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Bifurcations of ordinary differential equations of Clairaut type

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1 Introduction

In the classical theory of first order differential equations the notion of classical complete solutions plays an important role. The Clairaut equation is one of the typical examples of first order differential equations with complete solutions. In [8], a characterization and a classification of holonomic systems of differential equations of Clairaut type have been given. The next purpose is to study bifurcations of differential equations of Clairaut type as usual in singularity theory. In this paper we study bifurcations of ordinary differential equations of Clairaut type. If we consider the bifurcations of partial differential equations of Clairaut type, the calculation is technically difficalt. Therefore, we stick to ordinary differential equations in this paper.

Let $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ be the projective cotangent bundle over $\mathbb{R}^2$. We have a local coordinate (x, y, p) of $PT^*\mathbb{R}^2$ so that (x, y) gives the canonical coordinate of $\mathbb{R}^2$ and (x, y, p) is the line in $T_{(x,y)}\mathbb{R}^2$ given by $dy - pdx = 0$ this coordinate is called *the canonical coordinate* of $PT^*\mathbb{R}^2$. *The canonical contact form* on $PT^*\mathbb{R}^2$ is given by $\theta = dy - pdx$ on the canonical coordinate.

In the classical theory, *a first order differential equations* (or, briefly, *an equation*) is written in the form,

$$F(x, y, p) = 0, \ p = dy/dx.$$

It is naturally interpreted as being a subset of $PT^*\mathbb{R}^2$ (more precisely, $F^{-1}(0) \subset PT^*\mathbb{R}^2$). In this paper, we only consider the case when $F^{-1}(0)$ is a smooth submanifold like as usual (cf.[9]) Since we only consider local propaties, an equation is parametrized by a germ of an immersion $f : (\mathbb{R}^2, 0) \to PT^*\mathbb{R}^2$ (cf.[5]). We say that f is *completely integrable* if there exists a germ of a submersion $\mu : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that $d\mu \wedge f^*\theta = 0$. We call μ a *complete* *integral* of f and the pair $(\mu, f) : (\mathbb{R}^2, 0) \to \mathbb{R} \times PT^*\mathbb{R}^2$ is called *an equation with complete integral.* If $\pi \circ f|_{\mu^{-1}(s)}$ are non-singular maps for each $s \in (\mathbb{R}, 0)$, then f is called *a Clairaut type equation.*

One-parameter family of first order ordinary differential equation germ (or, briefly, *one-parameter family of equations*) is defined to be a germ of a map $f : (\mathbb{R}^2 \times \mathbb{R}, 0) \to PT^*\mathbb{R}^2$ such that f_t is an immersion for each $t \in (\mathbb{R}, 0)$ where $f_t(u_1, u_2) = f(u_1, u_2, t)$. We also say that f is *one-parameter completely integrable* if there exist a germ of a map $\mu : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R}$ such that μ_t is submersion, where $\mu_t(u_1, u_2) = \mu(u_1, u_2, t)$ and $d\mu_t \wedge f_t^*\theta = 0$. The pair $(\mu, f) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R} \times PT^*\mathbb{R}^2$ is called *a one-parameter family of ordinary differential equation with complete integral* (or, briefly, *one-parameter family of equations with complete integral*). If $\pi \circ f_t|_{\mu_t^{-1}(s)}$ are non-singular maps for each $t, s \in (\mathbb{R}, 0)$ then f is called *a one-parameter family of Clairaut type equations.* Let (μ, g) be a pair of germs of maps $g : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ and $\mu : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that μ_t is submersion for each $t \in (\mathbb{R}, 0)$. Then the diagram

$$(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^2 \times \mathbb{R}, 0) \xrightarrow{g} (\mathbb{R}^2, 0)$$

or briefly (μ, g), is called *a one-parameter family of integral diagrams* if there exists a one-parameter family of equations $f : (\mathbb{R}^2 \times \mathbb{R}, 0) \to PT^*\mathbb{R}$ such that (μ, f) is a one-parameter family of equations with complete integral and $\pi \circ f = g$.

Define $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to PT^*\mathbb{R} \times \mathbb{R}$ by $F(u_1, u_2, t) = (f(u_1, u_2, t), t)$ for a one-parameter family of equations f which is called *a one-parameter unfolding of equation associated to f.* We also define $\hat{\mu} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ by $\hat{\mu}(u_1, u_2, t) = (\mu(u_1, u_2, t), t)$. The pair $(\hat{\mu}, F)$ or (μ, F) is called *a one-parameter unfolding of equation with complete integral* and let $(\hat{\mu}, G)$ be the pair of a germ of a map $G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ and a germ of a map $\hat{\mu} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ such that μ_t is a submersion for each $t \in (\mathbb{R}, 0)$. Then the diagram

$$(\mathbb{R} \times \mathbb{R}, 0) \xleftarrow{\hat{\mu}} (\mathbb{R}^2 \times \mathbb{R}, 0) \xrightarrow{G} (\mathbb{R}^2 \times \mathbb{R}, 0)$$

or briefly $(\hat{\mu}, G)$, is called *a one-parameter unfolding of integral diagram* if there exists a one-parameter family of equations f such that $(\hat{\mu}, F)$ is a one-parameter unfolding of equation with complete integral and $(\pi \times id) \circ F = G$ where F is one-parameter unfolding of equations associated to f. If (μ, f) is a one-parameter family of Clairaut type equations, then $(\hat{\mu}, F)$ or (μ, F) is called *a one-parameter unfolding of Clairaut type equation.* Furthermore we introduce an equivalence relation among one-parameter unfoldings of integral diagrams. Let $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ be one-parameter unfoldings of integral diagrams. Then $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ are *equivalent as one-parameter unfolding of* *integral diagram* (respectively, *strictly equivalent*) if the diagram

$$\begin{tikzcd}
(\mathbb{R}\times\mathbb{R},0) \arrow[d, "\kappa"'] & (\mathbb{R}^2\times\mathbb{R},0) \arrow[l, "\hat{\mu}"'] \arrow[r, "G"] \arrow[d, "\psi"'] & (\mathbb{R}^2\times\mathbb{R},0) \arrow[d, "\phi"'] \\
(\mathbb{R}\times\mathbb{R},0) & (\mathbb{R}^2\times\mathbb{R},0) \arrow[l, "\hat{\mu'}"] \arrow[r, "G'"'] & (\mathbb{R}^2\times\mathbb{R},0)
\end{tikzcd}$$

commutes for some germs of diffeomorphism κ, ψ and ϕ of the form

$$\kappa(s,t) = (\kappa_1(s,t), \varphi(t)) \ (respectively, \kappa_1(s,t) = s),$$

$$\psi(u_1,u_2,t) = (\psi_1(u_1,u_2,t), \varphi(t)) \text{ and } \phi(x_1,x_2,t) = (\phi_1(x_1,x_2,t), \varphi(t)).$$

If $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ are strictly equivalent, we say that (μ, G) and (μ', G) are strictly equivalent.

