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a stochastic approximation and PDE**

Hitoshi Ishii and Toshio Mikami

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Convexified Gauss curvature flow of bounded open sets in an anisotropic external field: a stochastic approximation and PDE

Hitoshi Ishii*
Waseda University

Toshio Mikami†
Hokkaido University

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Abstract

We construct a random crystalline (or polyhedral) approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. We also show that a weak solution to the PDE which describes the motion of a bounded open set is unique and is a viscosity solution of it.

1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see e.g. [2, 3, 6, 7, 11, 14, 24]).

In the last few years we have been generalizing the theory of Gauss curvature flow to a class of nonconvex sets.

*Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan, supported in part by the Grant-in-Aid for Scientific Research, No. 12440044 and No. 12304006 JSPS.

†Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan, supported in part by the Grant-in-Aid for Scientific Research, No. 12440044 and 13640096, JSPS.

In [17] we studied the existence and the uniqueness of a viscosity solution to the PDE that describes the time evolution of a nonconvex graph by a convexified Gauss curvature (see (1.10) for PDE).

In [19] we proposed and studied the discrete stochastic approximations of evolving functions which are generalizations of those considered in [17], and proved the existence and the uniqueness of a weak solution to the PDE which appears in the continuum limit of discrete stochastic processes, and discussed under what conditions a weak solution to the PDE is a viscosity solution of it.

In [20] we studied the existence and the uniqueness of the motion (or time evolution) of a nonconvex compact set which evolves by a convexified Gauss curvature in $\mathbf{R}^N$ ($N \geq 2$), by the level set approach in the theory of viscosity solutions (see e.g. [5, 10, 23] for the level set approach).

We introduce the notion of the motion of a smooth oriented closed hypersurface by a convexified Gauss curvature.

Let M be a smooth oriented closed hypersurface in $\mathbf{R}^N$ and e be a smooth vector field over M of unit normal vectors. For $x \in M$, let $T_x M$ denote the tangent space of M at x , and let $A_x : T_x M \mapsto T_x M$ denote the Weingarten map at x defined by the following:

$$A_x(v) = -D_v e \quad \text{for } v \in T_x M, \quad (1.1)$$

where $D_v e$ denotes the derivative of e with respect to v . Recall that the principal curvatures $\kappa_1, \dots, \kappa_n$ ($n := N - 1$) of M at x are the eigenvalues of the symmetric map A_x and the Gauss curvature $K(x)$ of M at x is given by $\det A_x$.

Let C be the convex hull $\text{co } M$ of M . We define $\sigma : M \mapsto \{0, 1\}$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in M \cap \partial C, \\ 0 & \text{otherwise,} \end{cases} \quad (1.2)$$

and call $\sigma(x)K(x)$ the *convexified* Gauss curvature of M at x .

The motion of a smooth oriented closed hypersurface by a convexified Gauss curvature is the curvature flow:

$$v = -\sigma K \nu, \quad (1.3)$$

where ν denotes the unit outward normal vector on the surface and v denotes the velocity of the surface.

Let $(A_x)_+$ denote the positive part of the symmetric map A_x . $K_+(x) := \det\{(A_x)_+\}$ is called the *positive part* of the Gauss curvature of M at x , and the following holds:

$$\sigma(x)K(x) = \sigma(x)K_+(x). \quad (1.4)$$

Remark 1 For $x \in M$,

$$\det\{(A_x)_+\} = \begin{cases} \det A_x & \text{if } A_x \text{ is nonnegative definite,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.5)$$

The crystalline (or polyhedral) approximation of a smooth simple closed convex curve which evolves as the curvature flow was considered by Girão and is useful in the numerical analysis (see [13] and the references therein). We refer to [12] and the references therein for the recent development of this topics.

When $N = 2$, the discrete stochastic approximation of the curvature flow of smooth simple closed convex curves was given in [18] where the model and the approach are completely different from those in this paper.

In this paper we propose and study the discrete stochastic approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. Our result in this paper is the first one in case $N \geq 3$, among random and nonrandom results, which gives a crystalline approximation of the motion of a bounded open set in $\mathbf{R}^N$ by Gauss curvature.

We briefly describe what we proved in [19], and then discuss the results in this paper more precisely to compare a convexified Gauss curvature flow of graphs with that of closed hypersurfaces.

