Characteristic classes of singular varieties

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CHARACTERISTIC CLASSES OF SINGULAR VARIETIES

TATSUO SUWA

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In this article we review recent developements on the characteristic classes of singular varieties, in particular the theory of Milnor classes and related topics.

For a complex manifold M, its Chern class $c^*(M)$ is defined, in the cohomology $H^*(M)$, as the Chern class of the holomorphic tangent bundle TM and it has been playing an important role in the theory of complex manifolds. To generalize this notion to the case of a compact singular variety V, several classes have been introduced. Among them are the Mather class $c_*^M(V)$, the Schwartz-MacPherson class $c_*(V)$ (cf. [Sc1] and [Ma]) and the Fulton-Johnson class $c_*^{FJ}(V)$ (cf. [FJ] and [F]). These are all defined in the homology $H_*(V)$ of V and, when V is non-singular, they coincide with the Poincaré dual of $c^*(V)$, i.e., the cap product $c^*(V) \frown [V]$ with the fundamental class $[V]$. Each of these classes is defined in different context and has its own advantages. For instance, the Mather class is defined by "extending" the tangent bundle taking the Nash modification of V and is relatively easy to understand geometrically. As for the Schwartz-MacPherson class, it was first defined by M.-H. Schwartz in the relative cohomology $H^*(W, W \setminus V)$, where W is an ambient manifold, using some special vector fields, called radial vector fields (see Section 4 (A) below). Then it was defined by R. MacPherson in $H_*(V)$ combining the theory of Mather classes and the notion of local Euler obstructions and was shown to have all the nice properties we can expect, as conjectured earlier by A. Grothendieck and P. Deligne. The two classes can be identified via the Alexander duality $H^*(W, W \setminus V) \simeq H_*(V)$ (cf. [BS]) and it is called the Schwartz-MacPherson class nowadays. For a singular variety V, we may define the Poincaré homomorphism $H^*(V) \to H_*(V)$ by taking the cap product with $[V]$ (cf. [Br1]). When V is a local complete intersection, we have the virtual tangent bundle τ_V of V and, in this case, the Fulton-Johnson class of V coincides with $c^*(\tau_V) \frown [V]$ and it can be calculated relatively easily.

The differences between these classes are deeply related with some important invariants of the singular set of the variety. For instance, when V is a local

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complete intersection with only isolated singularities, the difference of $c_*(V)$ and $c_*^{FJ}(V)$ can be expressed as the sum of the Milnor numbers at the singular points (see Theorem 1.13 below). The notion of Milnor class is introduced to express the difference when the singularities may not be isolated. It has been studied by a number of authors from different approaches, for hypersurfaces in [A] and [PP2], for local complete intersections in [BLSS] and for mappings in [BY]. Also its properties are studied in [OY] and [Y3].

In this article, we discuss the Milnor classes along the line of [BLSS]. We also review, as a related topic, the theory of characteristic classes of coherent sheaves on singular varieties introduced in [Su4], which allows us, in particular, to define a new characteristic class of a singular variety V, as the (homology) Chern class of the tangent sheaf of V.

As review articles on the Milnor classes, there are also [Br2] and [Y2]. As introductory articles related to the subjects discussed here, we list [Su5, 6]. The recent article [IS] deals with the multiplicities of functions on singular varieties from the approach explained in this article.

In the following Section 1, we review the case of local complete intersections with isolated singularities. Taking the Poincaré-Hopf theorem as a model, we consider its generalizations in two directions, introducing the Schwartz index and the virtual index of a vector field on a singular variety. The difference between the two indices turns out to be the Milnor number at the singular point and we obtain the result mentioned above comparing the two generalizations. In Section 2, we summarize the theory of localization of characteristic classes, which is the fundamental idea in the theories explained in this article. To deal with the case of possibly non-isolated singularities, we recall, in Section 3, the classical obstruction theory in the case of the Chern class of the tangent bundle of a complex manifold. From the construction, we see the following. If we are given a (continuous) r-tangent field $F^{(r)}$ (Definition 3.2) on a complex manifold M of dimension n, for each compact connected component S of the singular set of $F^{(r)}$, we may naturally define the Poincaré-Hopf class $\mathrm{PH}(F^{(r)}, S)$ in $H_{2r-2}(S)$. In the global situation where M is compact, the sum of these classes coincides with the Poincaré dual of the p-th Chern class $c^p(M)$ of M, $p = n - r + 1$ (Theorem 3.5).

Let V be a compact subvariety of dimension n in a complex manifold W of dimension $n + k$. In Section 4, we first recall the construction by M.-H. Schwartz of the q-th Schwartz class of V in the relative cohomology $H^{2q}(W, W \setminus V)$ using a radial r-tangent field, $q = n + k - r + 1$. Its image $c_{r-1}(V)$ by the Alexander isomorphism $H^{2q}(W, W \setminus V) \xrightarrow{\sim} H_{2r-2}(V)$ coincides with the $(r-1)$-st MacPherson class of V, as mentioned above. We then introduce the (local) Schwartz class of a tangent frame. In the sequel, we let $\mathrm{Sing}(V)$ denote the singular set of V and $V_0 = V \setminus \mathrm{Sing}(V)$ the non-singular part. Let $F^{(r)}$ be an r-tangent field on V_0 and Σ the union of $\mathrm{Sing}(V)$ and the singular set of $F^{(r)}$. For each compact connected component S of Σ, we define the Schwartz class $\mathrm{Sch}(F^{(r)}, S)$ in $H_{2r-2}(S)$ using a radial tangent frame as a reference frame. Then, in the global situation, the sum of these classes is equal to $c_{r-1}(V)$ (Theorem 4.5). In Section 5, we consider the case where V is a local complete intersection in W defined by a section of a holomorphic vector bundle N over W. In this case, we may consider the "virtual tangent bundle" $\tau_V = (TW - N)|_V$ of V. For an r-tangent field $F^{(r)}$ on V_0 and a connected component S of the set Σ as above, by localizing the p-th Chern class $c^p(\tau_V)$ of τ_V, we define the virtual class $\mathrm{Vir}(F^{(r)}, S)$ in $H_{2r-2}(S)$, $p = n - r + 1$. In the global situation, the sum of these classes is equal to the image of $c^p(\tau_V)$ by the Poincaré homomorphism $H^{2p}(V) \to H_{2r-2}(V)$ (Theorem 5.1). In the case we are considering, this coincides with the $(r-1)$-st Fulton-Johnson class $c^{FJ}_{r-1}(V)$.

In the above construction, when S is in the non-singular part V_0, all these classes are identical; $\mathrm{Sch}(F^{(r)}, S) = \mathrm{Vir}(F^{(r)}, S) = \mathrm{PH}(F^{(r)}, S)$. In Section 6, when S is a connected component of $\mathrm{Sing}(V)$, for each non-negative integer r, we define the r-th Milnor class $\mu_r(V, S)$ of V at S to be the difference of the Schwartz class and the virtual class of an $(r+1)$-tangent frame $F^{(r+1)}$ (Definition 6.1). This is a class in $H_{2r}(S)$ and does not depend on the choice of $F^{(r+1)}$. When r is greater than the (complex) dimension of S, we have $\mu_r(V, S) = 0$. In particular, when S consists of a point p, only $\mu_0(V, p)$ is non-zero and it coincides with the usual Milnor number which was introduced in [Mi] in the case of hypersurfaces and in [Ha] in the case of complete intersections. Also, when V is a hypersurface, it can be shown that $\mu_0(V, S)$ coincides with the generalized Milnor number defined in [Par]. For a local complete intersection V with arbitrary singularities, the difference of $c_*(V)$ and $c^{FJ}_*(V)$ is expressed as the sum of the Milnor classes (Theorem 6.2). When S is a non-singular connected component of $\mathrm{Sing}(V)$, we give an explicit formula for $\mu_*(V, S)$ (Theorem 6.3).

As mentioned before, the Chern class of a complex manifold was defined to be the Chern class of the holomorphic tangent bundle. In the case of a singular variety V, it would be a natural question to ask if it is possible to define some meaningful characteristic classes from, for example, some coherent sheaves on V. In particular we ask what is the characteristic class of the tangent sheaf Θ_V of V. In general, for a coherent sheaf $\mathcal{F}$ on a possibly singular variety V, the (cohomology) Chern character $\mathrm{ch}^*(\mathcal{F})$ or the Chern class $c^*(\mathcal{F})$ is classically defined either when V is non-singular or when $\mathcal{F}$ is locally free. In Section 7, we define, for a coherent sheaf $\mathcal{F}$ on a local complete intersection V, the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ and the Chern class $c_*(\mathcal{F})$ (Definition 7.3). If $\mathcal{F}$ is locally free, the class $\mathrm{ch}_*(\mathcal{F})$ coincides with the image of $\mathrm{ch}^*(\mathcal{F})$ by the Poincaré homomorphism. This is a consequence of the Riemann-Roch theorem for the embedding of V in W (Theorem 7.2). When V has only isolated singularities, it is possible to obtain explicit formulas for the Chern character and the Chern class of the tangent sheaf Θ_V of V (Theorem 7.6).

