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Norm of a linear combination of two operators on a Hilbert space

by

Takahiko Nakazi* and Takanori Yamamoto*

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Abstract

Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. If A and B are bounded linear operators on the Hilbert space H such that $\gamma A + \delta B$ is right invertible then we study the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ using the angle ϕ between two subspaces $\text{ran } A$ and $\text{ran } B$ or the angle $\psi = \psi(A, B)$ between two operators A and B where

$$\cos \psi(A, B) = \sup\{|\langle Af, Bf \rangle| / (\|Af\| \cdot \|Bf\|) ; f \in H, Af \neq 0, Bf \neq 0\}.$$

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Keywords: Operator norm; Angle; Hilbert space; Idempotent operator.

1. Introduction

Let $B(H)$ be the set of all bounded linear operators on the Hilbert space H . Let $P \in B(H)$ satisfy $P^2 = P$ and let $Q = I - P$ where I denotes the identity operator on H . Denote by $\phi(H_1, H_2)$ the minimal angle between two subspaces H_1 and H_2 of H :

$$\cos \phi(H_1, H_2) = \sup_{0 \neq f \in H_1, 0 \neq g \in H_2} \frac{|\langle f, g \rangle|}{\|f\| \cdot \|g\|}.$$

Then $0 \leq \phi(H_1, H_2) \leq \frac{\pi}{2}$. Let $\text{ran } P$ denote the range of P . If $\phi = \phi(\text{ran } P, \text{ran } Q) > 0$ then

$$\|P\| = \|Q\| = \csc \phi = \frac{1}{\sin \phi}$$

(see [2, p.339]). Let $J = P - Q$. Then

$$\|P\| = \|Q\| = \frac{1}{2} \left(\|J\| + \frac{1}{\|J\|} \right)$$

(see [6, Lemma 2], [1]). Hence

$$\|J\| = \|P\| + \sqrt{\|P\|^2 - 1} = (\csc + \cot)\phi = \cot \frac{\phi}{2}.$$

Let α and β be complex numbers, and let t be a nonnegative number. Then we define a function $F(\alpha, \beta, t)$ which is the generalization of $\max(|\alpha|, |\beta|) = F(\alpha, \beta, 0)$.

Definition 1. Let

$$F(\alpha, \beta, t) = \sqrt{\left| \frac{\alpha - \beta}{2} \right|^2 t + \left(\frac{|\alpha| + |\beta|}{2} \right)^2} + \sqrt{\left| \frac{\alpha - \beta}{2} \right|^2 t + \left(\frac{|\alpha| - |\beta|}{2} \right)^2}.$$

Then $F(\alpha, \beta, t)$ is a nondecreasing function of t , $t \geq 0$ and satisfies

$$\max(|\alpha|, |\beta|) = F(\alpha, \beta, 0) \leq F(\alpha, \beta, t) < \infty, \quad (t \geq 0).$$

Feldman, Krupnik and Markus [1] established the following formula.

Feldman–Krupnik–Markus formula. Let $P \in B(H)$ satisfy $P \neq 0, I$ and $P^2 = P$. Let $Q = I - P$. Let $\alpha, \beta \in \mathbf{C}$. Then

$$\|\alpha P + \beta Q\| = F(\alpha, \beta, \|P\|^2 - 1).$$

Let $\phi = \phi(\text{ran } P, \text{ran } Q)$. Since $\|P\| = \csc \phi$, it follows that $\|P\|^2 - 1 = \cot^2 \phi$. Hence

$$\|\alpha P + \beta Q\| = F(\alpha, \beta, \cot^2 \phi).$$

Definition 2. For two nonzero operators A, B on H , let $\psi(A, B)$ satisfy $0 \leq \psi(A, B) \leq \frac{\pi}{2}$ and

$$\cos \psi(A, B) = \sup_{Af \neq 0, Bf \neq 0} \frac{|\langle Af, Bf \rangle|}{\|Af\| \cdot \|Bf\|}.$$

Since $\cos \psi(A, B) \leq \cos \phi(\text{ran } A, \text{ran } B)$, it follows that $\psi(A, B) \geq \phi(\text{ran } A, \text{ran } B)$. We call $\psi(A, B)$ as the angle between two operators A and B . If $P^2 = P (\neq 0, I)$ and $Q = I - P$ then $\phi(\text{ran } P, \text{ran } Q) = \psi(P, Q)$, because if $h = Pf + Qg$ then

$$\frac{|\langle Pf, Qg \rangle|}{\|Pf\| \cdot \|Qg\|} = \frac{|\langle Ph, Qh \rangle|}{\|Ph\| \cdot \|Qh\|}.$$

In this paper, we shall study the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$. We use $\phi(\text{ran } A, \text{ran } B)$ in Section 2, and we use $\psi(A, B)$ in Section 4. Let $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$. In Section 2, we shall study in the case when $K \neq \overline{\text{ran } A}$ and $K \neq \overline{\text{ran } B}$. In Theorem 1, we shall use the Feldman–Krupnik–Markus formula [1] and Lemma 1 to establish the formula of the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ using the angle $\phi(\text{ran } A, \text{ran } B)$ in the case when $K = \{0\}$ and $\gamma A + \delta B$ is right invertible. In Theorem 2, we shall use Theorem 1 to estimate the norm from below using the angle $\phi(\overline{\text{ran } A \ominus K}, \overline{\text{ran } B \ominus K})$ in the case when K is a nonzero invariant subspace of $A(\gamma A + \delta B)^{-1}$. In Section 3, we shall study in the case when $K = \overline{\text{ran } B}$. We shall consider the nilpotent operator B on H . The results in Sections 2 and 3 follow from the Feldman–Krupnik–Markus formula. In Section 4, if $\psi(A, B) > 0$ and $\gamma A + \delta B$ is right invertible or left invertible then we shall estimate $\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\|$ from above. In Theorem 3, we do not assume the boundedness of two operators A and B . As a corollary of Theorem 3, we shall show that if $\|A + B\| < \infty$ and $\psi(A, B) > 0$ then $\|A\| < \infty$ and $\|B\| < \infty$. The results in Section 4 do not follow from the Feldman–Krupnik–Markus formula.

