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ON CONTROLLING THE CONDUCTOR**

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ON GOUVÊA'S CONJECTURE ON CONTROLLING THE CONDUCTOR

ATSUSHI YAMAGAMI

ABSTRACT. In this article, Gouvêa's conjecture on controlling the conductor is proven in a special case.

0. Introduction

Let $p \geq 7$ be a prime number, and $G_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$. For an absolutely irreducible residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_p),$$

we denote by $N = N(\bar{\rho})$ the conductor (for the definition of the conductor, see Section 2). We assume that $\bar{\rho}$ is associated to a classical eigenform f of level N defined over $\mathbb{Z}_p$. Let $S = \{ \text{the prime divisors } l \text{ of } Np \} \cup \{\infty\}$. Then we know that $\bar{\rho}$ factors through the Galois group G_S of the maximal Galois extension of $\mathbb{Q}$ unramified outside S , and we consider the deformation problem of the residual representation

$$\bar{\rho} : G_S \rightarrow GL_2(\mathbb{F}_p).$$

In [10], Mazur showed that there exist a complete Noetherian local ring $R(\bar{\rho}, S)$ with residue field k and a deformation of $\bar{\rho}$

$$\rho^{\text{univ}} : G_S \rightarrow GL_2(R(\bar{\rho}, S))$$

such that any deformation of $\bar{\rho}$

$$\rho : G_S \rightarrow GL_2(A)$$

is obtained from ρ^{univ} via a unique homomorphism $R(\bar{\rho}, S) \rightarrow A$.

On the other hand, we regard f as being in $V_{\text{par}}(\mathbb{Z}_p, N)$ which is the ring of parabolic p -adic modular functions of tame level N defined over $\mathbb{Z}_p$. Let $T_0^*(\mathbb{Z}_p, N)$ be the restricted Hecke algebra on $V_{\text{par}}(\mathbb{Z}_p, N)$ (cf. [5]). We denote by $T(\bar{\rho}, N)$ the completion of $T_0^*(\mathbb{Z}_p, N)$ at the maximal ideal $\mathfrak{m}_f$ which is the kernel of the map $T_0^*(\mathbb{Z}_p, N) \rightarrow \mathbb{F}_p$ determined by f . Then it was shown in [5] that there exists a deformation of $\bar{\rho}$ to $T(\bar{\rho}, N)$

$$\rho^{\text{mod}} : G_S \rightarrow GL_2(T(\bar{\rho}, N))$$

which is universal among deformations of $\bar{\rho}$ associated to p -adic modular functions. Thus we have a natural surjective homomorphism

$$R(\bar{\rho}, S) \rightarrow T(\bar{\rho}, N).$$

For a residual modular representation $\bar{\rho}$ of level $N = N(\bar{\rho})$, Gouvêa's conjecture on controlling the conductor is stated as follows (for the details, see [6]). Let S be as above and S^0 the subset of S consisting of primes $l \mid N$ at which $\bar{\rho}$ is " l -ordinary." In [6], Gouvêa showed that the conductor of any " S^0 -ordinary" deformation of $\bar{\rho}$ is N . Then he defined the level N universal deformation ring $R(\bar{\rho}, N)$ to be the universal S^0 -ordinary deformation ring (for which, see Section 1). He also showed that the surjective homomorphism

$$R(\bar{\rho}, S) \rightarrow T(\bar{\rho}, N)$$

factors through the universal level N deformation ring:

$$\begin{array}{ccc} R(\bar{\rho}, S) & \longrightarrow & T(\bar{\rho}, N) \\ & \searrow \quad \nearrow & \\ & R(\bar{\rho}, N) & \end{array}$$

and he gave the following

Conjecture ([6], Section 4). The surjective homomorphism

$$R(\bar{\rho}, N) \rightarrow T(\bar{\rho}, N)$$

is an isomorphism.

