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Abstract

A perturbed Dirac operator $Q(\alpha)$ on the abstract Boson-Fermion Fock space is considered, where $\alpha \in \mathbb{C}$ is a perturbation (coupling) parameter and the unperturbed operator $Q(0)$ is taken to be a free infinite dimensional Dirac operator introduced by the author (A. Arai, J. Funct. Anal. **105**(1992), 342–408). The following results are reported: (i) Under some conditions, the kernel of $Q(\alpha)$ is one dimensional for all $\alpha \neq \alpha_0$ with some $\alpha_0 \neq 0$ and degenerate at $\alpha = \alpha_0$, while, under another condition, the kernel of $Q(\alpha)$ is one dimensional for all $\alpha \in \mathbb{C}$. (ii) There are cases where, for all sufficiently large $|\alpha|$ with $\alpha < 0$, $Q(\alpha)$ has infinitely many non-zero eigenvalues even if $Q(0)$ has no non-zero eigenvalues. This is a strong coupling effect. (iii) Fredholm property of $Q(\alpha)$ also depends on the coupling parameter α.

Key words: Boson-Fermion Fock space, supersymmetric quantum field, infinite dimensional Dirac operator, non-regular perturbation, kernel, spectrum, Fredholm property, strong coupling effect

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1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces. Then the abstract Boson-Fermion Fock space $\mathcal{F}(\mathcal{H},\mathcal{K})$ associated with the pair $\langle\mathcal{H},\mathcal{K}\rangle$ is defined by

$$\mathcal{F}(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{f}}(\mathcal{K}), \tag{1.1}$$

where $\mathcal{F}_{\mathrm{b}}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{H}$ is the Boson Fock space over $\mathcal{H}$ ($\otimes_{\mathrm{s}}^{n}\mathcal{H}$ denotes the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$; $\otimes_{\mathrm{s}}^{0}\mathcal{H} := \mathbb{C}$) and $\mathcal{F}_{\mathrm{f}}(\mathcal{K}) := \oplus_{p=0}^{\infty} \wedge^{p} \mathcal{K}$ is the Fermion Fock space over $\mathcal{K}$ ($\wedge^{p}\mathcal{K}$ denotes the p-fold anti-symmetric tensor product Hilbert space of $\mathcal{K}$; $\wedge^{0}\mathcal{K} := \mathbb{C}$).

Let $\mathsf{C}(\mathcal{H},\mathcal{K})$ be the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then, for each $A \in \mathsf{C}(\mathcal{H},\mathcal{K})$, one can define a Dirac-type operator Q_A on $\mathcal{F}(\mathcal{H},\mathcal{K})$ [Ar92] (for the definition of Q_A, Section 2 below). The operator Q_A is an infinite dimensional version of free Dirac operators on finite dimensional spaces.

In [Ar92] some fundamental properties of Q_A were established. Moreover, a perturbed Dirac operator of the form $Q_A(V) := Q_A + V$ was considered in view of index theory, where V is a symmetric operator on $\mathcal{F}(\mathcal{H},\mathcal{K})$, and a functional integral representation for the index of $Q_A(V)$ restricted to a subspace, called the "bosonic subspace", was derived (for related aspects and further developments, see [Ar89], [Ar91], [Ar93], [Ar93], [Ar94], [Ar96], [Ar97], [AM91], [AM93]). It still remains, however, as an important problem, to investigate spectral properties of $Q_A(V)$. In this paper, as a first step towards this direction, we present a perturbation of Q_A which is not of the form V considered in [Ar92] and rather simple, but, gives rise to interesting *nonperturbative instability phenomena* in spectral and Fredholm properties. For proofs on the results reported in this paper, see [Ar00].

2 A Class of Perturbed Dirac Operators

We denote by $a(f)$ ($f \in \mathcal{H}$) and $b(u)$ ($u \in \mathcal{K}$) the annihilation operators on $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ and on $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ respectively (e.g., [BR97, §5.2]). Let $\Omega_{\mathrm{b}} := \{1,0,0,\cdots\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{H})$ (resp. $\Omega_{\mathrm{f}} := \{1,0,0,\cdots\} \in \mathcal{F}_{\mathrm{f}}(\mathcal{K})$) be the Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ (resp. $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$). Let $A \in \mathsf{C}(\mathcal{H},\mathcal{K})$ and