Let F,F' be one-parameter unfoldings of equation associated to f,f' respectively. Then F and F' are *equivalent* if the diagram

$$\begin{tikzcd}
(\mathbb{R}^2\times\mathbb{R},0) \arrow[r, "F"] \arrow[d, "\psi"'] & (PT^*\mathbb{R}^2\times\mathbb{R},(z,0)) \arrow[d, "\Phi"] \\
(\mathbb{R}^2\times\mathbb{R},0) \arrow[r, "F'"'] & (PT^*\mathbb{R}^2\times\mathbb{R},(z',0))
\end{tikzcd}$$

commutes for some germs of diffeomorphism ψ and Φ of the form $\psi(u_1,u_2,t) = (\psi_1(u_1,u_2,t), \varphi(t))$ and $\Phi(x_1,x_2,x_3,t) = (\hat{\phi}_t(x_1,x_2,x_3), \varphi(t))$ where $\hat{\phi}_t$ is the unique contact lift of ϕ_t where $\phi_t = \phi|_{\mathbb{R}^2\times\{t\}} : (\mathbb{R}^2\times\{t\},0) \to (\mathbb{R}^2\times\{\varphi(t)\},0)$ and $\phi : (\mathbb{R}^2\times\mathbb{R},0) \to (\mathbb{R}^2\times\mathbb{R},0)$ is a germ of a diffeomorphism. We shall show that two one-parameter unfoldings equation F and F' are equivalent if and only if induced one-parameter unfoldings of integral diagram $(\hat{\mu}, (\pi\times id)\circ F)$ and $(\hat{\mu'}, (\pi\times id)\circ F')$ are equivalent for generic (μ, F) and (μ', F') (c.f, Theorem 3.3). The main result in this paper is the following theorem which gives a generic classification of one-parameter families of Clairaut type equations:

Theorem 1.1 *For a generic one-parameter family of Clairaut type equations*

$$(\mu, f) : (\mathbb{R}^2\times\mathbb{R},0) \to \mathbb{R}\times PT^*\mathbb{R}^2,$$

let F be the one-parameter unfolding of equation associated to f. Then the one-parameter unfolding of integral diagram $(\mu, (\pi\times id)\circ F)$ is strictly equivalent to one of germs in the following list:

(1) $\mu = u_2,$
$G = (u_1, u_2, t).$

(2) $\mu = u_2 - \frac{1}{2}u_1,$
$G = (u_1, {u_2}^2, t).$

(3) $\mu = u_2 + \alpha \circ G \quad for\ \alpha \in \mathfrak{M}_{(x,y,t)},$
$G = (u_1, u_2^3 + u_1u_2 + tu_2^2, t).$

(4) $\mu = u_2 + \alpha \circ G \quad for\ \alpha \in \mathfrak{M}_{(x,y,t)},$
$G = (u_1, u_2^4 + u_1u_2 + \beta(u_1, t)u_2^2, t) \quad for\ \beta \in \mathfrak{M}_{(u_1,t)}, \frac{\partial \beta}{\partial t}(0) \neq 0.$

We call the germ of function α which appears in the normal form (3) and (4), *a one-parameter functional moduli* and the germ of function β which appears in the normal form (4), *a second functional moduli*. In the final remark (cf. §5), we will describe why the second functional moduli appears in the normal form (4). The meaning of the genericity in the above theorem will be described in §2 (cf. Theorem 2.2). We now give typical examples of one-parameter families of Clairaut type equations as follows :

Example 1.2 (One-parameter families of Clairaut equations) We consider a one-parameter family of equations:

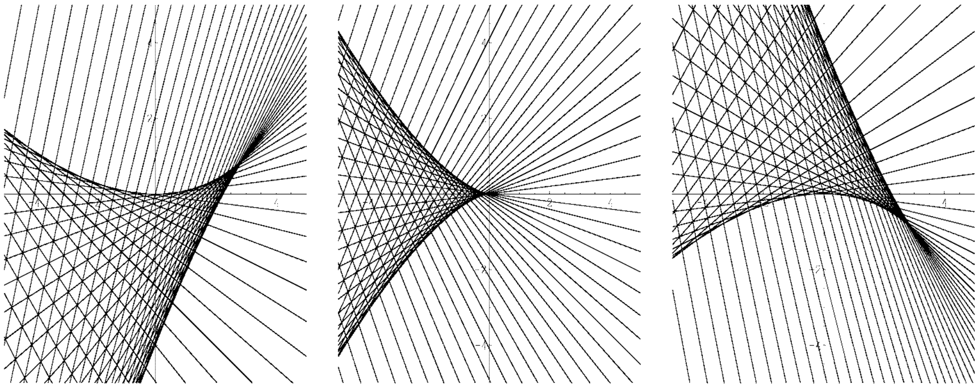
$$y = \frac{dy}{dx}x + g_t(\frac{dy}{dx}),$$

where $g(t,p)$ is a C^∞-function. In this case, we consider that germs of map $f : (\mathbb{R}^2 \times \mathbb{R}, 0) \to J^1(\mathbb{R}, \mathbb{R})$ given by

$$f(u_1, u_2, t) = (u_1, u_1u_2 + g_t(u_2), u_2)$$

and $\mu : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R}$ given by $\mu(u_1, u_2, t) = u_2$. Then we can easily show that (μ, f) is a one-parameter family of Clairaut type equations. In this case, the complete solusion is a family of straight lines.

In the case when $g(t,p) = p^3 + tp^2$, the one-parameter family of integral diagram corresponds to the normal form (3) for the case $\alpha = 0$ in Theorem 1.1. We can draw the picture of the bifurcation of the phase portrait $\{\pi \circ f_t({\mu_t}^{-1}(c))\}_{c \in \mathbb{R}}$ see Figure 1 :

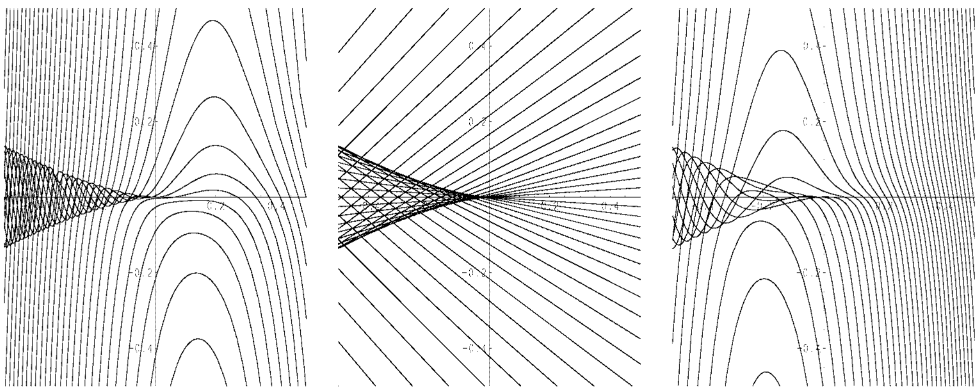


t<0 t=0 t>0

Figure 1

We can show that the above three cases ($t > 0, t = 0, t < 0$) are equivalent. Therefore the bifurcation does not occur in Fig 1.

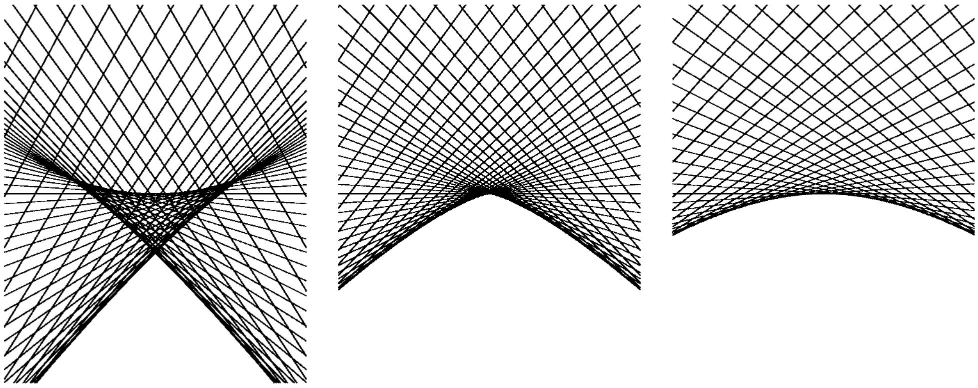
But, for example, we take $\alpha(x, y, t) = -tx - t^2x^2$ in the normal form (3) in Theorem1.1, we exactly draw the picture of the bifurcation of the phase portrait (see Figure 2).



t<0 t=0 t>0

Figure 2

If we consider the case $g(t, p) = p^4 + tp^2$, then the one-parameter family of integral diagram corresponds to the normal form (4) in the case $\alpha = 0, \beta = t$ in Theorem 1.1. We can also draw the picture of the bifurcation of the phase portrait $\{\pi \circ f_t({\mu_t}^{-1}(c))\}_{c\in\mathbb{R}}$ see Figure 3:



t<0 t=0 t>0

Figure 3

We can observe that bifurcations of Web structures appear in our situation.