2. Norm formula using the angle ϕ between $\text{ran } A$ and $\text{ran } B$

Let $A, B \in B(H)$, $A \neq 0$, $B \neq 0$ and let $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$. In this section we shall study in the case when $K \neq \overline{\text{ran } A}$ and $K \neq \overline{\text{ran } B}$. In Theorem 1, if $K = \{0\}$ and $\text{ran}(\gamma A + \delta B) = H$ then we shall use the Feldman–Krupnik–Markus formula [1], and establish the norm formula of $\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\|$ using the angle $\phi(\text{ran } A, \text{ran } B)$. In Theorem 2, if K is an invariant subspace of $A(\gamma A + \delta B)^{-1}$ then we shall estimate the norm from below using $\phi(\overline{\text{ran } A \ominus K}, \overline{\text{ran } B \ominus K})$.

The operator $X \in B(H)$ is said to be right invertible if there exists an operator $Y \in B(H)$ such that $XY = I$. The operator Y is called the right inverse to X and is denoted by X^{-1} . Then $X^{-1} \in B(H)$ is not uniquely defined (see [3, Vol.I, p.63]). If $\gamma A + \delta B$ is right invertible then

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| = \sup_{(\gamma A + \delta B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(\gamma A + \delta B)f\|},$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.

Lemma 1. Let $A, B \in B(H)$ satisfy $A \neq 0$, $B \neq 0$ and $\text{ran}(A + B) = H$.

The following assertions are equivalent:

- (1) $\text{ran } A \cap \text{ran } B = \{0\}$.
- (2) $\overline{\text{ran } A} \cap \overline{\text{ran } B} = \{0\}$.
- (3) $\phi(\text{ran } A, \text{ran } B) > 0$.

Suppose (1) $\sim$ (3) hold. Let $(A + B)^{-1}$ denote one of the right inverses to $A + B$. Let $P = A(A + B)^{-1}$ and let $Q = B(A + B)^{-1}$. Then P and Q do not depend on the choice of $(A + B)^{-1}$. Then $P^2 = P \neq 0, I$, $P + Q = I$, $\text{ran } P = \text{ran } A$ and $\text{ran } Q = \text{ran } B$.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3): Let $H_1 = \ker(A + B)$, and let $H_2 = H_1^\perp$. Then $H = H_1 \oplus H_2$. Since $(A + B)|_{H_1} = 0$, it follows that $A|_{H_1} = -B|_{H_1}$. By (1), $A|_{H_1} = B|_{H_1} = 0$. Let $T = (A + B)|_{H_2}$. Then $T \in B(H_2, H)$ and $\ker T = \{0\}$. Since $\text{ran}(A + B) = H$, it follows that $\text{ran } T = H$. By the open mapping theorem, there exists $S \in B(H, H_2)$ such that $ST = I_{H_2}$ and $TS = I_H$. Hence $(A + B)S = TS = I_H = I$. Hence S is a right inverse to $A + B$. Let C be one of the right inverses to $A + B$. Then $P + Q = (A + B)C = I$. Hence $A(C - S) = -B(C - S)$. By (1), $A(C - S) = -B(C - S) = 0$. Hence $P = AC = AS$ and $Q = BC = BS$. Hence P and Q do not depend on the choice of $(A + B)^{-1}$. By (1),

$$\text{ran } P \cap \text{ran}(I - P) = \text{ran } P \cap \text{ran } Q \subset \text{ran } A \cap \text{ran } B = \{0\}.$$

Since $P(I - P) = (I - P)P$, this implies that $P^2 = P$. Suppose $P = 0$. Then $AS = 0$ and hence $A|_{H_2} = AST = 0$. Since $A|_{H_1} = 0$, it follows that $A = 0$. This is a contradiction. Hence $P \neq 0$. Suppose $P = I$. Then $BS = Q = I - P = 0$ and hence $B|_{H_2} = BST = 0$. Since $B|_{H_1} = 0$, it follows that $B = 0$. This is a contradiction. Hence $P \neq I$. Since $P \neq 0, I$ and $Q = I - P$, it follows from Gohberg Krein [2, p.339] that $\|P\| = \csc \phi(\text{ran } P, \text{ran } Q)$. Hence

$$\cos \phi(\text{ran } P, \text{ran } Q) = \frac{\sqrt{\|P\|^2 - 1}}{\|P\|} < 1.$$