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1. Isolated singularities case

(A) Poincaré-Hopf Theorem

Let v be a a continuous vector field on a C^∞ manifold M. For an isolated zero (singularity) p of v, we have the Poincaré-Hopf index, which is defined, for instance, as the mapping degree of the map induced by v on a small sphere centered at p. It will be denoted by $\mathrm{PH}(v,p)$ hereafter. When M is a compact oriented manifold (without boundary) and when v admits only isolated singular points $p_1,\dots,p_r$, the Poincaré-Hopf theorem says that

$$\sum_{i=1}^{r} \mathrm{PH}(v,p_i) = \chi(M), \tag{1.1}$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M. This formula can be interpreted in various ways. One way to think of it is as follows. There is a fundamental characteristic class of a vector bundle, which is called the Euler class. The Euler class $e(M)$ of M is the Euler class of its tangent bundle and is in the cohomology of the top dimension. If we use this, the formulas (1.1) may be decomposed as

$$\sum_{i=1}^{r} \mathrm{PH}(v,p_i) = \int_M e(M), \tag{1.2}$$

$$\int_M e(M) = \chi(M). \tag{1.3}$$

One way to interpret the class $e(M)$ is as the obstruction to constructing a vector field without singularities and the identity (1.2) explicitly shows how, for a given vector field v, the class $e(M)$ is "localized" at its singularities. On the other hand, the equality (1.2) can be seen as a special case of the Atiyah-Singer index theorem.

Note that the formula (1.1) holds also when the manifold M has a boundary ∂M, provided that the vector field v is everywhere outbound from ∂M.

Suppose now that M is a complex manifold of dimension n. Then its holomorphic tangent bundle TM is isomorphic, as a real bundle, with the tangent bundle of M considered as a C^∞ manifold. In this case, the Euler class $e(M)$ coincides with the top Chern class $c^n(M)$ of M. Thus, when M is compact, (1.3) may be written as

$$\int_M c^n(M) = \chi(M). \tag{1.4}$$

If we take a connection for the bundle TM, we may represent the class $c^n(M)$ by a differential form determined by its curvature and then we may think of (1.4) as a Gauss-Bonnet type theorem.

In the sequel, if we simply say a vector field, it means a continuous vector field. We consider it as a section of TM, identifying it with the real tangent bundle.

We now try to generalize the above formulas to the case of singular varieties. For this purpose, we generalize the Poincaré-Hopf index to the case of vector fields on (the non-singular part of) a singular variety V. There are several ways to do so. First, we introduce the "Schwartz index" in order to have an identity as (1.1). Also, if V is a local complete intersection, we may consider the "virtual index" as a localization of the Chern class of its virtual tangent bundle by the vector field. Then we have a formula corresponding to (1.2). This type of index has the advantage of being stable under "perturbations". Then an identity as (1.4) does not hold any more and, when V admits only isolated singularities, the Milnor numbers of V at the singularities appear as the difference.

(B) Schwartz index

Let V be a variety (reduced complex analytic space) of pure dimension n. Suppose that V has at most isolated singularity at a point p. We may assume that there exists a neighborhood U of p which can be embedded in a complex euclidean space $\mathbb{C}^m$. For a non-singular vector field v on $U \setminus \{p\}$, we define the Schwartz index at p as follows.

First, for two non-singular vector fields v_1 and v_2 on $U \setminus \{p\}$, we define the difference (of the indices) $d(v_1, v_2)$ as follows. Take two small spheres S_i, $i = 1, 2$, in $\mathbb{C}^m$ with center p and radii ε_i with $\varepsilon_1 < \varepsilon_2$, and set $K_i = V \cap S_i$. We consider a vector field on a neighborhood of the (disjoint) union of K_1 and K_2 which coincides with v_i in a neighborhood of K_i, for each $i = 1, 2$, and try to extend it to the region between K_1 and K_2. It is always possible to have an extension w with a finite number of singular points $p_1, \ldots, p_r$. Then we set

$$d(v_1, v_2) = \sum_{i=1}^{r} \mathrm{PH}(w, p_i),$$

which is independent of the extension w.

Next, we recall that there exists a vector field $\tilde{v}_0$ on a neighborhood of p in $\mathbb{C}^m$ satisfying the following (see for example [Mi, §2]) :
(i) $\tilde{v}_0$ is non-singular and outbound everywhere except at p,
(ii) $\tilde{v}_0$ is tangent to $V \setminus \{p\}$.

Such a vector field $\tilde{v}_0$ is a special case of "radial vector fields", which will be discussed in Section 4 below. Let v_0 denote the restriction of $\tilde{v}_0$ to $U \setminus \{p\}$.

With these ingredients, for a non-singular vector field v on $U \setminus \{p\}$, we define the *Schwartz index* $\mathrm{Sch}(v, p)$ of v at p by

$$\mathrm{Sch}(v, p) = 1 + d(v_0, v).$$

In particular, for v_0, we have $\mathrm{Sch}(v_0, p) = 1$, which can be interpreted as the Euler-Poincaré characteristic of the point p (see Section 4). When p is a non-singular point of V, we have $\mathrm{Sch}(v, p) = \mathrm{PH}(v, p)$. Also, for two vector fields v_1 and

v_2 as above, we have

$$\mathrm{Sch}(v_2, p) = \mathrm{Sch}(v_1, p) + d(v_1, v_2). \tag{1.5}$$

We have the following formula for these indices :

Theorem 1.6 [SS2]. *Let V be a compact variety with only isolated singular points $p_1, \ldots, p_s$ and v a vector field on $V_0 = V \setminus \{p_1, \ldots, p_s\}$ with only isolated singular points $p_{s+1}, \ldots, p_r$. Then*

$$\sum_{i=1}^{r} \mathrm{Sch}(v, p_i) = \chi(V).$$

This generalizes the Poincaré-Hopf theorem to the case of vector fields on singular varieties.

(C) Virtual index

We have another type of index, which is called the "virtual index". A singular variety V does not have a tangent bundle in the usual sense. However, if V is of the following type, we may consider the "virtual tangent bundle" and we may define the virtual index of a vector field by localizing its Chern class by the vector field.

Let W be a complex manifold of dimension $n+k$ and N a holomorphic vector bundle of rank k over W. Also, let s be a holomorphic section of N generically transverse to the zero section and set $V = s^{-1}(0)$. In this case, V is a variety of dimension n and is locally expressed as follows. Let $(s_1, \ldots, s_k)$ be a frame of N on an open set $\tilde{U}$ of W and write $s = \sum_{i=1}^{k} f_i s_i$ with f_i holomorphic functions on $\tilde{U}$. Then we may express

$$\begin{aligned} V \cap \tilde{U} &= \{\, p \in \tilde{U} \mid f_1(p) = \cdots = f_k(p) = 0 \,\}, \\ \mathrm{Sing}(V) \cap \tilde{U} &= \{\, p \in \tilde{U} \mid df_1 \wedge \cdots \wedge df_k(p) = 0 \,\}. \end{aligned}$$

Moreover, in this case, $f_1 \ldots, f_k$ generate the ideal of functions vanishing on V in $\tilde{U}$ and V is a local complete intersection (cf. [T]). In the sequel, we call a variety as above a *local complete intersection defined by a section.*

Here are some examples of varieties as above. Let V be an arbitrary hypersurface in W. Then there exist a natural line bundle N and its section s so that V is defined by s. This way V is a variety as above. Also, a complete intersection in the projective space $\mathbb{CP}^{n+k}$ (an algebraic variety of dimension n defined by k homogeneous polynomials P_i, $i = 1, \ldots, k$) is an example of such a variety. In this case, let H denote the hyperplane bundle and let d_i be the degree of P_i. Then we may take the vector bundle $H^{\otimes d_1} \oplus \cdots \oplus H^{\otimes d_k}$ as N. A local complete intersection defined by a section is a "strong" local complete intersection in the sense of [LS].

Let V be a local complete intersection defined by a section of a bundle N and set $V_0 = V \setminus \mathrm{Sing}(V)$. Then the restriction $N|_{V_0}$ coincides with the normal bundle N_{V_0} of V_0 in W and we have the exact sequence

$$0 \longrightarrow TV_0 \longrightarrow TW|_{V_0} \longrightarrow N_{V_0} \longrightarrow 0. \tag{1.7}$$

We set $\tau_V = (TW - N)|_V$ and call it the virtual tangent bundle of V. Since the Chern class of $TW - N$ is given by $c^*(TW - N) = c^*(TW)/c^*(N)$, the characteristic classes of the virtual tangent bundle can be calculated relatively easily. As in (B) above, suppose V admits at most an isolated singularity at p and let U be a neighborhood of p in V. For a non-singular vector field v on $U \setminus \{p\}$, we define the *virtual index* $\mathrm{Vir}(v,p)$ as the localization of $c^n(\tau_V)$ by v. This is defined differential geometrically, combining the localizations of types (I) and (II) of Section 2 (B) below (see [LSS], [SS2] and [Su2, Ch.IV, 3] for details). It can be defined similarly in the case of non-isolated singularities. In the case of isolated singularities, a geometric interpretation of this index is given in (D) below (cf. (1.10)). It shows that the virtual index coincides with the "GSV-index", which was introduced earlier ([GSV], [SS1]). It also shows that the virtual index at an isolated singularity is an integer.

