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And
A Helson-Szegö Weight**

Takahiko Nakazi and Takanori Yamamoto

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The Real Part Of An Outer Function
And
A Helson-Szegö Weight

by

Takahiko Nakazi* and Takanori Yamamoto*

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Abstract. Suppose F is a nonzero function in the Hardy space H^1 . We study the set $\{f ; f \text{ is outer and } |F| \leq \operatorname{Re} f \text{ a.e. on } \partial D\}$ where ∂D is a unit circle. When F is a strongly outer function in H^1 and γ is a positive constant, we describe the set $\{f ; f \text{ is outer, } |F| \leq \gamma \operatorname{Re} f \text{ and } |F^{-1}| \leq \gamma \operatorname{Re}(f^{-1}) \text{ a.e. on } \partial D\}$. Suppose W is a Helson-Szegö weight. As an application, we parametrize real valued functions v in $L^\infty(\partial D)$ such that the difference between $\log W$ and the harmonic conjugate function $\tilde{v}$ of v belongs to $L^\infty(\partial D)$ and $\|v\|_\infty$ is strictly less than $\pi/2$ using a contractive function α in H^∞ such that $(1 + \alpha)/(1 - \alpha)$ is equal to the Herglotz integral of W .

1. Introduction

Let D be the open unit disc in the complex plane and let ∂D be the boundary of D . An analytic function f on D is said to be of class N if the integrals

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta$$

are bounded for $r < 1$. If f is in N , then $f(e^{i\theta})$ which we define to be $\lim_{r \rightarrow 1} f(re^{i\theta})$ exists almost everywhere on ∂D . If

$$\lim_{r \rightarrow 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| d\theta$$

then f is said to be of class N_+ . The set of all boundary functions in N or N_+ is denoted by N or N_+ , respectively. For $0 < p \leq \infty$, the Hardy space H^p is defined by $N_+ \cap L^p$. Hence any function in H^p has an analytic extension to D .

A function h in N_+ is called outer if h is invertible in N_+ . A function g in H^1 is called strongly outer if the only functions $f \in H^1$ such that $\frac{f}{g} \geq 0$ a.e. on ∂D are scalar multiples of g . If g is strongly outer then outer. Suppose F is a nonzero function in H^1 . Define α by

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} |F(e^{i\theta})| d\theta \quad (z \in D).$$

The right hand side is the Herglotz integral of $|F|$. Then α is a contractive function in H^∞ . Let $f_0 = \frac{1 + \alpha}{1 - \alpha}$. Then $\operatorname{Re} f_0(z) > 0$ ($z \in D$),

$$|F| = \operatorname{Re} f_0 = \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D,$$

and $f_0 \in \bigcap_{p < 1} H^p$ by a theorem of Kolmogorov (c.f. [1, Theorem 4.2]). Since $\operatorname{Re} f_0(z) > 0$,

$f_0 = c e^{\tilde{v} - iv}$ where c is a positive constant and $\|v\|_\infty \leq \frac{\pi}{2}$, $\tilde{v}$ is a harmonic conjugate function of v satisfying $\tilde{v}(0) = 0$. By a theorem of Kolmogorov, $\tilde{v} - iv \in \bigcap_{p < \infty} H^p$,

$$|F| = e^{u + \tilde{v}} \quad \text{and} \quad e^u = c \cos v \quad \text{a.e. on } \partial D$$

where u is a real valued function. In Section 2, when F is strongly outer we study an outer function f in N_+ such that $|F| \leq \operatorname{Re} f$ a.e. on ∂D . We then show that $|F| \leq \gamma \operatorname{Re} F$ if and only if α^2 is a γ -Stolz function, where γ is a positive constant. If β is a contractive function in H^∞ and $|1 - \beta| \leq \gamma(1 - |\beta|)$ a.e. on ∂D then we call β a γ -Stolz function. Suppose W is a Helson-Szegö weight (cf. [3]). In Section 3, using Theorem 1 in Section 2, we parametrize real valued functions v such that $\log W - \tilde{v} \in L^\infty$ and $\|v\|_\infty < \frac{\pi}{2}$.