Hence $\phi(\text{ran } P, \text{ran } Q) > 0$. Since $A|_{H_1} = B|_{H_1} = 0$, it follows that

$$\text{ran } P = \text{ran } AS = \text{ran } A|_{H_2} = \text{ran } A,$$

$$\text{ran } Q = \text{ran } BS = \text{ran } B|_{H_2} = \text{ran } B.$$

Hence

$$\phi(\text{ran } A, \text{ran } B) = \phi(\text{ran } P, \text{ran } Q) > 0.$$

(3) $\Rightarrow$ (2): Suppose $\overline{\text{ran } A} \cap \overline{\text{ran } B} \neq \{0\}$. Then there exists a $h \in H$ and sequences $\{f_n\}, \{g_n\} \subset H$ such that $h \neq 0$, $\|Af_n - h\| \rightarrow 0$, $\|Bg_n - h\| \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\cos \phi(\text{ran } A, \text{ran } B) = \sup_{f, g \in H} \frac{|\langle Af, Bg \rangle|}{\|Af\| \cdot \|Bg\|} \geq \lim_{n \rightarrow \infty} \frac{|\langle Af_n, Bg_n \rangle|}{\|Af_n\| \cdot \|Bg_n\|} = \frac{|\langle h, h \rangle|}{\|h\| \cdot \|h\|} = 1.$$

Hence $\phi(\text{ran } A, \text{ran } B) = 0$. Lemma 1 is proved. $\square$

The assertions in Lemma 1 is equivalent to the formula:

$$\|A(A+B)^{-1}\| = \csc \phi(\text{ran } A, \text{ran } B).$$

If $\alpha = \gamma = \delta = 1$ and $\beta = 0$ then the following Theorem 1 implies this formula. Let P_{H_1} (resp. P_{H_2}) denote the orthogonal projection from H onto H_1 (resp. H_2). By Lemma 1, if $H_1 \cap H_2 = \{0\}$ and $\text{ran}(P_{H_1} + P_{H_2}) = H$ then $\phi(H_1, H_2) > 0$.

Theorem 1. *Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Let $A, B \in B(H)$ satisfy $A \neq 0, B \neq 0, \text{ran}(\gamma A + \delta B) = H$ and*

$$\text{ran } A \cap \text{ran } B = \{0\}$$

Then

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| = F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \phi\right).$$

where $\phi = \phi(\text{ran } A, \text{ran } B) > 0$ and $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.

Proof. It is sufficient to prove when $\gamma = \delta = 1$. Let $P = A(A+B)^{-1}$ and let $Q = B(A+B)^{-1}$, where $(A+B)^{-1}$ denote one of the right inverses to $A+B$. Then $P+Q=I$. By Lemma 1, if $A \neq 0$ and $B \neq 0$ then $P^2 = P \neq 0, I$, $\text{ran } P = \text{ran } A$ and $\text{ran } Q = \text{ran } B$. Since $\|P\| = \csc \phi(\text{ran } P, \text{ran } Q)$, it follows from the Feldman Krupnik Markus formula [1] that

$$\begin{aligned} \|(\alpha A + \beta B)(A+B)^{-1}\| &= \|\alpha P + \beta Q\| \\ &= F\left(\alpha, \beta, \|P\|^2 - 1\right) \\ &= F\left(\alpha, \beta, \cot^2 \phi(\text{ran } P, \text{ran } Q)\right) \\ &= F\left(\alpha, \beta, \cot^2 \phi(\text{ran } A, \text{ran } B)\right). \end{aligned}$$

Theorem 1 is proved. $\square$

In Theorem 1, if $AB = BA$ then

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| = \|(\gamma A + \delta B)(\alpha A + \beta B)^{-1}\| = F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \phi\right).$$

We have assumed that $\text{ran } A \cap \text{ran } B = \{0\}$. This is equivalent to the assertions in Lemma 1. If $(\alpha\delta - \beta\gamma)\gamma\delta \neq 0$ then $\frac{\alpha}{\gamma} - \frac{\beta}{\delta} \neq 0$ and hence the norm formula in Theorem 1 is equivalent to the assertions in Lemma 1. By Theorem 1, if $P \in B(H)$ satisfy $P^2 = P \neq I, 0$ and $Q = I - P$ then

$$\|P\| = \|Q\| = \csc \phi(\text{ran } P, \text{ran } Q),$$

which is in the book of Gohberg and Krein [2, p.339]. If $A = P$ and $B = Q$ then Theorem 1 becomes the Feldman Krupnik Markus formula [1].

In Theorem 1, if $A = P$ and $B = Q^*$ then $A + B$ becomes invertible and we can compute the norm $\|(\alpha A + \beta B)(A + B)^{-1}\|$ as the following. For all $f \in H$,

$$\begin{aligned} \langle (A^*A + B^*B)f, f \rangle &= \|Af\|^2 + \|Bf\|^2 \\ &= \|Af\|^2 + \|f - A^*f\|^2 \\ &= \|f\|^2 + \|Af - A^*f\|^2 \\ &\geq \|f\|^2. \end{aligned}$$

Hence $A^*A + B^*B$ is invertible. Similarly, $AA^* + BB^*$ is invertible. Since

$$(A + B)(A^* + B^*) = AA^* + BB^*$$

and

$$(A^* + B^*)(A + B) = A^*A + B^*B,$$

it follows that $A + B$ is invertible. Since $\text{ran } A \perp \text{ran } B$, it follows that $\phi(\text{ran } A, \text{ran } B) = \frac{\pi}{2}$. By Theorem 1,

$$\|(\alpha A + \beta B)(A + B)^{-1}\| = F(\alpha, \beta, 0) = \max(|\alpha|, |\beta|).$$

Let $P = A(A + B)^{-1}$ and $Q = B(A + B)^{-1}$. Since $(A + B)^*A = A^*(A + B)$, it follows that P and Q are selfadjoint. By Lemma 1, P and Q are selfadjoint idempotent.