For $x \in \mathbf{R}^n$ and $u : \mathbf{R}^n \mapsto \mathbf{R}$, the following set is called the subdifferential of u at x :

$$\partial u(x) := \{p \in \mathbf{R}^n : u(y) - u(x) \geq p \cdot (y - x) \text{ for all } y \in \mathbf{R}^n\}, \quad (1.6)$$

where $\cdot$ denotes the inner product in $\mathbf{R}^n$.

Alexandrov-Bakelman's generalized curvature introduced in the following played a crucial role in [19].

Definition 1 (see e.g. [4, section 9.6]). Let $R \in L^1(\mathbf{R}^n : [0, \infty), dx)$ and $u \in C(\mathbf{R}^n)$. For $A \in B(\mathbf{R}^n) (= \text{Borel } \sigma\text{-field of } \mathbf{R}^n)$, put

$$w(R, u, A) := \int_{\cup_{x \in A} \partial u(x)} R(y) dy \quad (A \in B(\mathbf{R}^n)). \quad (1.7)$$

Let $T \in [0, \infty]$ and $R \in L^1(\mathbf{R}^n : [0, \infty), dx)$. We showed the existence and the uniqueness of a solution $u \in C([0, T) \times \mathbf{R}^n)$ to the following equation (see [19, Theorem 1]): for any $\varphi \in C_o(\mathbf{R}^n)$ and any $t \in [0, T)$,

$$\int_{\mathbf{R}^n} \varphi(x)(u(t, x) - u(0, x))dx = \int_0^t ds \int_{\mathbf{R}^n} \varphi(x)w(R, u(s, \cdot), dx). \quad (1.8)$$

The existence of a continuous solution to (1.8) was given by the continuum limit of the infinite particle systems $\{(Z_m(t, z))_{z \in \mathbf{Z}^n/m}\}_{t \geq 0}$ that satisfies the following: for any $t \geq 0$ and any $z \in \mathbf{Z}^n/m$,

$$P(Z_m(t + \Delta t, z) - Z_m(t, z) > 0) = m^n E[w(R, \hat{Z}_m(t, \cdot), \{z\})] \Delta t + o(\Delta t) \quad (1.9)$$

as $\Delta t \rightarrow 0$ ($m \geq 1$), where $\hat{Z}_m(t, \cdot)$ denotes a convex envelope of the function $z \mapsto Z_m(t, z)$, i.e., the graph of the boundary of the convex hull, in $\mathbf{R}^N$, of the set $\{(z, y) | z \in \mathbf{Z}^n/m, y \geq Z_m(t, z)\}$.

In [19, Theorem 2], we proved that a continuous solution u to (1.8) sweeps in time $t > 0$ a region with volume given by $t \cdot w(R, u(0, \cdot), \mathbf{R}^n)$, and that, for continuous solutions u and v to (1.8) with $v(0, \cdot) = \hat{u}(0, \cdot)$, $\hat{u}(t, \cdot)$ is different from $v(t, \cdot)$ at time $t > 0$ in general if $u(0, \cdot) \neq \hat{u}(0, \cdot)$.

We also showed that a continuous solution to (1.8) is a viscosity solution of the following PDE (see [19, Theorem 3]):

$$\partial_t u(t, x) = \chi(u, Du(t, x), t, x) R(Du(t, x)) \text{Det}_+(D^2 u(t, x)) \quad (1.10)$$

$((t, x) \in (0, \infty) \times \mathbf{R}^n)$, where $Du(t, x) := (\partial u(t, x)/\partial x_i)_{i=1}^n$, $D^2 u(t, x) := (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^n$,

$$\chi(u, p, t, x) := \begin{cases} 1 & \text{if } p \in \partial u(t, x), \\ 0 & \text{otherwise} \end{cases}$$

$(\partial u(t, x))$ denotes the subdifferential of the function $x \mapsto u(t, x)$. Conversely, we discussed under what conditions a viscosity solution to (1.10) is a solution to (1.8).

Remark 2 When $R(p) = (1 + |p|^2)^{-(n+1)/2}$,

$$(1 + |Du(t, x)|^2)^{-1/2} \chi(u, Du(t, x), t, x) R(Du(t, x)) \text{Det}_+(D^2u(t, x))$$

can be considered as the convexified Gauss curvature of $\{(y, u(t, y)) | y \in \mathbf{R}^N\}$ at x if we consider $\{(y, z) | y \in \mathbf{R}^N, z \geq u(t, y)\}$ as the inside of the hypersurface $\{(y, u(t, y)) | y \in \mathbf{R}^N\}$.