2. The Real Part of an Outer Function

In this section, we study the inequality : $|F| \leq \gamma \operatorname{Re} F$ a.e. on ∂D when F is a nonzero function in H^1 . The first author [4] studied the inequality : $|F| \leq \gamma \operatorname{Re} f$ a.e. on ∂D when F is strongly outer and f is outer in N_+ . We give necessary and sufficient conditions of this inequality. We study two inequalities : $|F| \leq \gamma \operatorname{Re} f$ and $|F^{-1}| \leq \gamma \operatorname{Re}(f^{-1})$ a.e. on ∂D when F is strongly outer and f is in N_+ . Results in this section will be used in the latter section.

Proposition 1. *Suppose F is a nonzero function in H^1 and γ is a constant satisfying $\gamma \geq 1$. Then the following (1) $\sim$ (3) are equivalent.*

- (1) $|F| \leq \gamma \operatorname{Re} F$ a.e. on ∂D .
- (2) $F = \frac{1+\alpha}{1-\alpha}$ a.e. on ∂D for a contractive function α in H^∞ such that α^2 is a γ -Stolz function.
- (3) $F = c e^{\tilde{v}-iv}$ a.e. on ∂D , where c is a positive constant and v is a real function in L^∞ satisfying $\|v\|_\infty \leq \cos^{-1} \left(\frac{1}{\gamma} \right) < \frac{\pi}{2}$.

Proof. (1) $\Leftrightarrow$ (2): Since $F \in H^1$ and $\operatorname{Re} F \geq 0$ a.e. on ∂D , it follows that

$$\operatorname{Re} F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} \operatorname{Re} F(e^{i\theta}) d\theta \geq 0 \quad (z \in D).$$

Hence $F = \frac{1+\alpha}{1-\alpha}$ for a contractive function α in H^∞ . Since $|F| \leq \gamma \operatorname{Re} F$ a.e. on ∂D ,

$$\left| \frac{1+\alpha}{1-\alpha} \right| \leq \gamma \operatorname{Re} \left(\frac{1+\alpha}{1-\alpha} \right) = \gamma \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial D.$$

Hence $|1-\alpha^2| \leq \gamma(1-|\alpha|^2)$ and so α^2 is a γ -Stolz function. The converse is clear.

(2) $\Rightarrow$ (3): Since $\|\alpha\|_\infty \leq 1$, $\operatorname{Re} F = \frac{1-|\alpha|^2}{|1-\alpha|^2} \geq 0$ a.e. on ∂D . Since $F \in H^1$, this implies that $\operatorname{Re} F(z) \geq 0$ ($z \in D$). Hence $F = c e^{\tilde{v}-iv}$ and $|v| \leq \frac{\pi}{2}$ a.e. on ∂D . Since α^2 is a γ -Stolz function, it follows that

$$|F| = \left| \frac{1+\alpha}{1-\alpha} \right| = \frac{|1-\alpha^2|}{|1-\alpha|^2} \leq \gamma \frac{1-|\alpha|^2}{|1-\alpha|^2} = \gamma \operatorname{Re} F \quad \text{a.e. on } \partial D.$$

Hence $1 \leq \gamma \cos v$. Since $|v| \leq \frac{\pi}{2}$, this implies that $\|v\|_\infty \leq \cos^{-1} \left(\frac{1}{\gamma} \right) < \frac{\pi}{2}$.

(3) $\Rightarrow$ (1): By (3), $|F| = c e^{\tilde{v}} \leq \gamma c e^{\tilde{v}} \cos v = \gamma \operatorname{Re} F$. This implies (1). $\square$

By (3) in Proposition 1 and Corollary 2.6 in [2, Chapter III], if $|F| \leq \gamma \operatorname{Re} F$ a.e. on ∂D then both F and F^{-1} belong to H^p for some $p > 1$.

