1, 2, \cdots$ satisfies (3.2), then we see that (x_j, t_j) subsequentially converges to $(\hat{y}, \hat{t})$ with $\hat{y} \in Y$ by [GG5, Lemma 2.16]. (i) If $\hat{y}$ is not an end point of $R(f)(\hat{x})$, then (3.4) holds for large j by the similar argument as Case B-1. Sending j to the infinity we obtain that

$$\begin{aligned} 0 &\geq \tilde{\varphi}_t(\hat{y}, \hat{t}) - |\nabla\tilde{\varphi}(\hat{y}, \hat{t})| g(\hat{\mathrm{m}},\ \Lambda_\gamma(\tilde{\varphi}(\cdot, \hat{t}))(\hat{y})) \\ &= \varphi_t(\hat{x}, \hat{t}) - |\nabla\varphi(\hat{x}, \hat{t})| g(\hat{\mathrm{m}},\ \Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x})), \end{aligned} \tag{3.5}$$

since $L(\tilde{f})(\hat{y}) = L(\tilde{f})(\hat{x}) = L(f)(\hat{x})$. (ii) If $\hat{y}$ is an end point of $R(f)(\hat{x})$ and if there exists a sebsequence $\{i(j)\} \subset \{j\}$ with $-\nabla\tilde{\varphi}(x_i, t_i)/|\nabla\tilde{\varphi}(x_i, t_i)| = \hat{\mathrm{m}}$, then the same argument as (i) for the subsequence $\{(x_i, t_i)\}$ yields (3.5). (iii) Otherwise, $\hat{y}$ is an end point of $R(f)(\hat{x})$ and $\nabla\tilde{\varphi}(x_j, t_j) \neq 0$ and $\mathbf{m}_j := -\nabla\tilde{\varphi}(x_j, t_j)/|\nabla\tilde{\varphi}(x_j, t_j)| \notin \mathcal{N}$ for large j by the definition of $C^2_\gamma(\tilde{U})$. This implies that (3.3) holds and $\Lambda_\gamma(\tilde{\varphi}(\cdot, t_j))(x_j) = (\tilde{\gamma}''(\theta_j) + \tilde{\gamma}(\theta_j))\kappa(\tilde{f})(x_j)$, where $\kappa(\tilde{f})(x)$ is the (usual) curvature of level set of $\tilde{f}$ at x with orientation $\mathbf{m}_j = (\cos\theta_j, \sin\theta_j)$ and $\tilde{\gamma}(\theta_j) = \gamma(\mathbf{m}_j)$. Since $\Lambda_\gamma(f)(\hat{x}) > 0$, we observe that $\kappa(\tilde{f})(\hat{y}) = 0$ and $\kappa(\tilde{f})(x) > 0$ for x ($\notin R(f)(\hat{x})$) in small neighborhood of the point $\hat{y}$ of $R(f)(\hat{x})$ with orientation $\mathbf{m} := -\nabla\tilde{f}(x)/|\nabla\tilde{f}(x)| \neq \hat{\mathbf{m}}$. Since γ and $\nabla\nabla\gamma$ are bounded on $S^1 \setminus \mathcal{N}$ by the definition of $\mathcal{I}$, $\Lambda_\gamma(\tilde{\varphi}(\cdot, t_j))(x_j)$ tends to zero as j tends to the infinity. Sending j to the infinity in (3.3), we have

$$
\begin{aligned}
0 &\geq \tilde{\varphi}_t(\hat{y}, \hat{t}) - |\nabla\varphi(\hat{y}, \hat{t})| g(\hat{\mathbf{m}}, 0) \\
&\geq \varphi_t(\hat{x}, \hat{t}) - |\nabla\varphi(\hat{x}, \hat{t})| g(\hat{\mathbf{m}}, \Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x}))
\end{aligned}
$$

by the monotonicity of $g \in \mathcal{G}_0$ in the second variable.

Case B-3 ($\Lambda_\gamma(\varphi(\cdot, \hat{t}))(\hat{x}) = 0$). Let y_1 denote an end point of the segment $R(f)(\hat{x})$ and let y_2 denote another end point of $R(f)(\hat{x})$. We may assume that $\kappa(f)(x)$ for x near y_1 is nonpositive and $\kappa(f)(x)$ for x near y_2 is nonpositive with orientation $-\nabla h(x)/|\nabla h(x)|$. There is an open neighborhood $\tilde{U} \subset U$ of $\hat{R} := R(f)(\hat{x})$ and $\tilde{f} \in C^2_\gamma(\tilde{U})$ such that $\Lambda_\gamma(\tilde{f})(\hat{x}) = 0$, $\tilde{f} \geq f$ in $\tilde{U}$, $\tilde{f} = f$ in $\hat{R}$, $\tilde{f} > f$ in $\{x \in \tilde{U}; f(x) = f(\hat{x})\} \setminus \hat{R}$, $\nabla\tilde{f}(\hat{x}) = \nabla f(\hat{x})$, $R(\tilde{f})(\hat{x}) \supset \hat{R}$, $L(\tilde{f})(\hat{x}) = L(f)(\hat{x}) + \delta$ with small $\delta > 0$ and that an end point of segment $R(\tilde{f})(\hat{x})$ equals y_2. Setting $\tilde{\varphi}(x, t) = \tilde{f}(x) + k(t) + a(t - \hat{t})^2$ with some $a > 0$ for $(x, t) \in \tilde{D} := \tilde{U} \times (t_1, t_2)$, we see that $\tilde{\varphi} \in A_\gamma(\tilde{D})$ is a test function of v in $\tilde{D}$ at $(\hat{x}, \hat{t})$.

Suppose that $(x_j, t_j) \in \tilde{D}$, $j = 1, 2, \cdots$ satisfies (3.2), then we see that (x_j, t_j) subsequentially converges to $(\hat{y}, \hat{t})$ with $\hat{y} \in Y$ by [GG5, Lemma 2.16]. (i) If $\hat{y} \neq y_2$, then

$$
0 \geq \tilde{\varphi}_t(x_j, t_j) - |\nabla\tilde{\varphi}(x_j, t_j)| g(\hat{\mathbf{m}}, 0) \tag{3.6}
$$

holds for large j since $-\nabla\tilde{\varphi}(x_j,t_j)/|\nabla\tilde{\varphi}(x_j,t_j)|$ equals $\hat{\mathbf{m}}$ and $\Lambda_\gamma(\varphi(\cdot,t_j))(x_j)=0$. Sending j to the infinity we have

$$\begin{aligned} 0 &\geq \tilde{\varphi}_t(\hat{y},\hat{t}) - |\nabla\tilde{\varphi}(\hat{y},\hat{t})|g(\hat{\mathbf{m}},0) \\ &= \varphi_t(\hat{x},\hat{t}) - |\nabla\varphi(\hat{x},\hat{t})|g(\hat{\mathbf{m}},\Lambda_\gamma(\tilde{\varphi}(\cdot,\hat{t}))(\hat{x})). \end{aligned} \tag{3.7}$$

(ii) If $\hat{y}=y_2$ and there exists a subsequence $\{i(j)\}\subset\{j\}$ with $-\nabla\tilde{\varphi}(x_i,t_i)/|\nabla\tilde{\varphi}(x_i,\ t_i)|=\hat{\mathbf{m}}$, then the same argument as (i) for the subsequence $\{(x_i,t_i)\}$ yields (3.7). (iii) Otherwise $\hat{y}=y_2$, $\nabla\tilde{\varphi}(x_j,t_j)\neq 0$ and $\mathbf{m}_j:=-\nabla\tilde{\varphi}(x_j,t_j)/|\nabla\tilde{\varphi}(x_j,t_j)\notin\mathcal{N}$ for large j. The similar argument as (iii) of Case B-2 yields (3.7).

Case C ($\nabla\varphi(\hat{x},\hat{t})=0$ and $\nabla\nabla\varphi(\hat{x},\hat{t})=0$). This case is contained in the case 2 of the proof of Theorem 7.1, so we omit it. Here, the assumption (1.5) is invoked. ∎

Proof of Lemma 3.1. Assume that u is a subsolution of (L) in Q. We approximate θ by $\theta_j\in C^2(\mathbf{R})$ with $\theta_j'>0$ by Proposition 3.4. Since $v_j=\theta_j\circ u^*$ is a subsolution of (L) in Q by Proposition 3.3 and since $v:=\theta\circ u^*=\limsup\limits_{j\to\infty}{}^* v_j$, we see that v is a subsolution of (L) in Q by Lemma 3.5. The proof is symmetric when u is a supersolution. ∎

Proof of Remark 3.2. Since an upper semicontinuous function θ can be approximated from above by $\theta_j\in C^2(\mathbf{R})$ with $\theta_j'>0$ [GSa], we have $v:=\theta\circ u^*=\limsup\limits_{j\to\infty}{}^* v_j$. The results of Lemma 3.1 extend for such θ when u is a subsolution.

The proofs do not require any special structure of Q so one can extend Lemma 3.1 when Q is a relatively open set of $\mathbf{R}^2\times(0,T]$. ∎

4. Properties of graph-like solutions

We recall the notion of solution in [GG3] for

$$u_t + F(u_x, \Lambda_W(u)) = 0 \tag{E}$$

in Q, where Q is a relatively open set in $(0, T] \times \mathbf{R}$ and $u_x = \partial u / \partial x$. Here W is a convex function belonging to $\mathcal{E}$ defined in [GG5] i.e.,

$$\begin{aligned}\mathcal{E} = &\{W : \mathbf{R} \to \mathbf{R} \text{ convex; The smallest set } P \text{ such that } W \in C^2(\mathbf{R} \setminus P) \text{ is} \\ &\text{discrete and } \sup_{K \setminus P} W'' < \infty \text{ for every compact set } K \text{ in } \mathbf{R}\}.\end{aligned}$$

The quantity $\Lambda_W(f)$ is formally equals $W''(f_x) f_{xx}$ unless f_x does not belong to the jump discontinuity P of W'. If $f_x(x_0) \in P$, then $\Lambda_W(f)(x_0)$ is a nonlocal quantity defined by $\chi\Delta(p)/L$, where L denotes the length of faceted region (of f) containing x_0 and $\Delta(p) = W'(p + 0) - W'(p - 0)$, $p = f_x(x_0)$; χ is the transition number. For detailed definitions the reader is refered to [GG5].

The goal of this section is to give a version of maximum principle for sub- and supersolutions of (E) which is a key to get comparison principle for (L) as well as (E). Such a maximum principle is essentially obtained in [GG3] but we give a modified version which is more convenient to apply.

In [GG3], [GG5] new notions of solutions for (E) are introduced when Q is cylindrical and does not contain points on $t = T$. The definition is easily extended for general Q.

Definition 4.1. For $T > 0$ let Q be a relatively open set in $(0, T] \times \mathbf{R}$. Assume that $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous and that $F(p, X) \geq F(p, Y)$ for $Y \geq X$. Assume that $W \in \mathcal{E}$ with singularity P. A real-valued function u on Q is a *subsolution* of (E) in Q if $u^* < \infty$ in $\overline{Q}$ and

$$\psi_t(\hat{t}, \hat{x}) + F(\psi_x(\hat{t}, \hat{x}),\ \Lambda_W(\psi(\hat{t}, \cdot))(\hat{x})) \leq 0 \tag{4.1}$$

wherever $(\psi, (\hat{t}, \hat{x})) \in A_P(Q') \times Q$ fulfills

$$\max_{Q'}(u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x})$$

and Q' is an arbitrary cylindrical and relatively open set in Q containing $(\hat{t}, \hat{x})$. For given α, β if (4.1) holds only when $\alpha < u^*(\hat{t}, \hat{x}) < \beta$, we say u is a *subsolution* of (E) in $Q \cap \{\alpha < u < \beta\}$. (Note that the set $\{\alpha < u^* < \beta\}$ may not be open.)

For $Q' = I \times \Omega'$ the set $A_P(Q')$ denotes the space of all functions of form

$$f(x) + h(t)$$

with $f \in C_P^2(\Omega')$ and $h \in C^1(I)$, where $I = (a, b)$ or $(a, T]$ with $0 < a < b < T$. (For the definition of C_P^2 see [GG3] or [GG5].) The above definition includes the case that Q contains the set $\{t = T\}$ and corresponds to the local version of the definition in [GG3]. When Q does not touch $\{t = T\}$ and is cylindrical, i.e., $Q = (t_0, t_1) \times \Omega (\subset (0, T] \times \mathbf{R})$, our assumption on $u^* < \infty$ on $\overline{Q}$ is stronger than the one in [GG3] where $u^* < \infty$ is not assumed at $t = t_1$. However, such a difference does not affect the developement of the theory. For example Comparison, Stability principles for (E) holds when $Q = (0, T) \times \Omega$, where Ω is bounded as stated in [GG3], [GG5].

By definition it is clear that u is a subsolution of (E) in any cylindical domain (relatively open in Q provided that u is a subsolution of (E) in Q).