In this article, we prove this conjecture in a special case. We obtain the following

Main Theorem. *Let $p \geq 7$ be a prime number, $k \geq 2$ an integer. For an absolutely irreducible residual representation*

$$\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_p),$$

we denote by $N = N(\bar{\rho})$ the conductor. We assume that $\bar{\rho}$ is associated to a classical eigenform f of type $(N, k, 1)_{\mathbb{Z}_p}$. Let $S = \{ \text{the prime divisors } l \text{ of } Np \} \cup \{\infty\}$, and we consider the deformation problem of the residual representation

$$\bar{\rho} : G_S \rightarrow GL_2(\mathbb{F}_p)$$

where G_S is the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S . We denote by A_p the T_p -eigenvalue of f and suppose that A_p is not a unit of $\mathbb{Z}_p$. Suppose further that the polynomial

$$X^2 - A_p X + p^{k-1}$$

has simple roots in $\mathbb{Z}_p$. If the deformation problem for $\bar{\rho}$ is unobstructed, then the conjecture on controlling the conductor is true for $\bar{\rho}$, i.e., the surjective homomorphism

$$R(\bar{\rho}, N) \rightarrow T(\bar{\rho}, N)$$

is an isomorphism.

Here we mean by “the deformation problem for $\bar{\rho}$ is unobstructed” that

$$\dim_{\mathbb{F}_p} H^2(G_S, \text{Ad}(\bar{\rho})) = 0,$$

where $\text{Ad}(\bar{\rho})$ is the $\mathbb{F}_p$ -vector space of 2×2 matrices with entries in $\mathbb{F}_p$, on which G_S -action is given by

$$\sigma \cdot M = \bar{\rho}(\sigma) M \bar{\rho}(\sigma)^{-1} \quad (\sigma \in G_S, M \in \text{Ad}(\bar{\rho})).$$

Remark 0.1. Let S' be a finite set of rational places containing S . Using “Inflation-Restriction sequence” of Galois cohomology, we can show that if the deformation problem of $\bar{\rho}$ is unobstructed for S' , then the deformation problem of $\bar{\rho}$ is unobstructed for S .

Remark 0.2. (1) We can easily check that if $0 < \text{ord}_p(A_p) < (k-1)/2$ then the polynomial $X^2 - A_p X + p^{k-1}$ has simple roots in $\mathbb{Z}_p$ (cf. [13], Theorem II. 2.1).

(2) In the case that k is odd and $\text{ord}_p(A_p) > (k-1)/2$, we see that the polynomial has simple roots in $\mathbb{Z}_p$ if and only if $p \equiv 1 \pmod{4}$ (cf. [13], Theorem II. 3.3).

(3) Coleman and Edixhoven showed that if the crystalline Frobenius of the Grothendieck motive $M(f)$ over $\mathbb{Q}_p$ is semi-simple, then the characteristic polynomial $X^2 - A_p X + p^{k-1}$ has simple roots (which are not necessarily in $\mathbb{Z}_p$, cf. [4]).

The proof of the main theorem is divided into two steps. The first step is to show that the natural homomorphism

$$R(\bar{\rho}, S) \rightarrow R(\bar{\rho}, N)$$

is surjective. In Section 1, we show in a general situation that the universal S^0 -ordinary deformation ring is a quotient of the universal full-deformation ring using the Lenstra-de Smit construction of universal deformation rings.

The second step is to show that the natural surjective homomorphism

$$R(\bar{\rho}, S) \rightarrow T(\bar{\rho}, N)$$

is an isomorphism. This is a generalization of the main theorem of [8] to the case of an arbitrary tame level. In Section 2, we regard our $\bar{\rho}$ as being associated to the “twins” of level Np coming from f . We then

reduce the proof of the isomorphism to the arguments of Gouvêa-Mazur in [8] which use the theory of “infinite ferns.”

1. The universal S^0 -ordinary deformation ring

In this section, we consider “ S^0 -ordinary” deformations in a general situation. Let $p \geq 7$ be a prime, $\mathbf{k}$ a finite field of characteristic p , S a finite set of rational places containing p and ∞ , and G_S the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S . Let

$$\bar{\rho} : G_S \rightarrow GL_2(\mathbf{k})$$

be an absolutely irreducible residual representation.