Proposition 2. Suppose F is a strongly outer function in H^1 . Define α by

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} |F(e^{i\theta})| d\theta \quad (z \in D).$$

For f in N_+ , (1) $\sim$ (3) are equivalent.

(1) $|F| \leq \operatorname{Re} f$ a.e. on ∂D and f is an outer function.

(2) $f = \frac{1 + \alpha}{1 - \alpha} + \frac{1 + \beta}{1 - \beta}$ a.e. on ∂D for some contractive function β in H^∞ .

(3) $|F| = e^{u+\bar{v}}$, $|v| < \frac{\pi}{2}$, $e^u \leq c \cos v$ and $f = c e^{\bar{v}-iv}$ a.e. on ∂D

where c is a positive constant and u and v are real functions.

Proof. The following proof is similar to one of Theorem 6 in the first author's paper [4]. (1) $\Rightarrow$ (3): Let $\operatorname{Arg} f$ denote the argument of f restricted to $-\pi < \operatorname{Arg} f \leq \pi$. Let $v = -\operatorname{Arg} f$. Then $|v| \leq \pi$ and $f = |f|e^{-iv}$. Since $0 < |F| \leq \operatorname{Re} f$, $|v| < \frac{\pi}{2}$. By the proof of Lemma 5.4 in [2, Chapter IV], if $|v| \leq \frac{\pi}{2}$ then $e^{\bar{v}} \cos v \in L^1$. Let $g = e^{iv-\bar{v}}$. Then $fg = |f|e^{-\bar{v}} > 0$. Since f is outer, $F/fg \in N_+$. Since

$$\left| \frac{F}{fg} \right| \leq \frac{\operatorname{Re} f}{|fg|} = \frac{\cos v}{|g|} = e^{\bar{v}} \cos v \in L^1,$$

it follows that $F/fg \in H^1$. Since F is strongly outer, F/fg is a scalar multiple of F . Hence $fg = c$ for some positive constant c . Hence $f = c e^{\bar{v}-iv}$, and hence $|F| \leq c e^{\bar{v}} \cos v$. Define u by $|F| = e^{u+\bar{v}}$. Then $e^u \leq c \cos v$. This implies (3).

(3) $\Rightarrow$ (2): In the following we do not assume that F is strongly outer. We assume that F is a nonzero function in H^1 . By (3), $|F| \leq \operatorname{Re} f$ and $\operatorname{Re} f \in L^1$. Let $(\bar{v} - iv)(z)$ denote the Poisson transform of $(\bar{v} - iv)(e^{i\theta})$. Let $g(z) = c e^{(\bar{v}-iv)(z)}$. Then $\operatorname{Re} g(z) \geq 0$ ($z \in D$), $\lim_{r \rightarrow 1} g(re^{i\theta}) = f(e^{i\theta})$ a.e. on ∂D , and

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} g(re^{i\theta}) d\theta = \operatorname{Re} g(0) < \infty.$$

Hence

$$\begin{aligned} \operatorname{Re} g(z) &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \operatorname{Re} f(e^{i\theta}) d\theta \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} |F(e^{i\theta})| d\theta = \operatorname{Re} \left(\frac{1 + \alpha(z)}{1 - \alpha(z)} \right) \quad (z \in D). \end{aligned}$$

Hence there exists a contractive function β in H^∞ such that

$$g(z) = \frac{1 + \alpha(z)}{1 - \alpha(z)} + \frac{1 + \beta(z)}{1 - \beta(z)} \quad (z \in D).$$

Since $\lim_{r \rightarrow 1} g(re^{i\theta}) = f(e^{i\theta})$ a.e. on ∂D , this implies (2).

(2) $\Rightarrow$ (1): Since $|\beta| \leq 1$, $\operatorname{Re} \frac{1 + \beta}{1 - \beta} \geq 0$. Hence

$$|F| = \operatorname{Re} \frac{1 + \alpha}{1 - \alpha} \leq \operatorname{Re} \left(\frac{1 + \alpha}{1 - \alpha} + \frac{1 + \beta}{1 - \beta} \right) = \operatorname{Re} f \quad \text{a.e. on } \partial D.$$

This implies (1). □

By (3) in Proposition 2 and Corollary 2.6 in [2, Chapter III], if $|F| \leq \operatorname{Re} f$ a.e. on ∂D and f is an outer function then both f and f^{-1} belong to H^p for all $p < 1$.