Next we briefly discuss what we study in this paper.

Let F be a closed convex set in $\mathbf{R}^N$. For $x \in \partial F$, put

$$N_F(x) := \{p \in \mathbf{S}^{N-1} | F \subset \{y | \langle y - x, p \rangle \leq 0\}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^N$.

To consider a convexified Gauss curvature flow of bounded open sets by the level set approach, we introduce new types of measures.

Definition 2 Let u be a bounded function from a subset of $\mathbf{R}^N$ to $\mathbf{R}$, and $R \in L^1(\mathbf{S}^{N-1} : [0, \infty), d\mathcal{H}^{N-1})$, where $d\mathcal{H}^{N-1}$ denotes a $(N - 1)$ -dimensional Hausdorff outer measure.

(i). Let $r \in \mathbf{R}$. For $B \in B(\mathbf{R}^N)$, put

$$\omega_r(R, u, B) := \int_{N_{co\ u^{-1}([r, \infty))} - (\partial(co\ u^{-1}([r, \infty))) \cap B} R(p) d\mathcal{H}^{N-1}(p), \quad (1.11)$$

where A^- denotes the closure of the set A .

(ii). For $B \in B(\mathbf{R}^N)$, put

$$\mathbf{w}(R, u, B) := \int_{\mathbf{R}} dr \omega_r(R, u, B), \quad (1.12)$$

provided the right hand side is well defined.

When it is not confusing, we write $\omega_r(R, u, dx) = \omega_r(u, dx)$ and $\mathbf{w}(R, u, dx) = \mathbf{w}(u, dx)$ for the sake of simplicity.

The existence and the uniqueness of a solution to the following equation is given in section 2.

Definition 3 Let $T \in [0, \infty]$ and $R \in L^1(\mathbf{S}^{N-1} : [0, \infty), d\mathcal{H}^{N-1})$. A family of bounded open sets $\{D(t)\}_{t \in [0, T)}$ in $\mathbf{R}^N$ is called a convexified Gauss curvature flow in an (R) -anisotropic external field on $[0, T)$ if

$$D(t) = (co D(t)) \cap D(0) \quad \text{for } t \in [0, T], \quad (1.13)$$

and if the following holds: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \in [0, T]$,

$$\int_{\mathbf{R}^N} \varphi(x)(I_{D(0)}(x) - I_{D(t)}(x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\omega_1(R, I_{D(s)}(\cdot), dx). \quad (1.14)$$

We also show the existence and the uniqueness of a solution $u \in C_b([0, T] \times \mathbf{R}^N)$ to the following: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \in [0, T]$,

$$\int_{\mathbf{R}^N} \varphi(x)(u(0, x) - u(t, x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\mathbf{w}(R, u(s, \cdot), dx). \quad (1.15)$$

The existence of $\{I_{D(t)}\}_{t \geq 0}$ in Definition 3 is given by the continuum limit of a class of particle systems $\{(Y_m(t, z))_{z \in \mathbf{Z}^N/m}\}_{t \geq 0}$ that satisfies the follows: for any $t \geq 0$ and any $z \in \mathbf{Z}^N/m$,

$$P(Y_m(t + \Delta t, z) - Y_m(t, z) < 0) = m^N E[\omega_1(Y_m(t, \cdot), \{z\})]\Delta t + o(\Delta t) \quad (1.16)$$

as $\Delta t \rightarrow 0$ ($m \geq 1$) (see Theorem 1 in section 2).

The existence and the uniqueness of a solution to (1.15) will be given by the continuum limit of the linear combinations of solutions to (1.14) with $D(0) = u(0, \cdot)^{-1}((r, \infty))$ for $r \in \mathbf{R}$ (see Corollary 2 in section 2).

We also discuss the properties of $\{D(t)\}_{t \geq 0}$ in Definition 3 (see Theorem 2 in section 2).

For $p \in \mathbf{R}^N$ and a $N \times N$ -symmetric real matrix X , put

$$G(p, X) := \begin{cases} |p| \det_+ \left(-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p} \right) & \text{if } p \neq o, \\ 0 & \text{if } p = o \end{cases} \quad (1.17)$$

(see (1.4) for notation), where $\bar{p} := p/|p|$.