Corollary 1. *Let $A, B \in B(H)$ satisfy $\text{ran}(A + B) = H$. Let α and β be complex numbers. Let p, q, r, s be complex numbers satisfying $p + r = q + s = 1$, $ps - qr \neq 0$, $\text{ran}(pA + qB) \neq H$, $\text{ran}(rA + sB) \neq H$ and $\text{ran}(pA + qB) \cap \text{ran}(rA + sB) = \{0\}$. Then*

$$\|(\alpha A + \beta B)(A + B)^{-1}\| = F\left(\frac{\alpha s - \beta r}{ps - qr}, \frac{p\beta - q\alpha}{ps - qr}, \cot^2 \phi\right),$$

where $\phi = \phi(\text{ran}(pA + qB), \text{ran}(rA + sB))$, and $(A + B)^{-1}$ denote one of the right inverses to $A + B$.

Proof. Let $A' = pA + qB$ and let $B' = rA + sB$. Then $A + B = A' + B'$. Define α' and β' by

$$\alpha' = \frac{\alpha s - \beta r}{ps - qr} \quad \text{and} \quad \beta' = \frac{p\beta - q\alpha}{ps - qr}.$$

Then $\alpha A + \beta B = \alpha' A' + \beta' B'$. Since $\text{ran } A' \neq H$, $\text{ran } B' \neq H$ and $\text{ran } A' + \text{ran } B' = \text{ran}(A + B) = H$, it follows from Theorem 1 that

$$\begin{aligned} \|(\alpha A + \beta B)(A + B)^{-1}\| &= \|(\alpha' A' + \beta' B')(A' + B')^{-1}\| \\ &= F(\alpha', \beta', \cot^2 \phi(A', B')). \end{aligned}$$

Corollary 1 is proved. $\square$

There are many operators $A, B \in B(H)$ such that $\rho(A, B) = 1$ and $\rho(pA + qB, rA + sB) < 1$. If $p = s = 1$ and $q = r = 0$ then Corollary 1 implies Theorem 1. Let $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$.

In Theorem 1, we have established the norm formula in the case when $K = \{0\}$. Next we shall consider the case when $K \neq \{0\}$. Then $\phi(\text{ran } A, \text{ran } B) = 0$. In the following Theorem 2, we shall estimate the norm $\|(\alpha A + \beta B)(A + B)^{-1}\|$ from below in the case when $K \neq \overline{\text{ran } A}$ and $K \neq \overline{\text{ran } B}$.

Theorem 2. *Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Let $A, B \in B(H)$. Let $\gamma A + \delta B$ be invertible. Let $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$ be an invariant subspace of $A(\gamma A + \delta B)^{-1}$ such that $K \neq \overline{\text{ran } A}$ and $K \neq \overline{\text{ran } B}$. Then*

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| \geq F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \phi\right),$$

where $\phi = \phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K) > 0$.

Proof. It is sufficient to prove when $\gamma = \delta = 1$:

$$\|(\alpha A + \beta B)(A + B)^{-1}\| \geq F(\alpha, \beta, \cot^2 \phi).$$

Let $X = A(A + B)^{-1}$ and let $Y = B(A + B)^{-1}$. Then $X + Y = I$. Let P_K denote the orthogonal projection from H onto K and let $P_{K^\perp}$ denote the orthogonal projection from H onto $K^\perp$. Let $X_1 = P_{K^\perp}X|_{K^\perp}$ and let $Y_1 = P_{K^\perp}Y|_{K^\perp}$. We shall show that $\text{ran } X_1 \cap \text{ran } Y_1 = \{0\}$. Let $h \in \text{ran } X_1 \cap \text{ran } Y_1$. Then $h \in K^\perp$. There exist $f, g \in K^\perp$ such that $h = P_{K^\perp}Xf = P_{K^\perp}Yg$. Then $Xf = h + P_KXf$ and $Yg = h + P_KYg$. Hence $Xf - Yg \in K$. Since $Yg \in \text{ran } B$, it follows that $Xf \in \text{ran } A \cap \overline{\text{ran } B} \subset K$. Hence $h \in K \cap K^\perp = \{0\}$. Therefore $\text{ran } X_1 \cap \text{ran } Y_1 = \{0\}$ and $X_1 + Y_1 = I|_{K^\perp}$. By Lemma 1, this implies that X_1 and Y_1 are idempotent operators on $K^\perp$, and $\phi(\text{ran } X_1, \text{ran } Y_1) > 0$. Then $\text{ran } X_1$ and $\text{ran } Y_1$ are closed subspaces of H . By Theorem 1,

$$\begin{aligned} \|(\alpha A + \beta B)(A + B)^{-1}\| &= \|\alpha X + \beta Y\| \\ &\geq \sup_{f \in K^\perp} \frac{\|P_{K^\perp}(\alpha X + \beta Y)f\|}{\|f\|} = \sup_{f \in K^\perp} \frac{\|(\alpha X_1 + \beta Y_1)f\|}{\|f\|} \\ &= \|\alpha X_1 + \beta Y_1\|_{K^\perp} \\ &= F(\alpha, \beta, \cot^2 \phi(\text{ran } X_1, \text{ran } Y_1)). \end{aligned}$$