Lemma 4.2 (zero curvature). *Let u be a subsolution of (E) in $Q = (t_0, t_1] \times \Omega$, where Ω is an open set in $\mathbf{R}$. Assume that $\varphi \in A_P(Q)$ and that*

$$\max_Q(u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x})$$

for $(\hat{t}, \hat{x}) \in Q$. If $\hat{x}$ is an end point of a faceted region I of $\varphi(\hat{t}, \cdot)$ with $\varphi_x(\hat{t}, \hat{x}) \in P$ and

$$(u^* - \varphi)(\hat{t}, x) < (u^* - \varphi)(\hat{t}, \hat{x})$$

for all $x \in I$ *near* $\hat{x}$, *then*

$$\varphi_t(\hat{t},\hat{x}) + F(\varphi_x(\hat{t},\hat{x}),0) \le 0.$$

Proof. Since $F(p,X) \ge F(p,Y)$ for $Y \ge X$ by the assumption, the last inequality holds if $\chi(\varphi(\hat{t},\cdot),\hat{x}) \le 0$. We may assume $\chi(\varphi(\hat{t},\cdot),\hat{x}) = 1$. By the assumption of $A_P(Q)$, $\varphi(t,x)$ is of form $f(x)+h(t)$ with $f \in C^2_P(\Omega)$ and $h \in C^1(t_0,t_1)$. We may assume $f(x) = \varphi(\hat{t},x)$ and $f'(\hat{x}) = 0 \in P$ by adding on affine function. We may also assume $\hat{x} = \sup I$, so that $I = [a,\hat{x}]$ with some $a \in \Omega$. Modifying f, we can construct $\tilde{f} \in C^2_P(\Omega)$ (Figure 1) such that

$$\begin{aligned} \tilde{f} &< f \quad \text{in } J', \\ \tilde{f} &= f \quad \text{in } \Omega \setminus J' \end{aligned}$$

with some open set J'; $\overline{J}' \subset \operatorname{int} J$ and such that

$$(u^* - \tilde{\varphi})(\hat{t},x) < (u^* - \tilde{\varphi})(\hat{t},\hat{x}) \quad \text{for } x \in J,$$

where $\tilde{\varphi}(x,t) = \tilde{f}(x) + h(t)$. This implies that $\chi(\tilde{f},\hat{x}) = 0$, and that there exist $\tilde{t}_0$ and $\tilde{t}_1$ such that $t_0 \le \tilde{t}_0 < \hat{t} < \tilde{t}_1 \le t_1$ and

$$\max_{\tilde{Q}}(u^* - \tilde{\varphi}) = (u^* - \tilde{\varphi})(\hat{t},\hat{x})$$

with $\tilde{Q} = (\tilde{t}_0,\tilde{t}_1] \times \Omega$. Since $\tilde{\varphi} \in A_P(\tilde{Q})$, the definition of the subsolution of (E) implies that

$$\begin{aligned} &\varphi_t(\hat{t},\hat{x}) + F(\varphi_x(\hat{t},\hat{x}),0) \\ =&\tilde{\varphi}_t(\hat{t},\hat{x}) + F(\tilde{\varphi}_x(\hat{t},\hat{x}),\ \chi(\tilde{\varphi}(\hat{t},\cdot),\hat{x}) \le 0 \quad \blacksquare \end{aligned}$$

It is convenient to recall an infinitesimal version of definition of solutions as in [GG3]. Because of zero curvature lemma (Lemma 4.2), the definition is simplified.

Let Q be a relatively open set in $(0,T] \times \mathbf{R}$. Let $\varphi : Q \to \mathbf{R}$ be an upper semicontinuous function. Let $(\hat{t},\hat{x})$ be a point in Q. Assume that $\varphi(t,\cdot) \in C(\Omega)$ for t near $\hat{t}$. We say φ is an (infinitesimally) *admissible superfunction* at $(\hat{t},\hat{x})$ in Q if one of following three conditions holds.

(A) The function $\varphi(\hat{t},\cdot)$ is P-faceted at $\hat{x}$ in $\Omega(\hat{t}) = \{x \in \mathbf{R};\ (\hat{t},x) \in Q\}$ and $\hat{x} \in \operatorname{int} R(\varphi(\hat{t},\cdot),\hat{x})$, where $R(\varphi(\hat{t},\cdot),\hat{x})$ denotes the faceted region of $\varphi(\hat{t},\cdot)$ at $\hat{x}$ as in [GG3]. The set of upper time derivatives $\mathcal{T}_P^+\varphi(\hat{t},\hat{x})$ is nonempty. The definition of $\mathcal{T}_P^+$ is found in [GG3, §6.4]. However, if $\hat{t} = T$, we should replace $(\hat{t}-\delta,\hat{t}+\delta)$ by $(\hat{t}-\delta,\hat{t}]$ in the definition of $\mathcal{T}_P^+$.

(B) There is $(\tau,p,X) \in \mathcal{P}^+\varphi(\hat{t},\hat{x})$ with $p \notin P$, where $\mathcal{P}^+$ denotes the parabolic superjet in Q [CIL], [GG3, §6.2]).

(C) The function $\varphi(\hat{t},\cdot)$ is P-faceted at $\hat{x} \in \Omega(\hat{t})$ but $\hat{x} \in \partial R(\varphi(\hat{t},\cdot),\hat{x})$. There is an element $(\tau,p,0)$ in $\mathcal{P}^+\varphi(\hat{t},\hat{x})$ for some $\tau \in \mathbf{R}$.

Our definition is the same for $Q = (0,t_0) \times \Omega$ as [GG3] except (C).

Theorem 4.3 (equivalent definition – infinitesimal version). *Assume the same hypotheses of Definition 4.1 concerning Q, F and W. A real-valued function u on Q is a subsolution of (E) in Q if and only if $u^* < \infty$ in $\overline{Q}$ and the following conditions are fulfilled. For each $(\hat{t},\hat{x}) \in Q$ let φ be an admissible superfunction at $(\hat{t},\hat{x})$ in Q such that $\max_Q(u^*-\varphi) = (u^*-\varphi)(\hat{t},\hat{x})$. Then*

(i) $\tau + F(\varphi_x(\hat{t},\hat{x}),\ \Lambda_W(\varphi(\hat{t},\cdot),\ \hat{x})) \le 0$ *for all* $\tau \in \mathcal{T}_P^+\varphi(\hat{t},\hat{x})$ *if (A) holds.*

(ii) $\tau + F(p, W''(p)X) \le 0$ *for all* $(\tau,p,X) \in \mathcal{P}^+\varphi(\hat{t},\hat{x})$ *if (B) holds.*

(iii) $\tau + F(p,0) \le 0$ *for all* $(\tau,p,0) \in \mathcal{P}^+\varphi(\hat{t},\hat{x})$ *if (C) holds and* $(u^*-\varphi)\ (\hat{t},x) < \max_Q\ (u^*-\varphi)$ *for all* $x \in R\ (\varphi(\hat{t},\cdot),\ \hat{x}) \setminus \{\hat{x}\}$ *near* $\hat{x}$.

Proof. The proof essentially parallels that of [GG3, Theorem 6.9]. In the proof of 'only if' part, (iii) follows from the zero curvature lemma (Lemma 4.2). We do not repeat the tedious details of the proof of 'only if' part. The proof of 'if' part is easy since ψ in Definition 4.1 can be regarded as an admissible superfunction at $(\hat{t}, \hat{x})$ in Q by extended ψ to Q' in a suitable way. The only problem is the case when $\hat{x} \in \partial R(\psi(\hat{t}, \cdot),\ \hat{x})$. If $\Lambda_W(\psi(\hat{t}, \cdot))(\hat{x}) < 0$, then we may assume $\hat{x} \in \operatorname{int} R(\psi(\hat{t}, \cdot), \hat{x})$ by replacing ψ by its upper canonical modification $\psi^{\#,\varepsilon}$ as in [GG3, §8.3-§8.4]. So we may assume that $\Lambda_W(\psi(\hat{t}, \cdot))(\hat{x}) \geq 0$. If there is $x_0 \in \operatorname{int} R(\psi(\hat{t}, \cdot), \hat{x})$ satisfying $(u^* - \psi)(\hat{t}, x_0) = (u^* - \psi)(\hat{t}, \hat{x})$, then (i) yields

$$\psi_t(\hat{t}, x_0) + F(\psi_x(\hat{t}, x_0),\ \Lambda_W(\psi(\hat{t}, \cdot))) \leq 0$$

which is the desired inequality since $\psi_t, \psi_x, \Lambda_W(\psi(\hat{t}, \cdot))$ is constant on $R(\psi(\hat{t}, \cdot), \hat{x})$. If there is no such x_0, then by (iii) we have

$$\psi_t(\hat{t}, \hat{x}) + F(\psi_x(\hat{t}, \hat{x}), 0) \leq 0.$$

Since $\Lambda_W(\psi(\hat{t}, \cdot))(\hat{x}) \geq 0$, by monotonicity of $F(p, X)$ in X, we have $\psi_t(\hat{t}, \hat{x}) + F(\psi_x(\hat{t}, \hat{x}),\ \Lambda_W(\psi(\hat{t}, \cdot))(\hat{x})) \leq 0$. ■

Lemma 4.4 (Faceted maximum principle). *Assume that* $u : [t_0, t_1] \times \mathbf{R} \to [-\infty, \infty)$ *and* $-v : [s_0, s_1] \times \mathbf{R} \to [-\infty, \infty)$ *are upper semicontinuous. Assume that there is* $R_0 > 0$ *such that* $u(t, x) = -\infty$ *for* $|x| \geq R_0$ *and* $t \in [t_0, t_1]$ *and that* $v(s, y) = +\infty$ *for* $|y| \geq R_0$ *and* $s \in [s_0, s_1]$. *Let* $(\hat{t}, \hat{x})$ *be a point in* $\mathcal{R}_u = (t_0, t_1] \times \mathbf{R}$ *and* $(\hat{s}, \hat{y})$ *be a point in* $\mathcal{R}_v = (s_0, s_1] \times \mathbf{R}$. *Assume that* $A : [t_0, t_1] \to \mathbf{R}$ *and* $B : [s_0, s_1] \to \mathbf{R}$ *are continuous and that* $u(\hat{t}, \hat{x}) < A(\hat{t})$ *and* $v(\hat{s}, \hat{y}) > B(\hat{s})$. *Assume that* 0 *belongs the singular set* P *of* W *in (E). Assume that* u *and* v *are sub- and supersolution of (E) in* Q_u *and* Q_v *respectively, with*

$$Q_u = \{(t, x) \in \mathcal{R}_u;\ u(t, x) < A(t)\},$$

$$Q_v = \{(s, y) \in \mathcal{R}_v;\ v(s, y) > B(t)\}.$$

Assume that $\varphi : \mathbf{R}^2 \to (-\infty, \infty]$ *is lower semicontinuous and* $\varphi(\tau, \cdot)$ *is faceted with slope zero at* $\hat{x} - \hat{y}$ *near* $\tau = \hat{t} - \hat{s}$ *and that* $R(\varphi(\tau, \cdot), \hat{x} - \hat{y})$ *moves* C^1 *in* τ *and* $\chi(\varphi(\tau, \cdot), \hat{x} - \hat{y}) = 1$ *near* $\tau = \hat{t} - \hat{s}$*. Moreover,* φ *is* C^1 *near* $\{\hat{t} - \hat{s}\} \times R(\varphi(\hat{t} - \hat{s}, \cdot), \hat{x} - \hat{y})$ *and* $\varphi(\tau, \cdot)$ *is* C^2 *near* $R(\varphi(\tau, \cdot), \hat{x} - \hat{y})$*. Assume that*

$$u(t, x) - v(s, y) - S(t, s) - \varphi(t - s,\ x - y)$$

attains its maximum at $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$ *over* $\overline{\mathcal{R}}_u \times \overline{\mathcal{R}}_v$ *for* $S \in C^1((t_0, t_1] \times (s_0, s_1])$*. Then*

$$\frac{\partial S}{\partial t}(\hat{t}, \hat{s}) + \frac{\partial S}{\partial s}(\hat{t}, \hat{s}) \leq 0.$$

Lemma 4.5 (Separable test function). *Assume that* $\varphi \in C^2((t_0, t_1) \times (a, b))$ *and that*

(i) $\varphi(\hat{t}, \cdot)$ *is faceted with slope zero at* $\hat{x}$*,*

(ii) $R(\varphi(t, \cdot), \hat{x})$ *moves 1/2- Hölder continuous in* $t \in (t_1, t_2)$*,*

(iii) $\varphi_t(\hat{t}, x) \equiv c$ *on* $x \in R(\varphi(\hat{t}, \cdot), \hat{x}) = [x_0, x_1]$*,*

where $(\hat{t}, \hat{x})$ *is a given point of* $(t_0, t_1) \times (a, b)$ *and* c *is a constant. Then there is* $f \in C^2(a', b')$ *and* $h \in C^1(t_0', t_1')$ *that satisfies*

(a) $\varphi(\hat{t}, x) = f(x)$*,* $x \in [x_0, x_1]$*,*

(b) $\varphi(\hat{t}, x) < f(x)$*,* $x \in (a', b') \backslash [x_0, x_1]$*,*

(c) $\varphi(t, x) \leq f(x) + h(t)$ *in* $(t_0', t_1') \times (a', b')$*,*

(d) $h(\hat{t}) = 0$*,* $h_t(\hat{t}) = c$*,*

where (a', b') *is some neighborhood of* $[x_0, x_1]$ *and* (t_0', t_1') *is some neighborhood of* $\hat{t}$*.*

Proof. We may assume that $c = 0$ by replacing φ by $\varphi - ct$. Since φ is C^2,

$$\begin{aligned} \varphi(t, x) \leq \varphi(\hat{t}, \hat{x}_i) &+ \omega(|x - x_i|^2)|x - x_i|^2 \\ &+ \omega(|t - \hat{t}|)|t - \hat{t}| \end{aligned}$$

near $(\hat{t}, x_i)$ for $i = 0, 1$, where ω is a modulus, i.e., $\omega \in C[0,\infty)$ is nondecreasing and $\omega(0) = 0$.