$$\begin{aligned}\mathcal{D}_A^{\infty} \; := \; & \mathcal{L}\Big\{a(f_1)^* \cdots a(f_n)^*\Omega_{\mathrm{b}} \otimes b(u_1)^* \cdots b(u_p)^*\Omega_{\mathrm{f}} \Big| n,p \geq 0, f_j \in C^{\infty}(A^*A), \\ & \qquad j = 1,\cdots,n,\; u_k \in C^{\infty}(AA^*), k = 1,\cdots,p\Big\},\end{aligned} \tag{2.1}$$

where $\mathcal{L}\{\cdots\}$ means the subspace algebraically spanned by the vectors in the set $\{\cdots\}$ and $C^{\infty}(T) := \cap_{n=1}^{\infty} D(T^n)$ for a linear operator T on a Hilbert space ($D(T^n)$ denotes the domain of T^n) . It follows that $\mathcal{D}_A^{\infty}$ is dense in $\mathcal{F}(\mathcal{H},\mathcal{K})$.

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system of $\mathcal{K}$ such that $e_n \in D(A^*), n \in \mathbb{N}$. Then it is shown that there is a unique densely defined closed linear operator d_A on $\mathcal{F}(\mathcal{H},\mathcal{K})$ such that (i) for all $\Psi \in \mathcal{D}_A^{\infty}$

$$d_A\Psi = \sum_{n=1}^{\infty} a(A^*e_n)b(e_n)^*\Psi \tag{2.2}$$

independently of the choice of $\{e_n\}_{n=1}^{\infty}$ and (ii) $\mathcal{D}_A^{\infty}$ is a core of d_A. It follows that $d_A^2 = 0$. The operator may be regarded as an infinite dimensional version of finite dimensional exterior differential operators. *A free Dirac operator* on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by

$$Q_A = d_A + d_A^*. \tag{2.3}$$

We have an orthogonal decomposition

$$\mathcal{F}(\mathcal{H}, \mathcal{K}) = \mathcal{F}_+(\mathcal{H}, \mathcal{K}) \oplus \mathcal{F}_-(\mathcal{H}, \mathcal{K}) \tag{2.4}$$

with

$$\mathcal{F}_+(\mathcal{H}, \mathcal{K}) := \oplus_{p=0}^{\infty} \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \wedge^{2p}(\mathcal{K}), \quad \mathcal{F}_-(\mathcal{H}, \mathcal{K}) := \oplus_{p=0}^{\infty} \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \wedge^{2p+1}(\mathcal{K}). \tag{2.5}$$

Let P_+ and P_- be the orthogonal projections onto $\mathcal{F}_+(\mathcal{H}, \mathcal{K})$ and $\mathcal{F}_-(\mathcal{H}, \mathcal{K})$ respectively and define

$$\Gamma := P_+ - P_-. \tag{2.6}$$

For a self-adjoint operator S on $\mathcal{H}$ (resp. $\mathcal{K}$), $d\Gamma_{\mathrm{b}}(S)$ (resp. $d\Gamma_{\mathrm{f}}(S)$) denotes the second quantization of S in the Boson Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ (resp. the Fermion Fock space $\mathcal{F}_{\mathrm{f}}(\mathcal{H})$) [BR97, §5.2].

Basic properties of Q_A are as follows [Ar92]:

(i) The operator Q_A is self-adjoint and essentially self-adjoint on $\mathcal{D}_A^{\infty}$.

(ii) The operator Γ leaves $D(Q_A)$ invariant and $\Gamma Q_A + Q_A \Gamma = 0$ on $D(Q_A)$.

(iii) The following operator equations hold :

$$Q_A^2 = d_A^* d_A + d_A d_A^* = d\Gamma_{\mathrm{b}}(A^*A) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(AA^*), \tag{2.7}$$

where I denotes identity.

We perturb Q_A through a perturbation of d_A. Let $g \in D(A)$, $g \neq 0$, and $v \in D(A^*), v \neq 0$ and define

$$d(\alpha) := d_A + \alpha a(g) \otimes b(v)^*. \tag{2.8}$$

where $\alpha \in \mathbb{C}$ is a coupling constant. It is easy to see that $D(d(\alpha)) \supset \mathcal{D}_A^{\infty}$ and $d(\alpha)|\mathcal{D}_A^{\infty}$ is closable. Let

$$\bar{d}(\alpha) := \overline{d(\alpha)|\mathcal{D}_A^{\infty}}, \tag{2.9}$$

the closure of $d(\alpha)|\mathcal{D}_A^{\infty}$, and

$$Q(\alpha) := \bar{d}(\alpha) + \bar{d}(\alpha)^*. \tag{2.10}$$

This is the perturbed Dirac operator considered in this paper.