As a next problem we should characterize one-parameter function moduli and second moduli relative to the equivalence. Also we would like to classify the bifurcation of first order partial differential equations with complete integral in general. However it seems that the calculations for these ploblems are so difficult that these are future problems for us.

In §3 and §4 we prepare some basic tools and describe the meaning of the genericity of properties for one-parameter unfolding of Clairaut type equations. In §6 Appendex, we introduce the unfolding theory of function germs which will be useful for the proof of Theorem 1.1. We give a proof of Theorem 1.1 in §5.

All germs of maps considered here are of class C^∞, unless stated otherwise.

2 Genericity

Our aim is to construct a family of Legendrian immersions depending on (μ, f). We therefore consider the projective cotangent bundle $PT^*(\mathbb{R}\times\mathbb{R}^2)$. We have a local coordinate $(\sigma, x, y, \rho, \xi)$ of $PT^*(\mathbb{R}\times\mathbb{R}^2)$, such that (σ, x, y) gives the canonical coordinate of $\mathbb{R}^3$ and $(\sigma, x, y, \rho, \xi)$ is the plane in $T_{(\sigma,x,y)}\mathbb{R}^3$ given by $dy - \xi dx - \rho d\sigma \ = 0$. This coordinate is called *the canonical coordinate* of $PT^*(\mathbb{R}\times\mathbb{R}^2)$ Then the canonical contact form is given by $\Theta = dy - \xi dx - \rho d\sigma$. Let (μ, f) be a one-parameter family of equations with complete integral. Then there exists a unique element $h_t \in \mathcal{E}_u$ such that $h_t d\mu_t = {f_t}^*\theta$, where $\mathcal{E}_u$ is the ring of function germs of u-variables. Define a map germ

$$\ell_{(\mu,f)} : (\mathbb{R}^2\times\mathbb{R}, 0) \to PT^*(\mathbb{R}\times\mathbb{R}^2).$$

by

$$\ell_{(\mu,f)}(u,t) = (\mu(u,t), x \circ f(u,t), y \circ f(u,t), h(u,t), p \circ f(u,t)),$$

where $h(u,t) = h_t(u)$. Then we have $\ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}{}^*\Theta = 0$.

We say that the pair $(\mu, \ell) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R} \times PT^*(\mathbb{R} \times \mathbb{R}^2)$ is *a Legendrian family* if $\ell_s = \ell|_{\mu^{-1}(s)}$ is a Legendrian immersion germ for each $s \in (\mathbb{R}, 0)$ (cf.[6]). Then we have the following simple but important lemma:

Lemma 2.1 [6] *Let (μ, ℓ) be a Legendrian family. Then there exists a unique element $k \in \mathcal{E}_{(u,t)}$ such that $\ell^*\Theta = kd\mu$.*

We apply Lemma 2.1 to $(\pi, \ell_{(\mu,f)})$, where $\pi : (\mathbb{R}^2 \times \mathbb{R}) \to (\mathbb{R}, 0); \pi(u,t) = t$, then there exists a unique element $k \in \mathcal{E}_{(u,t)}$ such that $\ell_{(\mu,f)}{}^*\Theta = kdt$. We also consider the projective cotangent bundle $PT^*(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$. Let $(\sigma, x, t, y, \rho, \xi, \tau)$ be the canonical coordinate on $PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$. Here the canonical 1-form is given $\widetilde{\Theta} = dy - \xi dx - \rho d\sigma - \tau dt = \Theta - \tau dt$. We define a map germ

$$\ell_{(\mu,F)} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$$

by

$$\ell_{(\mu,F)}(u,t) = (\mu(u,t), x \circ F(u,t), t, y \circ F(u,t), h(u,t), p \circ F(u,t), k(u,t)).$$

Then we can easily show that $\ell_{(\mu,F)}$ is Legendrian immersion germ. (*i.e.*, $\ell_{(\mu,F)}$ is an immersion germ with $\ell_{(\mu,F)}{}^*\widetilde{\Theta} = 0$). We call $\ell_{(\mu,F)}$ *a one-parameter complete Legendrian unfolding associated to* (μ, F). If F is one-parameter unfolding of equation associated to f then we also call $\ell_{(\mu,F)}$ *a one-parameter complete Legendrian unfolding associated to* (μ, f).

We now establish the notion of *the genericity*. Let $U \times V \subset \mathbb{R}^2 \times \mathbb{R}$ be an open set. We denote by $\mathrm{Int}(U\times V, \mathbb{R}\times PT^*\mathbb{R}^2\times\mathbb{R})$ the set of one-parameter unfolding of equations with complete integral $(\mu, F) : U \times V \to \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}$. We also define $L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$ to be the set of one-parameter complete Legendrian unfolding $\ell_{(\mu,F)} : U \times V \to PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$.

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows. Let P be a property of one-parameter unfolding of equation with complete integral (μ, F) : $U \times V \to \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}$ (respectively, one-parameter complete Legendrian unfolding $\ell_{(\mu,F)} : U\times V \to PT^*(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R})$). For an open set $U\times V \subset \mathbb{R}^2 \times \mathbb{R}$, we define $\mathcal{P}(U \times V)$ to be the set of $(\mu, F) \in \mathrm{Int}(U \times V, \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R})$ (respectively, $\ell_{(\mu,F)} \in L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}))$) such that the germ at (u,t)

whose representative is given by (μ, F) (respectively, $\ell_{(\mu,F)}$) has property P for any $(u,t) \in U \times V$.

The property P is said to be *generic* if for some neighbourhood $U \times V$ of 0 in $\mathbb{R}^2 \times \mathbb{R}$, the set $\mathcal{P}(U \times V)$ is a generic subset in $\mathrm{Int}(U \times V, \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R})$ (respectively, $L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})) \to \mathrm{Int}(U \times V, \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R})$$

defined by $(\Pi_1)_*(\ell_{(\mu,F)}) = \Pi_1 \circ \ell_{(\mu,F)} = (\mu, F)$, where $\Pi_1 : PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}) \to \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}$ is canonical projection. Then we have the following fundamental theorem.

Theorem 2.2 *The continuous map*

$$(\Pi_1)_* : L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})) \to \mathrm{Int}(U \times V, \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R})$$

is a homeomorphism.

The proof follows from a direct analogy of the proof for Theorem 4.4 in [7], so that we omit it.

This theorem asserts that the genericity of a property of one-parameter unfolding of equation with complete integral can be interpreted by the genericity of the corresponding property of one-parameter complete Legendrian unfolding.