We take $f_0 \in C^2(a', b')$ such that $f_0(x) = 0$ on $[x_0, x_1]$

$$f_0(x) \geq \omega(|x - x_i|^2)|x - x_i|^2, \quad i = 0, 1$$

for x between x_i and x_i' for x_i' close to x_i. (Such f_0 always exist by defining $f_0(x) = \rho_0(x - x_1) + \rho_0(x_0 - x)$, where $\rho_0(x) = \int_x^{2x} d\tau \int_\tau^{2\tau} \omega(\rho)d\rho$ for $x \geq 0$ and $\rho_0(x) = 0$ for $x \leq 0$.) Similarly we take $h_0 \in C^1(t_0', t_1')$ such that

$$h_0(t) \geq \omega(|t - \hat{t}|)|t - \hat{t}|$$

such that $h_0(0) = h_0'(0) = 0$ by taking t_0', t_1' close to $\hat{t}$. Then

$$\varphi(t, x) \leq \varphi(\hat{t}, \hat{x}) + f_0(x) + h_0(t) \quad \text{for} \quad x \in (x_0', x_1') \backslash (x_0, x_1), \ t \in (t_0', t_1').$$

For $x \geq x_0$ if $\varphi(t, x) > \varphi(\hat{t}, \hat{x})$ and x is close to x_0, then $|x - x_0| \leq c_0 |t - \hat{t}|^{1/2}$ with some constant c_0 by 1/2- Hölder continuity of $R(\varphi(t, \cdot), \hat{x})$. Thus

$$\omega(|x - x_0|^2)|x - x_0| \leq \omega(c_0^2 |t - \hat{t}|)|t - \hat{t}|$$

for $x \geq x_0$ close to x_0 with the property that $\varphi(t, x) > \varphi(\hat{t}, \hat{x})$. We take $h_1 \in C^1(t_0', t_1')$ such that

$$\omega(c_0^2 |t - \hat{t}|)(|t - \hat{t}|) \leq h_1(t) \quad \text{on} \quad (t_0', t_1')$$

with $h_1(\hat{t}) = h_{1t}(\hat{t}) = 0$. Similar argument near x_1 yields

$$\varphi(t, x) \leq \varphi(\hat{t}, \hat{x}) + h_1(t) + h_0(t), \quad t \in (t_0', t_1'), \ x \in [x_0, x_1].$$

We set

$$f(x) = \varphi(\hat{t}, \hat{x}) + f_0(x), \ h(t) = h_1(t) + h_0(t)$$

to get (a)–(b). ■

Proof of Lemma 4.4. By Lemma 4.5 we may assume that φ is of form $\varphi(t,x) = f(x)+h(t)$. By the zero curvature lemma (Lemma 4.2) we may assume that $\hat{x}-\hat{y} \in \operatorname{int} R(\varphi(\tau,\cdot),\hat{x}-\hat{y})$ near $\tau = \hat{t}-\hat{s}$. We take $\lambda > 0$ small so that $f(x) \le \vartheta(x-\eta,\lambda)$ with $\eta = \hat{x}-\hat{y}$, where

$$\vartheta(x,\lambda) = \begin{cases} (x-\lambda)^2/\lambda, & x > \lambda, \\ 0, & |x| \le \lambda, \\ (x+\lambda)^2/\lambda, & x < -\lambda. \end{cases}$$

Thus we may replace f by ϑ. We may assume that u and v are sub- and supersolutions in $Q = I \times \Omega$ and $Q' = I' \times \Omega'$ where I, I', Ω, Ω' are open intervals containing $\hat{t}, \hat{x}, \hat{s}, \hat{y}$, respectively and $R(\varphi(\tau,\cdot),\hat{x}-\hat{y}) \subset \Omega - \{\hat{y}\}$, $R(\varphi(\tau,\cdot),\hat{x}-\hat{y}) \subset \{\hat{x}\} - \Omega'$. Arguing as in [GG3, Propositions 7.13-7.15] there is an admissible superfunction U_1 at $(\hat{t},\hat{x})$ in Q and an admissible subfunction U_2 at $(\hat{s},\hat{y})$ in Q' such that

$$L_{U_1} = L_{U_2},\ \chi_{U_1} \le \chi_{U_2}$$

where $L_{U_i} = L(U_i(\hat{t},\cdot),\hat{x}), \chi_{U_i} = \chi(U_i(\hat{t},\cdot),\hat{x})$ for $i = 1,2$. Since u is a subsolution of (E) in Q, we have, by definition of infinitesimal version,

$$h_t(\hat{t}-\hat{s}) + S_t(\hat{t},\hat{s}) + F(0, \Delta\chi_{U_1}/L_{U_1}) \le 0,$$

where $\Delta = W'(+0) - W'(-0)$. Similarly,

$$h_t(\hat{t}-\hat{s}) - S_s(\hat{t},\hat{s}) + F(0, \Delta\chi_{U_2}/L_{U_2}) \ge 0$$

Since F is nonincreasing in the second variable, we subtract the second inequality from the first to get

$$S_t(\hat{t},\hat{s}) + S_s(\hat{t},\hat{s}) \le 0. \qquad ■$$

5. Level set and graph-like solutions

The goal of this section is to give the slicing lemma which is crucial in this paper.

If a curve Γ_t is given by the graph of a function $w = w(t, x_1)$, the equation (I) is written in the form

$$(G) \qquad w_t + F(w_{x_1}, \ \Lambda_W(w)) = 0$$

with

$$F(q, Y) = -\sqrt{1+q^2}g(-\frac{q}{\sqrt{1+q^2}}, \ \frac{1}{\sqrt{1+q^2}}, \ Y),$$
$$W(q) = \gamma(-q, 1), \ q, \ Y \in \mathbf{R},$$

where the orientation $\mathbf{n}$ is taken upward. The solution w is often called the graph-like solution of (I). The equation (G) is also obtained if we set $\psi(x_1, x_2, t) = w(t_1 x_1) - x_2$. We call (G) the graph version of (I) (or (L)). For a solution ψ of (L) if $\partial\psi/\partial x_2 < 0$ on the zero level set of ψ, then it is formally clear that the zero level set is of form $\{x_2 = w(t_1, x_1)\}$ and w solves (G), i.e., w is a graph-like solution of (I). Our slicing lemma asserts that a 'height' function $u^\#$ of a super level set $\{\psi \geq c\}$ is a graph-like subsolution if ψ solves (L)(Figure 2).

Lemma 5.1 (Slicing). *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. Assume that u is an upper semicontinuous subsolution (resp. lower semicontinuous supersolution) of the level set equation (L) in $\mathcal{R} = \mathbf{R}^2 \times (0, T]$ or $\mathbf{R}^2 \times (0, T)$. Let ℓ be a continuous function in $\mathcal{R}$. For $c \in \mathbf{R}$ let $u^\#$ be of form*

$$u^\#(t, x_1) = \sup\{x_2; \ u(x_1, x_2, t) \geq c, \ x_2 \leq \ell(t, x_1), (x, t) \in \mathcal{R}\}.$$
$$(\text{resp.} \quad u_\#(t, x_1) = \inf\{x_2; \ u(x_1, x_2, t) \leq c, \ x_2 \geq \ell(t, x_1), (x, t) \in \mathcal{R}\}.)$$

Then $u^{\#}$ (resp. $u_{\#}$) is a subsolution of the graph version (G) of (L) in

$$\begin{aligned} Q &= \{(t,x_1) \in \mathcal{R}_0;\ u(x_1,\ell(t,x_1),t) < c\} \\ &= \{(t,x_1) \in \mathcal{R}_0;\ u^{\#}(t,x_1) < \ell(t,x_1)\}, \\ (\text{resp.}\quad Q &= \{(t,x_1) \in \mathcal{R}_0;\ u(x_1,\ell(t,x_1),t) < c\} \\ &= \{(t,x_1) \in \mathcal{R}_0;\ u^{\#}(t,x_1) > \ell(t,x_1)\},) \end{aligned}$$

where $\mathcal{R}_0 = \{(t,x_1) \in (0,T] \times \mathbf{R};\ (x_1,u^{\#}(t,x_1),t) \in \mathcal{R}\}$. (Here we set $u^{\#}(t,x_1) = -\infty$ if the set of x_2 defined by $\{u(x_1,x_2,t) \geq c,\ x_2 \leq \ell(t,x_1), (x_1,x_2,t) \in \mathcal{R}\}$ is empty.)

Remark 5.2. One may replace ℓ as a lower semicontinuous function with values in $\mathbf{R}^{\cup}\{+\infty\}$. In particular, we may take $\ell \equiv +\infty$. In this case $u^{\#}$ is the height function of $\{u \geq c\}$:

$$u^{\#}(t,x_1) = \sup\{x_2;\ u(x_1,x_2,t) \geq c\}$$

and $Q = \{u^{\#}(t,x_1) < \infty\}$. One may replace $\mathbf{R}^2$ by $\Omega \times (\alpha,\beta)$ for an open set Ω in $\mathbf{R}$. In this case $u^{\#}$ is a subsolution of (G) in $Q \cap \{\alpha < u < \beta\}$.

Proof. By the invariance lemma (Lemma 3.1 and Remark 3.2) $I^- \circ u$ is a subsolution of (L), where $I^-(\sigma) = 0$ for $\sigma \geq 0$ and $I^-(\sigma) = -\infty$ for $\sigma < 0$. By definition $u^{\#}$ is upper semicontinuous in $\mathcal{R}$. Assume that

$$\max_{Q'}(u^{\#} - \varphi) = (u^{\#} - \varphi)(\hat{t},\hat{x}_1) = 0$$

for $(\varphi,(\hat{t},\hat{x}_1)) \in A_P(Q') \times Q$, where $Q'(\ni (\hat{t},\hat{x}_1))$ is a cylindrical open set in Q and P is the discontinuity of $W(q) = \gamma(-q,1)$. Note that $\gamma \in \mathcal{I}$ implies $W \in \mathcal{E}$ and that $g \in \mathcal{G}_0$ implies the monotonicity of F in (G), i.e., $F(q,X) \leq F(q,Y)$ for

$X \geq Y$, so the notion of subsolution (Definition 4.1) is defined for (G). We set $\tilde{\varphi}(x_1, x_2, t) = \varphi(t, x_1) - x_2$ to get

$$\tilde{\varphi} \in A_\gamma(U_T) \quad \text{with} \quad U_T = \{(x_1, x_2, t);\ x_2 \in \mathbf{R},\ (t, x_1) \in Q'\}.$$

We also observe that $I^- \circ u - \tilde{\varphi}$ attains its maximum 0 over $Z = \{(x_1, x_2, t);\ (t, x_1) \in Q',\ x_2 < \ell(t, x_1)\}$ at $(\hat{x}_1, \hat{x}_2, \hat{t})$ with $\hat{x}_2 = u^{\#}(\hat{t}, \hat{x}_1)$. Since $I^- \circ u$ is a subsolution of (L) in Z and $\nabla\tilde{\varphi} = (\varphi_{x_1}, -1)$, $\Lambda_\gamma(\tilde{\varphi}(\cdot, \hat{t}))(\hat{x}) = \Lambda_W(\varphi(\hat{t}, \cdot))(\hat{x}_1)$, we have

$$\varphi_t(\hat{t}, \hat{x}_1) + F(\varphi_{x_1}(\hat{t}, \hat{x}_1),\ \Lambda_W(\varphi(\hat{t}, \cdot))(\hat{x}_1)) \leq 0,$$

where F and W is defined in (G). We have thus proved that $u^{\#}$ is a subsolution of (G) in Q. ■

The slicing lemma is considered as a general principle for solutions of the level set equation. We shall state its typical form applicable to motion by local curvatures. We consider an evolution equation for $u = u(x, t)$ of form

$$u_t + F(x, t, Du, DDu) = 0, \quad (x, t) \in Q_0, \tag{5.1}$$

where Q_0 is an open set in $\mathbf{R}^N \times (0, T]$ and $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_N)$; DDu denotes the Hessian matrix. We assume that $F = F(x, t, p, X)$ is defined on $Z = Q_0 \times \mathbf{R}^N \times \mathbf{S}^N$, where $\mathbf{S}^N$ denotes the space of all $N \times N$ real symmetric matrices. We set for $(x, t, q, Y) \in Q_0 \times \mathbf{R}^{N-1} \times \mathbf{S}^{N-1}$

$$G(x, t, q, Y) = F(x, t, q, -1, X) \quad \text{with} \quad X = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}.$$

If $u(t, x) = w(t, x') - x_N$, $x' = (x_1, \dots, x_{N-1})$ and u solves (5.1), then clearly w solves

$$w_t + G(x', w, t, \nabla w, \nabla\nabla w) = 0 \tag{5.2}$$

for (x', t) such that $(x', w(t, x'), t) \in Q_0$, where ∇ denotes the gradient with respect to x' variables.

Theorem 5.3 (Slicing). *Assume that* $F : Q_0 \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ *is continuous and geometric i.e.,* $F(x, t, \lambda p, \lambda X + \sigma p \otimes p) = \lambda F(x, t, p, X)$ *for all* $(x, t) \in Q_0$, $p \in \mathbf{R}^N \setminus \{0\}$, $X \in \mathbf{S}^N$ *and* $\lambda > 0$, $\sigma \in \mathbf{R}$. *Assume that* Q_0 *is of form* $Q_0 = \{(x', x_N, t); (x', t) \in Q_0', \alpha < x_N < \beta\}$ *and that* Q_0' *is an open set in* $\mathbf{R}^{N-1} \times (0, T]$ *and* $\alpha < \beta$. *Assume that* u *is an upper (resp. lower) semicontinuous subsolution (resp. supersolution) of (5.1) in* Q_0. *For* $c \in \mathbf{R}$ *let* $u^\#$ *be of form*

$$u^\#(t, x') = \sup\{x_N;\ u(x, t) \geq c, \quad (x, t) \in Q_0\}$$
$$(\text{resp.} \quad u_\#(t, x') = \inf\{x_N;\ u(x, t) \leq c, \quad (x, t) \in Q_0\}).$$

Then $u^\#$ *(resp.* $u_\#$*) is an upper (resp. lower) semicontinuous subsolution (resp. supersolution) of (5.2) in* $Q_0 \cap \{\alpha < u^\# < \beta\}$ *(resp.* $Q_0 \cap \{\alpha < u_\# < \beta\}$*).*

The proof is based on the invariance principle for geometric equation [CGG1], [ES1], [G1]. It is basically the same as that of Lemma 5.1, so it is safely omitted.

Remark 5.4. There are several ways to define viscosity solutions when F is singular at $Du = 0$. However, to see that $u^\#$ is a subsolution we need no condition at $Du = 0$.

Remark 5.5. As pointed out in [GGo] the geometric equation (5.1) is not necessarily a level set equation of some interface equations. However if F is degenerate elliptic, (5.1) is interpreted as a level set equation for some interface equation (I). Above theorem asserts that the 'height function' of the slice $\{u \geq c_1\}$ is a graph-like subsolution of (I) if u is a subsolution of (L).

Remark 5.6. In [GSa] it is shown that if u is continuous and solves (5.1), then $u^\#$ is also a supersolution of (5.2) when $x_N \mapsto u(x', x_N, t)$ is nonincreasing at least for first order equation. The similar method applies in the situation of Lemma 5.1 and Theorem 5.3. Thus for a continuous solution u of (L) and (5.1), $u^{\#}$ is a supersolution of (G) and (5.2), respectively provided that $x_N \mapsto u(x', x_N, t)$ is nonincreasing. Note that the monotonicity assumption cannot be removed. Indeed, if u is a continuous solution of $u_t - |Du| = 0$ in $\mathbf{R}^2 \times (0, \infty)$ and $u(x, 0) = 0$ for $x \in A$ and $u(x, 0) < 0$ for $x \in A$ where A is a closed set, then $u(x, t) = 0$ if and only if the distance between x and A is less than t. We take $A = \{x_2 \leq 0\} \cup \{0 \leq x_2 \leq 1,\ x_2 \geq x_1\}$. Then the height function of $\{u \geq 0\}$ is of form

$$u^{\#}(t, x_1) = \sup\{x_2;\ \mathrm{dist}((x_1, x_2), A) \leq t\}, \quad t > 0,\ x_1 \in \mathbf{R}.$$

As elementary observation shows that $u^{\#}$ is nonincreasing in x_1 and it has only one jump at $x_1 = t + 1$. However, if $u^{\#}$ has such a jump, it is not a supersolution of (5.2):

$$w_t - \sqrt{1 + w_{x_1}^2} = 0.$$

In this example $u^{\#}$ and $u_{\#}$ does not agree and $u^{\#}$ has jump discontinuity. If $u^{\#} = u_{\#}$ then evidently $u^{\#}$ is a solution of (5.2). We shall give a sufficient condition to guarantee that $u^{\#}$ is a solution other than the monotonicity of u in x_N. The proof is easy so is omitted.

