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Title	Remarks on the uniqueness of bounded solutions of the NavierStokes equations
Author(s)	Giga, Y.; Inui, K.; Kato, J. et al.
Citation	Hokkaido University Preprint Series in Mathematics, 497, 1-4
Issue Date	2000-10-01
DOI	https://doi.org/10.14943/83643
Doc URL	https://hdl.handle.net/2115/69247
Type	departmental bulletin paper
File Information	re497.pdf



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Series #497. October 2000

HOKKAIDO UNIVERSITY
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Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations

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1 Introduction

This is an announcement of our results on the uniqueness of a bounded solution $u = (u_1, \dots, u_n)$ of the Navier-Stokes equations:

$$(1) \quad u_t - \Delta u + (u, \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbf{R}^n \times (0, T).$$

with initial data $u|_{t=0} = u_0$. The detailed argument as well as improvement will be published in [8].

It is by now well-known that for initial data $u_0 \in L^\infty(\mathbf{R}^n)$ the equation (1) admits a unique local-in-time (regular) solution u with

$$(2) \quad p = \sum_{i,j=1}^n R_i R_j u_i u_j,$$

where $R_j = (-\Delta)^{-1/2} \partial / \partial x_j$ is the Riesz operator [1], [11], [2], [6]. (Very recently, it turns out that this solution can be extended globally-in-time when the space dimension is two [7].) It is also well-known that for L^r -initial data ($n \leq r < \infty$) the equation (1) admits a

¹Party supported by the Grant-in-Aid for Scientific Research No.10304010, the Japan Society for the Promotion of Science

⁴Party supported by the Grant-in-Aid for Scientific Research No.11640211, the Japan Society for the Promotion of Science

unique local-in-time solution u with some p [5], [9], [4]... Because of decay at the space infinity of u the relation (2) follow (up to constant) a posteriori for L^r -data ($n \leq r < \infty$).

For L^∞ -initial data the constructed solution u is bounded and may not decay at the space infinity. So even if u solves (1) with some p the relation (2) may not follow. In fact, if we consider $u = g(t)$, then (u, p) always solves (1) with $p = -g'(t) \cdot x$ no matter what function g is. Here $\cdot$ denotes the inner product of $\mathbf{R}^n$. This says the solution u with a constant initial data is not unique without assuming (2). This example suggests that contrary to L^r -case ($n \leq r < \infty$) we need to impose some condition on p to derive uniqueness other than on u .

In this paper we give a new sufficient condition for the uniqueness of solution (u, p) when u is bounded.

Theorem. *Assume that (u, p) is a smooth solution of (1) in $\mathbf{R}^n \times (0, T)$ and that u is bounded. Assume that u is continuous on $[0, T)$ with values in BUC , the space of all bounded uniformly continuous functions. Let u_0 denote the initial data of u . If p is of the form*

$$(3) \quad p = \sum_{1 \leq i, j \leq n} R_i R_j \pi_{ij} + \pi_0$$

with some $\pi_{ij}, \pi_0 \in L^1_{loc}(0, T; L^\infty(\mathbf{R}^n))$, then $(u, \nabla p)$ is uniquely determined by u_0 and p fulfills (2) up to a constant.

Recently, the third author improved the result by weakening the assumption on pressure (3) by

$$(4) \quad p \in L^1_{loc}(0, T; BMO),$$

where BMO is the space of all functions of bounded mean oscillation. By the Calderón-Zygmund inequality (3) implies (4). By a theorem of A. Uchiyama [13] if a function $f \in BMO$ is compactly supported, then it is of the form $f = \sum_{1 \leq i, j \leq n} R_i R_j \eta_{ij} + \eta_0$ with some $\eta_{ij}, \eta_0 \in L^\infty(\mathbf{R}^n)$. Thus the difference of (3) and (4) is small.

The assumption of continuity on u is to determine initial data and can be weakened. The regularity assumption on (u, p) is also can be weakened by using a notion of weak solutions. The reader is referred to a recent paper by the third author [8] for improvement of Theorem.