3 One-parameter unfoldings of equations

In this section we prepare some results on one-parameter unfoldings of equation with complete integral which will be used in the later sections. We also use the notion of Legendrian immersions here. For the definition and basic properties of Legendrian immersions, see [1,11]. We can easily show the following lemma:

Lemma 3.1 *Let* $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*\mathbb{R}^2 \times \mathbb{R}, (z, 0))$ *and* $F' : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*\mathbb{R}^2 \times \mathbb{R}, (z', 0))$ *be germs of one-parameter unfoldings of equations. Then* F *and* F' *are equivalent if and only if the diagram*

$$\begin{tikzcd}
(\mathbb{R}^2, 0) \arrow[r, "f_t"] \arrow[d, "\psi_t"'] & (PT^*\mathbb{R}^2, z_t) \arrow[r, "\pi"] \arrow[d, "\hat{\phi}_t"'] & (\mathbb{R}^2, \pi(z_t)) \arrow[d, "\phi_t"'] \\
(\mathbb{R}^2, 0) \arrow[r, "f'_{\varphi(t)}"'] & (PT^*\mathbb{R}^2, z'_{\varphi(t)}) \arrow[r, "\pi"'] & (\mathbb{R}^2, \pi(z'_{\varphi(t)}))
\end{tikzcd}$$

commutes for some germs of diffeomorphisms φ, ψ_t, ϕ_t and the unique contact lift $\hat{\phi}_t$ of ϕ_t at each $t \in (\mathbb{R}, 0)$.

We also prove the following lemma which is a one-parameter version of [7, Lemma 2.1].

Lemma 3.2 *Let $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*\mathbb{R}^2 \times \mathbb{R}, (z, 0))$ and $F' : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*\mathbb{R}^2 \times \mathbb{R}, (z', 0))$ be germs of one-parameter unfoldings of equations such that the sets of singular points of $\pi \circ f_t$ and $\pi \circ f'_t$ are closed sets without interior points except for isolated t. Then F and F' are equivalent if and only if there exist a germ of a diffeomorphism $\psi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\psi(u_1, u_2, t) = (\psi_1(u_1, u_2, t), \varphi(t))$ and a germ of a diffeomorphism $\phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\phi(x_1, x_2, t) = (\psi_1(x_1, x_2, t), \varphi(t))$ such that $\psi_t^* \omega'_{\varphi(t)} \wedge \omega_t = 0$ for each t and the following diagram commutes:*

$$\begin{array}{ccc}
(\mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow{(\pi \times id) \circ F} & (\mathbb{R}^2 \times \mathbb{R}, 0) \\
\psi \downarrow & & \downarrow \phi \\
(\mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow[(\pi \times id) \circ F']{} & (\mathbb{R}^2 \times \mathbb{R}, 0),
\end{array}$$

where $\omega_t = {f_t}^* \theta$ and $\omega'_t = {f'_t}^* \theta$.

Proof. Since $\hat{\phi}_t$ is a contact diffeomorphism, the necessity of this condition is clear.

We now prove the sufficiency of this condition. By the assumption, we have $(f'_{\varphi(t)} \circ \psi_t)^* \theta \wedge f_t^* \theta = 0$. Since $\hat{\phi}_t$ is a contact diffeomorphism, we have $(\hat{\phi}_t \circ f_t)^* \theta \wedge {f_t}^* \theta = 0$. It follows from the above conditions that $(\hat{\phi}_t \circ f_t)^* \theta \wedge (f'_{\varphi(t)} \circ \psi_t)^* \theta = 0$. The assumption $\phi_t \circ \pi \circ f_t = \pi \circ f'_{\varphi(t)} \circ \psi_t$ guarantees that coordinate representations are given by

$$\hat{\phi}_t \circ f_t(u) = (X_t(u), Y_t(u), P_t(u))$$

and

$$f'_{\varphi(t)} \circ \psi_t(u) = (X_t(u), Y_t(u), P'_t(u)),$$

so that we can translate the last condition into

$$(P_t(u) - P'_t(u)) dX_t(u) \wedge dY_t(u) = 0.$$

Since the sets of singular points of $\pi \circ f_t$ and $\pi \circ f'_t$ are closed sets without interior points except for isolated t, $dX_t(u) \wedge dY_t(u)$ is linearly independent on open dense subset of neighbourhood of the origin in $\mathbb{R}^2$, then we have $P_t(u) = P'_t(u)$ on such a subset. Thus we assert that $\hat{\phi}_t \circ f_t(u) = f'_{\varphi(t)} \circ \psi_t(u)$

for almost all t. By the continuity, this equality holds for all t. By Lemma 3.1, this completes the proof of Lemma 3.2. $\square$

We can assert the following theorem which reduces the equivalence problem for one-parameter unfoldings of equations with complete integral to that for the corresponding induced one-parameter unfolding of integral diagrams.

Theorem 3.3 *Let* $(\mu, F) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}, (0, z, 0))$ *and* $(\mu', F') : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}, (0, z', 0))$ *be one-parameter unfoldings of equations with complete integral such that the sets of singular points of* $\pi \circ f_t$ *and* $\pi \circ f'_t$ *are closed sets without interior points except for isolate* t*. Then the following are equivalent:*

(1) F *and* F' *are equivalent*

(2) $(\hat{\mu}, (\pi \times id) \circ F)$ *and* $(\hat{\mu'}, (\pi \times id) \circ F')$ *are equivalent as one-parameter unfoldings of integral diagram.*

Proof. We prove that (2) implies (1). If $(\hat{\mu}, (\pi \times id) \circ F)$ and $(\hat{\mu'}, (\pi \times id) \circ F')$ are equivalent as one-parameter unfoldings of integral diagram, then there exist germs of diffeomorphism $\kappa : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$, $\psi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ such that $\hat{\mu'} \circ \psi = \kappa \circ \hat{\mu}$ and $\phi \circ (\pi \times id) \circ F = (\pi \times id) \circ F' \circ \psi$. Therefore we have $\mu'_{\varphi(t)} \circ \psi_t = \kappa_t \circ \mu_t$ and $\phi_t \circ \pi \circ f_t = \pi \circ f'_{\varphi(t)} \circ \psi_t$ for each $t \in (\mathbb{R}, 0)$.

Since $d\mu_t$ and $d\mu'_t$ are non-zero at 0, the existence of such a diffeomorphism κ is equivalent to the condition ${\psi_t}^* d\mu'_{\varphi(t)} \wedge d\mu_t = 0$.

On the other hand, (μ, F) and (μ', F') are one-parameter unfoldings of equations with complete integral, so we have $d\mu_t \wedge \omega_t = d\mu'_t \wedge \omega'_t = 0$, where $\omega_t = {f_t}^* \theta$ and $\omega'_t = f'^*_t \theta$ for each $t \in (\mathbb{R}, 0)$. Moreover ω_t and ω'_t do not vanish outside the set of critical points of $\pi \circ f_t$ and $\pi \circ f'_t$, respectively. These facts assert that $\psi^*_t \omega'_{\varphi(t)} \wedge \omega_t = 0$ and $\phi \circ (\pi \times id) \circ F = (\pi \times id) \circ F' \circ \psi$. By Lemma 3.2, F and F' are equivalent as one-parameter unfolding of equation.

We also prove that (1) implies (2). By Lemma 3.2, if F and F' are equivalent, then there exist diffeomorphism germs $\psi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ such that ${\psi_t}^* \omega'_{\varphi(t)} \wedge \omega_t = 0$ and $\phi \circ (\pi \times id) \circ F = (\pi \times id) \circ F' \circ \psi$.

By the same reason as in the above argument, it is enough to prove that ${\psi_t}^* d\mu'_{\varphi(t)} \wedge d\mu_t = 0$. However, such a formula is easy to prove. $\square$

We remark that in assumption of Lemma 3.2 and Theorem 3.3, the condition singular points of $\pi \circ f_t$ and $\pi \circ f'_t$ are closed sets without interior points except for isolated t is satisfied for generic equations.