Corollary 5.7. *Under the assumption of Theorem 5.3 assume that the level set $\{u = c\}$ is given as the graph of a continuous function $w(t, x')$ on Q_0' which is open in $\mathbf{R}^{N-1} \times (0, T]$. Assume that $Q_0 = \{(x', x_N, t), (x', t) \in Q_0', \alpha < x_N < \beta\}$ and $\alpha < w < \beta$ on Q_0'. Assume that $u < c$ if $x_N > w(t, x')$ and $u > c$ if $x_N < w(t, x')$. Then w is a solution of (5.2) in Q_0'.*

Remark 5.8. Theorem 5.3 applies the level set equation of the higher dimensional analog of (I) with continuous g satisfying (1.2) provided that the interfacial energy

$\gamma : \mathbf{R}^N \to (0, \infty)$ is convex and C^2 except the origin; we always assume $\gamma(\lambda p) = \lambda\gamma(p)$, $\lambda > 0$, $p \in \mathbf{R}^N \setminus \{0\}$.

For the level set mean curvature flow equation

$$u_t - |Du|\mathrm{div}(Du/|Du|) = 0 \tag{5.3}$$

L. C. Evans and J. Spruck [ES2] proved that if the level set $\{u = c\}$ is given as the graph of a continuous function $w(t, x')$, then w solves the graph version

$$w_t - \sqrt{1 + |\nabla w|^2}\mathrm{div}(\nabla w/\sqrt{1 + |\nabla w|^2}) = 0. \tag{5.4}$$

Their proof needs a priori estimates for quasilinear parabolic equation (5.4). Modifying Corollary 5.7, we easily recover their result without appealing hard estimates. Indeed, since $V = -\mathrm{div}\ \mathbf{n}$ is orientation free, we may assume that $u \geq c$ and observe that $2c - u$ is also a solution of (5.4). If $\{u = c\}$ is given as the graph of continuous function $u_{\#} = w$ and $(2c - u)^{\#} = w$, so by Theorem 5.3 w is a solution of (5.4).

Remark 5.9. Very recently, in [Sc] R. Schätzle showed that the height function of the limit varifold of solution Γ_m of $-\mathrm{div}\ \mathbf{n} = h_m$ as $m \to \infty$ satisfies the graph version of the limit equation $-\mathrm{div}\ \mathbf{n} = h$ in viscosity sense. However, his situation is totally different from ours since he discusses varifolds instead of the slice of level set sub- and supersolutions.

6. Comparison principle

We shall prove a crucial comparison principle for the level set equation

$$u_t - |\nabla u| g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) = 0. \tag{L}$$

Theorem 6.1 (Comparison principle). *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. For $T > 0$ let u and v be, respectively, sub- and supersolution of (L) in $\mathbf{R}^2 \times (0, T)$. Assume that spt $(u - \alpha)$ and spt $(v - \beta)$ are compact in $\mathbf{R}^2 \times [0, T)$ for some $\alpha, \beta \in \mathbf{R}$. If $u^* \le v_*$ at $t = 0$, then $u^* \le v_*$ in $\mathbf{R}^2 \times [0, T)$.*

The main idea for the proof is to regard a level set of solutions as the graph of a function and reduce the problem to the graph-version (G) of $V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n}))$ by using the slicing lemma. We shall first discuss relation of maximum of a function and its slice.

Lemma 6.2. *For $k = 1, 2$ assume that $\Psi \in C^k(\mathbf{R}^d)$ with $d \ge 2$. Assume that for a given $\hat{\xi} \in \mathbf{R}^d$ the set*

$$C = \{z \in \mathbf{R}^d;\ \Psi(z) < \Psi(\hat{\xi})\}$$

is bounded and convex. Assume that $\nabla\Psi$ does not vanish on ∂C and that $\nabla\Psi(\hat{\xi}) = (\alpha, -q)$ for some $q > 0$ and $\alpha \in \mathbf{R}^{d-1}$. Let $\psi : \mathbf{R}^{d-1} \to (-\infty, \infty]$ denote

$$\psi(z') = \inf\{z_d;\ z = (z', z_d) \in C\},\ \ z' = (z_1, \dots, z_{d-1}) \in \mathbf{R}^{d-1}$$

and dom $\psi = \{z' \in \mathbf{R}^{d-1}; \psi(z') < \infty\}$. Then the following two properties hold.

(i) ψ is convex, dom ψ is bounded and $\psi(z') - \alpha \cdot (z' - \hat{\xi}')/q$ takes its minimum $\hat{\xi}_d$ at $z' = \hat{\xi}' = (\hat{\xi}_1, \dots, \hat{\xi}_{d-1})$, where $\hat{\xi} = (\hat{\xi}', \hat{\xi}_d) \in \mathbf{R}^{d-1} \times \mathbf{R}$ and dom ψ is bounded;

(ii) there is $m > 0$ such that following two properties hold:

(a) ψ is C^k on the set $\{z' \in \mathbf{R}^{d-1}; \psi(z') - \alpha \cdot (z' - \hat{\xi}')/q < \psi(\hat{\xi}') + m\}$;

(b) $\psi(z') < z_d \le \psi(\hat{\xi}') + \alpha \cdot (z' - \hat{\xi}')/q + m$ for $z = (z', z_d)$ implies $z \in$ int C.

Proof. The first part of (i) follows from the convexity of C and the assumption on $\nabla\Psi(\hat{\xi})$. The compactness of C implies that dom ψ is bounded.

The part (ii) follows from the implicit function theorem and the convexity of ψ (Figure 3). ■

Lemma 6.3. *Let $\Psi \in C^k(\mathbf{R}^d)$, k, d, $\hat{\xi}$, ψ and m be as in Lemma 6.2. Assume that U and $-V$ are upper semicontinuous with values in $[-\infty, \infty)$. Assume that*

$$U(z) - V(\zeta) - \Psi(z-\zeta) \leq 0 \quad \text{for all} \quad (z,\zeta) \in \mathbf{R}^d \times \mathbf{R}^d \tag{6.1}$$

and that the equality holds at $(\hat{z}, \hat{\zeta})$ with $\hat{\xi} = \hat{z} - \hat{\zeta}$. Set

$$U^{\#}(z') = \sup\{z_d;\ U(z', z_d) \geq U(\hat{z}),\ z_d - \hat{\zeta}_d \leq \psi(\hat{\xi}') + \alpha \cdot (z' - \hat{z}')/q + m/2\},$$
$$V_{\#}(\zeta') = \inf\{\zeta_d;\ V(\zeta', \zeta_d) \leq V(\hat{\zeta}),\ \hat{z}_d - \zeta_d \leq \psi(\hat{\xi}') + \alpha \cdot (\hat{\zeta}' - \zeta')/q + m/2\},$$

where $\hat{z} = (\hat{z}', \hat{z}_d)$, $\hat{\zeta} = (\hat{\zeta}', \hat{\zeta}_d) \in \mathbf{R}^{d-1} \times \mathbf{R}$. Then $\hat{z}_d = U^{\#}(\hat{z}')$, $\hat{\zeta}_d = V_{\#}(\hat{\zeta}')$ and

$$U^{\#}(z') - V_{\#}(\zeta') - \psi(z' - \zeta') \leq 0 \quad \text{for all} \quad (z', \zeta') \in \mathbf{R}^{d-1} \times \mathbf{R}^{d-1}. \tag{6.2}$$

The equality holds at $(\hat{z}', \hat{\zeta}')$. If the equality of (6.2) is valid at $(\tilde{z}', \tilde{\zeta}')$, then the equality of (6.1) holds for $(\tilde{z}, \tilde{\zeta})$ with $z = (\tilde{z}', U^{\#}(\tilde{z}'))$, $\zeta = (\tilde{\zeta}', V_{\#}(\tilde{\zeta}'))$. Moreover, $U(\tilde{z}) = U(\hat{z})$ and $V(\tilde{\zeta}) = V(\hat{\zeta})$.

Proof. To prove (6.2) we may assume that $U^{\#}(z') > -\infty$, $V_{\#}(\zeta') < \infty$ and $\psi(z' - \zeta') < \infty$. Since U and $-V$ are upper semicontinuous, $z_d^0 = U^{\#}(z')$ and $\zeta_d^0 = V_{\#}(\zeta')$ satisfy

$$U(z', z_d^0) \geq U(\hat{z}),\ V(\zeta', \zeta_d^0) \leq V(\hat{\zeta}) \tag{6.3}$$

and

$$\begin{aligned} z_d^0 - \hat{\zeta}_d &\leq \psi(\hat{\xi}') + \alpha \cdot (z' - \hat{z}')/q + m/2, \\ \hat{z}_d - \zeta_d^0 &\leq \psi(\hat{\xi}') + \alpha \cdot (\hat{\zeta}' - \zeta')/q + m/2. \end{aligned} \tag{6.4}$$

By (6.4) and $\hat{\xi}_d = \psi(\hat{\xi}')$ we observe that

(6.5)
$$\begin{aligned} z_d^0 - \zeta_d^0 &= z_d^0 - \hat{\zeta}_d + \hat{z}_d - \zeta_d^0 - (\hat{z}_d - \hat{\zeta}_d) \\ &\leq \psi(\hat{\xi}') + \alpha \cdot (z' - \hat{z}')/q + m/2 + \psi(\hat{\xi}') + \alpha \cdot (\hat{\zeta}' - \zeta')/q + m/2 - \psi(\hat{\xi}') \\ &= \psi(\hat{\xi}') + \alpha \cdot (z' - \zeta' - \hat{\xi}')/q + m. \end{aligned}$$

Thus, we have (6.2), i.e., $z_d^0 - \zeta_d^0 \leq \psi(z' - \zeta')$ since otherwise $(z', z_d^0) - (\zeta', \zeta_d^0) \in \operatorname{int} C$ by Lemma 6.2 (ii)(b) which contradicts (6.1) because of (6.3).

Since $\hat{\xi} \in \partial C$, we see, by definition of ψ, that

$$\hat{z}_d - \hat{\zeta}_d = \psi(\hat{z}' - \hat{\zeta}').$$

By definition $\hat{z}_d \leq U^{\#}(\hat{z}')$ and $\hat{\zeta}_d \geq V^{\#}(\hat{\zeta}')$ so we obtain

$$\begin{aligned} 0 &= \hat{z}_d - \hat{\zeta}_d - \psi(\hat{z}' - \hat{\zeta}') \\ &\leq U^{\#}(\hat{z}') - V_{\#}(\hat{\zeta}') - \psi(\hat{z}' - \hat{\zeta}') \leq 0. \end{aligned}$$

We have thus proved $\hat{z}_d = U^{\#}(\hat{z}')$, $\hat{\zeta}_d = V_{\#}(\hat{\zeta}')$ as well as $U^{\#}(\hat{z}') - V_{\#}(\hat{\zeta}') - \psi(\hat{z}' - \hat{\zeta}') = 0$.

If the equality of (6.2) holds for $(\tilde{z}', \tilde{\zeta}')$ i.e.,

$$\tilde{z}_d - \tilde{\zeta}_d = \psi(\tilde{z}' - \tilde{\zeta}')$$

with $\tilde{z}_d = U^{\#}(\tilde{z}')$, $\tilde{\zeta}_d = V_{\#}(\tilde{\zeta}')$, then $\tilde{z} - \tilde{\zeta} \in \partial C$, i.e., $\Psi(\tilde{z} - \tilde{\zeta}) = \Psi(\hat{z} - \hat{\zeta})$. This together with (6.3) yields

$$\begin{aligned} &U(\tilde{z}) - V(\tilde{\zeta}) - \Psi(\tilde{z} - \tilde{\zeta}) \\ &\geq U(\hat{z}) - V(\hat{\zeta}) - \Psi(\hat{z} - \hat{\zeta}) = 0. \end{aligned}$$

Since we assume (6.1), this implies that the equality of (6.1) holds for $(\tilde{z}, \tilde{\zeta})$ and that $U(\tilde{z}) = U(\hat{z})$ and $V(\tilde{\zeta}) = V(\hat{\zeta})$. ■

Proof of Comparison principle (Theorem 6.1).

Step 1 (Doubling variables). By Lemma 2.6 on localization we may assume that u and $-v$ is a sub- and supersolution of (L) in $\mathbf{R}^2 \times (0,T]$. We may assume that u and $-v$ are upper semicontinuous in $\mathbf{R}^2 \times [0,T]$ with values in $[-\infty,\infty)$ by considering u^* and v_* instead of u and v. For $\sigma > 0$ we set $u_\sigma(x,t) = u(x,t) - \sigma t$, $v^\sigma(y,s) = v(s,y) + \sigma s$. By definition u_σ is a subsolution of

$$u_t - |\nabla u| g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) + \sigma = 0 \quad \text{in} \quad \mathbf{R}^2 \times (0,T] \tag{6.6}$$

and v^σ is a supersolution of

$$v_s - |\nabla v| g(-\nabla v/|\nabla v|,\ \Lambda_\gamma(v)) - \sigma = 0 \quad \text{in} \quad \mathbf{R}^2 \times (0,T]. \tag{6.7}$$

Here we define subsolution of (6.6) (resp. supersolution of (6.7)) by replacing $\varphi_t(\hat{x},\hat{t})$ in the inequality (i) and (ii) of Definition 2.1 by $\varphi_t(\hat{x},\hat{t})+\sigma$ (resp. $\varphi_t(\hat{x},\hat{t})-\sigma$). (See Remark 2.5 for solutions in $\mathbf{R}^2 \times (0,T]$.) Let K be a closed convex set in $\mathbf{R}^2$ such that ∂K (with inward orientation) is C^2_γ-curve and that int K contains the origin of $\mathbf{R}^2$. Let P_K be its Minkowski function, i.e.,

$$P_K(x) = \inf\{\lambda > 0;\ x/\lambda \in K\}.$$

We set

$$\varphi(x,t) = (P_K(x))^4 + t^2, \quad t \in \mathbf{R},\ x \in \mathbf{R}^2.$$

Evidently, φ belongs to $A_\gamma(\mathbf{R}^2 \times \mathbf{R})$.

