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Title	New special curves and developable surfaces
Author(s)	Izumiya, S.; Takeuchi, N.
Citation	Hokkaido University Preprint Series in Mathematics, 555, 1-10
Issue Date	2002-07
DOI	https://doi.org/10.14943/83700
Doc URL	https://hdl.handle.net/2115/69304
Type	departmental bulletin paper
File Information	re555.pdf

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NEW SPECIAL CURVES AND DEVELOPABLE SURFACES

Shyuichi Izumiya and Nobuko Takeuchi

Series #555. July 2002

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New special curves and developable surfaces

Shyuichi Izumiya and Nobuko Takeuchi

We define new special curves in Euclidean 3-space which we call slant helices and conical geodesic curves. Those notions are generalizations of the notion of cylindrical helices. One of the results in this paper gives a classification of special developable surfaces under the condition of the existence of such a special curve as a geodesic. As a result, we consider geometric invariants of space curves. By using these invariants, we can estimate the order of contact with those special curves for general space curves. All arguments in this paper are elementary and classical. However, there have been no papers which have investigated slant helices and conical geodesic curves so far as we know.

Mathematics Subject classification(2000): 53A05, 53A25, 58C27
Key Words and Phrases: Helix, Darboux vector, developable surfaces, singularities

1 Introduction

In [3] we have studied singularities of the rectifying developable (surface) of a space curve. The rectifying developable is defined to be the envelope of the family of rectifying planes along the curve. We have also studied the Darboux developable of a space curve whose singularities are given by the locus of the end points of modified Darboux vectors of the curve [3, 5, 6].

In this paper we define the notion of slant helices and conical geodesic curves which are generalizations of the notion of helices (cf., §2). We introduce the notion of the *tangential Darboux developable* of a space curve which is defined by the Darboux developable of the tangent indicatrix of the space curve (cf., §3). We

study singularities of the tangential Darboux developable of a space curve as an application of the classification theorem of developable surfaces in [6]. We find a geometric invariant of a space curve which is deeply related to the singularities of the tangential Darboux developable of the original curve.

In §2 we describe basic notions and properties of space curves. We review the classification of singularities of the rectifying developable and the Darboux developable of a space curve in §3. In §4 we study slant helices and conical geodesic curves as curves on developable surfaces. We can construct several examples of slant helices. However, we need a large part of pages for drawing those pictures, so we give only one example in §5.

This is one of the papers of the authors joint project entitled "Geometry of ruled surfaces and line congruence".

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2 Basic notions and properties

We now review some basic concepts on classical differential geometry of space curves in Euclidean space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, we denote $\boldsymbol{x} \cdot \boldsymbol{y}$ as the standard inner product. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a curve with $\dot{\boldsymbol{\gamma}}(t) \neq 0$, where $\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt(t)$. We also denote the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|$. The *arc-length parameter* s of a curve $\boldsymbol{\gamma}$ is determined such that $\|\boldsymbol{\gamma}'(s)\| = 1$, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds(s)$. Let us denote $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of $\boldsymbol{\gamma}$ at s. We define the *curvature* of $\boldsymbol{\gamma}$ by $\kappa(s) = \sqrt{\|\boldsymbol{\gamma}''(s)\|}$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\boldsymbol{n}(s)$ of the curve $\boldsymbol{\gamma}$ at s is given by $\boldsymbol{\gamma}''(s) = \kappa(s)\boldsymbol{n}(s)$. The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$ is called the *unit binormal vector* of $\boldsymbol{\gamma}$ at s. Then we have the Frenet-Serret formulae:

$$\begin{cases} \boldsymbol{t}'(s) &= \kappa(s)\boldsymbol{n}(s) \\ \boldsymbol{n}'(s) &= -\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) &= -\tau(s)\boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve $\boldsymbol{\gamma}$ at s. For any unit speed curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$, we define a vector field $\widetilde{\boldsymbol{D}}(s) = (\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)$ along $\boldsymbol{\gamma}$ under the condition that $\kappa(s) \neq 0$ and we call it *the modified Darboux vector field* of $\boldsymbol{\gamma}$. We also denote the unit Darboux vector field by $\bar{\boldsymbol{D}}(s) = (1/\sqrt{\tau^2 + \kappa^2})(s)(\tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s))$ (cf., [8], Section 5.2).

A curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *cylindrical helix* if the tangent lines of $\boldsymbol{\gamma}$ make a constant angle with a fixed direction. It has been known that the curve $\boldsymbol{\gamma}(s)$ is a cylindrical helix if and only if $(\tau/\kappa)(s) =$ constant. If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is, of course, a cylindrical helix. We call such a curve a *circular helix*. We now define new special curves as follows: A curve $\boldsymbol{\gamma}$ with $\kappa(s) \neq 0$ is

called a *slant helix* if the principal normal lines of $\boldsymbol{\gamma}$ make a constant angle with a fixed direction. We remark that the principal normal lines of a cylindrical helix is perpendicular to a fixed direction, so that a cylindrical helix is a slant helix. We also call a curve $\boldsymbol{\gamma}$ with $\kappa(s) \neq 0$ a *conical geodesic curve* if $(\tau/\kappa)'(s)$ is a constant function.

We have the following characterization of slant helices.

Proposition 2.1 *Let $\boldsymbol{\gamma}$ be a unit speed space curve with $\kappa(s) \neq 0$. Then $\boldsymbol{\gamma}$ is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is a constant function.

Proof. We can show that $\boldsymbol{\gamma}$ is a slant helix if and only if the spherical image of the principal normal $\boldsymbol{n} : I \longrightarrow S^2$ is a part of a circle in S^2. By a straightforward calculation, the geodesic curvature of $\boldsymbol{n}$ in S^2 is given by $\sigma(s)$. Therefore, the image of $\boldsymbol{n}$ is a part of a circle in S^2 if and only if $\sigma(s)$ is a constant function. □

If $\sigma(s) \equiv 0$, then we have $(\tau/\kappa)'(s) \equiv 0$. It follows that $\boldsymbol{\gamma}$ is a cylindrical helix. We have already studied geometric properties of cylindrical helices in [3, 5], so that we are interested in the case when $(\tau/\kappa)'(s) \neq 0$. We now consider the meaning of the above invariants $(\tau/\kappa)'(s)$ and $\sigma(s)$ for a general curve. Let $\boldsymbol{\gamma}_i : I_i \longrightarrow \mathbb{R}^3$ $(i = 1, 2)$ be regular curves. We say that $\boldsymbol{\gamma}_1(s_0)$ and $\boldsymbol{\gamma}_2(t_0)$ have *at least* $(k+1)$*-point contact* if $\boldsymbol{\gamma}_1^{(p)}(s_0) = \boldsymbol{\gamma}_2^{(p)}(t_0)$ for $0 \leq p \leq k$. We also say that $\boldsymbol{\gamma}_1(s_0)$ and $\boldsymbol{\gamma}_2(t_0)$ have $(k+1)$*-point contact* if they have at least $(k+1)$-point contact and $\boldsymbol{\gamma}_1^{(k+1)}(s_0) \neq \boldsymbol{\gamma}_2^{(k+1)}(t_0)$.

Proposition 2.2 *For a regular curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s_0) \neq 0$ and $(\tau/\kappa)'(s_0) \neq 0$, we have the following assertions:*

(1) *There exists an open interval $s_0 \in J \subset I$ and a unique conical geodesic curve $\boldsymbol{\delta} : J \longrightarrow \mathbb{R}^3$ such that $\boldsymbol{\delta}(s_0) = \boldsymbol{\gamma}(s_0)$, the curvature of $\boldsymbol{\delta}(s)$ is $\kappa(s)$, the torsion of $\boldsymbol{\delta}$ at s_0 is $\tau(s_0)$ and $\boldsymbol{\gamma}(s_0), \boldsymbol{\delta}(s_0)$ have at least 5-point contact.*