When p is a non-singular point of V, we have $\mathrm{Vir}(v,p) = \mathrm{PH}(v,p)$. Also, as in the case of Schwartz index, for two vector fields v_1 and v_2 as above, we have

$$\mathrm{Vir}(v_2,p) = \mathrm{Vir}(v_1,p) + d(v_1,v_2). \tag{1.8}$$

From the construction of the virtual index, we have the following :

Theorem 1.9. *Let V be a local complete intersection defined by a section with isolated singular points $p_1, \ldots, p_s$ and let v be a vector field on V_0 with isolated singular points at $p_{s+1}, \ldots, p_r$. Then*

$$\sum_{i=1}^{r} \mathrm{Vir}(v,p_i) = \int_V c^n(\tau_V).$$

This can be thought of as a generalization of (1.2).

(D) Milnor number

We recall the definition of the Milnor number of a complete intersection at an isolated singular point. Let $f = (f_1, \ldots, f_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$ be a holomorphic map defined in a neighborhood of the origin 0 and suppose that $V = f^{-1}(0)$ is a complete intersection with at most an isolated singularity at $p = 0$. For a general point t in $\mathbb{C}^k$, $V_t = f^{-1}(t)$ is non-singular. Moreover, the following is also known ([Mi] when $k = 1$ and [Ha] for general k, see also [Lo]). Let B be a small ball in $\mathbb{C}^{n+k}$ with center 0, then for a general point t sufficiently close to 0 in $\mathbb{C}^k$, the topological type of $F = V_t \cap B$ is constant and F has the homotopy type of a bouquet of

S^n. We call F the Milnor fiber and the number of spheres S^n which appear in the bouquet is the *Milnor number* $\mu(V,p)$ of V at p. When p is a non-singular point of V, $\mu(V,p) = 0$. The Milnor number can be also computed algebraically from the Jacobian ideal of f (cf. [G], [Lê], [Lo], [Pal]).

Let U be a neighborhood of p in V and v a non-singular vector field on $U \setminus \{p\}$. We may interpret the virtual index $\mathrm{Vir}(v,p)$ as follows. We take a vector field v' near the boundary of the Milnor fiber F, which is tangent to F and is "close" to v and try to extend it to the interior of F. We always have an extension w with a finite number of singularities $p_1, \dots p_r$. Then, $\mathrm{Vir}(v,p)$ is given by the following formula ([LSS], see also [SS2]) :

$$\mathrm{Vir}(v,p) = \sum_{i=1}^{r} \mathrm{PH}(w,p_i). \tag{1.10}$$

If v' is outbound everywhere on the boundary of F, the right hand side above is the Euler-Poincaré characteristic of F, which is equal to $1 + (-1)^n\mu(V,p)$. Therefore, for the vector field v_0 in (B) above, we have $\mathrm{Vir}(v_0,p) = 1+(-1)^n\mu(V,p)$. Since the Schwartz and virtual indices satisfy similar difference formulas (1.5) and (1.8), for an arbitrary non-singular vector field v on $U \setminus \{p\}$,

$$\mathrm{Sch}(v,p) - \mathrm{Vir}(v,p) = (-1)^{n+1}\mu(V,p). \tag{1.11}$$

From the above, using a vector field on V as a mediator, we have :

Theorem 1.12 [SS2]. *Let V be a compact local complete intersection defined by a section. If V admits only isolated singularities $p_1, \dots, p_s$, then*

$$\chi(V) = \int_V c^n(\tau_V) + (-1)^{n+1}\sum_{i=1}^{s}\mu(V,p_i).$$

We may think of the above as a generalization of the Gauss-Bonnet formula (cf. (1.4)).

The following formula for the Schwartz-MacPherson class $c_*(V)$ of V (see Section 4) is proved using the resolution of singularities of V, the naturality of the Schwartz-MacPherson class, the localization argument of type (II) in Section 2 (B) below and the above formula.

Theorem 1.13 [Su1]. *Let V be a compact local complete intersection defined by a section. If V admits only isolated singularities $p_1, \dots, p_s$, then*

$$c_*(V) = c^*(\tau_V) \frown [V] + (-1)^{n+1}\sum_{i=1}^{s}\mu(V,p_i)[p_i].$$

We recall that, for a local complete intersection V, the class $c^*(\tau_V) \frown [V]$ coincides with the Fulton-Johnson class $c_*^{FJ}(V)$ of V. In [OSY], a comparison formula of the Mather class $c_*^M(V)$ and $c_*^{FJ}(V)$ is given for a variety V as above.

2. Localization of characteristic classes

We already mentioned the localization of characteristic classes in the previous section. Here we discuss the theory in more details. As general references for characteristic classes, we list [BB], [Hi], [MS] and [St].

(A) Chern classes and related characteristic classes

Let M be a C^∞ manifold and E a complex vector bundle of rank e over M. For $i = 1, \dots, e$, we have the i-th Chern class $c^i(E)$ of E in $H^{2i}(M)$. The total Chern class $c^*(E)$ of E is given by $c^*(E) = 1 + c^1(E) + \cdots + c^e(E)$.

Topologically, $c^i(E)$ is defined to be the primary obstruction to constructing $e - i + 1$ sections of E everywhere linearly independent and is defined in the integral cohomology $H^{2i}(M, \mathbb{Z})$ (see Section 3). Differential geometrically, it is defined in $H^{2i}(M, \mathbb{R})$ by the Chern-Weil theory. Namely, for a connection ∇ for E and the i-th elementary symmetric polynomial σ_i, we may define canonically a closed $2i$-form $\sigma_i(\nabla)$ on M and we set

$$c^i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla).$$

Then its class $[c^i(\nabla)]$ in the de Rham cohomology $H^{2i}_{\mathrm{dR}}(M) \simeq H^{2i}(M, \mathbb{R})$ is equal to $c^i(E)$.

Let E_j, $j = 0, \dots, \nu$, be complex vector bundles. For the virtual bundle $\xi = \sum_{j=0}^{\nu} (-1)^j E_j$, its Chern class is determined by

$$c^*(\xi) = \prod_{j=0}^{\nu} c^*(E_j)^{\epsilon(j)}, \qquad \epsilon(j) = (-1)^j.$$

We may also define the Chern class for a coherent sheaf on a complex manifold M. Let $\mathcal{O}_M$ denote the sheaf of germs of holomorphic functions on M. For a coherent $\mathcal{O}_M$-module $\mathcal{F}$, we take its resolution by complex vector bundles E_j, $j = 0, \dots, \nu$ (cf. [AH]) and define its Chern class by $c^*(\mathcal{F}) = c^*(\xi)$, $\xi = \sum_{j=0}^{\nu} (-1)^j E_j$, which is independent of the choice of the resolution.

Thus we may define the Chern class $c^*(\xi)$ when ξ is a complex vector bundle, a (complex) virtual bundle or a coherent sheaf. More generally, let φ be a symmetric polynomial or a symmetric power series. Since we may express as $\varphi = P(\sigma_1, \sigma_2, \dots)$ for a suitable polynomial or a power series P, we may define the characteristic class $\varphi(\xi)$ of ξ with respect to φ by

$$\varphi(\xi) = P(c^1(\xi), c^2(\xi), \dots).$$

For example, the Chern character $\mathrm{ch}(\xi)$ and the Todd class $\mathrm{td}(\xi)$ are classes arising in this manner (see for example [Hi, §10]). We may also represent $\varphi(\xi)$ by the characteristic form $\varphi(\nabla_\bullet)$ determined from a family of connections $\nabla_\bullet = (\nabla_j)_j$ obtained by taking a connection ∇_j for each vector bundle E_j involved in the expression of ξ.

(B) Localization

We now think of localizing characteristic classes. Let M be a C^∞ manifold of dimension m and Σ a closed set in M. It sometimes happens that, by some reason, we have $\varphi(\xi) = 0$ on $M \setminus \Sigma$ and there is a class $\tilde{\varphi}(\xi)$ in $H^*(M, M \setminus \Sigma)$ which is sent to $\varphi(\xi)$ by the canonical homomorphism $H^*(M, M \setminus \Sigma) \to H^*(M)$. We then call $\tilde{\varphi}(\xi)$ a localization of $\varphi(\xi)$ at Σ. In the cases we consider below, the vanishing $\varphi(\xi) = 0$ occurs on the level of cocycles and, using this fact, we may define a "canonical" localization $\tilde{\varphi}(\xi)$.

Suppose moreover that Σ is compact and that it admits a regular neighborhood. This last condition is satisfied, for example, if M admits a triangulation compatible with Σ, in particular, if M is a complex manifold and Σ is an analytic set. In this case, by the Alexander duality

$$H^*(M, M \setminus \Sigma) \xrightarrow{\sim} H_*(\Sigma),$$

$\tilde{\varphi}(\xi)$ defines a class in $H_*(\Sigma)$. If we let $(S_\lambda)_\lambda$ be the connected components of Σ, we have $H_*(\Sigma) = \oplus_\lambda H_*(S_\lambda)$ and, for each λ, $\tilde{\varphi}(\xi)$ defines a class in $H_*(S_\lambda)$, which is denoted by $\mathrm{Res}_\varphi(\xi, S_\lambda)$ and is called the residue of $\varphi(\xi)$ at S_λ. Denoting by $\iota_\lambda : S_\lambda \hookrightarrow M$ the inclusion map, if M is compact, we have the "residue formula" (see the commutative diagram in Section 3 (A)) :

$$\sum_\lambda (\iota_\lambda)_* \mathrm{Res}_\varphi(\xi, S_\lambda) = \varphi(\xi) \frown [M]. \tag{2.1}$$

The above arguments work also when we replace M by a possibly singular variety, with some modifications.