By (1), the set of all functions f satisfying one of the conditions (1) $\sim$ (3) is a convex subset of N_+ . If F is a nonzero function in H^1 , then (3) $\Rightarrow$ (2) $\Rightarrow$ (1) holds in Proposition 2. But by Theorem 6 in [4], (1) $\Rightarrow$ (3) does not hold in general.

Theorem 1. Suppose F is a strongly outer function in H^1 . Define α by

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} |F(e^{i\theta})| d\theta \quad (z \in D).$$

For f in N_+ , (1) $\sim$ (4) are equivalent. $\gamma_1, \dots, \gamma_5$ are positive appropriate constants.

- (1) $|F| \leq \gamma_1 \operatorname{Re} f$ and $|F^{-1}| \leq \gamma_1 \operatorname{Re} (f^{-1})$ a.e. on ∂D .
- (2) $\frac{1}{\gamma_2} \operatorname{Re} f \leq |F| \leq \gamma_2 \operatorname{Re} f$ and $|f| \leq \gamma_2 \operatorname{Re} f$ a.e. on ∂D and f is in H^1 .
- (3) There exists a contractive function β in H^∞ such that

$$\gamma_3 f = \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)} \quad \text{and} \quad \frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma_4 \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D.$$

- (4) There exists a constant $c > 0$ and real functions u, v in L^∞ such that

$$|F| = e^{u+\bar{v}}, \quad \|v\|_\infty \leq \cos^{-1} \gamma_5 < \frac{\pi}{2} \quad \text{and} \quad f = c e^{\bar{v}-iv} \quad \text{a.e. on } \partial D.$$

Proof. (1) $\Rightarrow$ (2): By (1),

$$(\operatorname{Re} f)^2 \leq |f|^2 \leq \gamma_1 (\operatorname{Re} f) |F| \leq \gamma_1^2 (\operatorname{Re} f)^2.$$

Hence $|f| \leq \gamma_1 \operatorname{Re} f \leq \gamma_1^2 |F| \in L^1$. This implies (2) with $\gamma_2 = \gamma_1$.

(2) $\Rightarrow$ (1): By (2),

$$\frac{1}{|F|} \leq \gamma_2 \frac{1}{\operatorname{Re} f} \leq \gamma_2^3 \frac{\operatorname{Re} f}{|f|^2} = \gamma_2^3 \operatorname{Re} \frac{1}{f}.$$

This implies (1) with $\gamma_1 = \gamma_2^3$.

(2) $\Rightarrow$ (3): Since $f \in H^1$ and $\operatorname{Re} f \geq 0$ a.e. on ∂D , $\operatorname{Re} f(z) > 0$ ($z \in D$). Hence f is an outer function. Since $|F| \leq \gamma_2 \operatorname{Re} f$, by Proposition 2,

$$\gamma_2 f = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} = \frac{2(1-\alpha\beta)}{(1-\alpha)(1-\beta)}$$

for some contractive function β in H^∞ . Since $|f| \leq \gamma_2 \operatorname{Re} f \leq \gamma_2^2 |F|$,

$$\frac{2|1-\alpha\beta|}{|1-\alpha| \cdot |1-\beta|} = \left| \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} \right| = \gamma_2 |f| \leq \gamma_2^3 |F| = \gamma_2^3 \frac{1-|\alpha|^2}{|1-\alpha|^2}.$$

This implies (3) with $\gamma_3 = \gamma_2/2$ and $\gamma_4 = \gamma_2^3/2$.