(2) *There exists an open interval $s_0 \in J \subset I$ and a unique slant helix $\boldsymbol{\delta} : J \longrightarrow \mathbb{R}^3$ such that $\boldsymbol{\delta}(s_0) = \boldsymbol{\gamma}(s_0)$, the curvature of $\boldsymbol{\delta}(s)$ is $\kappa(s)$, the torsion of $\boldsymbol{\delta}$ at s_0 is $\tau(s_0)$ and $\boldsymbol{\gamma}(s_0), \boldsymbol{\delta}(s_0)$ have at least 5-point contact.*

Proof. (1) We denote that $(\tau/\kappa(s_0) = b$ and $(\tau/\kappa)'(s_0) = c$. We consider the natural equation:

$$\begin{cases} \kappa_\delta(s) &= \kappa(s) \\ \tau_\delta(s) &= \kappa(s)(cs+a), \end{cases}$$

where $a = b - cs_0$ under the initial conditions $\boldsymbol{\delta}^{(p)}(s_0) = \boldsymbol{\gamma}^{(p)}(s_0)$. The solution curve $\boldsymbol{\delta} : J \longrightarrow \mathbb{R}^3$ satisfies the required conditions.

(2) We denote that $\sigma(s_0) = d$. We also consider the natural equation:

$$\begin{cases} \kappa_\delta(s) & = & \kappa(s) \\ \left(\frac{\tau_\delta}{\kappa_\delta}\right)'(s) & = & d\kappa_\delta(s)\left(1+\left(\frac{\tau_\delta}{\kappa_\delta}\right)^2(s)\right)^{3/2}, \end{cases}$$

under the initial conditions $\boldsymbol{\delta}^{(p)}(s_0) = \boldsymbol{\gamma}^{(p)}(s_0)$. The solution curve $\boldsymbol{\delta} : J \longrightarrow \mathbb{R}^3$ satisfies the required conditions. □

We call the conical geodesic curve (respectively, the slant helix) given in Proposition 2.2 the *osculating conical geodesic curve* (respectively, the *osculating slant helix*) of $\boldsymbol{\gamma}$ at s_0.

3 Developable surfaces associated to a space curve

In this section we consider three developable surfaces associated to a space curve. Developable surfaces are ruled surfaces. A *ruled surface* in $\mathbb{R}^3$ is (locally) the map $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(t,u) = \boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t)$, where $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, $\boldsymbol{\delta} : I \to \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ are smooth mappings and I is an open interval or a unit circle S^1. We call $\boldsymbol{\gamma}$ the *base curve* and $\boldsymbol{\delta}$ the *director curve*. The straight lines $u \mapsto \boldsymbol{\gamma}(t)+u\boldsymbol{\delta}(t)$ are called *rulings*.

Let $\boldsymbol{\gamma}$ be a unit speed space curve with $\kappa(s) \neq 0$. A ruled surface $F_{(\gamma,\widetilde{D})}(s,u) = \boldsymbol{\gamma}(s)+u\widetilde{\boldsymbol{D}}(s)$ is called the *rectifying developable of* $\boldsymbol{\gamma}$. We also define a ruled surface $F_{(b,t)}(s,u) = \boldsymbol{b}(s)+u\boldsymbol{t}(s)$ which is called the *Darboux developable of* $\boldsymbol{\gamma}$. We consider the Darboux developable of the unit tangent vector $\boldsymbol{n}(s)$ of $\boldsymbol{\gamma}$ which is given by $F_{(\bar{D},n)}(s,u) = \bar{\boldsymbol{D}}(s) + u\boldsymbol{n}(s)$. We call it the *tangential Darboux developable* of $\boldsymbol{\gamma}$.

First, we consider the rectifying developable of a unit speed space curve $\boldsymbol{\gamma}(s)$ with $\kappa(s) \neq 0$. We can calculate that $\widetilde{\boldsymbol{D}}'(s) = (\tau/\kappa)'(s)\boldsymbol{t}(s)$, so that (s_0,u_0) is a singular point of $F_{(\gamma,\widetilde{D})}$ if and only if $(\tau/\kappa)'(s_0) \neq 0$ and $u_0 = -1/((\tau/\kappa)'(s_0))$.