First we briefly recall the deformation theory of Galois representations. Let A be a complete Noetherian local ring with residue field $\mathbf{k}$. We shall consider homomorphisms $\rho : G_S \rightarrow GL_2(A)$ which make the following diagram commutative:

$$\begin{array}{ccc} G_S & \xrightarrow{\rho} & GL_2(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & GL_2(\mathbf{k}), \end{array}$$

where the homomorphism $GL_2(A) \rightarrow GL_2(\mathbf{k})$ is induced from the natural reduction homomorphism $A \rightarrow \mathbf{k}$. We say two such lifts are *strictly equivalent* if they are conjugate by an element of $\ker(GL_2(A) \rightarrow GL_2(\mathbf{k}))$, and we refer to a strict equivalence class of lifts of $\bar{\rho}$ as *deformation* of $\bar{\rho}$ to A .

Mazur showed in [10] the following

Theorem 1.1 ([10], Proposition 1). *Let a residual representation*

$$\bar{\rho} : G_S \rightarrow GL_2(\mathbf{k})$$

be absolutely irreducible. Then there exist a complete Noetherian local ring $\mathbf{R}(\bar{\rho}, S)$ with residue field $\mathbf{k}$ and a deformation of $\bar{\rho}$

$$\rho^{\text{univ}} : G_S \rightarrow GL_2(\mathbf{R}(\bar{\rho}, S))$$

such that any deformation ρ of $\bar{\rho}$ to a complete Noetherian local ring A with residue field $\mathbf{k}$ is obtained from ρ^{univ} via a unique homomorphism $\mathbf{R}(\bar{\rho}, S) \rightarrow A$.

The ring $\mathbf{R}(\bar{\rho}, S)$ depends only on $\bar{\rho}$ up to “twist-equivalence,” i.e., if $\bar{\rho}'$ is equivalent to a twist of $\bar{\rho}$ by a one-dimensional character, then there exists a canonical isomorphism

$$r(\bar{\rho}, \bar{\rho}') : \mathbf{R}(\bar{\rho}, S) \xrightarrow{\sim} \mathbf{R}(\bar{\rho}', S).$$

Now we consider the “ordinary” deformations. Let $\rho : G_S \rightarrow GL_2(A)$ be a Galois representation and $M = A \times A$ with G_S -action by ρ . Let l be a prime number, and I an inertia group at l . We say ρ is l -ordinary if the submodule $M^I \subset M$ of invariants under I is a free A -module of rank one and a direct summand of M .

Let $S^0 \subset S$, and assume that $\bar{\rho} : G_S \rightarrow GL_2(\mathbf{k})$ is S^0 -ordinary, i.e., $\bar{\rho}$ is l -ordinary for all $l \in S^0$. Then we can consider S^0 -ordinary deformations of $\bar{\rho}$. The following theorem asserts that the universal S^0 -ordinary deformation ring exists and it is a quotient of $\mathbf{R}(\bar{\rho}, S)$.

Theorem 1.2 (cf. [6], Theorem 2). *Let $S^0 \subset S$, and assume that $\bar{\rho} : G_S \rightarrow GL_2(\mathbf{k})$ is absolutely irreducible and S^0 -ordinary. Then there exist a complete Noetherian local ring $\mathbf{R}(\bar{\rho}, S, S^0)$ with residue field $\mathbf{k}$ and an S^0 -ordinary deformation of $\bar{\rho}$*

$$\rho(S, S^0) : G_S \rightarrow GL_2(\mathbf{R}(\bar{\rho}, S, S^0))$$

such that any S^0 -ordinary deformation ρ of $\bar{\rho}$ to a complete Noetherian local ring A with residue field $\mathbf{k}$ is obtained from $\rho(S, S^0)$ via a unique homomorphism $\mathbf{R}(\bar{\rho}, S, S^0) \rightarrow A$.

Moreover, the natural homomorphism

$$\mathbf{R}(\bar{\rho}, S) \rightarrow \mathbf{R}(\bar{\rho}, S, S^0)$$

is surjective.

Proof. We use the Lenstra-de Smit construction of universal deformation rings to show the existence of the universal S^0 -ordinary deformation ring (cf. [9]).

Note that Lenstra and de Smit considered the deformation problem of residual representations in the category of complete local rings (not necessarily Noetherian). However, by Theorem (2.3)(3), (4) in [9], the universal deformation ring that they constructed in [9] is isomorphic to the one that is constructed in the category of complete Noetherian local rings, since $\dim_{\mathbb{F}_p} H^1(G_S, \text{Ad}(\bar{\rho})) < \infty$.