(3) $\Rightarrow$ (4): By (3), f is outer, since α and β are contractive. Since

$$|F| = \operatorname{Re} \left(\frac{1+\alpha}{1-\alpha} \right) \leq 2\gamma_3 \operatorname{Re} f,$$

by Proposition 2, $|F| = e^{u+\bar{v}}$, $|v| < \frac{\pi}{2}$, $e^u \leq c_0 \cos v$ and $2\gamma_3 f = c_0 e^{\bar{v}-iv}$, where c_0 is a positive constant and u, v are real functions. Hence

$$\begin{aligned} c_0 e^{\bar{v}} &= 2\gamma_3 |f| = \frac{2|1-\alpha\beta|}{|1-\alpha| \cdot |1-\beta|} \\ &\leq 2\gamma_4 \frac{1-|\alpha|^2}{|1-\alpha|^2} = 2\gamma_4 |F| = 2\gamma_4 e^{u+\bar{v}} \\ &\leq 2c_0 \gamma_4 e^{\bar{v}} \cos v \leq 2c_0 \gamma_4 e^{\bar{v}}. \end{aligned}$$

Hence $\frac{c_0}{2\gamma_4} \leq e^u \leq c_0$ and $\cos v \geq \frac{1}{2\gamma_4} > 0$. Hence $u, v \in L^\infty$ and $\|v\|_\infty \leq \cos^{-1} \left(\frac{1}{2\gamma_4} \right) < \frac{\pi}{2}$. This implies (4) with $c = \frac{c_0}{2\gamma_3}$ and $\gamma_5 = \frac{1}{2\gamma_4}$.

(4) $\Rightarrow$ (1): Since $\cos v \geq \gamma_5$,

$$|F| = e^{u+\bar{v}} \leq \frac{1}{\gamma_5} e^{\|u\|_\infty} e^{\bar{v}} \cos v = \frac{1}{c\gamma_5} e^{\|u\|_\infty} \operatorname{Re} f,$$

and

$$\frac{1}{|F|} = e^{-u-\bar{v}} \leq \frac{c}{\gamma_5} e^{\|u\|_\infty} e^{-\bar{v}} \cos v = \frac{c}{\gamma_5} e^{\|u\|_\infty} \operatorname{Re} \frac{1}{f}.$$

This implies (1) with $\gamma_1 = \frac{1}{\gamma_5} \max \left(c, \frac{1}{c} \right) e^{\|u\|_\infty}$. □

By (2) in Theorem 1, the set of all functions f satisfying one of the conditions (1) $\sim$ (4) is a convex subset of H^1 .

3. Helson-Szegö Weight

Let W be a positive function in L^1 and $\log W$ is in L^1 . For each $\varepsilon > 0$, put

$$\mathcal{E}_{W,\varepsilon} = \left\{ v \in \text{Re } L^\infty ; \quad \log W - \tilde{v} \in L^\infty \quad \text{and} \quad \|v\|_\infty \leq \frac{\pi}{2} - \varepsilon \right\}$$

and $\mathcal{E}_W = \bigcup_{\varepsilon > 0} \mathcal{E}_{W,\varepsilon}$. $\mathcal{E}_{W,\varepsilon}$ and $\mathcal{E}_W$ are convex subsets of $\text{Re } L^\infty$. When $\mathcal{E}_W$ is nonempty, W is called a Helson-Szegö weight. Then for each v in $\mathcal{E}_W$ there exists a $u \in \text{Re } L^\infty$ such that $\log W = u + \tilde{v}$. In this section, we study two problems about a Helson-Szegö weight. In Theorem 2 we describe $\mathcal{E}_W$. Theorem 3 follows from Theorem 2 immediately.

Theorem 2. *Let W be a positive function in L^1 . Define α by*

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} W(e^{i\theta}) d\theta \quad (z \in D).$$

Then v belongs to $\mathcal{E}_W$ if and only if

$$v = - \text{Arg} \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)} \quad \text{a.e. on } \partial D,$$

where β is a contractive function in H^∞ satisfying

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D$$

for some constant $\gamma > 0$.