In [3] we have studied singularities of the rectifying developable of $\boldsymbol{\gamma}$ and given a local classification. However, we have not be able to interpret the geometric meaning of singular points of rectifying developable. Here, we can say that the singular point of the rectifying developable of $\boldsymbol{\gamma}$ corresponds to the point s_0 that $\boldsymbol{\gamma}$ and the osculating conical geodesic curve at s_0 have at least 6-point contact. Moreover, we have the following proposition:

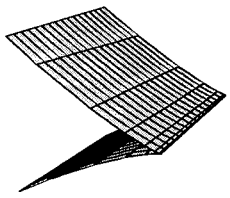
Proposition 3.1 *For a unit speed curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$, the following are equivalent.*

(1) *The rectifying developable* $F_{(\gamma,\widetilde{D})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ *of* γ *is a conical surface.*
(2) γ *is a conical geodesic curve.*

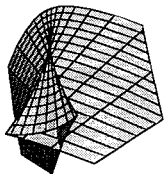
Proof. The singular locus of $F_{(\gamma,\widetilde{D})}$ is given by $\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s) - (1/(\tau/\kappa)')(s)\widetilde{\boldsymbol{D}}(s)$. By the Frenet-Serret formulae, we have $\boldsymbol{\sigma}'(s) = (\tau/\kappa)''(s)((\tau/\kappa)')^{-2}(s)\widetilde{\boldsymbol{D}}(s)$. Therefore, $\boldsymbol{\sigma}'(s) \equiv \mathbf{0}$ if and only if $(\tau/\kappa)''(s) \equiv 0$. We can easily show that $F_{(\gamma,\widetilde{D})}$ is a conical surface if and only if $\boldsymbol{\sigma}(s) \equiv \mathbf{0}$. This completes the proof. □

Secondly, a classification of singularities of the Darboux developable of a space curve is given as follows:

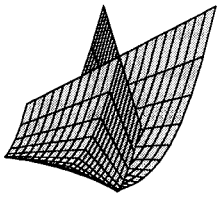
Theorem 3.2 **[3, 6, 7]** *Let* $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ *be a unit speed curve with* $\kappa(s) \neq 0$. *Then we have the following:*
(1) *The Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *at* $F_{(b,t)}(s_0, u_0)$ *if and only if* $\tau(s_0) \neq 0$, $(\tau/\kappa)'(s_0) \neq 0$ *and* $u_0 = (\tau/\kappa)(s_0)$, *where* $C \times \mathbb{R} = \{(x_1, x_2) \mid {x_1}^2 = {x_2}^3 \} \times \mathbb{R}$.
(2) *The Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the swallowtail SW at* $F_{(b,t)}(s_0, u_0)$ *if and only if* $\tau(s_0) \neq 0$, $(\tau/\kappa)'(s_0) = 0$, $(\tau/\kappa)''(s_0) \neq 0$, *and* $u_0 = (\tau/\kappa)(s_0)$, *where* $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v \}$.
(3) *The Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal crosscap CCR at* $F_{(b,t)}(s_0, u_0)$ *if and only if* $u_0 = \tau(s_0) = 0$, $(\tau/\kappa)'(s_0) \neq 0$, *where* $CCR = \{(x_1, x_2, x_3) \mid x_1 = u^3, x_2 = u^3 v^3, x_3 = v^2 \}$.



cuspidal edge



swallowtail



cuspidal crosscap

Fig.1

As a corollary of Theorem 3.2, we have the following local classification theorem of the tangential Darboux developable of a generic space curve.

Theorem 3.3 *Let* $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ *be a unit speed curve with* $\kappa(s) \neq 0$. *Then we have the following:*
(1) *The tangential Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to a cuspidaledge* $C \times \mathbb{R}$ *at* $F_{(\bar{D},n)}(s_0, u_0)$ *if and only if* $u_0 = \sigma(s_0) \neq 0$, $\sigma'(s_0) \neq 0$.