Let us mention a few known results closely related to our uniqueness results. It was shown in [3] that if $n = 3$ and u with ∇u is bounded in $\mathbf{R}^3 \times (0, T)$, then the uniqueness holds provided that for some $C > 0$ and some $\varepsilon \in (0, 1)$ the inequality

$$(5) \quad |p(x, t)| \leq C(|x|^{1-\varepsilon} + 1)$$

holds for all $(x, t) \in \mathbf{R}^3 \times (0, T)$. Later it was shown in [12], [10] that if $n = 2, 3$ and ∇u is bounded in $\mathbf{R}^3 \times (0, T)$, then the uniqueness holds provided that (5) holds with $\varepsilon = n/2$. (The argument is valid for general n). Our assumptions (3) and (4) do not imply (5) so it is not comparable with those results.

2 Sketch of the proof

The main idea is to prove

$$(6) \quad \nabla p = \sum_{1 \leq i, j \leq n} \nabla R_i R_j u_i u_j$$

which yields (2) so that u solves that integral equation without p whose solution is known [6] to be unique.

A formal idea to prove (6) is to apply $R_i R_j$ to the first j -th equation and add with respect to j . Since $\sum_{j=1}^n R_j f_j = 0$ for divergence free $f = (f_1, \dots, f_n)$, we come to

$$(7) \quad \sum_{1 \leq j, \ell \leq n} R_i R_j (u_\ell \partial_{x_\ell} u_j) + \sum_{j=1}^n R_i R_j \partial_{x_j} p = 0, \quad i = 1, \dots, n,$$

where $\partial_{x_j} = \partial/\partial x_j$. Since $\operatorname{div} u = 0$ and $R_i R_j \partial_{x_\ell} = \partial_{x_i} R_j R_\ell$ the first term of (7) equals $\sum_{j, \ell} \partial_{x_i} R_j R_\ell u_j u_\ell$. Since $\sum_j R_i R_j \partial_{x_j} p = -\partial_{x_i} p$, (7) now yields (6). Unfortunately, the above argument is formal since it is not clear in what way we operate $R_i R_j$ to $\partial_{x_j} p$ when p is merely in BMO .

To overcome this inconvenience we approximate $R_i R_j$ by R_{ij}^ε defined by

$$R_{ij}^\varepsilon f = (\partial_{x_i} \partial_{x_j} k_\varepsilon) * f.$$

Here k_ε is obtained by a truncation of the fundamental solution k of $-\Delta$ and is defined by

$$k_\varepsilon = \psi_\varepsilon \lambda_\varepsilon k$$

with $\psi_\varepsilon(x) = \psi(x/\varepsilon)$, $\lambda_\varepsilon(x) = \lambda(\varepsilon x)$, where ψ is a radial function satisfying $\psi = 0$ for $|x| \leq 1$ and $\psi(x) = 1$ for $|x| \geq 2$ and $\lambda = 1 - \psi$.

Instead of applying $R_i R_j$ we apply R_{ij}^ε to (1) and send $\varepsilon \rightarrow 0$ which realizes the formal idea and obtain (6). A key convergence results is

$$(8) \quad R_{ij}^\varepsilon \partial_{x_j} g \rightarrow -\partial_{x_i} g \quad (1 \leq i \leq n)$$

in the sense of tempered distributions for $g \in BMO$. To prove Theorem it suffices to prove the above convergence for g of the form $g = \sum_{ij} R_i R_j \pi_{ij} + \pi_0$ with some π_{ij} and π_0 . The convergence (8) follows from the observation:

$$(9) \quad (-\Delta) k_\varepsilon * \psi \rightarrow \psi$$

in $\mathcal{H}^1$, the Hardy space for $\psi \in C_0^\infty(\mathbf{R}^n)$ satisfying $\int_{\mathbf{R}^n} \psi dx = 0$. Again to prove (8) for restricted g for Theorem it suffices to prove (9) in L^1 -sense and almost everywhere sense. We won't go into more detail in the present paper and refer to a paper [8] for the detail.

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