Assume that the conclusion were false. Then $u(x_0,t_0) > v(x_0,t_0)$ for some $x_0 \in \mathbf{R}^2$, $t_0 \in (0,T]$. For sufficiently small $\sigma > 0$ we would have $u_\sigma(x_0,t_0) > v^\sigma(x_0,t_0)$. We then double "variables" and consider

$$\Phi(x,t,y,s) = u_\sigma(x,t) - v^\sigma(y,s) - \eta\varphi(x-y,t-s)$$

for $\eta > 0$. The maximum $\max_{\mathbf{R}^2\times[0,T]} \Phi$ would be positive since $\Phi(x_0, t_0, x_0, t_0) > 0$. (By assumptions on behaviour of space infinity for u and v the maximum is always attained.) Since we assume that $u_\sigma \leq v^\sigma$ at $t = 0$, by a standard argument [CIL] for sufficiently large η a maximizer $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ of Φ (over $\mathbf{R}^2 \times [0, T]$) always satisfies $\hat{t} > 0$ and $\hat{s} > 0$. We shall fix such $\eta > 0$. There are two cases.

Case 1. $\nabla\varphi(\hat{x} - \hat{y}, \hat{t} - \hat{s}) = 0$ (which implies $\hat{x} = \hat{y}$ and $\nabla\nabla\varphi(\hat{x} - \hat{y}, \hat{t} - \hat{s}) = 0$).

Case 2. $\nabla\varphi(\hat{x} - \hat{y}, \hat{t} - \hat{s}) \neq 0$ (i.e., $\hat{x} \neq \hat{y}$).

In the first case we should handle the singularity at $\nabla u = 0$ of the level set equations. Fortunately, it turns out that this case is easily handled as in classical reference for the level set method including [ES1], [CGG1], [GGIS], [IS], [G1]. In fact since u_σ is a subsolution of (6.6) and

$$(x, t) \to u_\sigma(x, t) - v^\sigma(\hat{y}, \hat{s}) - \varphi(x - \hat{y}, t - \hat{s})$$

takes its maximum over $\mathbf{R}^2 \times [0, T]$ at $(\hat{x}, \hat{t})$ we have

$$\varphi_t(\hat{x} - \hat{y},\ \hat{t} - \hat{s}) + \sigma \leq 0.$$

Similarly, since v^σ is a supersolution of (6.7) we have

$$\varphi_t(\hat{x} - \hat{y},\ \hat{t} - \hat{s}) - \sigma \geq 0.$$

These two inequalities imply $2\sigma \leq 0$ which is a contradiction. So Case 1 does not occur.

Case 2 is divided into two subcases.

(i) $\nabla P_K(\hat{x} - \hat{y})/|\nabla P_K(\hat{x} - \hat{y})| \notin -\mathcal{N}$ (ii) $\nabla P_K(\hat{x} - \hat{y})/|\nabla P_K(\hat{x} - \hat{y})| \in -\mathcal{N}$

and from (ii) it follows a contradiction by a standard argument. So the only case for which a new argument is necesssary is the case (ii). However, since our argument applies to both (i) and (ii), we do not distinguish these two subcases.

Step 2 (Reduction to graph-like solutions). We may assume that Φ does not take its maximum at $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ if $\overline{t} - \overline{s} \neq \hat{t} - \hat{s}$ by replacing φ by

$$\varphi(x, t) = (P_K(x))^4 + t^2 + (t - (\hat{t} - \hat{s}))^2.$$

We may also assume that

$$\nabla P_K(\hat{x} - \hat{y}) / |\nabla P_K(\hat{x} - \hat{y})| = (0, -1), \quad \hat{x} \neq \hat{y}$$

by a rotation. By taking K slightly smaller but keeping the value $P_K(\hat{x} - \hat{y})$ unchanged we may assume that Φ does not take its maximum at $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ unless $\overline{t} - \overline{s} = \hat{t} - \hat{s}$, $\overline{x}_2 - \overline{y}_2 = \hat{x}_2 - \hat{y}_2$ and $P_K(\hat{x} - \hat{y}) = P_K(\overline{x} - \overline{y})$, where $\overline{x} = (\overline{x}_1, \overline{x}_2)$, $\overline{y} = (\overline{y}_1, \overline{y}_2) \in \mathbf{R}^2$.

For $\rho > 0$ we set

$$u_{\rho\sigma}(x, t) = u_\sigma(x_1, x_2 + \rho t, t).$$

This shift in the direction of x_2 is crucial in our proof. Since Φ does not take its maximum at $(\overline{x}, \overline{t}, \overline{y}, \overline{s})$ unless $\overline{t} - \overline{s} = \hat{t} - \hat{s}$, $\overline{x}_2 - \overline{y}_2 = \hat{x}_2 - \hat{y}_2$, $P_K(\hat{x} - \hat{y}) = P_K(\overline{x} - \overline{y})$, for sufficiently small $\rho > 0$

$$\Phi_\rho(x, t, y, s) = u_{\rho\sigma}(x, t) - v^\sigma(y, s) - \eta\varphi(x - y, t - s)$$

takes its maximum at $(x_\rho, t_\rho, y_\rho, s_\rho)$ so that $t_\rho - s_\rho$, $x_\rho - y_\rho$ and $P_K(x_\rho - y_\rho)$ are close to $\hat{t} - \hat{s}$, $\hat{x}_2 - \hat{y}_2$, $P_K(\hat{x} - \hat{y})$, respetively. By taking $\rho > 0$ smaller so that $\nabla P_K(x_\rho - y_\rho) / |\nabla P_K(x_\rho - y_\rho)|$ is close to $(0, -1)$ we may assume that

$$\partial_{x_2} P_K(x_\rho - y_\rho) < 0$$

and $\nabla P_K(x_\rho - y_\rho) \notin -\mathcal{N}$ unless $\partial_{x_1} P_K(x_\rho - y_\rho) = 0$. We shall fix $\rho > 0$ and denote $(x_\rho, t_\rho, y_\rho, s_\rho)$ simply by $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$.

We set

$$\Psi(t,x_1,x_2) = \eta\varphi(x_1,x_2,t) + \beta,\ \ \beta = \Phi_\rho(\hat{x},\hat{t},\hat{y},\hat{s})$$

in Lemma 6.2 with $d = 3$ and $\hat{\xi} = (\hat{t}-\hat{s}, \hat{x}_1 - \hat{y}_1, \hat{x}_2 - \hat{y}_2)$ and observe that

$$(\partial_t, \nabla)\Psi(\hat{\xi}) = (\alpha_1, \alpha_2, -q)$$

with $\alpha_1 = 2\eta(\hat{t}-\hat{s})$, $(\alpha_2, -q) = 4\eta(P_K(\hat{x}-\hat{y}))^3\nabla\ P_K(\hat{x}-\hat{y})$. We choose m as in Lemma 6.2 and define

(6.8)
$$\begin{aligned} u_\rho^\#(t,x_1) = \sup\{x_2;\ & u_{\rho\sigma}(x_1,x_2,t) \geq u_{\rho\sigma}(\hat{x},\hat{t}), \\ & x_2 - \hat{y}_2 \leq \psi(\hat{\xi}') + \frac{\alpha_1}{q}(t-\hat{t}) + \frac{1}{q}\alpha_2(x_1 - \hat{x}_1) + \frac{m}{2}\}, \end{aligned}$$

(6.9)
$$\begin{aligned} v_\#(s,y_1) = \inf\{y_2;\ & v^\sigma(y_1,y_2,t) \leq v^\sigma(\hat{y},\hat{s}), \\ & \hat{x}_2 - y_2 \leq \psi(\hat{\xi}') + \frac{\alpha_1}{q}(\hat{s}-s) + \frac{1}{q}\alpha_2(\hat{y}_1 - y_1) + \frac{m}{2}\}. \end{aligned}$$

Since Φ_ρ takes maximum at $(\hat{x},\hat{t},\hat{y},\hat{s})$,

$$u_{\rho\sigma}(x,t) - v^\sigma(y,s) - \Psi(t-s, x-y) \leq 0 \tag{6.10}$$

and the inequality holds for $(\hat{x},\hat{t},\hat{y},\hat{s})$. Applying Lemma 6.3 yields

$$u_\rho^\#(t,x_1) - v_\#(s,y_1) - \psi(t-s, x_1-y_1) \leq 0 \tag{6.11}$$

and the equality holds for $(\hat{t},\hat{x}_1,\hat{s},\hat{y}_1)$.

We would like to prove that $u_\rho^\# + \rho t$ and $v_\#$ are sub- and supersolutions of the graph version (G) of (L) near $(\hat{t},\hat{x}_1)$ and $(\hat{s},\hat{y}_1)$, respectively. Since u_σ is a subsolution of

$$u_t - |\nabla u| g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) = 0 \quad \text{in } \mathbf{R}^2 \times (0,T],$$

$u_{\rho\sigma}$ is a subsolution of

$$u_t - |\nabla u|\{g(-\nabla u/|\nabla u|,\ \Lambda_\gamma(u)) + \rho\partial_{x_2}u/|\nabla u|\} = 0 \quad \text{in} \quad \mathbf{R}^2 \times (0,T]$$

which is the level set equation of

$$V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n})) - \rho n_2, \tag{6.12}$$

where $\mathbf{n} = (n_1, n_2)$. By the slicing lemma (Lemma 5.1) $u^\#_\rho$ is a subsolution of the graph version of (6.12) in

$$Q_1 = \{(t,x_1) \in (0,T] \times \mathbf{R};\ u^\#_\rho(t,x_1) - \hat{y}_2 < \psi(\hat{\xi}') + \frac{\alpha_1}{q}(t-\hat{t}) + \frac{\alpha_2}{q}(x_1 - \hat{x}_1) + \frac{m}{2}\}.$$

Thus $u^\#_\rho + \rho t$ is a subsolution of the graph version (G) of $V = g(\mathbf{n}, \Lambda_\gamma(\mathbf{n}))$. Similarly by the slicing lemma $v_\#$ is a supersolution of (G) in

$$Q_2 = \{(s,y_1) \in (0,T] \times \mathbf{R};\ \hat{x}_2 - v_\#(s,y_1) < \psi(\hat{\xi}') + \frac{\alpha_1}{q}(\hat{s}-s) + \frac{\alpha_2}{q}(\hat{y}_1 - y_1) + \frac{m}{2}\}.$$

Both Q_1 and Q_2 are relatively open in $(0,T] \times \mathbf{R}$ and containing $(\hat{t},\hat{x}_1)$ and $(\hat{s},\hat{y}_1)$ respectively.

Step 3 (Application of faceted maximum principle). We consider

$$\begin{aligned}\tilde{\Phi}(t,x_1,s,y_1) &= u^\#_\rho(t,x_1) - v_\#(s,y_1) - \psi(t-s, x_1 - y_1)\\ &= u^\#(t,x_1) - v_\#(s,y_1) - \psi(t-s,x_1-y_1) - \rho t,\end{aligned}$$

where $u^\# = u^\#_\rho + \rho t$. As we know Φ is maximized at $(\hat{t},\hat{x}_1,\hat{s},\hat{y}_1)$ in $Q_1 \times Q_2$.

Assume that $\partial_{x_1}\psi(\hat{t}-\hat{s},\hat{x}_1-\hat{y}_1) = 0$ does not belong to the singular direction $\mathcal{N}$. Since $u^\#$ and $v_\#$ are sub- and supersolutions of (G) in Q_1 and Q_2 respectively, a standard argument implies $\rho \le 0$ which is a contradiction.

We may assume that $\partial_{x_1}\psi(\hat{t}-\hat{s},\hat{x}_1-\hat{y}_1) = 0$ belongs to the singular direction so that $\psi(\hat{t}-\hat{s},\cdot)$ is faceted at $\hat{x}_1-\hat{y}_1$. By our choice of ρ we may assume that

$\psi_{x_1}(\hat{t}-\hat{s},\cdot)=0$ on the faceted region I containing $\hat{x}_1-\hat{y}_1$. Note that in this case $I=\{x_1;\psi(\hat{t}-\hat{s},\hat{x}_1-\hat{y}_1)=\psi(\hat{t}-\hat{s},x_1)\}$. We apply the faceted maximum principle (Lemma 4.4) for (G) with $S(t,s)=\rho t$ to get a contradiction $\rho\leq 0$. ■

Remark 6.4. As in [CGG1], [G1] the invariance (Lemma 3.1) and the comparison (Theorem 6.1) yields the uniqueness of level sets in the introduction. Thus Theorem 1 in the introduction has been proved except the item (iv) which will be proved in §8.

7. Stability

We shall prove the stability of solutions for the level set equation (L_ε)

$$(\mathrm{L}_\varepsilon) \qquad u_t-|\nabla u|g_\varepsilon(-\nabla u/|\nabla u|,\ \Lambda_{\gamma_\varepsilon}(u))=0$$

as ε tends to zero.

Theorem 7.1 (Stability). *Assume that $\gamma_\varepsilon\in\mathcal{I}$ and $g_\varepsilon\in\mathcal{G}_0$ for $\varepsilon>0$. Assume that $\gamma_\varepsilon\to\gamma$ locally uniformly on $\mathbf{R}^2$ and that $g_\varepsilon\to g$ locally uniformly on $S^1\times\mathbf{R}$ as $\varepsilon\to 0$. Assume moreover the uniform growth bound (1.6). For $T\in(0,\infty]$ let $u^\varepsilon(\varepsilon>0)$ be a subsolution (resp. supersolution) of (L_ε) in $Q=\Omega\times(0,T)$, where Ω is an open set in $\mathbf{R}^2$. Then the relaxed upper limit $\overline{u}=\limsup^*_{\varepsilon\to 0}u^\varepsilon$ (resp. relaxed lower limit $\underline{u}=\liminf\limits_{\varepsilon\to 0}{}_* u^\varepsilon$) is a subsolution (resp. supersolution) of (L) in $\Omega\times(0,T)$ provided that $\overline{u}<\infty$ (resp. $\underline{u}>-\infty$) on $\overline{\Omega}\times[0,T)$.*

Our strategy of the proof is to reduce the problem to graph-like solutions by using the slicing lemma (Lemma 5.1). For this purpose we prepare several properties of relaxed upper limit. We begin by recalling its definition. Let X_i be a metric space with metric d_i, $i=1,2$ and let $u:X_1\times X_2\to\mathbf{R}\cup\{\pm\infty\}$. For $\lambda\in X_2$

we set $u_\lambda(z) = u(z,\lambda)$. Then the relaxed upper limit of $\{u_\lambda\}$ as $\lambda \to \lambda_0$ is defined by

$$\begin{aligned}\overline{u}(z) &= \limsup_{\lambda\to\lambda_0}{}^* \ u_\lambda(z) \\ &= \lim_{\delta\downarrow 0} \sup\{u(y,\lambda); y \in X_1,\ \lambda \in X_2,\ d_1(y,z) < \delta,\ d_2(\lambda_0,\lambda) < \delta,\ \lambda \neq \lambda_0\}\end{aligned} \tag{7.1}$$

for $z \in X_1$. By definition $\overline{u}$ is upper semicontinuous on X_1. The relaxed lower limit $\underline{u}(z) = \liminf_{\lambda\to\lambda_0}{}_* \ u_\lambda(z)$ is defined by $-\limsup^*_{\lambda\to\lambda_0}(-u_\lambda(z))$. Let χ_E denote the characteristic function of a set $E \subset X_1$ i.e., $\chi_E(z) = 1$ if $z \in E$ and otherwise $\chi_E(z) = 0$.