3 Results

Theorem 3.1 (i) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint, and essentially self-adjoint on $\mathcal{D}_A^\infty$ with $Q(\alpha) = \overline{Q_A + V_{g,v}}$, where $V_{g,v} := \alpha a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v)$. Moreover, Γ leaves $D(Q(\alpha))$ invariant and $\Gamma Q(\alpha) + Q(\alpha)\Gamma = 0$ on $D(Q(\alpha))$.*

(ii) *Suppose that A is injective and $g \in D(|A|^{-1})$. Then, for all $|\alpha| < (\|v\| \, \| |A|^{-1}g\|)^{-1}$, $D(Q(\alpha)) = D(Q_A)$.*

(iii) *For all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.*

To describe the spectral properties of $Q(\alpha)$, we introduce a bounded linear operator $T_{g,v}$ from $\mathcal{H}$ to $\mathcal{K}$ by

$$T_{g,v} f := (g, f)v, \quad f \in \mathcal{H}, \tag{3.1}$$

where $(\cdot\,, \cdot)$ denotes inner product, and define

$$A(\alpha) := A + \alpha T_{g,v}. \tag{3.2}$$

For a linear operator T on a Hilbert space, we denote by $\sigma(T)$ (resp. $\sigma_{\mathrm{p}}(T)$) the (resp. point) spectrum of T. A general feature of the spectra of $Q(\alpha)$ is given in the following theorem:

Theorem 3.2 *For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_{\mathrm{p}}(Q(\alpha))$ are symmetric with respect to the origin and*

$$\sigma(Q(\alpha)) = \{0\} \bigcup \overline{\left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_j} \,\middle|\, \lambda_j \in \sigma(A(\alpha)^* A(\alpha)), j = 1, \cdots, n \right\} \right)}, \tag{3.3}$$

$$\sigma_{\mathrm{p}}(Q(\alpha)) = \{0\} \bigcup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^{n} \lambda_j} \,\middle|\, \lambda_j \in \sigma_{\mathrm{p}}(A(\alpha)^* A(\alpha)), j = 1, \cdots, n \right\} \right), \tag{3.4}$$

with

$$\dim \ker(Q(\alpha) - \lambda) = \dim \ker(Q(\alpha) + \lambda), \quad \lambda \in \sigma_{\mathrm{p}}(Q(\alpha)). \tag{3.5}$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^* A(\alpha)$.

To state properties of the kernel of $Q(\alpha)$, we introduce the following conditions on $\{A, g, v\}$:

(C.1) A is injective and $v \in D(A^{-1})$ with $(g, A^{-1}v) \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{(g, A^{-1}v)}. \tag{3.6}$$

(C.2) A^* is injective and $g \in D(A^{*-1})$ with $(A^{*-1}g, v) \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{(A^{*-1}g, v)}. \tag{3.7}$$

(C.3) A is injective, and $v \notin D(A^{-1})$ or $v \in D(A^{-1})$ with $(g, A^{-1}v) = 0$.

(C.4) A^* is injective, and $g \notin D(A^{*-1})$ or $g \in D(A^{*-1})$ with $(A^{*-1}g, v) = 0$.

For a linear operator T on a Hilbert space, we set $\mathrm{nul}\, T := \dim \ker T$.

Theorem 3.3 (i) *Suppose that (C.1) and (C.2) hold. Let*

$$\Psi_{n,j} := a(A^{-1}v)^{*^n}\Omega_{\mathrm{b}} \otimes b(A^{*-1}g)^{*^j}\Omega_{\mathrm{f}}, \quad n = 0, 1, 2, \cdots, \quad j = 0, 1. \tag{3.8}$$

Then $\mathrm{nul}\, Q(\alpha_0) = \infty$ *with* $\ker Q(\alpha_0) = \overline{\mathcal{L}\{\Psi_{n,j} | n \geq 0, j = 0, 1\}}$. *Moreover, for all* $\alpha \neq \alpha_0$,

$$\mathrm{nul}\, Q(\alpha) = 1, \quad \ker Q(\alpha) = \{c\Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}} | c \in \mathbb{C}\}. \tag{3.9}$$

(ii) *Suppose that (C.1) and (C.4) hold. Then* $\mathrm{nul}\, Q(\alpha_0) = \infty$ *with*

$$\ker Q(\alpha_0) = \overline{\mathcal{L}\{\Psi_{n,0} | n \geq 0\}}.$$

Moreover, for all $\alpha \neq \alpha_0$, (3.9) holds.