If $f \in \overline{\text{ran } X} \ominus K$ then there exists $g_n + h_n \in K \oplus K^\perp$ such that $\|f - X(g_n + h_n)\| \rightarrow 0$ as $n \rightarrow \infty$. Since K is an invariant subspace of X , it follows that $Xg_n \in K$. Since $f \in K^\perp$, $X_1h_n = P_{K^\perp}Xh_n$, $P_{K^\perp}Xg_n = 0$ and $\|P_{K^\perp}\| = 1$, it follows that

$$\|f - X_1h_n\| = \|P_{K^\perp}(f - X(g_n + h_n))\| \leq \|f - X(g_n + h_n)\|.$$

Hence $\overline{\text{ran } X} \ominus K \subset \overline{\text{ran } X_1} = \text{ran } X_1$. Since K is an invariant subspace of X and $X + Y = I$, it follows that K is an invariant subspace of Y . By the similar proof, $\overline{\text{ran } Y} \ominus K \subset \text{ran } Y_1$. Since $A + B$ is invertible, it follows that $\text{ran } X = \text{ran } A$ and $\text{ran } Y = \text{ran } B$. Hence

$$\begin{aligned} 1 &> \cos \phi(\text{ran } X_1, \text{ran } Y_1) \\ &\geq \cos \phi(\overline{\text{ran } X} \ominus K, \overline{\text{ran } Y} \ominus K) \\ &= \cos \phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K). \end{aligned}$$

Hence

$$0 < \phi(\text{ran } X_1, \text{ran } Y_1) \leq \phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K).$$

Hence

$$\infty > \cot \phi(\text{ran } X_1, \text{ran } Y_1) \geq \cot \phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K).$$

Since $F(\alpha, \beta, t)$ is a nondecreasing function of t , ($t \geq 0$), it follows that

$$\begin{aligned} \|(\alpha A + \beta B)(A + B)^{-1}\| &\geq F(\alpha, \beta, \cot^2 \phi(\text{ran } X_1, \text{ran } Y_1)) \\ &\geq F(\alpha, \beta, \cot^2 \phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K)). \end{aligned}$$

Theorem 2 is proved. $\square$

By Theorem 1, if $K = \{0\}$ then the equality holds in the inequality in Theorem 2. There are many operators $A, B \in B(H)$ such that $\phi(\text{ran } A, \text{ran } B) = 0$ and

$$\phi(\overline{\text{ran } A} \ominus K, \overline{\text{ran } B} \ominus K) > 0.$$

If $AB = BA$ then $A(\gamma A + \delta B)^{-1}B = BA(\gamma A + \delta B)^{-1}$ and hence $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$ is an invariant subspace of $A(\gamma A + \delta B)^{-1}$. Even when the conditions in Theorem 2 do not hold, the similar result holds as the followings. If $A, B \in B(H)$ satisfy $A \neq 0$, $B \neq 0$, $A + B$ is right invertible, and M is the closed subspace of H satisfying

$$\text{ran } A \cap \overline{\text{ran } B} \subset M \subset \overline{\text{ran } B},$$

then

$$\|(\alpha A + \beta B)(A + B)^{-1}\| \geq F(\alpha, \beta, \cot^2 \phi),$$

where $\phi = \phi(P_{M^\perp}A(A + B)^{-1}|_{M^\perp}, P_{M^\perp}B(A + B)^{-1}|_{M^\perp}) > 0$.

3. Norm formula when $A = I$ and $B^n = 0$

Let $K = \overline{\text{ran } A} \cap \overline{\text{ran } B}$. In Section 2, we have considered two operators $A, B \in B(H)$ satisfying $K \neq \overline{\text{ran } A}$ and $K \neq \overline{\text{ran } B}$. In this section we shall consider two operators A and B satisfying $K = \overline{\text{ran } B}$. In general, suppose $A_0, B_0 \in B(H)$, $\text{ran } B_0 \subset \text{ran } A_0$ and $\ker B_0 \neq \{0\}$. Then there are many $C_0 \in B(H)$ such that $C_0(\text{ran } B_0) \subset \ker B_0$. Let $A = C_0A_0$ and $B = C_0B_0$. Then

$$C_0(\alpha A_0 + \beta B_0) = \alpha A + \beta B.$$

Since $\text{ran } B_0 \subset \text{ran } A_0$, it follows that $\text{ran } B \subset \text{ran } A$ and hence $K = \overline{\text{ran } B}$. Since $C_0(\text{ran } B_0) \subset \ker B_0$, it follows that $B^2 = 0$. Hence, in many cases, the linear combination $\alpha A + \beta B$ for $A, B \in B(H)$ satisfying $B^2 = 0$ appears. We shall prove the following Proposition 1 using the Feldman Krupnik Markus formula [1]. The first author [5] proved it in the different way.

Proposition 1. *Let $B \in B(H)$ satisfy $B^2 = 0$. Let α and β be complex numbers. Then*

$$\|\alpha I + \beta B\| = \sqrt{\left|\frac{\beta}{2}\right|^2 \|B\|^2 + |\alpha|^2 + \left|\frac{\beta}{2}\right| \|B\|}.$$

Proof. It is sufficient to prove when $\alpha = -1$ and $\beta = 2$:

$$\|2B - I\| = \sqrt{\|B\|^2 + 1 + \|B\|}.$$

Let P_B denote the orthogonal projection from H onto $\overline{\text{ran } B}$. For $\varepsilon > 0$ let

$$P_\varepsilon = P_B - \frac{B}{\varepsilon}.$$

Then

$$P_\varepsilon^2 = P_B - \frac{P_B B}{\varepsilon} - \frac{B P_B}{\varepsilon} = P_B - \frac{B}{\varepsilon} = P_\varepsilon.$$