Suppose that a smooth oriented hypersurface M in $\mathbf{R}^N$ is given by $M = \{y \in \mathbf{R}^N \mid \varphi(y) = a, D\varphi(y) \neq o\}$ for some $\varphi \in C^2(\mathbf{R}^N)$ and $a \in \mathbf{R}$, and that the vector field e is given by $e_x = D\varphi(x)/|D\varphi(x)|$. Regard the tangent space, $T_x M$, as the orthogonal complement of e_x , and let $E_x := \text{span } e_x$ and id_{E_x} denote the identity map on E_x . Then the map

$$A_x \oplus \text{id}_{E_x} : \mathbf{R}^N \equiv T_x M \oplus E_x \rightarrow T_x M \oplus E_x$$

has a matrix representation

$$-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p},$$

with $p = D\varphi(x)$ and $X = D^2\varphi(x)$. Therefore,

$$K(x) = \det \left(-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p} \right), \quad (1.18)$$

$$K_+(x) = \frac{G(p, X)}{|p|}. \quad (1.19)$$

For $\{D(t)\}_{t \geq 0}$ in Definition 3, we show that $I_{D(t)}(x)$ and $I_{D(t)^-}(x)$ are respectively a viscosity supersolution and a viscosity subsolution of the following PDE (see Theorem 3 in section 2):

$$\partial_t u(t, x) + R \left(\frac{Du(t, x)}{|Du(t, x)|} \right) \sigma^-(u, Du(t, x), t, x) G(Du(t, x), D^2u(t, x)) = 0 \quad (1.20)$$

$((t, x) \in (0, \infty) \times \mathbf{R}^N)$. Here

$$\sigma^-(u, p, t, x) := \begin{cases} 1 & \text{if } u(t, \cdot) < u(t, x) \text{ on } H(p, x) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.21)$$

where

$$H(p, x) := \{y \in \mathbf{R}^N \setminus \{x\} \mid \langle y - x, p \rangle \leq 0\}. \quad (1.22)$$

Moreover, we show that a continuous solution to (1.15) is a viscosity solution of (1.20) (see Corollary 3 in section 2).

In [21], we will study the uniqueness of a viscosity solution to (1.20), from which we conclude that a viscosity solution to (1.20) with a bounded continuous initial data is a unique solution to (1.15).

Since $G(p, X)$ is singular at $p = o$, the standard definition of a viscosity solution (see [8]) is not appropriate for (1.20). We take the definition of a viscosity solution to (1.20) from [22].

We first introduce the set of admissible test functions. We denote by $\mathcal{F}$ the set of all functions $f \in C^2([0, \infty))$ for which $f'' > 0$ on $(0, \infty)$ and

$$\lim_{r \downarrow 0} \frac{f(r)}{r^N} = 0. \quad (1.23)$$

Let Ω be an open subset of $(0, \infty) \times \mathbf{R}^N$. A function $\varphi \in C^2(\Omega)$ is called admissible in Ω if for any $(\hat{t}, \hat{x}) \in \Omega$ for which $D\varphi$ vanishes, there exists $f \in \mathcal{F}$ such that as $(t, x) \rightarrow (\hat{t}, \hat{x})$,

$$|\varphi(t, x) - \varphi(\hat{t}, \hat{x}) - \partial_t \varphi(\hat{t}, \hat{x})(t - \hat{t})| \leq f(|x - \hat{x}|) + o(|t - \hat{t}|). \quad (1.24)$$

We denote by $\mathcal{A}(\Omega)$ the set of all admissible functions in Ω .

Remark 3 $f(r) = r^{N+1} \in \mathcal{F}$ and $\varphi(t, x) = f(|x - \hat{x}|) \in \mathcal{A}((0, \infty) \times \mathbf{R}^N)$ for any $\hat{x} \in \mathbf{R}^N$.

Definition 4 (Viscosity solution) Let $0 < T \leq \infty$ and set $\Omega := (0, T) \times \mathbf{R}^N$, and put $R(o/|o|) := 0$.

(i). A function $u \in \text{LSC}(\Omega)$ is called a viscosity supersolution of (1.20) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local minimum at (s, y) , then

$$\partial_t \varphi(s, y) + R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right) \sigma^+(u, D\varphi(s, y), s, y) G(D\varphi(s, y), D^2\varphi(s, y)) \geq 0, \quad (1.25)$$

where

$$\sigma^+(u, p, s, y) := \begin{cases} 1 & \text{if } u(s, \cdot) \leq u(s, y) \text{ on } H(p, y) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.26)$$

(ii). A function $u \in \text{USC}(\Omega)$ is called a viscosity subsolution of (1.20) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local maximum at (s, y) , then

$$\partial_t \varphi(s, y) + \sigma^-(u, D\varphi(s, y), s, y) R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right) G(D\varphi(s, y), D^2\varphi(s, y)) \leq 0. \quad (1.27)$$

(iii). A function $u \in C(\Omega)$ is called a viscosity solution of (1.20) in Ω if it is a viscosity supersolution and a viscosity subsolution of (1.20) in Ω .