4 Equivalence of unfoldings and generating functions

The main idea of the proof for Theorem 1.1 is to define an equivalence relation which can ignore one-parameter functional modulus and to do everything in terms of generating families for Legendrian unfoldings analogous to those of in [8]

Proposition 4.1 *Let $(\mu, F) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R} \times PT^*\mathbb{R}^2 \times \mathbb{R}$ be a one-parameter unfolding of equation with complete integral. Then (μ, F) is a one-parameter unfolding of Clairaut type equation if and only if $\ell_{(\mu,F)}$ is Legendrian non-singular.*

Proof. By definition, (μ, F) is a one-parameter unfolding of Clairaut type equation then (μ, f) is a one-parameter family of Clairaut type equation (see. Sect.1). We now have a decomposition of the tangent space as follows:

$$T_{(u_0,t)}(\mu_t^{-1}(s) \times \{t\}) \oplus V_t = T_{(u_0,t)}(\mathbb{R}^2 \times \{t\}).$$

Then

$$d(\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}) : T_{(u_0,t)}(\mu_t^{-1}(s) \times \{t\}) \to T_{(s,\alpha)}(\{s\} \times \mathbb{R}^2)$$

is an injective and

$$d(\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}) : V_t \to T_s\mathbb{R}$$

is an isomorphism. Thus $\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$ is an immersion at $u_0 \in (\mathbb{R}^2, 0)$ for each $t \in (\mathbb{R}, 0)$. We conclude that $\ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$ is Legendrian non-singular for each $t \in (\mathbb{R}, 0)$. Then it is easy to show that $\ell_{(\mu,F)}$ is Legendrian non-singular.

By definition $\ell_{(\mu,F)}$ is Legendrian non-singular then $\ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$ is Legendrian non-singular for each $t \in (\mathbb{R}, 0)$. This concludes that $\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$ is an immersion, we have

$$\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}(u) = (\mu_t(u), x \circ f_t(u), y \circ f_t(u)).$$

Let $P_1 : \mathbb{R}\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{R}$ be the canonical projection onto the first component. By definition $P_1 \circ \Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}} = \mu_t$ is a submersion.

Hence, if $\ell_{(\mu,f)}|\mathbb{R}^2 \times \{t\}$ is Legendrian non-singular, then

$$P_1| \text{ Image}\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$$

is a submersion. It follows that

$$P_1^{-1}(s) \cap \mathrm{Image}\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}} = \{s\} \times \mathbb{R} \times \mathbb{R} \cap \mathrm{Image}\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}$$

is a 1-dimensional submanifold of $\{s\} \times \mathbb{R} \times \mathbb{R}$.

We also define a mapping $P_2 : \mathbb{R}\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{R}\times\mathbb{R}$ as the canonical projection. Then

$$P_2(\{s\} \times \mathbb{R} \times \mathbb{R} \cap \mathrm{Image}\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}})$$

is a 1-dimensional submanifold of $\mathbb{R} \times \mathbb{R}$. It is easy to show that

$$P_2(\{s\} \times \mathbb{R} \times \mathbb{R} \cap \mathrm{Image}\Pi \circ \ell_{(\mu,f)}|_{\mathbb{R}^2\times\{t\}}) = \Pi \circ f_t(\mu_t^{-1}(s)).$$

This shows that $f_t|_{\mu_t^{-1}(s)}$ is Legendrian non-singular for each $t, s \in (\mathbb{R}, 0)$. Then (μ, f) is a one-parameter family of Clairaut type equation. $\square$

Since $\ell_{(\mu,F)}$ is a germ of a Legendrian immersion, there exists a generating family of $\ell_{(\mu,F)}$ by the theory of Legendrian singularities ([1,11]). Let $G : ((\mathbb{R}\times\mathbb{R}\times\mathbb{R})\times\mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ be a germ of a function such that $d_2G|_{0\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^k}$ is non-singular,where

$$d_2G(s, x, t, q) = \Big(\frac{\partial G}{\partial q_1}(s, x, t, q), ..., \frac{\partial G}{\partial q_k}(s, x, t, q)\Big).$$

Then $C(G) = d_2G^{-1}(0)$ is a germ of a smooth 3-manifold and $\pi_G : (C(G), 0) \to \mathbb{R}$ is a germ of a submersion, where $\pi_G(s, x, t, q) = s$. Define germs of maps

$$\widetilde{\mathcal{L}_G} : (C(G), 0) \to J^1(\mathbb{R}, \mathbb{R})$$

by

$$\widetilde{\mathcal{L}_G}(s, x, t, q) = \Big(x, G(s, x, t, q), \frac{\partial G}{\partial x}(s, x, t, q)\Big),$$

and

$$\mathcal{L}_G : (C(G), 0) \to J^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$$

by

$$\mathcal{L}_G(s, x, t, q) = \Big(s, x, t, G(s, x, t, q), \frac{\partial G}{\partial s}(s, x, t, q), \frac{\partial G}{\partial x}(s, x, t, q), \frac{\partial G}{\partial t}(s, x, t, q)\Big).$$

Since $\partial G/\partial q_i = 0$ on $C(G)$, we can easily show that

$$(\widetilde{\mathcal{L}_{Gt}}|_{\pi_G{}^{-1}(s)})^*\theta = 0.$$

By definition, $\mathcal{L}_G$ is a one-parameter complete Legendrian unfolding associated to $(\pi_G, \widetilde{\mathcal{L}_G})$. By the same method of the theory of [1,11], we can also show the following proposition.

Proposition 4.2 *All one-parameter Legendrian unfolding germs are constructed by the above method.*

We say that G is *a generalised phase family* of the complete Legendrian unfolding $\mathcal{L}_G$.

Let (μ, F) be a one-parameter unfolding of Clairaut type equation. By Proposition 4.1, $\ell_{(\mu,F)}$ is Legendrian non-singular. Then we can choose a family of germs of functions

$$G : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$$

such that Image $j^1 G_t = \ell_{(\mu,f)}|_{\mathbb{R}^2 \times \{t\}}$, Image $j^1 G_{(s,t)} = f_t({\mu_t}^{-1}(s))$ for any s,t $\in (\mathbb{R}, 0)$ and

$$j_1^1 G_t : (\mathbb{R} \times \mathbb{R}, 0) \to J^1(\mathbb{R}, \mathbb{R})$$

is a germ of an immersion, where $G_t(s,x) = G(s,x,t)$, $G_{(s,t)}(x) = G(s,x,t)$ and $j_1^1 G_t(s,x) = j^1 G_{(s,t)}(x)$. The fact that $j_1^1 G_t$ is an immersion leads us to the following equality:

$$\operatorname{rank}\left(\frac{\partial G}{\partial s} \frac{\partial^2 G}{\partial s \partial x}\right) = 1.$$

In this case we have $(C(G), 0) = (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0)$ and

$$\mathcal{L}_G = j^1 G : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to J^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),$$

so that it is a one-parameter complete Legendrian unfolding associated to $(\pi_G, j_1^1 G)$. Thus the generalised phase family of a one-parameter complete Legendrian unfolding of Clairaut type equation is given by the above germ. We define $\tilde{G} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ by $\tilde{G}(s,x,t,y) = G(s,x,t) - y$. We call $\tilde{G}$ *a generating family* of a one-parameter complete Legengrian unfolding of Clairaut type.