Let $Q_\varepsilon = I - P_\varepsilon$. Then

$$\begin{aligned} 2B - I &= 2\varepsilon(P_B - P_\varepsilon) - (P_\varepsilon + Q_\varepsilon) \\ &= 2\varepsilon P_B - (1 + 2\varepsilon)P_\varepsilon - Q_\varepsilon. \end{aligned}$$

Since $\|\varepsilon P_\varepsilon\| = \|\varepsilon P_B - B\| \rightarrow \|B\|$ as $\varepsilon \rightarrow 0$, it follows from the Feldman–Krupnik–Markus formula [1] that

$$\begin{aligned} \|2B - I\| &= \lim_{\varepsilon \rightarrow 0} \|(1 + 2\varepsilon)P_\varepsilon + Q_\varepsilon\| \\ &= \lim_{\varepsilon \rightarrow 0} F(1 + 2\varepsilon, 1, \|P_\varepsilon\|^2 - 1) \\ &= \lim_{\varepsilon \rightarrow 0} \left(\sqrt{\varepsilon^2(\|P_\varepsilon\|^2 - 1) + (1 + \varepsilon)^2} + \sqrt{\varepsilon^2(\|P_\varepsilon\|^2 - 1) + \varepsilon^2} \right) \\ &= \sqrt{\|B\|^2 + 1 + \|B\|}. \end{aligned}$$

Proposition 1 is proved. $\square$

Suppose A is invertible. Let $C = A^{-1}B$. If $B^2 = 0$ and $A^{-1}(\text{ran } B) \subset \ker B$ then $C^2 = 0$. In Proposition 1, we have considered $\|\alpha I + \beta C\|$. Then

$$\frac{\|\alpha I + \beta C\|}{\|A^{-1}\|} \leq \|\alpha A + \beta B\| \leq \|A\| \cdot \|\alpha I + \beta C\|.$$

For example, if A is invertible, $B^2 = 0$ and $AB = BA$ then $A^{-1}(\text{ran } B) \subset \ker B$. If $\|A\| = \|A^{-1}\|$ then inequalities become equalities. Then A is a unitary operator.

Proposition 2. *Let n be an integer satisfying $n \geq 2$. Let $B \in B(H)$ satisfy $B^n = 0$. Let α and β be complex numbers. Then*

$$\|\alpha I + \beta B\| \geq \sqrt{\left|\frac{\beta}{2}\right|^2 \left\|B|_{\overline{\text{ran } B^{n-2}}}\right\|^2 + |\alpha|^2 + \left|\frac{\beta}{2}\right| \left\|B|_{\overline{\text{ran } B^{n-2}}}\right\|}.$$

4. Norm formula using the angle ψ between A and B

In this section, we shall estimate the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ from above in the case when $\psi(A, B) > 0$ where ψ is defined in Introduction. We do not assume the boundedness of two operators A and B on H . In the proof of Theorem 3 (1), we do not use the linearity of A and B . We do not use the Feldman–Krupnik–Markus formula [1] in this section.

Definition 3. For $f, g \in H$, let

$$\rho(f, g) = \begin{cases} \frac{|\langle f, g \rangle|}{\|f\| \cdot \|g\|} & \text{if } f \neq 0 \text{ and } g \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\cos \phi(\text{ran } A, \text{ran } B) = \sup_{f, g \in H} \rho(Af, Bg) \geq \sup_{f \in H} \rho(Af, Bf) = \cos \psi(A, B).$$

Lemma 2. Let α and β be distinct complex numbers. Let x and ρ be real numbers satisfying $x \geq \max(|\alpha|, |\beta|)$ and $0 \leq \rho < 1$. The following assertions are equivalent:

$$(1) \quad x \geq F\left(\alpha, \beta, \frac{\rho^2}{1 - \rho^2}\right).$$

$$(2) \quad x^4 - \left(\frac{\rho^2}{1 - \rho^2}|\alpha - \beta|^2 + |\alpha|^2 + |\beta|^2\right)x^2 + |\alpha\beta|^2 \geq 0.$$

$$(3) \quad \frac{|x^2 - \alpha\bar{\beta}|}{|\alpha - \beta|x} \geq \frac{1}{\sqrt{1 - \rho^2}}.$$

$$(4) \quad (x^2 - |\alpha|^2)(x^2 - |\beta|^2) \geq \rho^2 |x^2 - \alpha\bar{\beta}|^2.$$

The equivalence holds for not only inequalities but also equalities.

Proof. (1) $\Leftrightarrow$ (2): Since $x \geq 0$, (1) is equivalent to

$$\begin{aligned} x^2 &\geq \frac{|\alpha - \beta|^2}{2} \frac{\rho^2}{1 - \rho^2} + \frac{|\alpha|^2 + |\beta|^2}{2} \\ &\quad + \sqrt{\left(\frac{|\alpha - \beta|^2}{2} \frac{\rho^2}{1 - \rho^2} + \frac{|\alpha|^2 + |\beta|^2}{2}\right)^2 - |\alpha\beta|^2}. \end{aligned}$$

Since $x \geq \max(|\alpha|, |\beta|)$, this is equivalent to (2).

(2) $\Rightarrow$ (3): By (2),

$$(x^2 - |\alpha|^2)(x^2 - |\beta|^2) \geq \frac{\rho^2}{1 - \rho^2} |\alpha - \beta|^2 x^2.$$

Hence

$$\left| x^2 - \alpha\bar{\beta} \right|^2 - |\alpha - \beta|^2 x^2 \geq \frac{\rho^2}{1 - \rho^2} |\alpha - \beta|^2 x^2.$$

This implies (3).