Proof. If $v \in \mathcal{E}_W$, then $v \in \mathcal{E}_{W,\varepsilon}$ for some constant $\varepsilon > 0$. Hence

$$W = e^{u+\tilde{v}}$$

where $u \in L^\infty$ and $\|v\|_\infty \leq \frac{\pi}{2} - \varepsilon$. Hence there exists a constant $\gamma > 0$ such that

$$W \leq \gamma e^{\tilde{v} \cos v} \quad \text{and} \quad W^{-1} \leq \gamma e^{-\tilde{v} \cos v}$$

where $e^{\|u\|_\infty} \leq \gamma \cos v$. Put $f = e^{\tilde{v}-iv}$ then $W \leq \gamma \text{Re } f$, $W^{-1} \leq \gamma \text{Re}(f^{-1})$ and $f \in H^1$. Since $W, W^{-1} \in L^1$, there exists an outer function F such that $|F| = W$ and $F, F^{-1} \in H^1$. Hence F is strongly outer. By Theorem 1, there exist constants $\gamma_3, \gamma_4 > 0$ and a contractive function $\beta \in H^\infty$ such that

$$\gamma_3 f = \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)}$$

and

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma_4 \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D.$$

Hence

$$v = -\operatorname{Arg} f = -\operatorname{Arg} \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)} \quad \text{a.e. on } \partial D.$$

This implies the 'only if' part. Conversely suppose v satisfies the condition. Define f by

$$f = \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)}.$$

Then

$$v = -\operatorname{Arg} f \quad \text{and} \quad |f| \leq \gamma \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D$$

for some constant $\gamma > 0$. Then f satisfies (3) of Theorem 1 and

$$W = \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \leq \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + \frac{1 - |\beta|^2}{|1 - \beta|^2} = 2 \operatorname{Re} f \leq 2|f| \leq 2\gamma \frac{1 - |\alpha|^2}{|1 - \alpha|^2} = 2\gamma W.$$

Since W is a positive function in L^1 , $\operatorname{Re} f \geq 0$ a.e. on ∂D and $f \in H^1$. Hence f is strongly outer. Since $\log W \in L^1$, there exists an outer function $F \in H^1$ such that $|F| = W$. Let k be any function satisfying $k \in H^1$ and $k/F \geq 0$ a.e. on ∂D . Since $f/F \in H^\infty$, $kf/F \in H^1$. Since f is strongly outer, $kf/F = cf$ for some constant c . Hence $k = cF$. Therefore F is strongly outer. By Theorem 1, there exists a constant $c > 0$ and real functions $u, v_0 \in L^\infty$ such that $\|v_0\|_\infty < \frac{\pi}{2}$, $W = e^{u+\tilde{v}_0}$ and $f = c e^{\tilde{v}_0 - iv_0}$ a.e. on ∂D . Hence

$$v_0 = -\operatorname{Arg} f = -\operatorname{Arg} \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)} = v.$$

Hence $W = e^{u+\tilde{v}}$ a.e. on ∂D and $\|v\|_\infty < \frac{\pi}{2}$. Hence v belongs to $\mathcal{E}_W$. $\square$

By Theorem 2, if $W = 1$ then $\alpha = 0$ and hence

$$\begin{aligned} \mathcal{E}_1 &= \left\{ v \in \operatorname{Re} L^\infty ; \quad \|v\|_\infty < \frac{\pi}{2} \quad \text{and} \quad \tilde{v} \in L^\infty \right\} \\ &= \left\{ -\operatorname{Arg} \frac{1}{1 - \beta} ; \beta \in H^\infty, \quad \|\beta\| \leq 1 \quad \text{and} \quad \frac{1}{1 - \beta} \in L^\infty \right\}. \end{aligned}$$

Theorem 3. Let W be a positive function in L^1 . Define α by

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} W(e^{i\theta}) d\theta \quad (z \in D).$$

(1) W is a Helson-Szegö weight, that is, $\mathcal{E}_W \neq \emptyset$ if and only if there exists a constant $\gamma > 0$ and a contractive function β in H^∞ such that

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D.$$

(2) If α is a Stolz function, then W is a Helson-Szegö weight, and W^{-1} belongs to L^∞ .