Remark 4 $\sigma^+(u, p, s, y) \geq \sigma^-(u, p, s, y)$ for all $u : \Omega \mapsto \mathbf{R}$ and all $(p, s, y) \in \mathbf{R}^N \times \Omega$.

Let $\mathcal{A}_0(\Omega)$ denote the set of all $\phi_1(t) + \phi_2(x) \in \mathcal{A}(\Omega)$ such that $x \mapsto G(D\phi_2(x), D^2\phi_2(x))$ is continuous in Ω . Then one can replace, in Definition 5, $\mathcal{A}(\Omega)$ by $\mathcal{A}_0(\Omega)$ (see [20]).

Remark 5 For any $f \in \mathcal{F}$ and $\hat{x} \in \mathbf{R}^N$, $\varphi(t, x) = f(|x - \hat{x}|) \in \mathcal{A}_0((0, \infty) \times \mathbf{R}^N)$.

In section 2 we state our main results which will be proved in section 4. In section 3 we give technical lemmas.

2 Main Result

In this section we give our main result.

We give two assumptions to state the stochastic process which approximates the solution to (1.13)-(1.14).

(A.0). D is a bounded open set in $\mathbf{R}^N$ such that $\text{Vol}(\partial D) :=$ the Lebesgue measure of the boundary ∂D of D is zero.

(A.1). $R \in L^1(\mathbf{S}^{N-1} : [0, \infty), d\mathcal{H}^{N-1})$ and $\|R\|_{L^1(\mathbf{S}^{N-1})} = 1$.

Take $K > 0$ so that $co D \subset [-K + 1, K - 1]^N$. For $m \geq 1$, put

$$\mathcal{S}_m := \{I_A : [-K, K]^N \cap (\mathbf{Z}^N/m) \mapsto \{0, 1\} | A \subset [-K, K]^N \cap \mathbf{Z}^N/m\}, \quad (2.1)$$

$$D_m := D \cap (\mathbf{Z}^N/m). \quad (2.2)$$

For $x, z \in \mathbf{Z}^N/m$ and $v \in \mathcal{S}_m$, put

$$v_{m,z}(x) := \begin{cases} v(x) & \text{if } x \neq z, \\ 0 & \text{if } x = z \end{cases}$$

; and for $f : \mathcal{S}_m \mapsto \mathbf{R}$, put

$$A_m f(v) := m^N \sum_{z \in [-K, K]^N \cap (\mathbf{Z}^N/m)} \omega_1(R, v, \{z\}) \{f(v_{m,z}) - f(v)\}. \quad (2.3)$$

Let $\{Y_m(t, \cdot)\}_{t \geq 0}$ be a Markov process on $\mathcal{S}_m$ ($m \geq 1$), with the generator A_m , such that $Y_m(0, z) = I_{D_m}(z)$ ($z \in [-K, K]^N \cap (\mathbf{Z}^N/m)$). For $(t, x) \in [0, \infty) \times [-K, K]^N$, put also

$$X_m(t, x) := I_{(co Y_m(t, \cdot)^{-1}(1))^{\circ} \cap D}(x), \quad (2.4)$$

where A° denotes the interior of the set $A \subset \mathbf{R}^N$.

Then $\{X_m(t, \cdot)\}_{t \geq 0}$ is a stochastic process on

$$\mathcal{S} := \{f \in L^2([-K, K]^N) : \|f\|_{L^2([-K, K]^N)} \leq (2K)^N\} \quad (2.5)$$

which is a complete separable metric space by the metric

$$d_{\mathcal{S}}(f, g) := \sum_{k=1}^{\infty} \frac{\max(|\langle f - g, e_k \rangle_{L^2([-K, K]^N)}|, 1)}{2^k}. \quad (2.6)$$

Here $\{e_k\}_{k \geq 1}$ denotes a complete orthonormal basis of $L^2([-K, K]^N)$.

The following is our main result.