We now consider an equivalence relation among one-parameter unfoldings of integral diagrams which ignore one-parameter functional moduli. Let $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ be one-parameter unfolding of integral diagrams. Then $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ are *one-parameter* $\mathcal{R}^+$*-equivalent* if there exist a germ of a diffeomorphism $\Psi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\Psi(s,x,t) = (s + \alpha(x,t), \psi(x,t), \varphi(t))$ and a germ of a diffeomorphism $\Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\Phi(u,t) = (\phi(u,t), \varphi(t))$ such that $\Psi \circ (\mu, G) = (\mu', G') \circ \Phi$. We remark that if $(\hat{\mu}, G)$ and $(\hat{\mu'}, G')$ are one-parameter $\mathcal{R}^+$- equivalent by the above diffeomorphisms, then we have $\mu(u,t) + \alpha \circ G(u,t) = \mu' \circ \Phi(u,t)$ and $(\psi, \varphi) \circ G(u,t) = G' \circ \Phi(u,t)$ for any $(u,t) \in (\mathbb{R}^2 \times \mathbb{R}, 0)$. Thus the one-parameter unfolding of integral diagram $(\mu + \alpha \circ G, G)$ is strictly equivalent to (μ', G').

We now define the corresponding equivalence relation among one-parameter Legendrian unfoldings. Let $\ell_{(\mu,F)} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z)$ and $\ell_{(\mu',F')} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z')$ be one-parameter complete Legendrian unfoldings. We say that $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are *one-parameter SP^+-Legendrian equivalent* (respectively, *one-parameter SP-Legendrian equivalent*) if there exist a germ of a contact diffeomorphism $K : (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z) \to (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z')$, a germ of a diffeomorphism $\Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\Phi(u, t) = (\phi(u, t), \varphi(t))$ and a germ of a diffeomorphism $\Psi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \Pi(z)) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \Pi(z'))$ of the form $\Psi(s, x, t) = (s+\alpha(x, t), \psi(x, t), \varphi(t))$ (respectively, $\Psi(s, x, t) = (s, \psi(x, t), \varphi(t))$), such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \ell_{(\mu,F)} = \ell_{(\mu',F')} \circ \Phi$. where $\Pi : (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \Pi(z))$ is projection. It is clear that if $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are one-parameter SP^+-Legendrian equivalent (respectively, one-parameter SP-Legendrian equivalent), then $(\mu, (\pi \times id) \circ F)$ and $(\mu', (\pi \times id) \circ F')$ are one-parameter R^+-equivalent (respectively, strictly equivalent). By [11, Theorem1.1], the converse is also true for generic $(\mu, (\pi \times id) \circ F)$ and $(\mu', (\pi \times id) \circ F')$.

We also say that $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are *one-parameter P-Legendrian equivalent* if there exist a germ of a contact diffeomorphism $K : (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z) \to (PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}), z')$, a germ of a diffeomorphism $\Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $\Phi(u, t) = (\phi(u, t), \varphi(t))$ and a germ of a diffeomorphism $\Psi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \Pi(z)) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, \Pi(z'))$ of the form $\Psi(s, x, t) = (q(s, x, t), \psi(x, t), \varphi(t))$, such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \ell_{(\mu,F)} = \ell_{(\mu',F')} \circ \Phi$.

The notion of *stability* of one-parameter complete Legendrian unfoldings with respect to one-parameter SP^+-Legendrian equivalence (respectively, SP-Legendrian equivalence and P-Legendrian equivalence) is analogous to the usual notion of the stability of germs of Legendrian immersions with respect to the Legendrian equivalence. (cf,[1, Part *III*]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. Let $\tilde{G}, \tilde{G}' : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be generating families of one-parameter Legendrian unfolding of Clairaut type, where $\tilde{G}(s, x, t, y) = G(s, x, t) - y$, $\tilde{G}'(s, x, t, y) = G'(s, x, t) - y$. We say that $\tilde{G}$ and $\tilde{G}'$ are *one-parameter P-$\mathcal{C}^+$-equivalent* (respectively, *one-parameter P-$\mathcal{C}$-equivalent*) if there exists a germ of a diffeomorphism $\Phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0)$ of the form

$$\Phi(s, x, t, y) = (s + \alpha(x, t, y), \phi_1(x, t, y), \varphi(t), \phi_2(x, t, y))$$

(respectively,

$$\Phi(s,x,t,y) = (s, \phi_1(x,t,y), \varphi(t), \phi_2(x,t,y)))$$

such that $\langle \tilde{G} \circ \Phi \rangle_{\mathcal{E}_{(s,x,t,y)}} = \langle \tilde{G'} \rangle_{\mathcal{E}_{(s,x,t,y)}}$, where $\langle \tilde{G'} \rangle_{\mathcal{E}_{(s,x,t,y)}}$ is the ideal generated by $\tilde{G'}$ in $\mathcal{E}_{(s,x,t,y)}$. We say that $\tilde{G}(s,x,t,y)$ is P-$\mathcal{C}^+$ (respectively, P-$\mathcal{C}$)*-versal deformation* of $g = \tilde{G}|_{t=0}$ if

$$\mathcal{E}_{(s,x,y)} = \langle \frac{\partial g}{\partial s} \rangle_{\mathcal{E}_{(x,y)}} + \langle g \rangle_{\mathcal{E}_{(s,x,y)}} + \langle \frac{\partial g}{\partial x}, 1 \rangle_{\mathcal{E}_{(x,y)}} + \langle \frac{\partial G}{\partial t}|_{t=0} \rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_{(s,x,y)} = \langle g \rangle_{\mathcal{E}_{(s,x,y)}} + \langle \frac{\partial g}{\partial x}, 1 \rangle_{\mathcal{E}_{(x,y)}} + \langle \frac{\partial G}{\partial t}|_{t=0} \rangle_{\mathbb{R}}).$$

Let $\tilde{G}, \tilde{G'} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be generating families of one-parameter Legendrian unfolding of Clairaut type. We also say that $\tilde{G}$ and $\tilde{G'}$ are *t-P-$\mathcal{K}$-equivalent* if there exists a germ of a diffeomorphism $\Phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0)$ of the form

$$\Phi(s,x,t,y) = (q(s,x,t,y), \phi_1(x,t,y), \varphi(t), \phi_2(x,t,y))$$

such that $\langle \tilde{G} \circ \Phi \rangle_{\mathcal{E}_{(s,x,t,y)}} = \langle \tilde{G'} \rangle_{\mathcal{E}_{(s,x,t,y)}}$ and $\tilde{G}(s,x,t,y)$ is *P-$\mathcal{K}$-versal deformation* of $g = \tilde{G}|_{t=0}$ if

$$\mathcal{E}_{(s,x,y)} = \langle \frac{\partial g}{\partial s}, g \rangle_{\mathcal{E}_{(s,x,y)}} + \langle \frac{\partial g}{\partial x}, 1 \rangle_{\mathcal{E}_{(x,y)}} + \langle \frac{\partial G}{\partial t}|_{t=0} \rangle_{\mathbb{R}}.$$

By the similar arguments like as those of [1, Theorems 20.8 and 21.4], we can show the following:

Theorem 4.3 *Let $\tilde{G}, \tilde{G'} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be generating families of one-parameter Legendrian unfoldings of Clairaut type $\Phi_G, \Phi_{G'}$ respectively. Then*

(1) *Φ_G and $\Phi_{G'}$ are one-parameter SP^+ (respectively, one-parameter SP)-Legendrian equivalent if and only if $\tilde{G}$ and $\tilde{G'}$ are one-parameter P-$\mathcal{C}^+$ (respectively, one-parameter P-$\mathcal{C}$)-equivalent.*

(2) *Φ_G is a one-parameter SP^+ (respectively, one-parameter SP)-Legendrian stable if and only if $\tilde{G}$ is a P-$\mathcal{C}^+$ (respectively, P-$\mathcal{C}$)-versal deformation of $g = \tilde{G}|_{t=0}$.*

(3) Φ_G *is a* P*-Legendrian stable if and only if* $\tilde{G}$ *is a* P*-*$\mathcal{K}$*-versal deformation of* $g = \tilde{G}|_{t=0}$.