In the sequel, we discuss some vanishing theorems and the corresponding localizations. In the case (I) below, both the obstruction theory and (a modification of) the Chern-Weil theory are effective. When we use the obstruction theory, the localization is defined in $H^*(M, M \setminus \Sigma; \mathbb{Z})$. In the case (II), it seems difficult to apply the obstruction theory directly. We use either the relative K-theory or the Chern-Weil theory, choosing appropriate connections. In the case (III), we have continuous invariants and the topological method seems to be impossible. The method using connections is effective.

(I) Localization by a frame.

If a complex vector bundle E admits, on $M \setminus \Sigma$, a family $\mathbf{s} = (s_1, \dots, s_r)$ of r linearly independent sections (we call such a family an r-frame), we have $c^i(E|_{M \setminus \Sigma}) = 0$ for $i = e - r + 1, \dots, e$. In this case, the corresponding localization $\tilde{c}^i(E)$ of $c^i(E)$ is denoted by $c^i(E, \mathbf{s})$. When we use the obstruction theory, the class $c^i(E, \mathbf{s})$ is determined naturally by the construction. When we use the differential geometry, the class $c^i(E, \mathbf{s})$ is naturally determined taking an "$\mathbf{s}$-trivial" connection on $M \setminus \Sigma$ (cf. [Su2, II, Proposition 9.1], [Su7]). For each connected component S_λ of Σ, we have the residue, now denoted by $\mathrm{Res}_{c^i}(\mathbf{s}, E; S_\lambda)$, in $H_{m-2i}(S_\lambda)$.

We refer to [S7] for an explicit computation of the residue when S_λ consists of a point.

In fact, the assumption that there exists an r-frame on $M \setminus \Sigma$ is rather strong. We take a triangulation of M compatible with Σ (or the cellular decomposition dual to it) and $c^{e-r+1}(E)$ is localized when we are given an r-frame on the $2(e-r+1)$-skeleton of $M \setminus \Sigma$ (see Section 3).

In particular, when M is a complex manifold of dimension n and $E = TM$, for a vector field v on M and a connected component S_λ of its singular set Σ, we have the residue $\mathrm{Res}_{c^n}(v, TM; S_\lambda)$ in $H_0(S_\lambda, \mathbb{Z}) = \mathbb{Z}$. When S_λ consists of a point p, this is exactly $\mathrm{PH}(v,p)$ and, when M is compact, (2.1) is the Poincaré-Hopf theorem.

(II) Localization by the exactness of a sequence

Suppose we have a complex of complex vector bundles over M :

$$0 \longrightarrow E_\nu \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0,$$

which is exact on $M \setminus \Sigma$. If we consider the virtual bundle $\xi = \sum_{j=0}^{\nu}(-1)^j E_j$, by K-theory, ξ defines an element in $K(M, M \setminus \Sigma)$ and, for each $i = 1, 2, \dots$, there exists a localization $c^i_\Sigma(\xi)$ in $H^{2i}(M, M \setminus \Sigma)$ of $c^i(\xi)$ (cf. [BFM], [I]). In differential geometry, we use the fact that, if we take a connection ∇_j for E_j on $M \setminus \Sigma$, for each j, so that $\nabla_\bullet = (\nabla_j)_j$ is "compatible" with the above sequence, then the characteristic form $c^i(\nabla_\bullet)$ becomes identically equal to zero (cf. [BB, (4.22) Lemma]). In this case, the localization is naturally determined on the cocycle level.

In particular, when M is a complex manifold and when $\mathcal{F}$ is a coherent $\mathcal{O}_M$-module, we have the local Chern class $c^i_\Sigma(\mathcal{F})$ with Σ the support of $\mathcal{F}$. This will be used in Section 7 below.

(III) Localization by a Bott type vanishing theorem

This type of vanishing occurs on some characteristic forms of a vector bundle which admits an action of an involutive subbundle of the tangent bundle.

In general, (the tangent sheaf of) a singular foliation on a complex manifold M is defined to be an involutive coherent subsheaf $\mathcal{F}$ of the tangent sheaf $\Theta_M = \mathcal{O}_M(TM)$ of M. Also, the singular set Σ of a foliation $\mathcal{F}$ is defined to be the set of points where the normal sheaf $\Theta_M/\mathcal{F}$ of the foliation fails to be $\mathcal{O}_M$-free. In particular, when $\mathcal{F}$ is (locally) generated by a single holomorphic vector field v, Σ coincides with the zero set of v. On $M' = M \setminus \Sigma$, there exists an involutive subbundle F of TM such that $\mathcal{F}|_{M'} = \mathcal{O}_{M'}(F)$. If we let the rank of F be p, F defines a non-singular foliation of dimension p on M'. Suppose now that we have a vector bundle E on which F "acts". Then we may take an "F-connection" for E on M'. The Bott type vanishing theorem says that, if φ is a symmetric homogeneous polynomial of degree greater than $n-p$ and if ∇ is an F-connection for E, then $\varphi(\nabla) = 0$ (for example [BB], see also [Su2, II, Theorem 9.11]).

The above also works for virtual bundles and using this, we can study various invariants associated with the singular set of a foliation. When $\mathcal{F}$ is generated by a single vector field v, we may take an F-connection which is also v-trivial. From this we see that these invariants include the Poincaré-Hopf indices and the virtual indices of holomorphic vector fields. These theories, together with the case of singular foliations on singular varieties, are systematically described in [Su2].

In any case of the above, when we use the differential geometric methods, we take some special connection ∇ on $M \setminus \Sigma$ first. As to the treatment after this, there are several ways :
(i) Extend ∇ to the whole M by modifying it in a neighborhood of Σ. Then we get a characteristic form with compact support on M (for example [BB]).
(ii) Consider ∇ as a singular connection on M and extend the characteristic form as a current on M (for example [HL]).
(iii) Let ∇ as it is and take another connection on a neighborhood of Σ. In this case the difference in the intersection becomes important. The localization is naturally represented by a Čech-de Rham cocycle (for example [LS]).

3. Chern classes via obstruction theory

In Sections 3 and 4, the homology and cohomology are with integral coefficients.

(A) Poincaré and Alexander dualities

We briefly review the classical case, which will be generalized to the case of singular varieties in Section 4 (B) below (cf. [Br1]). Let M be an oriented manifold of dimension m. We take a triangulation (K) of M and the cellular decomposition (D) dual to (K). Note that each cell of (D) consists of simplices of a barycentric subdivision (K') of (K). The groups of chains relative to (K) and (D) are denoted by $C_*^{(K)}(M)$ and $C_*^{(D)}(M)$, respectively. Also, the groups of cochains relative to (K) and (D) are denoted by $C^*_{(K)}(M)$ and $C^*_{(D)}(M)$, respectively. For an i-simplex σ of (K), the $(m-i)$-cell dual to it is denoted by $\check{\sigma}$.

First, when M is compact, we define a homomorphism

$$P : C^{m-i}_{(D)}(M) \to C_i^{(K)}(M) \qquad \text{by} \qquad P(c) = \sum_\sigma \langle c, \check{\sigma} \rangle \, \sigma \tag{3.1}$$

for an $(m-i)$-cochain c, where the sum is taken over all i-simplices σ of M. This induces the Poincaré isomorphism

$$P_M : H^{m-i}(M) \xrightarrow{\sim} H_i(M).$$

Next, let L be a compact (K)-subcomplex of M (M may not be compact) and let $C^*_{(D)}(M, M \setminus L)$ be the subgroup of $C^*_{(D)}(M)$ consisting of cochains which is zero on the cells not intersecting with L. We define a homomorphism

$$A : C^{m-i}_{(D)}(M, M \setminus L) \to C_i^{(K)}(L)$$

taking, in the sum in (3.1), only i-simplices of L. This induces the Alexander isomorphism

$$A_{M,L} : H^{m-i}(M, M \setminus L) \xrightarrow{\sim} H_i(L).$$

When M is compact, the following diagram is commutative :

$$\begin{array}{ccc} H^{m-i}(M, M \setminus L) & \xrightarrow{j^*} & H^{m-i}(M) \\ \wr\big\downarrow A_{M,L} & & \wr\big\downarrow P_M \\ H_i(L) & \xrightarrow{i_*} & H_i(M). \end{array}$$

(B) Chern classes of a complex manifold

We discuss the case of the Chern class of the tangent bundle TM of a complex manifold M of dimension n. Similar arguments apply to general complex vector bundles as well. For details we refer to [St, §41].

Definition 3.2. An *r-tangent field* on a subset A of M is an ordered family $F^{(r)} = (v_1, \dots, v_r)$ of r continuous vector fields on A. A singular point of $F^{(r)}$ is a point where (v_i) fail to be linearly independent (over $\mathbb{C}$). An *r-tangent frame* is a non-singular r-tangent field.