Theorem 1 Suppose that (A.0)-(A.1) hold. Then there exists a unique solution $\{D(t)\}_{t \geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0, \infty)$ such that $I_{D(\cdot)}(\cdot) \in C([0, \infty) : L^2([-K, K]^N))$ and that the following holds: for any $\gamma > 0$,

$$\lim_{m \rightarrow \infty} P(\sup_{t \geq 0} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma) = 0. \quad (2.7)$$

We recall Hausdorff metric of compact sets A and $B \subset \mathbf{R}^N$:

$$d_H(A, B) := \max(\max_{p \in A} \text{dist}(p, B), \max_{q \in B} \text{dist}(q, A)). \quad (2.8)$$

As a corollary, we obtain

Corollary 1 Suppose that (A.0)-(A.1) hold and that D is convex. Then for a unique solution $\{D(t)\}_{t \geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0, \infty)$, the following holds: for any $T \in [0, \text{Vol}(D))$ and any $\gamma > 0$,

$$\lim_{m \rightarrow \infty} P(\sup_{0 \leq t \leq T} d_H(co Y_m(t, \cdot)^{-1}(1), D(t)) \geq \gamma) = 0. \quad (2.9)$$

We introduce the assumption on the initial function in the equation (1.15).

(A.2). $h \in C_b(\mathbf{R}^N)$. For any $r \in \mathbf{R}^N$, the set $h^{-1}((r, \infty))$ is bounded or $\mathbf{R}^N$.

Then one can easily obtain the following from Theorem 1.

Corollary 2 Suppose that (A.1)-(A.2) hold. Then there exists a unique continuous solution $\{u(t, \cdot)\}_{t \geq 0}$ to (1.15) with $u(0, \cdot) = h(\cdot)$ on $[0, \infty)$. In addition, for any $r \in \mathbf{R}$, $\{u(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$ is a unique solution to (1.13)-(1.14) with $D(0) = h^{-1}((r, \infty))$ on $[0, \infty)$. In particular, $\{u(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$ only depends on the set $u(0, \cdot)^{-1}((r, \infty))$.

The following theorem collects some of elementary properties of solutions to (1.13)-(1.14).

Theorem 2 Suppose that (A.0)-(A.1) hold. Let $\{D(t)\}_{t \geq 0}$ be a unique solution to (1.13)-(1.14) with $D(0) = D$ on $[0, \infty)$. Then the following holds.

(a) $t \mapsto D(t)$ is nonincreasing on $[0, \infty)$.

(b) For any $t \leq T^* := \text{Vol}(D(0))$,

$$\text{Vol}(D(0) \setminus D(t)) = t. \quad (2.10)$$

(c) $D(t) = \emptyset$ for $t \geq T^*$.

(d) Let $\{D_1(t)\}_{t \geq 0}$ be a solution to (1.13)-(1.14) on $[0, \infty)$ such that $D_1(0)$ is a bounded, convex, open set which contains D . Then

$$D(t) \subset D_1(t) \quad \text{for all } t \geq 0, \quad (2.11)$$

where the equality holds if and only if $D(0) = D_1(0)$.

Under

(A.3). $R \in C(S^{N-1} : [0, \infty))$,

we give the relation between the solution to (1.13)-(1.14) and the viscosity solution of (1.20).

Theorem 3 Suppose that (A.0)-(A.1) and (A.3) hold. Then for a unique solution $\{D(t)\}_{t \geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0, \infty)$, $I_{D(t)}(x)$ and $I_{D(t)^-}(x)$ is a viscosity supersolution and a viscosity subsolution to (1.20) in $(0, \infty) \times \mathbf{R}^N$, respectively.

As a corollary, we obtain

Corollary 3 Suppose that (A.1)-(A.3) hold. Then a solution $\{u(t, \cdot)\}_{t \geq 0}$ to (1.15) with $u(0, \cdot) = h(\cdot)$ on $[0, \infty)$ is a viscosity solution to (1.20) in $(0, \infty) \times \mathbf{R}^N$.

3 Lemma

In this section we give lemmas which will be used in the next section.

We extend $Y_m(t, \cdot)$ as a function on $\mathbf{R}^N$ so that

$$\bar{Y}_m(t, x) = \begin{cases} 0 & (x \in D^c \cap (\mathbf{Z}^N/m)), \\ Y_m(t, [mx]/m) & (x = (x_i)_{i=1}^N \in \mathbf{R}^N), \end{cases} \quad (3.1)$$

where $[mx] := ([mx_i])_{i=1}^N$ and $[mx_i]$ denotes an integer part of mx_i .