Proof. By Theorem 2, (1) follows immediately. By Theorem 2 with $\beta = 0$, if α is a Stolz function, then

$$v = -\text{Arg} \frac{1}{1 - \alpha}$$

belongs to $\mathcal{E}_W$, and hence $\mathcal{E}_W \neq \emptyset$. By (1), W is a Helson-Szegö weight. Since

$$W = \frac{1 - |\alpha|^2}{|1 - \alpha|^2} = \frac{1 + |\alpha|}{|1 - \alpha|} \frac{1 - |\alpha|}{|1 - \alpha|} \quad \text{a.e. on } \partial D$$

and α is a Stolz function, it follows that $W^{-1} \in L^\infty$. $\square$

Note that if α is a Stolz function, then α^2 is also a Stolz function. In fact if α is a γ -Stolz function, then $|\alpha| \leq 1$ and

$$|1 - \alpha^2| \leq |1 - \alpha| + |\alpha(1 - \alpha)| \leq 2|1 - \alpha| \leq 2\gamma(1 - |\alpha|) \leq 2\gamma(1 - |\alpha|^2).$$

Let W be a positive function in L^1 . By Proposition 1, $W = c e^{\tilde{v}}$ for a constant $c > 0$ and a real function v with $\|v\|_\infty < \frac{\pi}{2}$ if and only if there exists an $\alpha \in H^\infty$ such that α^2 is a Stolz function and $W = \left| \frac{1 + \alpha}{1 - \alpha} \right|$. Then there exists a $u \in \text{Re } L^\infty$ such that

$$W = \frac{|1 - \alpha^2|}{1 - |\alpha|^2} \frac{1 - |\alpha|^2}{|1 - \alpha|^2} = e^u \frac{1 - |\alpha|^2}{|1 - \alpha|^2} = e^u \text{Re } F,$$

where $F = \frac{1 + \alpha}{1 - \alpha}$.

4. Remark

Put $B_r = \{\beta \in H^\infty; \|\beta\|_\infty \leq r\}$ and put

$$B^\alpha = \left\{ \beta \in B_1; \frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \text{ a.e. on } \partial D \text{ for some constant } \gamma > 0 \right\}$$

where α is a contractive function in H^∞ . The set B^α was important in Theorems 1, 2 and 3. Let W be a Helson-Szegö weight. Define α by

$$\frac{1 + \alpha(z)}{1 - \alpha(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} W(e^{i\theta}) d\theta.$$

Then by Theorem 2

$$\mathcal{E}_W = \left\{ v = -\text{Arg} \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)}; \beta \in B^\alpha \right\}.$$

If $W = 1$ then $\alpha = 0$ and

$$\mathcal{E}_1 = \left\{ -\text{Arg} \frac{1}{1 - \beta}; \beta \in B^0 \right\}.$$

In this section, we study such a set B^α . α is a Stolz function if and only if $0 \in B^\alpha$. α^2 is a Stolz function if and only if $\alpha \in B^\alpha$. Hence if $0 \in B^\alpha$ then $\alpha \in B^\alpha$. If α is a Stolz function and $\beta \in B_r$, $r < 1$, then for some constant $\gamma > 0$

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \frac{2}{(1 - r)|1 - \alpha|} \leq \frac{2\gamma(1 - |\alpha|^2)}{(1 - r)|1 - \alpha|^2} \text{ a.e. on } \partial D,$$

and hence $\beta \in B^\alpha$. Hence if α is a Stolz function then $B_r \subset B^\alpha$ ($r < 1$).

For two positive functions f and g on ∂D , if there exists a constant $\gamma > 0$ such that $\frac{1}{\gamma}g \leq f \leq \gamma g$ a.e. on ∂D , then we write $f \sim g$.