We denote by $W_r(\mathbb{C}^n)$ the complex Stiefel manifold of r-frames (ordered families of r linearly independent vectors) in $\mathbb{C}^n$. It is known (cf. [St, §25.7]) that it is $(2n-2r)$-connected and $\pi_{2n-2r+1}(W_r(\mathbb{C}^n)) \simeq \mathbb{Z}$. Note that there is a canonical generator of $\pi_{2n-2r+1}(W_r(\mathbb{C}^n))$. We denote by $W_r(TM)$ the bundle of r-tangent frames on M. This is a bundle associated with the tangent bundle TM and the fiber $W_r(T_xM)$ over each point x of M is diffeomorphic with $W_r(\mathbb{C}^n)$. In the following, we set $p = n - r + 1$. Then the p-th Chern class $c^p(M)$ of M is defined to be the primary obstruction to constructing a non-vanishing section of $W_r(TM)$. It will be defined a priori in the cohomology with local coefficient system formed by $\pi_{2p-1}(W_r(T_xM))$. Since the unitary group is connected, this system is in fact trivial and we may naturally think of the obstruction as being in the integral cohomology $H^{2p}(M)$.

We take a triangulation (K) of M and the dual cellular decomposition (D), as before. We try to construct an r-tangent frame on each (D)-skeleton inductively from the skeleton of dimension 0. First, it is always possible to construct a section $F^{(r)}$ of $W_r(TM)$ on each 0-cell. Let $\check{\sigma}$ be a j-cell contained in an open set $U \subset M$ on which the bundle $W_r(TM)$ is trivial. When a section $F^{(r)}$ of $W_r(TM)$ is given on the boundary $\partial\check{\sigma}$ of $\check{\sigma}$, it defines the map :

$$S^{j-1} \simeq \partial\check{\sigma} \xrightarrow{F^{(r)}} W_r(TM)|_U \simeq U \times W_r(\mathbb{C}^n) \longrightarrow W_r(\mathbb{C}^n),$$

where the last map is the projection onto the second factor. Thus $F^{(r)}$ defines an element in $\pi_{j-1}(W_r(\mathbb{C}^n))$. If $j \le 2n - 2r + 1 = 2p - 1$, the section $F^{(r)}$ can be extended to the interior of $\breve{\sigma}$, since the homotopy group vanishes. Continuing this process, we reach to an obstruction $I(F^{(r)},\breve{\sigma})$ in $\pi_{2p-1}(W_r(\mathbb{C}^n))$ when $j=2p$. From the remarks in the previous paragraph, it is a well-defined integer. If we denote by $\dot{\sigma}=\sigma\cap\breve{\sigma}$ the barycenter of σ, $F^{(r)}$ can be extended to an r-tangent field on $\breve{\sigma}$ with singularity only at $\dot{\sigma}$. Thus we denote $I(F^{(r)},\breve{\sigma})$ by $I(F^{(r)},\dot{\sigma})$ and call it the index of $F^{(r)}$ at $\dot{\sigma}$. In this way, we may define a cochain γ in $C^{2p}_{(D)}(M)$ by assigning $\gamma(\breve{\sigma})=I(F^{(r)},\dot{\sigma})$ to each $2p$-cell $\breve{\sigma}$ and then extending it linearly. This cochain is in fact a cocycle and represents $c^p(M)$ in $H^{2p}(M)$.

When M is compact, the Poincaré dual of $c^p(M)$ is represented in $H_{2r-2}(M)$ by the cycle

$$\sum_{\sigma} I(F^{(r)},\dot{\sigma})\,\sigma, \tag{3.3}$$

where the sum is taken over all the $2(r-1)$-simplices σ of M. In particular, when $r=1$, if we let $F^{(1)}=(v)$, $I(F^{(1)},\dot{\sigma})$ coincides with the Poincaré-Hopf index of the vector field v at $\dot{\sigma}$ and the Poincaré dual of $c^n(M)$ is the sum of the Poincaré-Hopf indices of v.

(C) Poincaré-Hopf class of a tangent frame

Let M be a complex manifold of dimension n and (D) the cellular decomposition dual to a triangulation of M, as in (B). In the following, we denote by D^j the j-skeleton of (D).

Let M' be a (D)-subcomplex of M. When we have an r-tangent frame $F^{(r)}$ on $M'\cap D^{2p}$, by arguments similar to the above, we may extend $F^{(r)}$ without singularities to D^{2p-1}. We obtain a cochain as the obstruction to extending this to D^{2p}. This is a cocycle which vanishes on M' and it represents the relative Chern class $c^p(M,M';F^{(r)})$ in $H^{2p}(M,M')$. Its image by the canonical homomorphism $j^*:H^{2p}(M,M')\to H^{2p}(M)$ is the usual Chern class $c^p(M)$. However, as a relative class, it depends on the choice of the tangent frame $F^{(r)}$ on M'.

For two r-tangent frames $F_1^{(r)}$ and $F_2^{(r)}$ on $M'\cap D^{2p}$, the difference of the corresponding two classes is given by the "difference cocycle". To be more precise, consider the product $M'\times[0,1]$ and the tangent frame $\tilde{F}^{(r)}$ on the ($2p$-skeleton of) its boundary which coincides with $F_1^{(r)}$ on $M'\times\{0\}$ and with $F_2^{(r)}$ on $M'\times\{1\}$. Then the difference cocycle $d(F_1^{(r)},F_2^{(r)})$ is an element in

$$H^{2p}(M'\times I, M'\times\{0\}\cup M'\times\{1\})\simeq H^{2p-1}(M')$$

which is the obstruction to extending $\tilde{F}^{(r)}$ to the interior. We have (see [St, §33]) :

$$c^p(M,M';F_2^{(r)})=c^p(M,M';F_1^{(r)})+\delta d(F_1^{(r)},F_2^{(r)}),$$

where $\delta: H^{2p-1}(M')\to H^{2p}(M,M')$ denotes the connecting homomorphism.

Let S be a compact (K)-subcomplex of M. We denote by $\mathcal{T}_S$ the union of cells dual to the simplices of S (cells intersectiong with S) and call it a cellular tube around S. Note that $\mathcal{T}_S \setminus \partial\mathcal{T}_S$ is a regular neighborhood of S. Let U be a neighborhood of S and let $F^{(r)}$ be an r-tangent frame on $(U \setminus S) \cap D^{2p}$. Taking a subdivision of (K), if necessary, we may assume that U contains $\mathcal{T}_S$. Then we have the relative Chern class $c^p(\mathcal{T}_S, \partial\mathcal{T}_S; F^{(r)})$ in $H^{2p}(\mathcal{T}_S, \partial\mathcal{T}_S)$. Its image by the Alexander isomorphism

$$A_{U,S} : H^{2p}(U, U \setminus S) \simeq H^{2p}(\mathcal{T}_S, \partial\mathcal{T}_S) \xrightarrow{\sim} H_{2r-2}(S)$$

is denoted by $\mathrm{PH}(F^{(r)}, S)$ and is called the Poincaré-Hopf class of $F^{(r)}$ at S. This class is represented by the cycle which is expressed in (3.3) taking the sum over all the $2(r-1)$-simplices σ of S. Let $F_1^{(r)}$ and $F_2^{(r)}$ be two tangent frames as above. If we denote $A_{U,S}(\delta d(F_1^{(r)}, F_2^{(r)}))$ by $d_S(F_1^{(r)}, F_2^{(r)})$, we have

$$\mathrm{PH}(F_2^{(r)}, S) = \mathrm{PH}(F_1^{(r)}, S) + d_S(F_1^{(r)}, F_2^{(r)}). \tag{3.4}$$

Now we consider the global situation. Let Σ be a compact (K)-subcomplex of M and let $(S_\lambda)_\lambda$ be its connected components. If we are given an r-tangent frame $F^{(r)}$ on $(M \setminus \Sigma) \cap D^{2p}$, we have the class $\mathrm{PH}(F^{(r)}, S_\lambda)$ for each λ. From the construction, we have :

Theorem 3.5. *Let M be a compact complex manifold and let Σ be a (K)-subcomplex of M. For an r-tangent frame $F^{(r)}$ on $(M \setminus \Sigma) \cap D^{2p}$, we have*

$$\sum_\lambda (i_\lambda)_* \mathrm{PH}(F^{(r)}, S_\lambda) = c^p(M) \frown [M],$$

where i_λ denotes the inclusion $S_\lambda \hookrightarrow M$ and the sum is taken over the connected components S_λ of Σ.

4. Schwartz classes

(A) Schwartz-MacPherson class of a singular variety

We refer to [BS] and [Sc1,3,4] for details of this subsection. Let W be a complex manifold of dimension $n+k$ and V a subvariety of dimension n in W. We take a Whitney stratification $\{X\}$ of W compatible with V. Thus the singular set $\mathrm{Sing}(V)$ of V is a union of strata. Moreover, let (K) be a triangulation of W compatible with the stratification $\{X\}$ and let (D) be the cellular decomposition of W dual to (K). Since each cell of (D) is transverse to each stratum, for a stratum X of complex dimension ℓ and a cell $\check{\sigma}$ of real dimension $2q = 2(n+k-r+1)$, $X \cap \check{\sigma}$ is a cell of dimension $2(\ell - r + 1)$. We denote by $\tilde{\mathcal{T}}_V$ the union of cells dual to the simplices of V and call it a cellular tube around V.

Definition 4.1. Let A be a subset of W. A *stratified vector field* on A is a vector field on A which is tangent to each stratum. A *stratified r-tangent field* on A is an r-tangent field consisting of stratified vector fields on A. A *stratified r-tangent frame* is a non-singular stratified r-tangent field.