Lemma 7.2 (Convergence of level set). *Let X_1 be a metric space. Assume that $\overline{u} = \limsup^*_{\varepsilon\to 0} \ u_\varepsilon$ and $u_\varepsilon : X_1 \to \mathbf{R}^\cup\{\pm\infty\}$, where $\varepsilon \in (0,1)$ is a parameter. Then, for each $c \in \mathbf{R}$*

$$\chi_{\overline{u}\geq c}(z) = \limsup_{\substack{\ell\to c\\ \varepsilon\to 0}}{}^* \ \chi_{u_\varepsilon \geq \ell}(z),\ z \in X_1. \tag{7.2}$$

Here $\chi_{f\geq c}$ denotes the characteristic function χ_E of the super level set $E = \{f(z) \geq c\}$ of a function f.

Proof. The condition $\chi_{\overline{u}\geq c}(z) = 1$ is equivalent to say that there exists a sequence $\varepsilon_j \to 0$, $z_j \to z$ such that $\liminf_{j\to\infty} u_{\varepsilon_j}(z_j) \geq c$. We observe the last inequality is equivalent to $\lim_{j\to\infty} \chi_{u_{\varepsilon_j}\geq \ell_j}(z_j) = 1$ with $\ell_j \to c$ by setting $\ell_j = \min(u_{\varepsilon_j}(z_j), c)$. Thus (7.2) follows. ■

Lemma 7.3 (Convergence of slices). *Let X, Z be metric spaces. For $\eta \in Z$ let E_η be a subset of $X_1 = X \times \mathbf{R}$. Assume that $E \subset X_1$ satisfies $\chi_E = \limsup^*_{\eta\to\eta_0} \chi_{E_\eta}$ for some $\eta_0 \in Z$. Let $U : X \to \mathbf{R}^\cup\{-\infty\}$, $U_\eta : X \to \mathbf{R}^\cup\{-\infty\}$ be*

$$U(x) = \sup\{y;\ (x,y) \in E\}, U_\eta(x) = \sup\{y;\ (x,y) \in E_\eta\}$$

for $x \in X$. Then

$$U(x) = \limsup_{\eta \to \eta_0}{}^* \ U_\eta(x),\ x \in X. \tag{7.3}$$

Proof. By definition the set E is closed so that $(x_0, U(x_0)) \in E$ for $x_0 \in X$. Since χ_E is the relaxed upper limit of χ_{E_η}, there is $(x_\eta, y_\eta) \in E_\eta$ such that $x_\eta \to x_0$, $y_\eta \to U(x_0)$ as $\eta \to \eta_0$. By the definition U_η we see that $y_\eta \le U_\eta(x_\eta)$. Thus

$$U(x_0) \le \limsup_{\eta \to \eta_0}{}^* \ U_\eta(x_0)(:= \alpha).$$

Clearly, $(x_0, \alpha) \in E$ so that $\alpha \le U(x_0)$. We thus obtain (7.3). ■

Proof of Theorem 7.1. We shall present a proof only for subsolutions since the proof for supersolutions is symmetric. Assume that $(\varphi, (\hat{x}, \hat{t})) \in C(Q) \times Q$ satisfies

$$\max_Q(\overline{u} - \varphi) = (\overline{u} - \varphi)(\hat{x}, \hat{t}) \tag{7.4}$$

and $\varphi \in A_\gamma(D \times (t_1, t_2))$ for some open set $D \ni \hat{x}$ and $t_1 < \hat{t} < t_2$. We shall prove (i), (ii) of Definition 2.1.

Case 1. $\nabla\varphi(\hat{x}, \hat{t}) \neq 0$. In this case we may assume that $\nabla\varphi(\hat{x}, \hat{t}) = (0, -q)$ for some $q > 0$ by rotation in $\mathbf{R}^2$. We may take D smaller so that $D = I \times (\hat{x}_2 - \delta, \hat{x}_2 + \delta)$, $\delta > 0$ and I is an open interval containing $\hat{x}_1$ so that the level set

$$\{(x, t) \in D \times (t_1, t_2);\ \varphi(x, t) = \varphi(\hat{x}, \hat{t})\}$$

is of form $x_2 = \psi(t, x_1)$ and ψ is C^2 in $(t_1, t_2) \times I$ and $\psi(t, \cdot)$ has one faceted region in I if $(0, -1) \in -\mathcal{N}$ and otherwise $\psi(t, \cdot)$ has no faceted regions in I. We may also assume that $\overline{u}(x_1, x_2, t) < \overline{u}(\hat{x}, \hat{t})$ for $x_2 > \psi(t, x_1)$, $x_1 \in I$ $t \in (t_1, t_2)$ for $x_2 \in (\hat{x}_2 - \delta, \hat{x}_2 + \delta)$ by taking δ smaller. We set

$$\overline{u}^{\#}(t, x_1) = \sup\{x_2;\ \overline{u}(x_1, x_2, t) \ge \overline{u}(\hat{x}, \hat{t}),\ (x_1, x_2, t) \in D \times (t_1, t_2)\}$$

for $(t, x_1) \in (t_1, t_2) \times I$. Since $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$, we see, by Lemmas 7.2 and 7.3, that

$$\overline{u}^\# = \limsup_{\substack{\varepsilon\to 0 \\ \ell\to c}}{}^* \; u^\#_{\varepsilon\ell} \quad \text{with} \quad c = \overline{u}(\hat{x}, \hat{t}),$$

where

$$u^\#_{\varepsilon\ell} = \sup\{x_2;\; u^\varepsilon(x_1, x_2, t) \geq \ell,\; (x_1, x_2, t) \in D \times (t_1, t_2)\}.$$

By the slicing lemma (Lemma 5.1 and Remark 5.2) the function $u^\#_{\varepsilon\ell}$ is a subsolution of the graph version (G_ε) of (L_ε) in $Z_\varepsilon = (t_1, t_2) \times I \cap \{\hat{x}_2 - \delta < u^\#_{\varepsilon\ell} < \hat{x}_2 + \delta\}$. By the stability result [GG5] for the graph-like solutions, $\overline{u}^\#$ is a subsolution of the graph version (G) of (L) in $Z = (t_1, t_2) \times I \cap \{\hat{x}_2 - \delta < u^\# < \hat{x}_2 + \delta\}$. (Note that usually stability result does not have the restriction $\{\hat{x}_2 - \delta < u^\#_{\varepsilon\ell} < \hat{x}_2 + \delta\}$, but the proof is still valid with this type of restriction.)

Since $\overline{u}^\#$ is a subsolution of (G), we have

$$\psi_t(\hat{t}, \hat{x}_1) + F(\psi_{x_1}(\hat{t}, \hat{x}_1),\; \Lambda_W(\psi(\hat{t}, \cdot))(\hat{x}_1)) \leq 0$$

where F and W is defined in (G). This is equivalent to (i) of Definition 2.1.

Case 2. $\nabla\varphi(\hat{x}, \hat{t}) = 0$, $\nabla\nabla\varphi(\hat{x}, \hat{t}) = 0$. In this case there is a C^2 functions ρ on $\mathbf{R}$ such that

$$f(x) - f(\hat{x}) \leq \rho(|x - \hat{x}|), \quad x \in \mathbf{R}^2$$

such that $\rho(0) = \rho'(0) = \rho''(0) = 0$, where $\varphi(x, \hat{t}) = f(x)$. Let $\hat{W}$ denote the Wulff shape of $\check{\gamma}$ where $\check{\gamma}(p) = \gamma(-p)$. Then on each facet of $\hat{W}$ the inward normal $\mathbf{n}$ belongs to $\mathcal{N}$. Let $\hat{W}_\varepsilon$ denote the Wulff shape of $\check{\gamma}_\varepsilon$, where $\check{\gamma}_\varepsilon(p) = \gamma_\varepsilon(-p)$. Since $\check{\gamma}_\varepsilon \to \check{\gamma}$ as stated in Lemma 7.4 in the end of this section, there is a sequence of convex closed sets $\{K_\varepsilon\}_{0<\varepsilon<1}$ such that $K_\varepsilon \to \hat{W}$ in Hausdorff distance topology and ∂K_ε (with inward orientation) is $C^2_{\gamma_\varepsilon}$-curve and that

$$\sup\{\Lambda_{\gamma_\varepsilon}(x;\; \partial K_\varepsilon),\; x \in K_\varepsilon,\; 0 < \varepsilon < 1\} < \infty.$$

We may assume that int $K_\varepsilon \ni 0$ so that its Minkowski function P_{K_ε} is well-defined in $\mathbf{R}^2$ and $\sup\{|\nabla P_{K_\varepsilon}(x)|;\, x \in \mathbf{R}^2,\, 0 < \varepsilon < 1\} < \infty$. We consider

$$\varphi_\varepsilon(x,t) = \rho(\lambda P_{K_\varepsilon}(x - \hat{x})) + h(t),\ \ h(t) = \varphi(x,t) - f(x),$$

where $\lambda > 0$ is a constant such that $|x| \leq \lambda P_{K_\varepsilon}(x)$ for $x \in \mathbf{R}^2$, $0 < \varepsilon < 1$. Since $K_\varepsilon \to \hat{W}$, we see that φ_ε convergenes to

$$\tilde{\varphi}(x,t) = \rho(\lambda P_{\hat{W}}(x - \hat{x})) + h(t)$$

locally uniformly in Q. Since we may assume that $\overline{u} - \tilde{\varphi}$ takes its strict maximum at $(\hat{x}, \hat{t})$, there is a sequence $(x_\varepsilon, t_\varepsilon) \to (\hat{x}, \hat{t})$ such that

$$\max_{Q}(u_\varepsilon - \varphi_\varepsilon) = (u_\varepsilon - \varphi_\varepsilon)(x_\varepsilon, t_\varepsilon)$$

by convergence of maximum points (e.g. [GG5, Lemma 2.16]). Since u_ε is a subsolution of (L_ε) in Q,

$$h'(t_\varepsilon) - |\nabla f_\varepsilon(x_\varepsilon)| g_\varepsilon(-\nabla f_\varepsilon(x_\varepsilon)/|\nabla f_\varepsilon(x_\varepsilon)|,\ \Lambda_{\gamma_\varepsilon}(f_\varepsilon)(x_\varepsilon)) \leq 0 \quad \text{if } x_\varepsilon \neq \hat{x} \tag{7.5}$$

and

$$h'(t_\varepsilon) \leq 0 \quad \text{if } x_\varepsilon = \hat{x}, \tag{7.6}$$

where $f_\varepsilon(x) = \rho(\lambda P_{K_\varepsilon}(x))$. From the construction of K_ε it follows that

$$\Lambda_{\gamma_\varepsilon}(x; \partial K_\varepsilon) = \Lambda_{\gamma_\varepsilon}(f_\varepsilon)(x_\varepsilon) P_{K_\varepsilon}(x_\varepsilon - \hat{x})$$

is bounded as $\varepsilon \to 0$ for $x_\varepsilon \neq \hat{x}$. Since $\nabla f_\varepsilon = \rho'(\lambda P_{K_\varepsilon}) \nabla P_{K_\varepsilon}$ and $\rho''(0) = 0$, for each $\delta > 0$

$$|\nabla f_\varepsilon(x_\varepsilon)| \leq \delta P_{K_\varepsilon}(x_\varepsilon - \hat{x})$$

for sufficiently small ε. Thus

$$|\nabla f_\varepsilon(x_\varepsilon)|\Lambda_{\gamma_\varepsilon}(f_\varepsilon)(x_\varepsilon) \to 0$$

as $\varepsilon \to 0$. Since g_ε satisfies (1.6), this implies that

$$|\nabla f_\varepsilon(x_\varepsilon)|g_\varepsilon(-\nabla f_\varepsilon(x_\varepsilon)/|\nabla f_\varepsilon(x_\varepsilon)|,\ \Lambda_{\gamma_\varepsilon}(f_\varepsilon)(x_\varepsilon)) \to 0$$

as $\varepsilon \to 0$ for $x_\varepsilon \neq \hat{x}$. Sending ε to zero in (7.5) and (7.6) we have $h'(\hat{t}) \leq 0$. ■

Lemma 7.4. *Assume that γ_ε and γ is convex, positively homogeneous of degree one in $\mathbf{R}^2$ and that $\gamma_\varepsilon(p) > 0$, $\gamma(p) > 0$ for $p \neq 0$, where $\varepsilon \in (0,1)$. Let W_{γ_ε} and W_γ be the Wulff shape of γ_ε and γ, respectively.*

(i) If $\gamma_\varepsilon \to \gamma$ locally uniformly, then W_{γ_ε} converges to W_γ in the Hausdorff distance topology, i.e. $d_H(W_{\gamma_\varepsilon}, W_\gamma) \to 0$, where d_H is the Hausdorff distance. Moreover,

$$\sup\{|\nabla P_{W_\varepsilon}(x)|;\ x \in \mathbf{R}^2,\ 0 < \varepsilon < 1\} < \infty.$$

(ii) If moreover $\gamma_\varepsilon, \gamma \in \mathcal{I}$, then there is a closed convex set K_ε such that $K_\varepsilon \to W_\gamma$ in the Hausdorff distance topology and

$$\sup\{\Lambda_{\gamma_\varepsilon}(x;\ \partial K_\varepsilon);\ x \in \partial K_\varepsilon,\ 0 < \varepsilon < 1\} < \infty,$$
$$\sup\{|\nabla P_{K_\varepsilon}(x)|;\ x \in \mathbf{R}^2,\ x \neq 0,\ 0 < \varepsilon < 1\} < \infty.$$

and that ∂K_ε is a $C^2_{\gamma_\varepsilon}$-curve with inward orientation.