Let $V = \mathbf{k} \times \mathbf{k}$ with G_S -action by $\bar{\rho}$. We denote by $\text{Def}(V, A)$ the set of deformations of $\bar{\rho}$ to A . Further we denote by $S^0(A)$ the set of S^0 -ordinary deformations and $S_l(A)$ the set of l -ordinary deformations for each $l \in S^0$. Then $S^0(A)$ and $S_l(A)$ are subsets of $\text{Def}(V, A)$, and we have

$$S^0(A) = \bigcap_{l \in S^0} S_l(A).$$

By Proposition (6.1) of [9], we know that the functor $S_l(\cdot)$ satisfies the conditions (1) – (3) in Section 6 of [9]. Therefore the functor $S^0(\cdot)$ also satisfies these conditions, and we know that this is represented by

a quotient ring $R(\bar{\rho}, S, S^0)$ of the universal deformation ring $R(\bar{\rho}, S)$. Namely, the natural homomorphism

$$R(\bar{\rho}, S) \rightarrow R(\bar{\rho}, S, S^0)$$

is surjective. $\square$

We return to Gouvêa's conjecture on controlling the conductor. In [6], Gouvêa defined the level N universal deformation ring $R(\bar{\rho}, N)$ to be the universal S^0 -ordinary deformation ring. So the natural homomorphism

$$R(\bar{\rho}, S) \rightarrow R(\bar{\rho}, N)$$

is surjective by the theorem above.

2. The proof of the main theorem

In Gouvêa's commutative diagram

$$\begin{array}{ccc} R(\bar{\rho}, S) & \longrightarrow & T(\bar{\rho}, N) \\ & \searrow \quad \nearrow & \\ & R(\bar{\rho}, N), & \end{array}$$

we know that the homomorphism

$$R(\bar{\rho}, S) \rightarrow R(\bar{\rho}, N)$$

is surjective by Section 1. To obtain the main theorem, in this section, we show that the natural surjective homomorphism

$$R(\bar{\rho}, S) \rightarrow T(\bar{\rho}, N)$$

is an isomorphism for $\bar{\rho}$ of the main theorem.

2.1. The conductor of a residual representation

First we recall the definition of the conductor of a residual representation, and state an essential lemma on modular representations.

Let $p \geq 7$ be a prime number, k a finite field of characteristic p . For a residual representation

$$\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(k),$$

the conductor $N(\bar{\rho})$ of $\bar{\rho}$ is defined to be

$$N(\bar{\rho}) = \prod_{l \neq p} l^{n(l, \bar{\rho})},$$

where the numbers $n(l, \bar{\rho})$ are defined as follows: choose a place of $\bar{\mathbb{Q}}$ over l , and let I be the corresponding inertia group. Let $\bar{V} = k \times k$

with $G_{\mathbb{Q}}$ -action given by $\bar{\rho}$, and $\bar{V}_0$ the subspace of $\bar{V}$ fixed by $\bar{\rho}(I)$. Then

$$n(l, \bar{\rho}) = 2 - \dim \bar{V}_0 + \text{sw}(\bar{\rho})$$

where $\text{sw}(\bar{\rho})$ is the Swan conductor of the restriction of $\bar{\rho}$ to I .

Note that $n(l, \bar{\rho}) = 0$ if and only if $\bar{\rho}$ is unramified at l . So the product in the definition is meaningful. Note also that $N(\bar{\rho})$ is prime to p . (For the definition of the conductor of a residual representation, see also [6], [14].)

Now we assume that $\bar{\rho}$ is absolutely irreducible and associated to a classical eigenform f of level $N(\bar{\rho})$ defined over a $W(\mathbf{k})$ -algebra B with residue field $\mathbf{k}$. Then Gouvêa showed the following lemma in [6]. It is very essential to proving $\mathbf{R}(\bar{\rho}, S) \cong \mathbf{T}(\bar{\rho}, N)$ in the next section.