The Schwartz class of V is the primary obstruction to constructing a special stratified tangent frame, which is called a radial tangent frame, on a neighborhood of V. A radial r-tangent field $\tilde{F}_0^{(r)}$ on $\tilde{\mathcal{T}}_V \cap D^{2q}$ defined by M.-H. Schwartz has the following properties :
(i) If we write $\tilde{F}_0^{(r)} = (\tilde{F}_0^{(r-1)}, \tilde{v}_0)$ individualizing the last vector $\tilde{v}_0$, then $\tilde{F}_0^{(r-1)}$ is non-singular on $\tilde{\mathcal{T}}_V \cap D^{2q}$. $\tilde{F}_0^{(r)}$ is non-singular on $\tilde{\mathcal{T}}_V \cap D^{2q-1}$ and, for each $2q$-cell $\check{\sigma}$, it admits at most an isolated singularity at the barycenter $\dot{\sigma}$ of $\check{\sigma}$, which is a singularity of $\tilde{v}_0$.
(ii) If $\tilde{F}_0^{(r)}$ has a singularity on a $2q$-cell $\check{\sigma}$ and if $\check{\sigma}$ intersects with several strata, the singularity is on the stratum X of the lowest dimension. Let the complex dimension of X be ℓ. If $\ell > r-1$, then $I(\tilde{F}_0^{(r)}, \dot{\sigma}) = I(\tilde{F}_0^{(r)}|_X, \dot{\sigma})$, and if $\ell = r-1$, then $I(\tilde{F}_0^{(r)}, \dot{\sigma}) = 1$.
(iii) $\tilde{v}_0$ is outbound from a cellular tube around V.

A radial r-tangent field $\tilde{F}_0^{(r)}$ as above is non-singular away from V and it defines a cocycle in $C^{2q}_{(D)}(\tilde{\mathcal{T}}_V, \partial\tilde{\mathcal{T}}_V)$ as the obstruction to extending it to a non-singular tangent field on $\tilde{\mathcal{T}}_V \cap D^{2q}$. The class $\tilde{c}^q(V)$ represented by this cocycle in $H^{2q}(\tilde{\mathcal{T}}_V, \partial\tilde{\mathcal{T}}_V) \simeq H^{2q}(W, W \setminus V)$ does not depend on the choice of the Whitney stratification, the triangulation or the radial r-tangent field. We call $\tilde{c}^q(V)$ the q-th Schwartz class of V. The image of $\tilde{c}^q(V)$ by the Alexander isomorphism $A_{W,V}$: $H^{2q}(W, W \setminus V) \xrightarrow{\sim} H_{2r-2}(V)$ is represented by the cycle

$$\sum_{\sigma} I(\tilde{F}_0^{(r)}, \dot{\sigma})\, \sigma, \tag{4.2}$$

where the sum is taken over all $2(r-1)$-simplices σ of V.

In summary, the q-th Schwartz class of V is the localization of the q-th Chern class of the ambient tangent bundle TW by a radial r-tangent frame.

On the other hand, if we let $c_*(V)$ denote the MacPherson class of V defined in [Ma] and $c_{r-1}(V) \in H_{2r-2}(V)$ its component of dimension $r-1$, we have

$$A_{W,V}(\tilde{c}^q(V)) = c_{r-1}(V).$$

Thus we call $c_{r-1}(V)$ the $(r-1)$-st Schwartz-MacPherson class.

In particular, when $r = 1$, $\tilde{F}_0^{(1)} = \{\tilde{v}_0\}$ consists of a single radial vector field $\tilde{v}_0$. Since this is outbound eveywhere on $\partial\tilde{\mathcal{T}}_V$, (4.2) gives the Euler-Poincaré characteristic of $\tilde{\mathcal{T}}_V$. Moreover, since V is a deformation retract of $\tilde{\mathcal{T}}_V$, we see that $c_0(V) = \chi(V)$.

When V is non-singular, $F_0^{(r)} = \tilde{F}_0^{(r)}|_V$ is an r-tangent field on V and from the property (ii) of radial tangent frames, (4.2) coincides with $\sum_\sigma I(F_0^{(r)}, \dot{\sigma})\, \sigma$. Hence we have $c_{r-1}(V) = c^p(V) \frown [V]$, $p = n - r + 1$.

(B) Poincaré and Alexander homomorphisms

We recall the Poincaré and Alexander homomorphisms for singular varieties as defined in [Br1]. Let W, V, (K), (K') and (D) be as before.

First, when V is compact, we define a homomorphism

$$(4.3) \qquad P : C^{2n-i}_{(K')}(V) \to C^{(K)}_i(V) \qquad \text{by} \qquad P(c) = \sum_\sigma \langle c, V \cap \breve{\sigma} \rangle\, \sigma$$

for a $(2n-i)$-cochain c and a $(2n+2k-i)$-cell $\breve{\sigma}$, where the sum is taken over all i-simplices σ of V. This induces a homomorphism

$$P_V : H^{2n-i}(V) \longrightarrow H_i(V),$$

which is called the Poincaré homomorphism. When V is non-singular, this is the Poincaré isomorphism.

Next, let S be a compact (K)-subcomplex of V (V may not be compact). We define a homomorphism

$$A : C^{2n-i}_{(K')}(V, V \setminus S) \to C^{(K)}_i(S)$$

taking, in the sum of (4.3), only i-simplices of S. Then this induces a homomorphism

$$A_{V,S} : H^{2n-i}(V, V \setminus S) \longrightarrow H_i(S),$$

which is called the Alexander homomorphism. When V is non-singular, this is the Alexander isomorphism. If V is compact, we have a commutative diagram similar to the one in Section 3 (A).

(C) Local Schwartz class of a tangent frame

In the previous situation, let S be either one of the following :
(i) a compact connected (K)-subcomplex of V in the non-singular part V_0, or
(ii) a compact connected component of the singular set $\mathrm{Sing}(V)$.

Let $\tilde{U}$ be an open neighborhood of S in W such that $U \setminus S$ is contained in V_0, where $U = \tilde{U} \cap V$. We may assume, taking a barycentric subdivision of (K) if necessary, that a cellular tube $\tilde{\mathcal{T}}_S$ of S in W is contained in $\tilde{U}$. Note that $\partial\tilde{\mathcal{T}}_S$ is transverse to V_0. We set $\mathcal{T}_S = \tilde{\mathcal{T}}_S \cap V$.

For an r-tangent frame $F^{(r)}$ on $(U \setminus S) \cap D^{2q}$, we define the Schwartz class $\mathrm{Sch}(F^{(r)}, S) \in H_{2r-2}(S)$ of $F^{(r)}$ at S as follows.

First, in the case (i), we have the class $\mathrm{PH}(F^{(r)}, S)$ in $H_{2r-2}(S)$ and we simply set $\mathrm{Sch}(F^{(r)}, S) = \mathrm{PH}(F^{(r)}, S)$.

Next, we consider the case (ii). Recall that for two r-tangent frames $F_1^{(r)}$ and $F_2^{(r)}$ on $(U \setminus S) \cap D^{2q}$, we have the difference $d(F_1^{(r)}, F_2^{(r)})$ in $H^{2p-1}(\partial \mathcal{T}_S)$. Letting $\delta : H^{2p-1}(\partial \mathcal{T}_S) \to H^{2p}(\mathcal{T}_S, \partial \mathcal{T}_S)$ be the connecting homomorphism and

$$A_{U,S} : H^{2p}(U, U \setminus S) \simeq H^{2p}(\mathcal{T}_S, \partial \mathcal{T}_S) \to H_{2r-2}(S)$$

the Alexander homomorphism, we set $d_S(F_1^{(r)}, F_2^{(r)}) = A_{U,S}\delta d(F_1^{(r)}, F_2^{(r)})$. Take a radial r-tangent field $\tilde{F}_0^{(r)}$ on $\tilde{U} \cap D^{2q}$ whose singularities are all in S and let $F_0^{(r)}$ denote the restriction of $\tilde{F}_0^{(r)}$ to $U \setminus S$.

Definition 4.4. For an r-tangent frame $F^{(r)}$ on $(U \setminus S) \cap D^{2q}$, we define the *Schwartz class* $\mathrm{Sch}(F^{(r)}, S)$ of $F^{(r)}$ at S to be the class in $H_{2r-2}(S)$ given by :

$$\mathrm{Sch}(F^{(r)}, S) = c_{r-1}(S) + d_S(F_0^{(r)}, F^{(r)}).$$

As in the case of Poincaré-Hopf classes, for two r-tangent frames $F_1^{(r)}$ and $F_2^{(r)}$ on $(U \setminus S) \cap D^{2q}$, we have a formula similar to (3.4). Let Σ be a compact (K)-subcomplex of V_0, disjoint from a neighborhood of $\mathrm{Sing}(V)$ and let (S_λ) be the connected components of $\mathrm{Sing}(V) \cup \Sigma$. Then we have :

Theorem 4.5 [BLSS]. *Let V be a compact subvariety of a complex manifold and Σ a subset in V_0 as above. For an r-tangent frame $F^{(r)}$ on $(V_0 \setminus \Sigma) \cap D^{2q}$, we have*

$$\sum_\lambda (i_\lambda)_* \mathrm{Sch}(F^{(r)}, S_\lambda) = c_{r-1}(V).$$

For differential geometric definition of the Schwartz class, we refer to [BLSS] and [Leh].