(2) *The tangential Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to a swallowtail SW at* $F_{(\bar{D},n)}(s_0,u_0)$ *if and only if* $u_0=\sigma(s_0)\neq 0$, $\sigma'(s_0)=0$, $\sigma''(s_0)\neq 0$.

(3) *The tangential Darboux developable of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to a cuspidal-cross cap CCR at* $F_{(\bar{D},n)}(s_0,u_0)$ *if and only if* $u_0=\sigma(s_0)=0$, $\sigma'(s_0)\neq 0$.

Proof. We respectively denote $K(s)$ and $T(s)$ as the curvature and the torsion of the space curve $\boldsymbol{t}(s)$. By a straightforward computation, we have

$$K(s)=(\sqrt{\kappa^2+\tau^2})(s),\quad T(s)=\left(\frac{\kappa^2}{\tau^2+\kappa^2}\left(\frac{\tau}{\kappa}\right)'\right)(s).$$

It follows that $\sigma(s)=(T/K)(s)$.

Applying the result in Theorem 3.2 to the space curve $\boldsymbol{t}(s)$, we complete the proof. □

We can also interpret that the swallowtail point of the tangential Darboux developable corresponds to the point s_0 that $\boldsymbol{\gamma}$ and the osculating slant helix at s_0 have 6-point contact. We also have the following proposition:

Proposition 3.4 *For a unit speed curve* $\boldsymbol{\gamma}: I\longrightarrow\mathbb{R}^3$ *with* $\kappa(s)\neq 0$, *the following are equivalent.*

(1) *The tangential Darboux developable* $F_{(\bar{D},n)}: I\times\mathbb{R}\longrightarrow\mathbb{R}^3$ *of* $\boldsymbol{\gamma}$ *is a conical surface.*

(2) $\boldsymbol{\gamma}$ *is a slant helix.*

Proof. The singular locus of the tangential Darboux developable $F_{(\bar{D},n)}(s,u)$ is given by $\boldsymbol{\sigma}(s)=\bar{\boldsymbol{D}}(s)+\sigma(s)\boldsymbol{n}(s)$. Therefore, $F_{(\bar{D},n)}(s,u)$ is a conical surface if and only if $\boldsymbol{\sigma}'(s)\equiv\boldsymbol{0}$. By the Frenet-Serret formulae, we can show that $\bar{\boldsymbol{D}}'(s)=-\sigma(s)\boldsymbol{n}'(s)$. Hence, $\boldsymbol{\sigma}'(s)\equiv\boldsymbol{0}$ if and only if $\sigma'(s)\equiv 0$. By Lemma 2.1, the assertion holds. □

4 Curves on developable surfaces

In this section we study slant helices and conical geodesic curves from the view point of the theory of curves on developable surfaces. In the previous sections, we have remarked that the rectifying developable of a cylindrical helix is a cylindrical surface. It has been classically known that $\boldsymbol{\gamma}$ is a geodesic of the rectifying developable of $\boldsymbol{\gamma}$ itself (cf.,[2]). The following proposition shows that the converse is also true.

Proposition 4.1 *Let S be a ruled surface and $\boldsymbol{\gamma}(s)$ a regular curve on S with non-vanishing curvature. Then the following conditions are equivalent :*

(1) *S is the rectifying developable of $\boldsymbol{\gamma}(s)$.*

(2) *$\boldsymbol{\gamma}(s)$ is a geodesic of S which is transversal to rulings and S is a developable surface.*

Proof. Suppose that S is a developable surface. Then the tangent plane of S at $p \in S$ is constant along the ruling through p. Let $\boldsymbol{\gamma}(s)$ be a geodesic of S which is transversal to rulings, then the principal normal of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$ is parallel to the normal of S at p. This means that the rectifying plane of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$ is the tangent plane of S at p. Since the tangent plane at p is constant along the ruling through p, S is an envelope of the family of rectifying planes of $\boldsymbol{\gamma}(s)$. This completes the proof. □

We have the following classification of special developable surfaces under the condition that the existence of a slant helix as a geodesic on the surface.