The part (i) is standard (cf. [A]). It can be proved as in [GG5, Appendix]. The part (ii) is obtained by mollifying the corner of W_{γ_ε}. By our assumptions W_{γ_ε} and W_γ are compact and contain the origin as an interior point. The Hausdorff distance d_H for two sets A, B in a metric space is defined by

$$d_H(A,B) = \max(\sup_{x \in A} \operatorname{dist}(x,B),\ \sup_{y \in B} \operatorname{dist}(y,A)).$$

8. Convergence and its applications

As an application of the stability result we prove convergence of solutions of the initial value problem (L_ε).

Theorem 8.1 (Convergence). *Assume that $\gamma_\varepsilon \in \mathcal{I}$ and $g_\varepsilon \in \mathcal{G}_0$ for $\varepsilon > 0$. Assume that $\gamma_\varepsilon \to \gamma \in \mathcal{I}$ locally uniformly in $\mathbf{R}^2$ and that $g_\varepsilon \to g \in \mathcal{G}_0$ locally uniformly in $S^1 \times \mathbf{R}$ as $\varepsilon \to 0$. Assume that (1.6) holds. For $T \in (0,\infty)$ let $u^\varepsilon \in \mathcal{K}_{\alpha_\varepsilon}(\mathbf{R}^2 \times [0,T))$ be a solution of (L_ε) in $\mathbf{R}^2 \times (0,T)$ with initial data $u_0^\varepsilon \in \mathcal{K}_{\alpha_\varepsilon}(\mathbf{R}^2)$ for $\varepsilon > 0$, where α_ε is some real number. Assume that $u_0^\varepsilon \to u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ locally uniformly as $\varepsilon \to 0$ and spt $(u_0^\varepsilon - \alpha_\varepsilon) \subset B_R$ with R independent of $\varepsilon > 0$. Then $u = \lim_{\varepsilon\to 0} u^\varepsilon$ exists and the convergence is uniform in $\mathbf{R}^2 \times [0,T)$. Moreover, $u \in \mathcal{K}_\alpha(\mathbf{R}^2 \times [0,T))$ is a solution of (L) in $\mathbf{R}^2 \times (0,T)$ with initial data u_0.*

For any $\gamma \in \mathcal{I}$ one can approximate γ by $\gamma_\varepsilon \in \mathcal{I}$ such that the inward curvature of ∂ (Frank γ_ε) is positive and ∂ (Frank γ_ε) is smooth. It is well-known that for such γ_ε the initial value problem for (L_ε) with $g \in \mathcal{G}_0$ admits a unique global solution so Theorem 8.1 yields the existence of a solution of the initial value problem for (L). The uniqueness follows from the comparison principle (Theorem 6.1).

Corollary 8.2 (Existence). *Assume that $\gamma \in \mathcal{I}$ and $g \in \mathcal{G}_0$. Let $T \in (0,\infty)$ and $\alpha \in \mathbf{R}$. For any $u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ there exists a unique solution $u \in \mathcal{K}_\alpha(\mathbf{R}^2 \times [0,T))$ of (L) in $\mathbf{R}^2 \times (0,T)$ with initial data u_0.*

Our convergence result (Theorem 8.1) yields the convergence of the level set

$$\{u^\varepsilon \geq \ell\} := \{(x,t) \in \mathbf{R}^2 \times [0.T'];\ u^\varepsilon(x,t) \geq \ell\}, \quad 0 < T' < T$$

provided that "no fattening" occurs for the limit. We shall state it without using u^ε explicitly.

By the uniqueness of level sets in Theorem 1 in the introduction (cf. Remark 6.4) we say that $E^\varepsilon = \{u^\varepsilon \geq \ell\}$ (resp. $E = \{u \geq \ell\}$) is a (closed) *generalized solution*

$$(\mathrm{I}_\varepsilon) \qquad V = g_\varepsilon(\mathbf{n},\ \Lambda_{\gamma_\varepsilon}(\mathbf{n}))$$

(resp. (I)) with initial data E_0^ε (resp. $E_0 = \{u_0 \geq \ell\}$). Corollary 8.2 guarantees that generalized solution of (I) exists for any compact initial data E_0 since there is $u_0 \in \mathcal{K}_\alpha(\mathbf{R}^2)$ that satisfies $E_0 = \{u_0 \geq \ell\}$ for some ℓ. We say that E is *regular* if $E = \overline{\{u > \ell\}}$, i.e., "no fattening" occurs. The set $\{u > \ell\}$ is called an (open) generalized solution with initial data $\{u_0 > \ell\}$. From Theorem 8.1 we obtain a convergence of generalized solutions.

Corollary 8.3 (Convergence of generalized solutions). *Assume the same hypotheses of Theorem 8.1 concerning γ_ε, γ, g_ε and g. Let E_0^ε and E_0 be compact sets in $\mathbf{R}^2$. Let E^ε and E ($\subset \mathbf{R}^2 \times [0, T'], T' < \infty$) generalized solutions of (I_ε) and (I) with initial data E_0^ε and E_0, respectively. Assume that $d_H(E_0^\varepsilon, E_0) \to 0$ as $\varepsilon \to 0$. Assume that E is regular. Then*

(i) $d'_H(E^\varepsilon, E) \to 0$ as $\varepsilon \to 0$ where d'_H denotes te Hausdorff distance in $\mathbf{R}^2 \times [0, T']$.

(ii) Assume that $t \mapsto E(t)$ is continuous as a set valued function on $[0, T']$, where $E(t)$ is the cross-section of E at time t, i.e. $E(t) = \{x \in \mathbf{R}^2; (x, t) \in E\}$. Assume that $E(t) = \overline{D(t)}$ for all $t \in [0, T']$, where D is the open generalized solution of (I) with initial data int E_0. Then $d_H(E^\varepsilon(t), E(t)) \to 0$ as $\varepsilon \to 0$ and the convergence is uniform in $t \in [0, T']$.

Proof of Corollary 8.3 admitting Theorem 8.1. For E_0^ε we take a signed distance function u_0^ε (truncated the value from below by -1), i.e. $u_0^\varepsilon(x) = \mathrm{dist}(x, E_0^\varepsilon)$ for $x \in E_0^\varepsilon$ and $u_0^\varepsilon(x) = -\min(\mathrm{dist}(x, E_0^\varepsilon), 1)$ for $x \notin E_0^\varepsilon$ so that $u_0^\varepsilon \in \mathcal{K}_{-1}(\mathbf{R}^2)$.

Since $d_H(E_0^\varepsilon, E_0) \to 0$ as $\varepsilon \to 0$, we see that u_0^ε converges to some $u_0 \in \mathcal{K}_{-1}(\mathbf{R}^2)$ uniformly and that u_0 is the truncated distance function of E_0. Let u^ε be the solution of (L_ε) with $u^\varepsilon|_{t=0} = u_0^\varepsilon$ and let u be the solution of (L) with $u|_{t=0} = u_0$. By Theorem 8.1 u^ε converges to u uniformly in $\mathbf{R}^2 \times [0, T']$ for every $T' < \infty$. Since $E^\varepsilon = \{u^\varepsilon \geq 0\}$, $E = \{u \geq 0\}$, Corollary 8.3 follows from an elementary facts stated below.

Lemma 8.4. *Let Z and Y be compact metric spaces. Assume that $f_\lambda^\varepsilon \in C(Z)$ converges to $f_\lambda \in C(Z)$ as $\varepsilon \to 0$ uniformly in $\lambda \in Y$, where $\varepsilon \in (0,1)$. For a given $\ell \in \mathbf{R}$ assume that $\{z \in Z; f_\lambda(z) \geq \ell\} := \{f_\lambda \geq \ell\} = \overline{\{f_\lambda > \ell\}}$. Assume that the sets $\{f_\lambda \geq \ell\}$ and $\{f_\lambda^\varepsilon \geq \ell\}$ are compact in Z for all $\lambda \in Y$. Assume that for small $\varepsilon > 0$ the set $\{f_\lambda^\varepsilon \geq \ell\}$ is continuous in $\lambda \in Y$ with respect to the Hausdorff metric d_H in Z. Then $d_H(\{f_\lambda^\varepsilon \geq \ell\}, \{f_\lambda \geq \ell\}) \to 0$ as $\varepsilon \to 0$ uniformly in $\lambda \in Y$.*

To prove Corollary 8.3 (i) we take $Z = B_R \times [0, T']$ so that $\{u^\varepsilon \geq 0\} \subset Z$. We set $Y = \emptyset$ and $f^\varepsilon = u^\varepsilon$ to get (i). To show (ii) it suffices to take $Z = B_R$, $Y = [0, T']$ and $f_\lambda^\varepsilon = u^\varepsilon(\cdot, \lambda)$. ■

Proof of Lemma 8.4. We first note that for each $\eta > 0$ there is $\varepsilon_0 \in (0,1)$ such that $\{f_\lambda^\varepsilon \geq \ell\} \subset \{f_\lambda \geq \ell\}_\eta$ for all $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in Y$, where $A_\eta = \{z \in Z; \mathrm{dist}(z, A) \leq \eta\}$ for a subset A of Z. If not, for some $\eta > 0$ there is a sequence $\varepsilon_j \to 0$, $\{\lambda_j\} \subset Y$, $\{z_j\} \subset Z$ that satisfies dist $(z_j, \{f_{\lambda_j} \geq \ell\}) > \eta$ and $z_j \in \{f_{\lambda_j}^{\varepsilon_j} \geq \ell\}$. We may assume that $\lambda_j \to \lambda_0$, $z_j \to z_0$ as $j \to \infty$ by taking a subsequence. Since f_λ^ε converges to f_λ uniformly in Z and for $\lambda \in Y$, we have $f_{\lambda_j}^{\varepsilon_j}(z_j) \to f_{\lambda_0}(z_0)$. Since $f_{\lambda_j}^{\varepsilon_j}(z_j) \geq \ell$, this implies $f_{\lambda_0}(z_0) \geq \ell$. This is absurd since dist $(z_0, \{f_{\lambda_0} \geq \ell\}) = \lim_{j\to\infty}$ dist $(z_j, \{f_{\lambda_j} \geq \ell\}) \geq \eta$ by continuity of $\{f_\lambda \geq \ell\}$ in $\lambda \in Y$. Thus, $\{f_\lambda^\varepsilon \geq \ell\} \subset \{f_\lambda \geq \ell\}_\eta$ for small $\varepsilon > 0$ uniformly in $\lambda \in Y$.

It remains to prove that $\{f_\lambda \geq \ell\} \subset \{f_\lambda^\varepsilon \geq \ell\}_\eta$ for sufficiently small ε (uniformly in $\lambda \in Y$). If not, there is $r > 0$ and λ_j, z_j with $z_j \in \{f_\lambda \geq \ell\}$ such that $\{f_{\lambda_j}^{\varepsilon_j} \geq \ell\} \cap B_r(z_j) = \emptyset$ for some sequence $\varepsilon_j \to 0$. By continuity of $\{f_\lambda \geq \ell\}$ in λ we may assume that $\lambda_j \to \lambda_0$ and $z_j \to z_0 \in \{f_{\lambda_0} \geq \ell\}$ so that $\{f_{\lambda_j}^{\varepsilon_j} \geq \ell\} \cap B_{r/2}(z_0) = \emptyset$. Since $f_\lambda^\varepsilon \to f_\lambda$ uniformly in Z and for $\lambda \in Y$, this implies that $f_{\lambda_0} \leq \ell$ on $B_{r/2}(z_0)$. This contradicts $\overline{\{f_{\lambda_0} > \ell\}} = \{f_{\lambda_0} \geq \ell\}$. ■

Remark 8.5 (i) As an application of Corollary 8.3 (i) we have the convergence of extinction time T^ε of E^ε to that of E as $\varepsilon \to 0$, where $T^\varepsilon = \sup\{t; E^\varepsilon(t) \neq \emptyset\}$, $T = \sup\{t; E(t) \neq \emptyset\}$.

(ii) It is possible to replace E^ε by the interface evolution $\Gamma^\varepsilon = E^\varepsilon \backslash D^\varepsilon$ in Corollary 8.3 with trivial modifications.

(iii) In Corollary 8.3 (ii) the assumption $E(t) = \overline{D(t)}$, $t \in [0, T']$ is actually stronger than the regularity $E = \overline{D}$. For example if $E(t)$ is a shrinking disk solving $V = \kappa$, then at the extinction time T the set $E(T)$ is a point and $D(T) = \emptyset$. This $E(t)$ is not continuous at T, so continuity in t does not follow from $E = \overline{D}$. Actually, the continuity of $E(t)$ in t follows from $E(t) = \overline{D(t)}$, $t \in [0, T']$. One can prove this fact by comparing evolution of a small ball which does not disappear instanteously.

(iv) In [Ca] the convergence of the level set has been discussed. However, his results need convergence of $\partial_\theta^k \tilde{\gamma}_\varepsilon$ to $\partial_\theta^k \tilde{\gamma}$ for $k = 0, 1, 2$ which is substantially different from ours.

The rest of this section is devoted to the proof of Theorem 8.1.

Lemma 8.6 (Special solutions). *Assume that $\gamma_\varepsilon \in \mathcal{I}$ and $g_\varepsilon \in \mathcal{G}_0$ for $\varepsilon \in (0,1)$ and that $\gamma_\varepsilon \to \gamma \in \mathcal{I}$ locally uniformly on $\mathbf{R}^2$ as $\varepsilon \to 0$. Assume that $g_\varepsilon \to g \in \mathcal{G}_0$ locally uniformly in $S^1 \times \mathbf{R}$ as $\varepsilon \to 0$. Let K_ε be as in Lemma 7.4* *(ii), where γ_ε should be replaced by $\hat{\gamma}_\varepsilon(p) = \gamma_\varepsilon(-p)$. For each $A > 0$ there is a constant $c_0 > 0$ such that*

$$v_\varepsilon(x,t) = c_0 t + A(P_{K_\varepsilon}(x))^2 \tag{8.1}$$

is a supersolution of (L_ε) in $\mathbf{R}^2 \times (0,\infty)$, where P_{K_ε} is the Minkowski functional of K_ε. Similarly, for each $A > 0$ there is a constant $c_0 > 0$ such that

$$v_\varepsilon(x,t) = c_0 t - A(P_{K_\varepsilon}(x))^2 \tag{8.2}$$

is a supersolution of (L_ε) in $\mathbf{R}^2 \times (0,\infty)$, where K_ε is defined by Lemma 7.4 (ii) for γ_ε.