Remark 6 For $z \in \mathbf{Z}^N/m$,

$$Y_m(t, z) = \frac{1}{m^N} \int_{\{x \in \mathbf{R}^N | [mx] = mz\}} \bar{Y}_m(t, x) dx.$$

Lemma 1 Suppose that (A.0)-(A.1) hold. Then $\{\bar{Y}_m(\cdot, \cdot)\}_{m \geq 1}$ is tight in $D([0, \infty) : \mathcal{S})$, and any weak limit point of $\{\bar{Y}_m(\cdot, \cdot)\}_{m \geq 1}$ belongs to the set $C([0, \infty) : \mathcal{S})$.

(Proof). Since $\mathcal{S}$ is compact and since $t \mapsto \bar{Y}_m(t, x)$ is nonincreasing for any $x \in \mathbf{R}^N$, we only have to show the following (see [9, p. 129, Corollary 7.4 and p. 148, Theorem 10.2]): for any $\eta > 0$ and $T > 0$, there exists $\delta > 0$ such that for any i for which $1 \leq i \leq [T/\delta] + 1$,

$$\lim_{m \rightarrow \infty} P(\|\bar{Y}_m(i\delta, \cdot) - \bar{Y}_m((i-1)\delta, \cdot)\|_{L^1([-K, K]^N)} \geq \eta) = 0. \quad (3.2)$$

Indeed, for any s and t for which $(i-1)\delta \leq s \leq t \leq i\delta$,

$$\bar{Y}_m(s, x) - \bar{Y}_m(t, x) = 0 \text{ or } 1,$$

and

$$\begin{aligned} d_{\mathcal{S}}(\bar{Y}_m(t, \cdot), \bar{Y}_m(s, \cdot))^2 &\leq \|\bar{Y}_m(t, \cdot) - \bar{Y}_m(s, \cdot)\|_{L^2([-K, K]^N)}^2 \\ &= \|\bar{Y}_m(i\delta, \cdot) - \bar{Y}_m((i-1)\delta, \cdot)\|_{L^1([-K, K]^N)}. \end{aligned}$$

For $\delta < \eta/2$ and $m \geq 1$, by Chebychev's inequality and Itô's formula (see [15]),

$$\begin{aligned}
& P(\|\bar{Y}_m(i\delta, \cdot) - \bar{Y}_m((i-1)\delta, \cdot)\|_{L^1([-K, K]^N)} \geq \eta) \quad (3.3) \\
& \leq \left(\frac{2}{\eta}\right)^2 E\left[\left|\sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i-1)\delta, z))\right| \frac{1}{m^N} \right. \\
& \quad \left. + \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m) ds\right]^2 \\
& = \left(\frac{2}{\eta}\right)^2 m^{-N} E\left[\int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m) ds\right] \\
& \leq \left(\frac{2}{\eta}\right)^2 m^{-N} \delta \rightarrow 0 \quad \text{as } m \rightarrow \infty
\end{aligned}$$

(see (2.2) for notation). Indeed,

$$\begin{aligned}
& \|\bar{Y}_m(i\delta, \cdot) - \bar{Y}_m((i-1)\delta, \cdot)\|_{L^1([-K, K]^N)} \\
& = - \sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i-1)\delta, z)) \frac{1}{m^N} - \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m) ds \\
& \quad + \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s, \cdot), D_m) ds.
\end{aligned}$$

Q. E. D.

Remark 7 In (3.3), if $Y_m(s, \cdot) \equiv 0$, then $\omega_1(Y_m(s, \cdot), D_m) = 0$.

Lemma 2 Suppose that (A.0)-(A.1) hold. Then there exist a subsequence $\{m_k\}_{k \geq 1} \subset \mathbf{N}$ and stochastic processes $\{\bar{Y}_{1, m_k}(\cdot, \cdot)\}_{k \geq 1}$ on a probability space $(\Omega_1, \mathbf{B}_1, P_1)$ such that the probability law of $\{\bar{Y}_{1, m_k}(\cdot, \cdot)\}_{k \geq 1}$ is the same as that of $\{\bar{Y}_{m_k}(\cdot, \cdot)\}_{k \geq 1}$, and such that $\{\bar{Y}_{1, m_k}(\cdot, \cdot)\}_{k \geq 1}$ is convergent in $D([0, \infty) : \mathcal{S})$, P_1 -almost surely, and such that the following holds P_1 -almost surely: for any $T > 0$ and $\varphi \in C([-K, K]^N)$

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left| \sum_{z \in D_{m_k}} \varphi(z) (Y_{1, m_k}(t, z) - Y_{1, m_k}(0, z)) \frac{1}{m_k^N} \right. \quad (3.4) \\
& \quad \left. + \int_0^t \sum_{z \in D_{m_k}} \varphi(z) \omega_1(Y_{1, m_k}(s, \cdot), \{z\}) ds \right| \rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Here Y_{1,m_k} is defined by $\bar{Y}_{1,m_k}$ in the same way as in Remark 6.