For each function germ $g : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, we set

$$P\text{-}\mathcal{C}^+\text{-cod}(g) = \dim_{\mathbb{R}} \mathcal{E}_{(s,x,y)} / \langle \frac{\partial g}{\partial s} \rangle_{\mathcal{E}_{(x,y)}} + \langle g \rangle_{\mathcal{E}_{(s,x,y)}} + \langle \frac{\partial g}{\partial x}, 1 \rangle_{\mathcal{E}_{(x,y)}}.$$

By the definition, P-$\mathcal{C}^+$-cod$(g) \leqq 1$ then $\mathcal{C}^+$-cod$(g_0) \leqq 3$, where $g_0 = g|_{\mathbb{R}\times 0}$.

And we set

$$P\text{-}\mathcal{K}\text{-cod}(g) = \dim_{\mathbb{R}} \mathcal{E}_{(s,x,y)} / \langle \frac{\partial g}{\partial s}, g \rangle_{\mathcal{E}_{(s,x,y)}} + \langle \frac{\partial g}{\partial x}, 1 \rangle_{\mathcal{E}_{(x,y)}}.$$

Since $g_0 = g|_{\mathbb{R}\times 0}$ is a function germ of one-variable, we have $\langle \frac{dg_0}{ds} \rangle_{\mathbb{R}} + \langle g_0 \rangle_{\mathcal{E}_s} = \langle \frac{dg_0}{ds}, g_0 \rangle_{\mathcal{E}_s}$, so P-$\mathcal{K}$-cod$(g) \leq 1$ then $\mathcal{C}^+$-cod$(g_0) \leq 3$.

5 Proof of Theorem 1.1

By Theorem A in [8], the set of P-Legendrian stable one-parameter complete unfoldings is an open and dense subset in $L(U \times V, PT^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}))$. Therefore by theorem 2.2, it gives a classification of P-Legendrian stable one-parameter complete Legendrian unfoldings under the one-parameter SP^+-Legendrian equivalence (or SP-Legendrian equivalence). Let (μ, f) be a one-parameter family of Clairaut type equation such that the corresponding one-parameter complete Legendrian unfolding $\ell_{(\mu,F)}$ is one-parameter P-Legendrian stable. By the assumption, the generating family $\tilde{G}$ of $\ell_{(\mu,F)}$ is P-$\mathcal{K}$-versal deformation of $g = \tilde{G}|_{t=0}$, then $\tilde{G}$ is $\mathcal{K}$-versal deformation of $g_0 = g|_{\mathbb{R}\times 0}$. We remark that $\tilde{G}$ is $\mathcal{K}$-versal deformation of g_0 if and only if $\tilde{G}$ is $\mathcal{C}^+$-versal deformation of g_0. By Lemma 3.1 and Theorem 3.2, $\tilde{G}$ is P-$\mathcal{C}^+$-equivalent to one of the germs in the following list:

(1) $s + u_1 + u_2 + u_3$,

(2) $s^2 + u_1 + u_2 + u_3 s$,

(3) $s^3 + u_1 + u_2 s + u_3 s^2$,

(4) $s^4 + u_1 + u_2 s + u_3 s^2$,

where $(s, u_1, u_2, u_3) \in (\mathbb{R} \times \mathbb{R}^3, 0)$. We would like to classify these germs by the one-parameter P-$\mathcal{C}^+$-equivalence. By the above normal forms, there exist a germ of a diffeomorphism $\phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ such that $\tilde{G}$ is one-parameter P-$\mathcal{C}^+$-equivalent to one of the germs in the following list:

$(1')\ s + u_1(x,t,y) + u_2(x,t,y) + u_3(x,t,y)$,

$(2')\ s^2 + u_1(x,t,y) + u_2(x,t,y) + u_3(x,t,y)s$,

$(3')\ s^3 + u_1(x,t,y) + u_2(x,t,y)s + u_3(x,t,y)s^2$,

$(4')\ s^4 + u_1(x,t,y) + u_2(x,t,y)s + u_3(x,t,y)s^2$.

Since $\tilde{G}$ is the form $G - y$, we assume that $\frac{\partial u_1}{\partial y}(0) \neq 0$. Furthermore, we perform local coordinate change, so that $u_1(x,t,y) = -y, u_2(x,t,y) = u_2(x,t)$ and $u_3(x,t,y) = u_3(x,t)$. Therefore we classify these germs by the one-parameter P-$\mathcal{C}^+$-equivalent under the condition P-$\mathcal{K}$-cod(g) ≤ 1 where $g = \tilde{G}|_{t=0}$. We remark that $j_1^1 G_t$ is a germ of an immersion.

Next we consider the t-P-$\mathcal{K}$-equivalence. Since the P-$\mathcal{C}^+$-equivalence is a stronger equivalence relation than the P-$\mathcal{K}$-equivalence, $\tilde{G}$ is P-$\mathcal{K}$-equivalent to one of the germs in the above list.

We now classify these germs by the t-P-$\mathcal{K}$-equivalence under the condition that P-$\mathcal{K}$-cod ≤ 1. We use the result of [10, Theorem 3.1] such that $j_1^1 G_t$ is a germ of an immersion (i.e. $\operatorname{rank}\begin{pmatrix} \frac{\partial G}{\partial s} & \frac{\partial^2 G}{\partial s \partial x} \end{pmatrix} = 1$), then such a germ is t-P-$\mathcal{K}$-equivalent to one of the following germs:

$(a)\ s - y + x + t$,

$(b)\ s^2 - y + t + xs$,

$(c)\ s^3 - y + xs + ts^2$,

$(d)\ s^4 - y + xs + ts^2$.

We remark that the above assertion has been also proven by [2, Theorem 1.2] or [11].

Now $\tilde{G}$ is t-P-$\mathcal{K}$-equivalent to one of the germs in the above list, then g is P-$\mathcal{K}$-equivalent to one of the following germs:

$(a')\ s-y+x,$

$(b')\ s^2-y+xs,$

$(c')\ s^3-y+xs,$

$(d')\ s^4-y+xs.$

For (a'), (b') and (c'), g is a $\mathcal{K}$-versal deformation of g_0, so g is a $\mathcal{C}^+$-versal deformation of g_0. We remark that g_0 and s^i $(i=1,2,3)$ are $\mathcal{K}$-equivalent, then g_0 and s^i $(i=1,2,3)$ are $\mathcal{C}$-equivalent. We use Theorem 3.2 that g is P-$\mathcal{C}^+$-equivalent to one of the following germs:

$$(a')\ s-y+x,\ (b')\ s^2-y+xs,\ (c')\ s^3-y+xs.$$

These germs satisfy the condition that P-$\mathcal{C}^+$-cod $=0$, so that we have P-$\mathcal{C}^+$-cod$(g)=0$. By definition, $\tilde{G}$ is a P-$\mathcal{C}^+$-versal deformation of g. By the uniqueness result of P-$\mathcal{C}^+$-versal deformation, $\tilde{G}$ is one-parameter P-$\mathcal{C}^+$-equivalent to one of the following germs:

$$(a)\ s-y+x+t,\ (b)\ s^2-y+t+xs,\ (c)\ s^3-y+xs+ts^2.$$

On the other hand, we assume that g is P-$\mathcal{K}$-equivalent to the germ (d'): s^4-y+xs. By the previous arguments, we consider the germ of the form $\tilde{G}(s,x,t,y)=s^4-y+u_2(x,t)s+u_3(x,t)s^2$. We remark that the P-$\mathcal{K}$-codimension of s^4-y+xs is 1, so that P-$\mathcal{K}$-cod$(g)=1$. If $\partial u_2/\partial x(0,0)=0$, then we can show that P-$\mathcal{K}$-cod$(g)\geq 2$. Therefore we have a germ of a diffeomorphism defined by $X=u_2(x,t), T=t$. It follows that the germ: $s^4-y+u_2(x,t)s+u_3(x,t)s^2$ is one-parameter P-$\mathcal{C}^+$-equivalent to the germ:

$$(d'')\quad s^4-y+xs+\beta(x,t)s^2.$$

Since these transformations are local coordinate changes, we have $\partial\beta/\partial t(0,0)\neq 0$.