Lemma 2.1 (cf. [6], Lemma 7). *Let $\bar{\rho}$ and f be as above. Then the eigenform f is a newform of level $N(\bar{\rho})$.*

2.2. Twins, Coleman's families and infinite ferns

Now we prove that the isomorphism $\mathbf{R}(\bar{\rho}, S) \cong \mathbf{T}(\bar{\rho}, N)$ for $\bar{\rho}$ of the main theorem and finish the proof of the main theorem. Let $\bar{\rho}: G_S \rightarrow GL_2(\mathbb{F}_p)$ and f be as in the assumption of the main theorem. Note that $N = N(\bar{\rho})$ is the conductor of $\bar{\rho}$. Then the classical eigenform f associated to $\bar{\rho}$ is of type $(N, k, 1)_{\mathbb{Z}_p}$, i.e., a cusp form of level N , weight k with trivial character whose Fourier expansion is lying in $\mathbb{Z}_p[[q]]$ and whose first Fourier coefficient is equal to 1, and which is an eigenform for the Hecke operators T_l for prime numbers l not dividing Np and for the Atkin operators U_q for prime numbers q dividing Np .

Remark 2.1. We denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. We fix once and for all an isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$. Given a classical modular form f , we transport the Fourier coefficients of f in $\mathbb{C}$ to $\mathbb{C}_p$ via our fixed isomorphism. In this way we regard f as being defined over $\mathbb{C}_p$.

Since the level N of f is the conductor of $\bar{\rho}$, f is a *newform* of level N by Lemma 2.1. From the relation

$$U_p = T_p - p^{k-1}B_p,$$

we know that the characteristic polynomial of the Hecke operator U_p acting on the two-dimensional space of oldforms of level Np spanned by f and $f|B_p$ over $\mathbb{C}_p$ is

$$X^2 - A_p X + p^{k-1}.$$

Here A_p is the T_p -eigenvalue of f , and B_p is the standard “degeneracy operator” which acts on the Fourier expansion of f as $(f|B_p)(q) = f(q^p)$.

By the assumption, this polynomial has simple roots in $\mathbb{Z}_p$. We denote these roots by λ_1, λ_2 . Then the following two eigenforms of type $(Np, k, 1)_{\mathbb{Z}_p}$

$$\begin{aligned} f_1 &= f - \lambda_2 \cdot f|B_p, \\ f_2 &= f - \lambda_1 \cdot f|B_p \end{aligned}$$

form a basis of that space. Their T_l -eigenvalues for l not dividing Np and U_q -eigenvalues for q dividing N are equal to those of f , and U_p -eigenvalues are λ_1 and λ_2 , respectively. In particular, from the equality of the T_l -eigenvalues for l not dividing Np , we may assume that $\bar{\rho}$ is associated to both of f_1 and f_2 . We call f_1 and f_2 the *twins* coming from f . Note that the twins are “new away from p .” (We say that a classical eigenform g of level Np is *new away from p* when g is a newform of level Np or g belongs to a space of oldforms spanned by φ and $\varphi|B_p$ for a newform φ of level N .)

Putting $g = f_1$, we may assume that $\bar{\rho}$ is associated to g which is a classical eigenform of type $(Np, k, 1)_{\mathbb{Z}_p}$ and new away from p . Since the U_p -eigenvalue λ_1 of g is a simple root of the polynomial $X^2 - A_p X + p^{k-1}$, we have $\lambda_1^2 \neq p^{k-1}$. We put $\alpha = \text{ord}_p(\lambda_1)$ and call it the *slope* of g . Since A_p is not a unit, we have $0 < \alpha < k - 1$, i.e., g is of *non-critical slope*.

Remark 2.2. Note that we do not need the level of f to be the conductor $N(\bar{\rho})$ of $\bar{\rho}$ except when we show that f is a newform in the above arguments concerning eigenforms.

Note also that the assumption that the polynomial $X^2 - A_p X + p^{k-1}$ has simple roots in $\mathbb{Z}_p$ implies that the eigenform g is defined over $\mathbb{Z}_p$. This fact enables us to use the same arguments as in [8] dealing with eigenforms defined over $\mathbb{Z}_p$.

Summing up the arguments above, we know that the main theorem follows from the following theorem which is a generalization of the main theorem of [8] to the case of an *arbitrary* tame level. (From now on, the symbol N denotes any positive integer prime to p .)