(iii) *Suppose that (C.2) and (C.3) hold. Then* $\mathrm{nul}\, Q(\beta_0) = 2$ *with* $\ker Q(\beta_0) = \mathcal{L}\{\Psi_{0,j} | j = 0, 1\}$.

(iv) *Suppose that (C.3) and (C.4) hold. Then, for all $\alpha \in \mathbb{C}$, (3.9) holds.*

As for non-zero eigenvalues of $Q(\alpha)$, we have the following result:

Theorem 3.4 *Consider the case where $\mathcal{H} = \mathcal{K}$, A is a nonnegative self-adjoint operator with* $\ker A = \{0\}$ *and $g = v \in D(A^{-1})$ (then $\alpha_0 = -1/(v, A^{-1}v) < 0$). Let $\alpha < \alpha_0$. Then, there exists a constant $x_0(\alpha) < 0$ such that $\alpha(v, (x_0(\alpha) - A)^{-1}v) = 1$, and for all $n \in \{0\} \cup \mathbb{N}$,*

$$\pm \sqrt{n}|x_0(\alpha)| \in \sigma_{\mathrm{p}}(Q(\alpha)). \tag{3.10}$$

Note that this theorem holds even if Q_A has no non-zero eigenvalues. As the condition that $\alpha < \alpha_0$ shows, this is a *strong coupling effect.*

By Theorem 3.1 (i)-(ii), the operator $Q_+(\alpha)$ defined by

$$D(Q_+(\alpha)) := D(Q(\alpha)) \cap \mathcal{F}_+(\mathcal{H}, \mathcal{K}), \quad Q_+(\alpha)\Psi := Q(\alpha)\Psi, \ \Psi \in D(Q_+(\alpha)) \tag{3.11}$$

is a densely defined closed linear operator from $\mathcal{F}_+(\mathcal{H}, \mathcal{K})$ to $\mathcal{F}_-(\mathcal{H}, \mathcal{K})$. We define an index of $Q(\alpha)$ by

$$\mathrm{ind}_\Gamma(Q(\alpha)) := \mathrm{nul} Q_+(\alpha) - \mathrm{nul} Q_+(\alpha)^*, \tag{3.12}$$

the index of $Q_+(\alpha)$, provided that $\mathrm{nul} Q_+(\alpha) < \infty$ or $\mathrm{nul} Q_+(\alpha)^* < \infty$.

Results on the (semi-)Fredholm property and the index $\mathrm{ind}_\Gamma Q(\alpha)$ are as follows:

Theorem 3.5 (i) *Suppose that (C.1) and (C.2) hold. Then $Q(\alpha_0)$ is not semi-Fredholm. Moreover, for all $\alpha \neq \alpha_0$, $Q(\alpha)$ is Fredholm with* $\mathrm{ind}_\Gamma Q(\alpha) = 1$.

(ii) *Suppose that (C.1) and (C.4) hold. Then* $Q(\alpha_0)$ *is semi-Fredholm with* $\mathrm{ind}_\Gamma Q(\alpha) = \infty$. *Moreover, for all* $\alpha \neq \alpha_0$, $Q(\alpha)$ *is Fredholm with* $\mathrm{ind}_\Gamma Q(\alpha) = 1$.

(iii) *Suppose that (C.2) and (C.3) hold. Then* $Q(\beta_0)$ *is Fredholm with* $\mathrm{ind}_\Gamma Q(\beta_0) = 0$. *Moreover, for all* $\alpha \neq \beta_0$, $Q(\alpha)$ *is Fredholm with* $\mathrm{ind}_\Gamma Q(\alpha) = 1$.

(iv) *Suppose that (C.3) and (C.4) hold. Then, for all* $\alpha \in \mathbb{C}$, $Q(\alpha)$ *is Fredholm with* $\mathrm{ind}_\Gamma Q(\alpha) = 1$.

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