Theorem 4.2 *Let S be a developable surface and $\boldsymbol{\gamma}(s)$ a regular curve on S with non-vanishing curvature. Suppose that $\boldsymbol{\gamma}(s)$ is a slant helix of S and a geodesic which is transversal to rulings. Then*

(1) *S is a part of a cylindrical surface if γ is a cylindrical helix.*

(2) *S is a part of a circular cone if γ is not a cylindrical helix and $(\tau/\kappa)''(s) \equiv 0$.*

(3) *S is a part of the tangent developable of a cylindrical helix if $\boldsymbol{\gamma}$ is not a cylindrical helix and $(\tau/\kappa)''(s) \neq 0$.*

Proof. Since $\boldsymbol{\gamma}$ is a geodesic of S, S is the rectifying developable of $\boldsymbol{\gamma}$ by Proposition 4.1. Here, $\boldsymbol{\gamma}(s)$ is a slant helix, then

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is equal to a constant number c. By the proof of Proposition 3.5, we have $\bar{\boldsymbol{D}}'(s) = -\sigma(s)\boldsymbol{n}'(s) = -c\boldsymbol{n}'(s)$, where $\bar{\boldsymbol{D}}(s)$ is the unit Darboux vector and $\boldsymbol{n}(s)$ the unit principal normal vector of $\boldsymbol{\gamma}(s)$. By definition, there exists a constant vector $\boldsymbol{a}$ such that $\boldsymbol{n}'(s) \cdot \boldsymbol{a} = 0$. It follows from the relation $\bar{\boldsymbol{D}}'(s) = -c\boldsymbol{n}'(s)$ that the Darboux vector also makes a constant angle with a fixed direction given by $\boldsymbol{a}$.

If $c = 0$, then $(\tau/\kappa)'(s) \equiv 0$. In this case, $\boldsymbol{\gamma}(s)$ is a cylindrical helix. It also follows that $\bar{\boldsymbol{D}}'(s) \equiv \boldsymbol{0}$. This means that the Darboux vector $\boldsymbol{D}(s)$ has a constant direction. Since the direction of rulings for the rectifying developable of $\boldsymbol{\gamma}(s)$ is given by the direction of the Darboux vector $\boldsymbol{D}(s)$, S is a part of a cylindrical surface. This completes the assertion (1).

On the other hand, if $\boldsymbol{\gamma}(s)$ is not a cylindrical helix, then $c \neq 0$. By Proposition 3.2, S is a part of a conical surface if $(\tau/\kappa)''(s) \equiv 0$. The direction of rulings of S

is given by the Darboux vector $\boldsymbol{D}(s)$ which makes a constant angle with a fixed direction. This means that the above conical surface is a circular conic.

For the proof of the assertion (3), we consider the locus of the singular points of the rectifying developable of $\boldsymbol{\gamma}$, which is given by

$$\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s) - \frac{1}{(\tau/\kappa)'(s)}\widetilde{\boldsymbol{D}}(s),$$

where $\widetilde{\boldsymbol{D}}(s)$ is the modified Darboux vector of $\boldsymbol{\gamma}(s)$. We can calculate that

$$\boldsymbol{\sigma}'(s) = -\frac{(\tau/\kappa)''(s)}{((\tau/\kappa)')^2(s)}\widetilde{\boldsymbol{D}}(s) \neq \boldsymbol{0},$$

then $\boldsymbol{\sigma}(s)$ is a regular space curve. It also follows from the previous arguments that the Darboux vector of $\boldsymbol{\gamma}(s)$ makes a constant angle with a fixed direction. Here, the Darboux vector gives the direction of rulings of S. Since $\sigma(s)$ is the locus of singular points of the rectifying developable of $\boldsymbol{\gamma}$, S can be consider a part of the tangent developable of $\sigma(s)$. This means that the tangent direction of $\sigma(s)$ is parallel to the direction of the Darboux vector of $\boldsymbol{\gamma}(s)$. Hence, $\boldsymbol{\sigma}(s)$ is a cylindrical helix. This completes the proof. □

The following proposition also holds as a corollary of Propositions 3.2 and 4.1.