5. Virtual class of a tangent frame

In Sections 5 and 6, the homology and cohomology will be with real coefficients.

Let V be a local complete intersection of dimension n in a complex manifold W of dimension $n+k$ defined by a section of a holomorphic vector bundle N of rank k over W. As in Section 1 (C), we call $\tau_V = (TW - N)|_V$ the virtual tangent bundle of V. If V is compact, the image of its Chern class $c^*(\tau_V)$ by the Poincaré homomorphism coincides with the Fulton-Johnson class $c_*^{FJ}(V)$ of V.

The *virtual class* of an r-tangent frame $F^{(r)}$ is the localization of $c^p(\tau_V)$, $p = n - r + 1$, by $F^{(r)}$ at its singular set. To be a little more precise, take a triangulation (K) and the dual cellular decomposition (D) of W, as in Section 4, and let S be as (i) or (ii) of (C) in Section 4. Let $\tilde{U}$ be a neighborhood of S in W such that $U \setminus S$ is in V_0, $U = \tilde{U} \cap V$, and let $F^{(r)}$ be an r-tangent frame on

$(U \setminus S) \cap D^{2q}$. Let ∇ be an $F^{(r)}$-trivial connection for TV_0 on (a neighborhood of) $(U \setminus S) \cap D^{2q}$. We take connections ∇_0 and ∇_0' for TW and N, respectively, so that $(\nabla_0', \nabla_0, \nabla)$ is compatible with (1.7). We also take connections for TW and N on $\tilde{U}$. If we represent the Chern class $c^p(\tau_V)$ by a Čech-de Rham cocycle using these connections, we see that it is localized at S and it naturally defines a class $\gamma^q(F^{(r)}, S)$ in $H^{2q}(\tilde{U}, \tilde{U} \setminus S)$, $q = n + k - r + 1$, see [BLSS, Section 4] for details. Then we define the *virtual class* $\mathrm{Vir}(F^{(r)}, S)$ of $F^{(r)}$ at S to be the image of $\gamma^q(F^{(r)}, S)$ by the Alexander isomorphism $A_{\tilde{U},S} : H^{2q}(\tilde{U}, \tilde{U} \setminus S) \xrightarrow{\sim} H_{2r-2}(S)$.

When S is in the non-singular part, we have $\mathrm{Vir}(F^{(r)}, S) = \mathrm{PH}(F^{(r)}, S)$. Also, for two r-tangent frames as above, we have a formula similar to (3.4). In the global situation, if we let Σ be a subset of V_0 as considered in Theorem 4.5, from the construction of the virtual class, we have :

Theorem 5.1 [BLSS]. *Let V be a compact local complete intersection defined by a section. For an r-tangent frame $F^{(r)}$ on $(V_0 \setminus \Sigma) \cap D^{2q}$, we have*

$$\sum_{\lambda} (i_\lambda)_* \, \mathrm{Vir}(F^{(r)}, S_\lambda) = c^p(\tau_V) \frown [V].$$

6. Milnor classes

Let V be a local complete intersection in W defined by a holomorphic section and S a compact connected component of $\mathrm{Sing}(V)$. We now introduce the Milnor class of V at S. Let $F^{(r+1)}$ be an $(r+1)$-tangent frame on $(U \setminus S) \cap D^{2(q-1)}$, where U is a small neighborhood of S.

Definition 6.1. For a non-negative integre r, the r-th *Milnor class* $\mu_r(V, S)$ of V at S is a class in $H_{2r}(S)$ defined by

$$\mu_r(V, S) = (-1)^{n+1} \left(\mathrm{Sch}(F^{(r+1)}, S) - \mathrm{Vir}(F^{(r+1)}, S) \right).$$

Since both the Schwartz and virtual classes of a tangent frame satisfy identities similar to (3.4), $\mu_r(V, S)$ does not depend on the choice of the $(r+1)$-tangent frame $F^{(r+1)}$. We call $\mu_*(V, S) = \sum_r \mu_r(V, S)$ the total Milnor class. By definition, if r is greater than the (complex) dimension of S, then $\mu_r(V, S) = 0$. In particular, when S consists of a point p, we have $\mu_*(V, S) = \mu_0(V, S)$, which coincides with the usual Milnor number $\mu(V, p)$ by (1.11). When V is a hypersurface ($k = 1$), $\mu_0(V, S)$ coincides with the "generalized Milnor number" introduced in [Par] (see [BLSS, Theorem 6.2]). From Theorems 4.5 and 5.1, using a tangent frame as a mediator, we obtain the following :

Theorem 6.2 [BLSS]. *Let V be a compact local complete intersection defined by a section. Then,*

$$c_*(V) = c^*(\tau_V) \frown [V] + (-1)^{n+1} \sum_S i_* \mu_*(V, S),$$

where the sum is taken over the connected components S of $\mathrm{Sing}(V)$ *and $i : S \hookrightarrow V$ denotes the inclusion.*

The above theorem reduces to Theorem 1.13 when the singularities of V are all isolated.

In the following, we give an explicit formula for the Milnor class at a non-singular connected component of the singular set. Thus let S be a connected component of $\mathrm{Sing}(V)$ which is non-singular and let ℓ denote its dimension. Moreover, let $\tilde{\rho} : \tilde{U} \longrightarrow S$ be a tubular neighborhood of S in W. We assume that V satisfies the Whitney condition along S. Thus the fibers of $\tilde{\rho}$ are all transverse to V. Let $U = \tilde{U} \cap V$, $U_0 = U \setminus S$ and let ρ and ρ_0 be the restrictions of $\tilde{\rho}$ to U and U_0, respectively. The bundle $T\tilde{\rho}$ of vectors in $T\tilde{U}$ tangent to the fibers of $\tilde{\rho}$ admits a complex structure, since it is C^∞ isomorphic with the normal bundle of the complex submanifold S in W. We have the following exact sequence of vector bundles on U_0 :

$$0 \longrightarrow T\rho_0 \longrightarrow T\tilde{\rho}|_{U_0} \longrightarrow N|_{U_0} \longrightarrow 0.$$

Now if we consider the Chern class of the virtual bundle $(T\tilde{\rho} - N)|_U$ over U, since the rank of $T\rho_0$ is $n - \ell$, there is a natural localization $c_S^{n-\ell+i}((T\tilde{\rho} - N)|_U)$ in $H^{2(n-\ell+i)}(U, U \setminus S)$ of $c^{n-\ell+i}((T\tilde{\rho} - N)|_U)$, for all $i > 0$. Consider the integration along the fibers of ρ (see for example [Su2, II, 5]) :

$$\rho_* : H^{2(n-\ell+i)}(U, U \setminus S) \longrightarrow H^{2i}(S)$$

and define the class α_i in $H^{2i}(S)$ by

$$\alpha_i = \rho_* c_S^{n-\ell+i}((T\tilde{\rho} - N)|_U).$$

Let p be a point in S and let H be a plane of dimension $m - \ell$ through p in W which is transverse to S. The intersection $V_p = V \cap H$ has an isolated singularity at p and, from the Whitney condition, the Milnor number $\mu(V_p, p)$ does not depend on p. Note that, since S is non-singular, its Schwartz-MacPherson class $c_*(S)$ coincides with $c^*(S) \frown [S]$.

Theorem 6.3 [BLSS]. *Let S be a non-singular compact connected component of* $\mathrm{Sing}(V)$ *such that V satisfies the Whitney condition along S. Then*

$$\mu_*(V, S) = \Big((-1)^\ell \mu(V_p, p) \cdot (c^*(N) - c^k(N)) + (-1)^n \sum_{i=1}^{k-1} \sum_{j=1}^{i} c^{i-j}(N) \alpha_j \Big) \cdot c^*(N)^{-1} \frown c_*(S).$$

In particular, when $k = 1$,

$$\mu_*(V,S) = (-1)^\ell \mu(V_p,p) \cdot c^*(N)^{-1} \frown c_*(S).$$

Also, for arbitrary k,

$$\mu_\ell(V,S) = (-1)^\ell \mu(V_p,p) \cdot [S].$$

Remarks 6.4. 1. In [PP2], for a hypersurface V, the Milnor class is defined by $\mu_*(V) = (-1)^n(c_*(V) - c^*(\tau_V) \frown [V])$ and a formula for this is given as a sum of the contributions from the strata of a stratification of V. This result was obtained earlier for the Milnor number $\mu_0(V)$ in [PP1] and, for the Milnor class, it was conjectured in [Y1]. When the stratum is a non-singular component of $\mathrm{Sing}(V)$, its contribution coincides with the one given in the second formula above.

2. In fact, the formulas in Theorem 6.3 hold under an assumption weaker than the Whitney condition. Namely, we only need that there is a Whitney stratification of W compatible with V and S such that the $2(\ell - r)$-skeleton $S \cap D^{2(q-1)}$ of S is in the top dimensional stratum of S. Accordingly, under this assumption, we have a formula for $\mu_r(V,S)$ taking the terms of corresponding dimension in the above formulas.

7. Characteristic classes of coherent sheaves on singular varieties

(A) Thom class

The Thom isomorphism for a vector bundle over a manifold or for a submanifold can be generalized to the case of singular subvarieties as follows (cf. [Br1]).