Proof. The proof for (8.2) is the same so we just prove that (8.1) is a supersolution. Since $v_\varepsilon \in A_\gamma(\mathbf{R}^2 \times (0,\infty))$ for any c_0 and A, it suffices to prove

$$\begin{aligned} &\sup_{0<\varepsilon<1} \sup_{\substack{x\in\mathbf{R}^2 \\ x\neq 0}} |\nabla h_\varepsilon(x)| g_\varepsilon\Big(-\frac{\nabla h_\varepsilon(x)}{|\nabla h_\varepsilon(x)|}, \Lambda_{\gamma_\varepsilon}(h_\varepsilon)(x)\Big) < \infty \\ &\text{for } \ h_\varepsilon(x) = A(P_{K_\varepsilon}(x))^2 \end{aligned} \tag{8.3}$$

by Remark 2.4. First, we note

$$\Lambda_{\gamma_\varepsilon}(h_\varepsilon)(x)) = \Lambda_{\gamma_\varepsilon}(x;\partial K_\varepsilon)/P_{K_\varepsilon}(x), \ \nabla h_\varepsilon(x) = 2AP_{K_\varepsilon}(x)\nabla P_{K_\varepsilon}.$$

Thanks to the assumption (1.6) and boundedness of $\Lambda_{\gamma_\varepsilon}(x;\partial K_\varepsilon)$, $\nabla P_{K_\varepsilon}(x)$ in Lemma 7.4 we have (8.3). ■

Proof of Theorem 8.1. We first note that there is $R' \geq R$ (independent of ε) such that

$$\text{spt } (u^\varepsilon - \alpha_\varepsilon) \subset B_{R'} \times [0,T). \tag{8.4}$$

Indeed, we take a constant $\sigma > 0$ such that

$$\sigma + v_\varepsilon(x,0) \geq u_0^\varepsilon(x), \quad \varepsilon \in (0,1),$$

where v_ε is defined by (8.2). By the invariance we see

$$w_\varepsilon(x,t) = \max(\sigma + v_\varepsilon(x,t), \alpha_\varepsilon)$$

is a supersolution of (L_ε) in $\mathbf{R}^2 \times (0,\infty)$. Since $\alpha_\varepsilon \to \alpha$ and K_ε converges to the Wulff shape of $\hat{\gamma}$, there is $R' \geq R$ such that

$$\mathrm{spt}(w_\varepsilon - \alpha_\varepsilon) \subset B_{R'} \times [0,\infty).$$

Since spt $(u_0^\varepsilon - \alpha_\varepsilon) \subset B_R$, we see $u_0^\varepsilon \leq w_\varepsilon|_{t=0}$. By the comparison (Theorem 6.1) we see that $u^\varepsilon \leq w_\varepsilon$ on $\mathbf{R}^2 \times [0,T)$. This implies that $u^\varepsilon(x,t) \leq \alpha_\varepsilon$ for $x, |x| \geq R'$, $t \in [0,T)$. A symmetric argument yields that $u^\varepsilon(x,t) \geq \alpha_\varepsilon$ for $x, |x| \geq R''$, $t \in [0,T)$ for some R''. We have thus proved (8.4).

We shall prove that

$$\limsup\nolimits^*_{\varepsilon \to 0} u^\varepsilon(\xi, 0) \leq u_0(\xi), \quad \xi \in \mathbf{R}^2. \tag{8.5}$$

Since u_0^ε converges to u_0 uniformly in $\mathbf{R}^2$ and spt $(u_0^\varepsilon - \alpha_\varepsilon) \subset B_R$, there is a modulus ω of continuity independent of ε such that

$$u_0^\varepsilon(x) - u_0^\varepsilon(\xi) \leq \omega(|x-\xi|), \quad \varepsilon \in (0,1),\ x \in \mathbf{R}^2.$$

For each $\delta > 0$ there is $A_\delta > 0$ such that $\omega(s) \leq \delta + A_\delta s^2$, $s \geq 0$. Let K_ε be defined in Lemma 7.4 (ii) with γ_ε replaced by $\hat{\gamma}_\varepsilon$. Since there is $\lambda > 0$ that satisfies $|x| \leq \lambda P_{K_\varepsilon}(x)$ for $x \in \mathbf{R}^2$, $\varepsilon \in (0,1)$, for each δ there is $A = A_\delta \lambda$ that satisfies

$$u_0^\varepsilon(x) \leq u_0^\varepsilon(\xi) + \delta + A(P_{K_\varepsilon}(x-\xi))^2.$$

By Lemma 8.6 for some $c_0 > 0$

$$w_\varepsilon(x,t) = u_0^\varepsilon(\xi) + \delta + A(P_{K_\varepsilon}(x-\xi))^2 + c_0 t$$

is a supersolution of (L_ε) in $\mathbf{R}^2 \times (0,\infty)$. Since a constant is always a solution of (L_ε), we have, by comparison principle (Theorem 6.1),

$$u^\varepsilon \leq \sup_{0<\varepsilon<1} \sup_{x\in\mathbf{R}^2} u_0^\varepsilon =: M.$$

By the invariance (Lemma 3.1) the function

$$\tilde{w}_\varepsilon(x,t) = \min(M,\ w_\varepsilon(x,t))$$

is also a supersolution of (L_ε) in $\mathbf{R}^2 \times (0,\infty)$. (Actually, this follows from more general principle that infimum of a class of supersolutions is a supersolution; we do not prove this in this paper.) This $\tilde{w}_\varepsilon$ equals to a constant near space infinity. Since (8.4) holds, we are now in position to apply the comparison principle (Theorem 6.1) for (L_ε) in $\mathbf{R}^2 \times [0,T')$ to get $u^\varepsilon(x,t) \leq \tilde{w}_\varepsilon(x,t)$, $x \in \mathbf{R}^2$, $t \in [0,T')$. Thus

$$\limsup^*_{\varepsilon\to 0} u^\varepsilon(\xi,0) \leq \lim_{\varepsilon\to 0} u_0^\varepsilon(\xi) + \delta + \lim_{\varepsilon\to 0} A(P_{K_\varepsilon}(x-\xi))^2,$$

which yields

$$\limsup^*_{\varepsilon\to 0} u^\varepsilon(\xi,0) \leq u_0(\xi) + \delta.$$

Since $\delta > 0$ is arbitrary, we now obtain (8.5).

A symmetric argument yields

$$\liminf_{\varepsilon\to 0}{}_* u^\varepsilon(\xi,0) \geq u_0(\xi)$$

so that $\overline{u}|_{t=0} = \underline{u}|_{t=0} = u_0$, where $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$, $\underline{u} = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon$. Since $|u^\varepsilon| \leq M$, by the stability (Theorem 7.1) $\overline{u}$ is a subsolution of (L) and $\underline{u}$ is a supersolution of (L). Since $\overline{u}|_{t=0} = \underline{u}|_{t=0}$ and (8.4) implies

$$\mathrm{spt}(\overline{u} - \alpha),\ \mathrm{spt}(\underline{u} - \alpha) \subset B_{R'} \times [0,T),$$

the comparison principle (Theorem 6.1) is applicable to get $\overline{u} \leq \underline{u}$. This implies $\overline{u} = \underline{u}$ and the convergence $u^\varepsilon \to \overline{u} = \underline{u}$ is locally uniformly in $\mathbf{R}^2 \times [0,T)$. By (8.4) this implies the uniform convergence in $\mathbf{R}^2 \times [0, T']$ for any $T' < T$. Since $\overline{u}(= \underline{u})$ is a sub- and supersolution of (L), the proof of Theorem 8.1 is now complete. ∎

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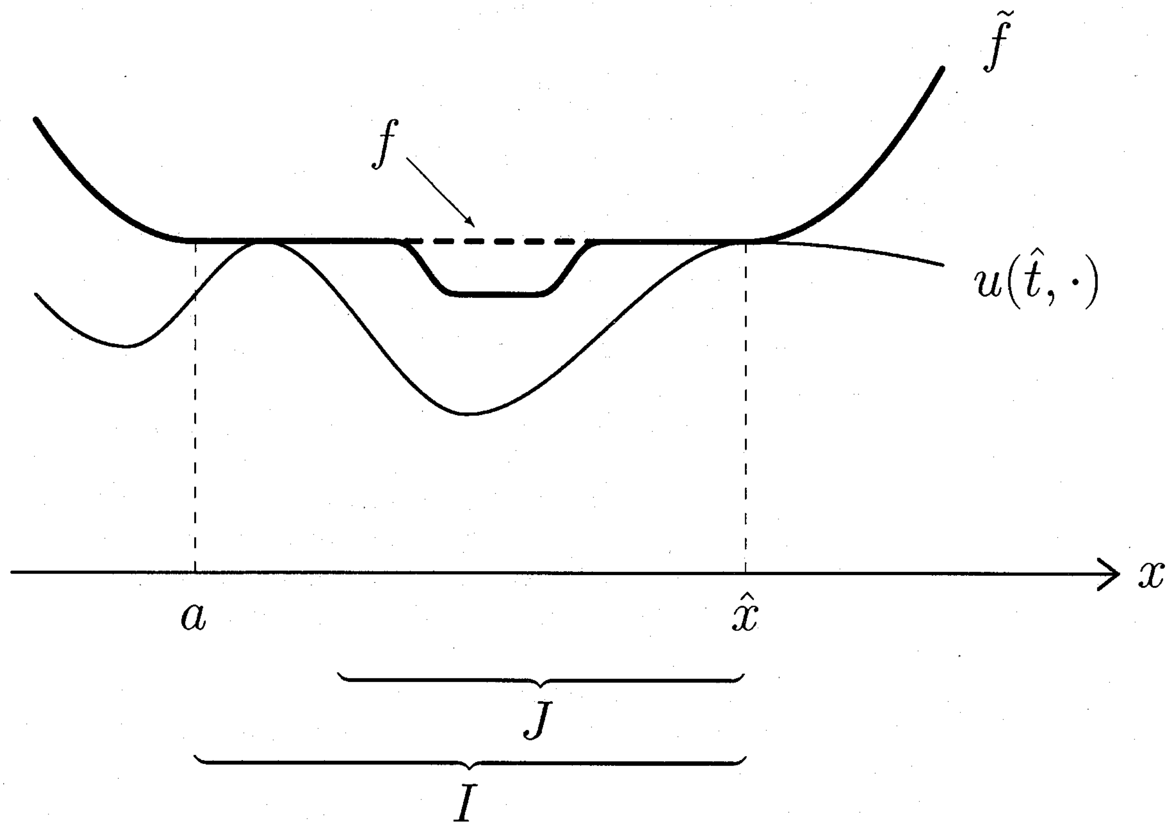

Figure 1. function f and $\tilde{f}$

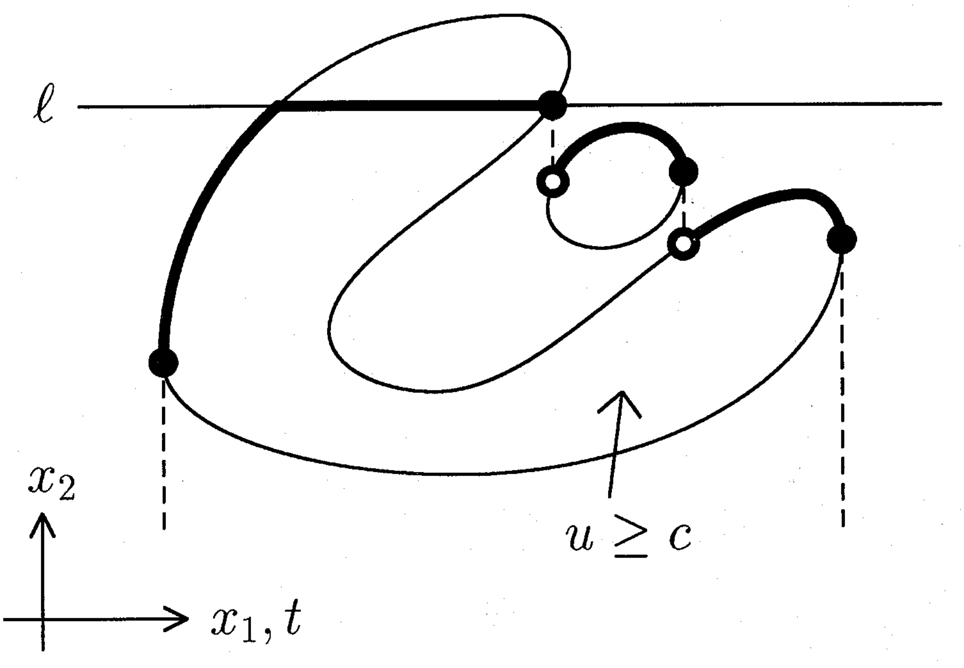

Figure 2. function $u^{\#}$

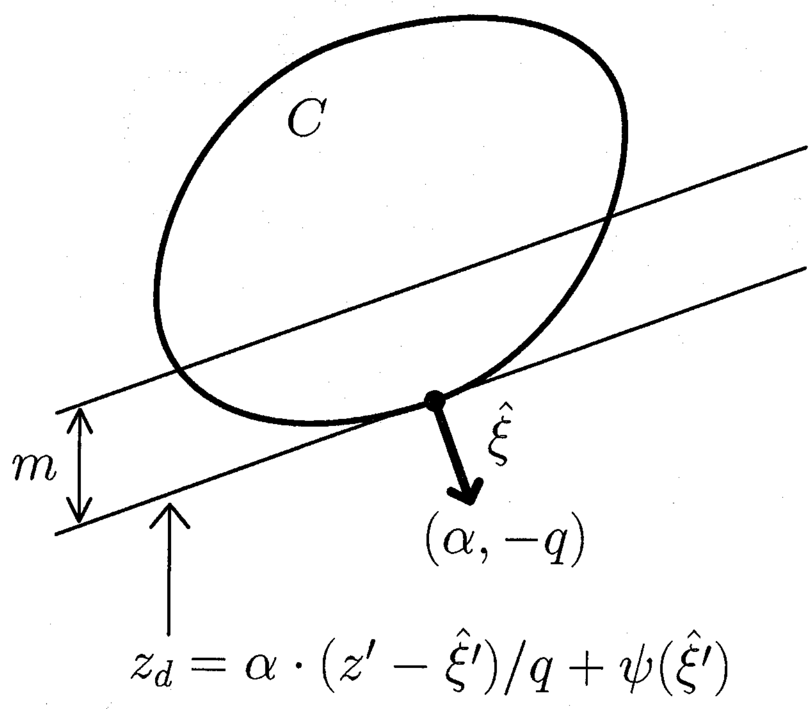

Figure 3. a choice of m