(3) $\Rightarrow$ (4): By (3),

$$\left| x^2 - \alpha\bar{\beta} \right|^2 - |\alpha - \beta|^2 x^2 \geq \rho^2 \left| x^2 - \alpha\bar{\beta} \right|^2.$$

This implies (4).

(4) $\Rightarrow$ (2): By (4),

$$(1 - \rho^2) (x^2 - |\alpha|^2) (x^2 - |\beta|^2) \geq \rho^2 |\alpha - \beta|^2 x^2.$$

This implies (2). Lemma 2 is proved. $\square$

Lemma 3. *Let $\alpha, \beta, \gamma, \delta$ be complex numbers satisfying $\gamma\delta \neq 0$, and let f, g be nonzero elements in the Hilbert space H satisfying $\rho = \rho(f, g) < 1$. Then*

$$\frac{\|\alpha f + \beta g\|}{\|\gamma f + \delta g\|} \leq F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \frac{\rho^2}{1 - \rho^2}\right).$$

Proof. It is sufficient to prove when $\alpha \neq \beta$ and $\gamma = \delta = 1$. Let $\rho = \rho(f, g)$, and let $x = F(\alpha, \beta, \frac{\rho^2}{1 - \rho^2})$. By Lemma 2, $(x^2 - |\alpha|^2)(x^2 - |\beta|^2) = \rho^2 |x^2 - \alpha\bar{\beta}|^2$. Hence

$$(x^2 - |\alpha|^2)(x^2 - |\beta|^2) \|f\|^2 \|g\|^2 = |x^2 - \alpha\bar{\beta}|^2 |\langle f, g \rangle|^2.$$

Hence

$$\begin{aligned} & (x^2 - |\alpha|^2) \|f\|^2 + (x^2 - |\beta|^2) \|g\|^2 + 2\operatorname{Re} \left((x^2 - \alpha\bar{\beta}) \langle f, g \rangle \right) \\ & \geq 2\sqrt{x^2 - |\alpha|^2} \sqrt{x^2 - |\beta|^2} \|f\| \cdot \|g\| - 2|x^2 - \alpha\bar{\beta}| \cdot |\langle f, g \rangle| = 0. \end{aligned}$$

Therefore

$$\|\alpha f + \beta g\|^2 \leq x^2 \|f + g\|^2.$$

Lemma 3 is proved. $\square$

Theorem 3. *Let A and B be nonzero linear operators on H satisfying $\psi(A, B) > 0$. Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

(1) $\gamma A + \delta B \neq 0$ and

$$\sup_{(\gamma A + \delta B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(\gamma A + \delta B)f\|} \leq F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A, B)\right).$$

(2) If $\gamma A + \delta B$ is right invertible then

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| \leq F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A, B)\right),$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.
(3) If $\gamma A + \delta B$ is left invertible then

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| \leq \|(\gamma A + \delta B)(\gamma A + \delta B)^{-1}\| F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A, B)\right),$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the left inverses to $\gamma A + \delta B$.

Proof. It is sufficient to prove when $\gamma = \delta = 1$. We shall prove (1). Suppose $f \in H$ satisfies $Af \neq 0$, $Bf \neq 0$ and $\rho(Af, Bf) < 1$. By Lemma 3,

$$\frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|} \leq F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right).$$

Since $y = F(\alpha, \beta, \frac{x^2}{1-x^2})$ is a nondecreasing function of x , $0 \leq x < 1$, and $y \geq \max(|\alpha|, |\beta|)$,

$$\begin{aligned} \sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|} &\leq \sup_{f \in H} F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right) \\ &= F\left(\alpha, \beta, \frac{\cos^2 \psi(A, B)}{1 - \cos^2 \psi(A, B)}\right) \\ &= F\left(\alpha, \beta, \cot^2 \psi(A, B)\right). \end{aligned}$$

We shall prove (2). For every $g \in H$, let $f = (A + B)^{-1}g$. Since $(A + B)^{-1}$ is the right inverse to $A + B$, it follows that

$$\sup_{g \neq 0} \frac{\|(\alpha A + \beta B)(A + B)^{-1}g\|}{\|g\|} \leq \sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|}.$$

By (1), this implies (2).

We shall prove (3). Let $c = \|(A + B)(A + B)^{-1}\|$ and let $f = (A + B)^{-1}g$. Since $(A + B)^{-1}$ is the left inverse to $A + B$, it follows that

$$\begin{aligned} \sup_{g \neq 0} \frac{\|(\alpha A + \beta B)(A + B)^{-1}g\|}{\|g\|} &\leq c \sup_{(A+B)(A+B)^{-1}g \neq 0} \frac{\|(\alpha A + \beta B)(A + B)^{-1}g\|}{\|(A + B)(A + B)^{-1}g\|} \\ &\leq c \sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|}. \end{aligned}$$

By (1), this implies (3). Theorem 3 is proved. $\square$

By Theorem 1, if $\psi(A, B) = \phi(\text{ran } A, \text{ran } B)$ then the equality holds in Theorem 3 (2). In many cases $\psi(A, B) = \phi(\text{ran } A, \text{ran } B)$. Let P be an analytic projection on the weighted L^2 space. Helson and Szegő [4] used the equivalence of $\psi(P, I - P) > 0$ and $\|P\| < \infty$.