Let W be a complex manifold of dimension $n+k$ and V a subvariety of dimension n of W. We denote by i the embedding $V \hookrightarrow W$. Let (K), (K') and (D) be as in Section 4. We define a homomorphism

$$T : C^j_{(K')} \longrightarrow C^{j+2k}_{(D)}(W, W \setminus V) \qquad \text{by} \qquad \langle T(c), \check\sigma\rangle = \langle c, V \cap \check\sigma\rangle$$

for a j-cochain c and a $(j+2k)$-cell $\check\sigma$. Then it induces a homomorphism

$$T_{V,W} : H^j(V) \longrightarrow H^{j+2k}(W, W \setminus V),$$

which we call the Thom homomorphism. When V is non-singular, this is the Thom isomorphism. When V is compact, let $A_{W,V}$ be the Alexander isomorphism and P_V the Poincaré homomorphism (Section 4 (B)). Then from the construction, we have $A_{W,V} \circ T_{V,W} = P_V$. For the class [1] in $H^0(V)$, we denote the class $T_{V,W}([1])$ in $H^{2k}(W, W \setminus V)$ by Ψ_V and call it the *Thom class* of V in W.

Let $\tilde U$ be a regular neighborhood of V in W and $\rho : \tilde U \longrightarrow V$ a continuous retraction. By excision, we have $H^*(W, W \setminus V) \simeq H^*(\tilde U, \tilde U \setminus V)$ and the Thom homomorphism is given by $T_{V,W}(\alpha) = \rho^*(\alpha) \smile \Psi_V$ for a class α in $H^j(V)$. we define the Gysin homomorphism i_* by the composition :

$$H^j(V) \xrightarrow{T_{V,W}} H^{j+2k}(W, W \setminus V) \xrightarrow{j^*} H^{j+2k}(W).$$

In the sequel, we consider the following two cases :
(i) V is non-singular,
(ii) V is a local complete intersection defined by a section.

In the case (i), let $p : N_V \to V$ be the normal bundle of V in W. Then, (W, V) and (N_V, V) are C^∞ diffeomorphic in a neighborhood of V and there is a natural isomorphism $H^*(W, W \setminus V) \simeq H^*(N_V, N_V \setminus V)$. Under this isomorphism, the Thom class Ψ_V of V corresponds to the Thom class Ψ_{N_V} of N_V. Let s_Δ be the diagonal section of the bundle p^*N_V over N_V, then its zero set is V, and we have the following identity (see for example [Su2, Ch.III, Theorem 4.4]) :

$$\Psi_{N_V} = c^k(p^*N_V, s_\Delta).$$

In the case (ii), let V be defined by a section s of a holomorphic vector bundle N over W. Then the localization $c^k(N, s)$ in $H^{2k}(W, W \setminus V)$ of $c^k(N)$ by s corresponds to $[V]$, under the Alexander isomorphism $H^{2k}(W, W \setminus V) \xrightarrow{\sim} H_{2n}(V)$ (see for example [Su3], and in the algebraic category, [F, §14.1]). Thus in this case, the Thom class is given by :

$$\Psi_V = c^k(N, s). \tag{7.1}$$

In the case (i) again, if we identify (N_V, V) with (W, V) and if we write p^*N_V by N and s_Δ by s, the Thom class of V is also expressed as (7.1).

(B) Riemann-Roch theorem for embeddings of singular varieties

Let V be a subvariety of a complex manifold W and let $\mathcal{I}_V$ be the ideal sheaf of germs of functions in $\mathcal{O}_W$ vanishing on V. Let $\mathcal{O}_V = \mathcal{O}_W/\mathcal{I}_V$ be the structure sheaf of V. For a coherent $\mathcal{O}_V$-module $\mathcal{F}$, its direct image $i_!\mathcal{F}$ is a coherent $\mathcal{O}_W$-module, which is obtained simply by extending $\mathcal{F}$ as zero on $W \setminus V$. Thus we have the local Chern character $\mathrm{ch}_V^*(i_!\mathcal{F})$ in $H^*(W, W \setminus V; \mathbb{Q})$ (Section 2 (B)). In the sequel, let V be as (i) or (ii) in (A) above. The following theorem is proved in [Su4] using the above representation of the Thom class of V and a fundamental relation (cf. [HL, III, Corollary 5.4]) among various characteristic forms of the bundle N. The proof is rather simple and is done on the Čech-de Rham cocycle level. We refer to [Su4, Remarks 3.6] for related references.

Theorem 7.2. *Let V be a compact subvariety of W and $\mathcal{F}$ a coherent $\mathcal{O}_V$-module. If either*
(i) V is non-singular, or

(ii) *V is a local complete intersection defined by a section and $\mathcal{F}$ is locally free, then we have*

$$\begin{aligned}\operatorname{ch}_V^*(i_!\mathcal{F}) &= T_{V,W}(\operatorname{ch}^*(\mathcal{F}) \smile \operatorname{td}(N_V)^{-1}),\\ \operatorname{ch}^*(i_!\mathcal{F}) &= i_*(\operatorname{ch}^*(\mathcal{F}) \smile \operatorname{td}(N_V)^{-1}).\end{aligned}$$

In the above, the first equality holds in $H^*(W, W \setminus V; \mathbb{Q})$ and the second in $H^*(W; \mathbb{Q})$.

(C) Homology Chern character and Chern class

Let V be a local complete intersection of codimension k in a complex manifold W. Thus the ideal sheaf $\mathcal{I}_V$ of V is locally generated by k functions and the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is locally free. We denote by N_V the associated vector bundle and let $\tau_V = TW|_V - N_V$ be the virtual tangent bundle of V.

Definition 7.3. For a coherent $\mathcal{O}_V$-module $\mathcal{F}$, we define the *homology Chern character* by

$$\operatorname{ch}_*(\mathcal{F}) = \operatorname{td}(N_V) \frown A_{W,V}(\operatorname{ch}_V^*(i_!\mathcal{F})).$$

This Chern character has various nice properties (see [Su4]). In particular, if we use Theorem 7.2, we have :

Proposition 7.4. *If V is non-singular, or if V is defined by a section and $\mathcal{F}$ is locally free,*

$$\operatorname{ch}_*(\mathcal{F}) = \operatorname{ch}^*(\mathcal{F}) \frown [V].$$

In particular, for the structure sheaf $\mathcal{O}_V$,

$$\operatorname{ch}_*(\mathcal{O}_V) = [V].$$

When $\operatorname{ch}_*(\mathcal{F})$ is in the image of the Poincaré homomorphism $H^*(V) \to H_*(V)$, we may define the homology Chern class $c_*(\mathcal{F})$ via the Newton formula (see for example [Hi, p.92]).

Remarks 7.5. 1. The above class $\operatorname{ch}_*(\mathcal{F})$ is related to the homology Todd class $\tau(\mathcal{F})$ of $\mathcal{F}$ in [BFM] by

$$\operatorname{ch}_*(\mathcal{F}) = \operatorname{td}(\tau_V)^{-1} \frown \tau(\mathcal{F}).$$

2. It would be an interesting problem to compare the homology Chern character $\operatorname{ch}_*(\mathcal{F})$ defined as above with the one in [Sc2] (see also [K]), which is defined using the Nash type modification of V with respect to $\mathcal{F}$.

(D) Characteristic classes of the tangent sheaf

Let V be a local complete intersection in a complex manifold W defined by a section. Let Ω_W and Ω_V denote, respectively, the sheaves of germs of holomorphic 1-forms on W and V. Then we have the exact sequence

$$0 \to \mathcal{I}_V/\mathcal{I}_V^2 \to \Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_V \to \Omega_V \to 0.$$

Let $\Theta_W = \mathcal{O}_W(TW)$ be the tangent sheaf of W and we define the tangent sheaf Θ_V of V by $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$. This does not depend on the embedding $V \hookrightarrow W$. From the above exact sequence, we obtain the exact sequence

$$0 \to \Theta_V \to \Theta_W \otimes_{\mathcal{O}_W} \mathcal{O}_V \to \mathcal{N}_V \to \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V) \to 0.$$

If we set $\mathcal{E} = \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, from Proposition 7.4, we have

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \mathrm{ch}_*(\mathcal{E}).$$

If p is an isolated singular point of V, applying Theorem 7.2 for the embedding $p \hookrightarrow W$, we have $\mathrm{ch}_*(\mathcal{E}) = \tau(V,p)[p]$, where $\tau(V,p) = \dim_{\mathbb{C}} \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)_p$ is the Tjurina number of V at p. Recall that it gives the dimension of the space of infinitesimal deformations of (V,p). From the above, we have :

Theorem 7.6 [Su4]. *Let V be a local complete intersection of dimension n defined by a section. If V admits only isolated singular points $p_1, \dots, p_s$, for the tangent sheaf Θ_V of V, we have*

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \sum_{i=1}^{s} \tau(V, p_i)\,[p_i],$$

$$c_*(\Theta_V) = c^*(\tau_V) \frown [V] + (-1)^{n+1}(n-1)! \sum_{i=1}^{s} \tau(V, p_i)\,[p_i].$$

By comparing the above theorem with Theorem 1.13, we see that, in some special cases, the class $c_*(\Theta_V)$ coincides with the Schwartz-MacPherson class $c_*(V)$ of V (cf. [Su4, Corollary 5.2]).

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