Proposition 4.3 *Let S be a developable surface and $\boldsymbol{\gamma}(s)$ is a geodesic on S with non-vanishing curvature which is transversal to rulings. Then S is a part of a conical surface if and only if $\boldsymbol{\gamma}(s)$ is a conical geodesic curve.*

5 Examples

In this section we give an example of slant helices and draw their pictures by using `Mathematica`. We can construct many examples of slant helices because the evolute of a cylindrical helix is a slant helix. The following example is, however, the example of a slant helix which is different from examples by such a construction.

Example 5.1 We consider a space curve defined by

$$\boldsymbol{\gamma}(\theta) = \left(-\frac{a^2-b^2}{2a}\left(\frac{\cos((a+b)\theta)}{(a+b)^2} + \frac{\cos((a-b)\theta)}{(a-b)^2}\right),\right.$$
$$\left. -\frac{(a^2-b^2)}{2a}\left(\frac{\sin((a+b)\theta)}{(a+b)^2} + \frac{\sin((a-b)\theta)}{(a-b)^2}\right), -\frac{\sqrt{(a^2-b^2)}}{ab}\cos(b\theta)\right).$$

We can calculate the curvature and the torsion as follows:

$$\kappa(\theta) = \sqrt{a^2-b^2}\cos(b\theta), \quad \tau(\theta) = \sqrt{a^2-b^2}\sin(b\theta).$$

Therefore, we have

$$\sigma(\theta) = \frac{-b}{a^2 - b^2}, \ \frac{\tau}{\kappa}(\theta) = \tan(b\theta),$$

$$\left(\frac{\tau}{\kappa}(\theta)\right)' = \frac{-b}{\cos^2(b\theta)} \neq 0, \text{ and } \left(\frac{\tau}{\kappa}(\theta)\right)'' = \frac{2b^2}{\cos^2(b\theta)} \neq 0.$$

Therefore $\gamma(\theta)$ is a slant helix and it is not a cylindircal helix. By Theorem 4.2, it is a geodesic of the tangent developable of a cylindrical helix. In fact, the corresponding tangent developable is the rectifying developable of $\gamma(\theta)$ by Proposition 4.1. We now draw the picture of $\gamma(\theta)$ $(a = 2, b = 1)$ in Fig.2a. We also draw the rectifying developable of $\gamma(\theta)$ in Fig.2b.

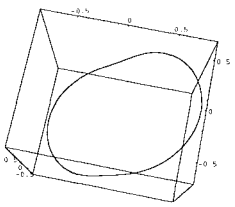


Fig.2a

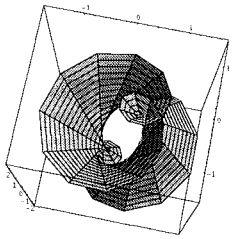


Fig.2b

By general theory of developable surface, a developable surface is the tangent developable of the singular locus. We can also draw the singular locus of the rectifying developable of $\gamma(\theta)$ in Fig.3a. Finally we draw both of $\gamma(\theta)$ and the singular locus of its rectifying developable in Fig.3b.

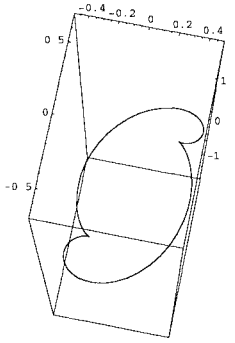


Fig.3a

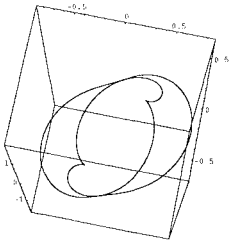


Fig.3b

Since we can easily construct a conical surface from a sheet of paper, we can draw conical geodesic curves. If we draw a line on the sheet of paper, the corresponding curve on the conical surface is a conical geodesic curve.

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