Theorem 2.2. *Let $p \geq 7$ be a prime number, $k \geq 2$ an integer, and N a positive integer prime to p . Let $S = \{ \text{the prime divisors } l \text{ of } Np \} \cup \{\infty\}$ and G_S the Galois group of the maximal Galois extension of $\mathbb{Q}$ unramified outside S . We assume that an absolutely irreducible residual representation*

$$\bar{\rho} : G_S \rightarrow GL_2(\mathbb{F}_p)$$

is associated to a classical eigenform g of type $(Np, k, 1)_{\mathbb{Z}_p}$, non-critical slope α and new away from p . Further, assume that $\lambda_p^2 \neq p^{k-1}$ where λ_p is the U_p -eigenvalue of g . If the deformation problem for $\bar{\rho}$ is unobstructed, then the natural surjective homomorphism

$$R(\bar{\rho}, S) \rightarrow T(\bar{\rho}, N)$$

is an isomorphism.

To prove this theorem, we need the following two propositions on “twins” and “Coleman’s families” respectively.

Proposition 2.3. *Let f be a classical eigenform of type $(Np, k, 1)_{\mathbb{C}_p}$. Suppose that the slope of f is equal to neither $(k-1)/2$ nor $(k-2)/2$ and that f is new away from p . Then f has a twin.*

Proof. If f is a newform, then its Hecke U_p -eigenvalue is $\pm p^{(k-2)/2}$ (cf. [1], Theorem 3). As we assume that the slope of f is not equal to $(k-2)/2$, f is an oldform of level Np . Since f is new away from p , f belongs to a space of oldforms spanned by φ and $\varphi|B_p$ for a newform φ of level N .

Let a_p be the U_p -eigenvalue of f . Then we can check easily that f and $f' = \varphi - a_p \cdot \varphi|B_p$ are the twins coming from φ , since the slope of f is not equal to $(k-1)/2$. $\square$

Remark 2.3. From the proof above, we know that if f is defined over $\mathbb{Z}_p$, then the twin f' of f is also defined over $\mathbb{Z}_p$. This fact is essential to constructing an “infinite fern” in the universal deformation space $X = \text{Hom}_{\mathbb{Z}_p\text{-alg}}(R(\bar{\rho}, S), \mathbb{Z}_p)$ (for which, see [12]).

Proposition 2.4 (cf. [3], Corollary B5. 7. 1). *Let g be a classical eigenform of type $(Np, k_0, 1)_{\mathbb{Z}_p}$ which is new away from p and slope $\alpha = \text{ord}_p(\lambda_p)$, where λ_p the U_p -eigenvalue of g . Suppose that $k_0 > \alpha + 1$ and that $\lambda_p^2 \neq p^{k_0-1}$.*

Then there exist a disk $D \subset \mathbb{Z}_p$ containing k_0 and p -adic analytic functions $a_n(s) : D \rightarrow \mathbb{Z}_p$ for $n \geq 1$ such that if $k \in D$ is an integer strictly greater than $\alpha + 1$, then

$$f_k(q) = \sum_{n \geq 1} a_n(k) q^n$$

is a classical eigenform of type $(Np, k, 1)_{\mathbb{Z}_p}$ and slope α which is new away from p , and $f_{k_0} = g$.

Proof. The proof of this proposition can be found in [3], Section B5. Note that Coleman constructed his families of eigenforms using the duality between the space of “ p' -new forms” and the Hecke algebra acting on it. This implies the eigenforms f_k are new away from p . $\square$

These propositions enable us to construct an “infinite fern” in the universal deformation space $X = \text{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathbf{R}(\bar{\rho}, S), \mathbb{Z}_p)$ by the same way as in [12]. The unobstructedness of the deformation problem for $\bar{\rho}$ implies that X has a structure of $\mathbb{Q}_p$ -analytic manifold of dimension 3. Gouvêa and Mazur showed in [8] the isomorphism of tame level 1,

$$\mathbf{R}(\bar{\rho}, \{p, \infty\}) \cong \mathbf{T}(\bar{\rho}, 1),$$

by showing the density of modular points in the universal deformation space. Recall that, in their arguments, the existence of an infinite fern of level p caused this density on the assumption that the universal deformation space is a 3-dimensional $\mathbb{Q}_p$ -analytic manifold. Now we have an infinite fern of level Np in our space X . Therefore Theorem 2.2 can be proven by the same arguments as in [8], and we omit the details of the proof here.

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