Corollary 2. *Let A and B be linear operators on H . If $\|A + B\| < \infty$ and $\psi(A, B) > 0$ then $\|A\| < \infty$ and $\|B\| < \infty$.*

Proof. By Theorem 3 (1), if $\psi(A, B) > 0$ and $\|A + B\| < \infty$ then

$$\|(\alpha A + \beta B)f\| \leq F(\alpha, \beta, \cot^2 \psi(A, B))\|(A + B)f\| \leq F(\alpha, \beta, \cot^2 \psi(A, B))\|A + B\| \cdot \|f\|,$$

for all $f \in H$. Hence

$$\|\alpha A + \beta B\| \leq F(\alpha, \beta, \cot^2 \psi(A, B))\|A + B\| < \infty,$$

for every complex numbers α and β . Corollary 2 is proved. $\square$

Lemma 4. *Let A and B be nonzero operators on H satisfying $\psi(A, B) > 0$. Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

$$\begin{aligned} \sup_{t \in \mathbf{C}} \sup_{(\gamma t A + \delta B)f \neq 0} \frac{\|(\alpha t A + \beta B)f\|}{\|(\gamma t A + \delta B)f\|} &= \sup_{t \in \mathbf{C}} \sup_{(\gamma A + \delta t B)f \neq 0} \frac{\|(\alpha A + \beta t B)f\|}{\|(\gamma A + \delta t B)f\|} \\ &= F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A, B)\right). \end{aligned}$$

Proof. It is sufficient to prove that

$$\sup_{t \in \mathbf{C}} \sup_{(tA+B)f \neq 0} \frac{\|(\alpha t A + \beta B)f\|}{\|(tA+B)f\|} = F(\alpha, \beta, \cot^2 \psi(A, B)).$$

Let

$$c = \sup_{t \in \mathbf{C}} \sup_{(tA+B)f \neq 0} \frac{\|(\alpha t A + \beta B)f\|}{\|(tA+B)f\|}.$$

We shall prove that $c = F(\alpha, \beta, \cot^2 \psi(A, B))$. By Theorem 3 (1), $c \leq F(\alpha, \beta, \cot^2 \psi(A, B))$. Hence it is sufficient to prove that $c \geq F(\alpha, \beta, \cot^2 \psi(A, B))$. Since

$$c \geq \frac{\|\alpha t A f + \beta B f\|}{\|t A f + B f\|}, \quad (t \in \mathbf{C}),$$

it follows that

$$|t|^2 (c^2 - |\alpha|^2) \|A f\|^2 + (c^2 - |\beta|^2) \|B f\|^2 + 2\operatorname{Re}(t(c^2 - \alpha\bar{\beta})\langle A f, B f \rangle) \geq 0,$$

for all $t \in \mathbf{C}$. Hence

$$|c^2 - \alpha\bar{\beta}|^2 |\langle A f, B f \rangle|^2 \leq (c^2 - |\alpha|^2) (c^2 - |\beta|^2) \|A f\|^2 \|B f\|^2.$$

Hence

$$\rho(A f, B f)^2 |c^2 - \alpha\bar{\beta}|^2 \leq (c^2 - |\alpha|^2) (c^2 - |\beta|^2).$$

By Lemma 2,

$$c \geq F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right),$$

for all $f \in H$. Hence

$$c \geq \sup_{f \in H} F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right).$$

Since $y = F(\alpha, \beta, \frac{x^2}{1-x^2})$ is a nondecreasing function of x , $0 \leq x < 1$, it follows that

$$\begin{aligned} c &\geq F\left(\alpha, \beta, \frac{\cos^2 \psi(A, B)}{1 - \cos^2 \psi(A, B)}\right) \\ &= F\left(\alpha, \beta, \cot^2 \psi(A, B)\right). \end{aligned}$$

Lemma 4 is proved. $\square$

Corollary 3. *Let A and B be nonzero linear operators on H satisfying $\psi(A, B) > 0$, $AB = BA = 0$ and $\gamma A + \delta B$ is right invertible. Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

$$\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\| = F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A, B)\right).$$

Proof. It is sufficient to prove when $\gamma = \delta = 1$. Suppose $A + B$ is right invertible. By Lemma 4, it is sufficient to prove the equality:

$$\sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|} = \sup_{t \in \mathbf{C}} \sup_{(tA+B)f \neq 0} \frac{\|(\alpha tA + \beta B)f\|}{\|(tA + B)f\|}.$$

Since $A + B$ is right invertible, it follows that $\text{ran } A + \text{ran } B = \text{ran}(A + B) = H$. Since A and B are linear operators satisfying $AB = BA = 0$, it follows that

$$\begin{aligned} \sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|} &\leq \sup_{t \in \mathbf{C}} \sup_{(tA+B)f \neq 0} \frac{\|(\alpha tA + \beta B)f\|}{\|(tA + B)f\|} \\ &\leq \sup_{Ag+Bh \neq 0} \frac{\|\alpha Ag + \beta Bh\|}{\|Ag + Bh\|} \\ &= \sup_{Ag+Bh \neq 0, g \in \text{ran } A, h \in \text{ran } B} \frac{\|\alpha Ag + \beta Bh\|}{\|Ag + Bh\|} \\ &= \sup_{(A+B)(g+h) \neq 0, g \in \text{ran } A, h \in \text{ran } B} \frac{\|(\alpha A + \beta B)(g + h)\|}{\|(A + B)(g + h)\|} \\ &= \sup_{(A+B)f \neq 0} \frac{\|(\alpha A + \beta B)f\|}{\|(A + B)f\|}. \end{aligned}$$

Corollary 3 is proved. $\square$

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