We now detect the corresponding normal forms of one-parameter integral diagrams as follows:

For the case (a), we can choose

$$G(s,x,t)=s+x+t,$$

as a generating family, so that

$$\mathcal{L}_G = (s, x, t, s + x + t, 1, 1, 1).$$

Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is strictly equivalent to

$$\begin{aligned}
&\mu = u_2,\\
&G = (u_1, u_2, t).
\end{aligned}$$

For the case (b), we can choose

$$G(s, x, t) = s^2 + xs + t,$$

as a generating family, so that

$$\mathcal{L}_G = (s, x, t, s^2 + xs + t, 2s + x, s, 1).$$

We have

$$\begin{aligned}
&\mu = u_2,\\
&G = (u_1, u_2^2 + u_1 u_2, t).
\end{aligned}$$

We now define a coordinate transformation by

$$U_1 = u_1,\ \ U_2 = u_2 + \tfrac{1}{2}u_1,$$

then (μ, G) is strictly equivalent to

$$\begin{aligned}
&\mu = u_2 - \frac{1}{2}u_1,\\
&G = (u_1, (u_2^2 - \frac{1}{4}u_1^2), t).
\end{aligned}$$

We also apply a coordinate transformation which is defined by

$$X = x,\ \ Y = y + \tfrac{1}{4}x^2,$$

then we have the normal form.

For the case (c), we can also choose

$$G(s, x, t) = s^3 + xs + ts^2$$

as a generating family, so that

$$\mathcal{L}_G = (s, x, t, s^3 + xs + ts^2, 3s^2 + x + 2ts, s, s^2).$$

Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is one-parameter R^+-equivalent to

$$\mu = u_2,$$
$$G = (u_1, u_2^3 + u_1u_2 + tu_2^2, t).$$

For the case (d''), we can also choose

$$G(s,x,t) = s^4 + xs + \beta(x,t)s^2$$

as a generating family, so that

$$\mathcal{L}_G = (s,x,t,s^4 + xs + \beta(x,t)s^2, 4s^3 + x + 2\beta(x,t)s, s + \frac{\partial \beta}{\partial x}(x,t), \frac{\partial \beta}{\partial t}(x,t)s^2).$$

Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is one-parameter R^+-equivalent to

$$\mu = u_2,$$
$$G = (u_1, u_2^4 + u_1u_2 + \beta(u_1,t)u_2^2, t).$$

Since each generating family for (c) and (d'') is one-parameter P-$\mathcal{C}^+$-equivalent to the normal form but these are not one-parameter P-$\mathcal{C}$-equivalent, the corresponding one-parameter integral diagram is strictly equivalent to the normal form (3) and (4) in Theorem 1.1.

This completes the proof of Theorem 1.1.

Remark. In the above proof, we can show that the P-$\mathcal{C}^+$-codimension of the germ: $s^4 - y + xs$ is infinite. This means that there appears the second functional moduli in the normal form (4) in Theorem 1.1.

6 Appendix

In this section we now give a quick review of the theory of unfoldings of function germs [1,3,4,7,8].

Let $f, f' : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of functions, $F, F' : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be unfoldings of f, f' respectively and $\tilde{F}, \tilde{F'} : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be germs of functions, where $\tilde{F}(s,x,y) = F(s,x) - y$, $\tilde{F'}(s,x,y) = F'(s,x) - y$. We say that $\tilde{F}$ and $\tilde{F'}$ are P-$\mathcal{C}^+$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if there exists a germ of a diffeomorphism $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(s,x,y) = (s + \alpha(x,y), \phi_1(x,y), \phi_2(x,y))$ (respectively, $\Phi(s,x,y) = (s, \phi_1(x,y), \phi_2(x,y)))$ such that $\langle \tilde{F} \circ \Phi \rangle_{\mathcal{E}_{(s,x,y)}} = \langle \tilde{F'} \rangle_{\mathcal{E}_{(s,x,y)}}$ where,

$\langle \tilde{F}' \rangle_{\mathcal{E}_{(s,x,y)}}$ is the ideal generated by $\tilde{F}'$ in $\mathcal{E}_{(s,x,y)}$. We also say that $\tilde{F}(s,x,y)$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|_{\mathbb{R}\times 0}$ if

$$\mathcal{E}_s = \langle \frac{df}{ds} \rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_s} + \langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}\times 0}, ..., \frac{\partial F}{\partial x_n}|_{\mathbb{R}\times 0}, 1 \rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_s = \langle f \rangle_{\mathcal{E}_s} + \langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}\times 0}, ..., \frac{\partial F}{\partial x_n}|_{\mathbb{R}\times 0}, 1 \rangle_{\mathbb{R}}).$$

Let $f, f' : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of functions. We say that f and f' are $\mathcal{C}$-*equivalent* if and only if $\langle f \rangle_{\mathcal{E}_s} = \langle f' \rangle_{\mathcal{E}_s}$. Then the classification theory of germs of functions by the $\mathcal{C}$-equivalence is quite useful for our purpose. For each germ of a function $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, we set

$$\begin{aligned}
\mathcal{C}\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_s / \langle f \rangle_{\mathcal{E}_s}, \\
\mathcal{C}^+\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_s / \langle f \rangle_{\mathcal{E}_s} + \langle \frac{df}{ds} \rangle_{\mathbb{R}}, \\
\mathcal{K}\text{-cod}(f) &= \dim_{\mathbb{R}} \mathcal{E}_s / \langle f \rangle_{\mathcal{E}_s} + \langle \frac{df}{ds} \rangle_{\mathcal{E}_s}.
\end{aligned}$$

Then we have the following well-known classification (cf.[4]).

Lemma 6.1 *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a germ of a function with $\mathcal{K}$-cod$(f) < \infty$. Then f is $\mathcal{C}$-equivalent to the germ $s^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By a direct calculation, we have

$$\begin{aligned}
\mathcal{C}\text{-cod}(s^{\ell+1}) &= \ell + 1, \\
\mathcal{C}^+\text{-cod}(s^{\ell+1}) &= \ell.
\end{aligned}$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^+$)-versal deformations of the above germs by using the usual method (cf. [8]):

-The $\mathcal{C}$-versal deformation: $s^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1} s^i$.

-The $\mathcal{C}^+$-versal deformation: $s^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1} s^i$.

The following theorem is useful and important for our purpose.

Theorem 6.2 *Let $\tilde{F}, \tilde{F}' : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be germs of functions such that $\tilde{F}, \tilde{F}'$ are $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformations of $f = F|_{\mathbb{R}\times 0}$, $f' = F'|_{\mathbb{R}\times 0}$ respectively. Then $\tilde{F}$ and $\tilde{F}'$ are P-$\mathcal{C}^+$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if and only if f and f' are $\mathcal{C}$-equivalent.*

Let $\tilde{F}(s, x, y)$ be a $\mathcal{C}^+$-versal deformation of $f = F|_{\mathbb{R}\times 0}$. By Lemma 3.1 and Theorem 3.2, $\tilde{F}(s, x, y)$ is P-$\mathcal{C}^+$-equivalent to one of the germs in the following list:

$$DA_\ell (1 \leqq \ell \leqq n+1) : s^\ell + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j - y,$$

$$\widetilde{DA}_{n+2} : s^{n+2} + \sum_{i=1}^{n} x_i s^i - y.$$

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