Lemma. Suppose α and β are contractive functions in H^∞ . Then the following (1) $\sim$ (5) are equivalent.

$$(1) \quad \left\| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right\|_\infty < 1.$$

$$(2) \quad |1 - \alpha\beta|^2 \leq \gamma_2(1 - |\alpha|^2)(1 - |\beta|^2) \text{ a.e. on } \partial D \text{ for some constant } \gamma_2 > 0.$$

$$(3) \quad \text{There exists a constant } \gamma_3 > 0 \text{ such that for any function } t > 0$$

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma_3 \left\{ t \frac{1 - |\alpha|^2}{|1 - \alpha|^2} + \frac{1}{t} \frac{1 - |\beta|^2}{|1 - \beta|^2} \right\} \text{ a.e. on } \partial D.$$

(4) There exists a constant $\gamma_4 > 0$ such that

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma_4 \frac{1 - |\alpha|^2}{|1 - \alpha|^2} \quad \text{a.e. on } \partial D$$

and

$$\frac{|1 - \alpha\beta|}{|1 - \alpha| \cdot |1 - \beta|} \leq \gamma_4 \frac{1 - |\beta|^2}{|1 - \beta|^2} \quad \text{a.e. on } \partial D.$$

(5) $|1 - \alpha| \sim |1 - \beta|$ and $1 - |\alpha| \sim 1 - |\beta| \sim |1 - \alpha\beta|$.

Proof. (1) and (2) are equivalent because

$$1 - \left| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right|^2 = \frac{(1 - |\alpha|^2)(1 - |\beta|^2)}{|1 - \alpha\beta|^2}.$$

(cf.[5, p.58]). (2) and (3) are equivalent because if $a, b > 0$ then $2\sqrt{ab} \leq a + b$ and the equality holds when $a = b$. (1) $\Rightarrow$ (5): Let $f = \frac{\bar{\alpha} - \beta}{1 - \alpha\beta}$. Then $\|f\|_\infty < 1$, $\beta = \frac{\bar{\alpha} - f}{1 - \alpha f}$ and

$$|1 - \beta| = \frac{|(1 - \bar{\alpha}) + f(1 - \alpha)|}{|1 - \alpha f|} \geq \frac{|1 - \alpha| - |f| \cdot |1 - \alpha|}{2} \geq \frac{1 - \|f\|_\infty}{2} |1 - \alpha|.$$

Let $g = \frac{\alpha - \bar{\beta}}{1 - \alpha\beta}$. Then $\|g\|_\infty = \|f\|_\infty < 1$, $\alpha = \frac{g + \bar{\beta}}{1 + g\beta}$ and

$$|1 - \alpha| = \frac{|(1 - \bar{\beta}) - g(1 - \beta)|}{|1 + g\beta|} \geq \frac{|1 - \beta| - |g| \cdot |1 - \beta|}{2} \geq \frac{1 - \|g\|_\infty}{2} |1 - \beta|.$$

Hence $|1 - \alpha| \sim |1 - \beta|$. Since $0 < 1 - \|f\|_\infty \leq |1 - \alpha f| \leq 2$ and

$$1 - |\beta|^2 = \frac{(1 - |\alpha|^2)(1 - |f|^2)}{|1 - \alpha f|^2},$$

$1 - |\alpha| \sim 1 - |\beta|$. Since $|1 - \alpha f| = \frac{1 - |\alpha|^2}{|1 - \alpha\beta|}$, $|1 - \alpha\beta| \sim 1 - |\alpha|$. It is clear that (5) implies (4). If we multiply both sides of two inequalities in (4), then (2) follows. $\square$

By the above lemma, Proposition 3 follows immediately.

Proposition 3. If $\alpha \in B_1$, then

$$B^\alpha \supset \left\{ \beta \in B_1 ; \quad \left\| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right\|_\infty < 1 \right\}.$$

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Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
E-mail: nakazi@math.sci.hokudai.ac.jp

and

Takanori Yamamoto
Department of Mathematics
Hokkai-Gakuen University
Sapporo 062-8605
Japan
E-mail: yamatk@hucc.hokudai.ac.jp