(Proof). By Lemma 1 and Skorohod's Theorem (see [9, p. 102, Theorem 1.8]), there exist a subsequence $\{m_{0,k}\}_{k \geq 1} \subset \mathbf{N}$ and stochastic processes $\{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1}$ on a probability space $(\Omega_1, \mathbf{B}_1, P_1)$ such that the probability law of $\{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1}$ is the same as that of $\{\bar{Y}_{m_{0,k}}(\cdot, \cdot)\}_{k \geq 1}$, and such that $\{\bar{Y}_{1,m_{0,k}}(\cdot, \cdot)\}_{k \geq 1}$ is convergent in $D([0, \infty) : \mathcal{S})$, P_1 -almost surely.

As in (3.3), by Doob-Kolmogorov's inequality (see [15]), for any $T > 0$ and $\varphi \in C([-K, K]^N)$

$$\begin{aligned} E_1[\sup_{0 \leq t \leq T} | \sum_{z \in D_{m_{0,k}}} \varphi(z) (Y_{1,m_{0,k}}(t, z) - Y_{1,m_{0,k}}(0, z)) \frac{1}{(m_{0,k})^N} \\ + \int_0^t \sum_{z \in D_{m_{0,k}}} \varphi(z) \omega_1(Y_{1,m_{0,k}}(s, \cdot), \{z\}) ds |^2] \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (3.5)$$

Since a L^2 -convergent sequence of random variables has an almost surely-convergent subsequence, and since $C([-K, K]^N)$ is separable, one can complete the proof by the diagonal method.

Q. E. D.

When it is not confusing, we write $\bar{Y}_{1,m_k} = \bar{Y}_{m_k}$ and $Y_{1,m_k} = Y_{m_k}$ on $(\Omega_1, \mathbf{B}_1, P_1)$ for the sake of simplicity.

Take $x_0 \in D$ and $r_0 > 0$ so that $U_{4r_0}(x_0) := \{y \in \mathbf{R}^N : |x_0 - y| < 4r_0\} \subset D$, and put $U_0 := U_{2r_0}(x_0)$. Then

$$V_0 := \inf_{x \in \partial U_0} \text{Vol}(U_{3r_0}(x_0) \cap H(x_0 - x, x)) > 0. \quad (3.6)$$

Put, on $(\Omega_1, \mathbf{B}_1, P_1)$,

$$\tau_m := \inf\{t > 0 | Y_{1,m}(t, z) = 0 \text{ for some } z \in (\mathbf{Z}^N/m) \cap U_0\}. \quad (3.7)$$

Then the following holds.

Lemma 3 Suppose that (A.0)-(A.1) hold. Then

$$P_1(V_0 \leq \liminf_{k \rightarrow \infty} \tau_{m_k} \leq \limsup_{k \rightarrow \infty} \tau_{m_k} \leq \text{Vol}(D)) = 1. \quad (3.8)$$

(Proof). By (3.4), for any $t > \text{Vol}(D)$,

$$\begin{aligned}
& \limsup_{k \rightarrow \infty} \{\min(\tau_{m_k}, t)\} \\
&= \limsup_{k \rightarrow \infty} \int_0^{\min(\tau_{m_k}, t)} \omega_1(Y_{m_k}(s, \cdot), D_{m_k}) ds \\
&\leq \limsup_{k \rightarrow \infty} \sum_{z \in D_{m_k}} (Y_{m_k}(0, z) - Y_{m_k}(\min(\tau_{m_k}, t), z)) \frac{1}{m_k^N} \leq \text{Vol}(D)
\end{aligned} \tag{3.9}$$

P_1 - almost surely. We also have

$$\begin{aligned}
V_0 &\leq \liminf_{k \rightarrow \infty} \sum_{z \in D_{m_k}} (Y_{m_k}(0, z) - Y_{m_k}(\tau_{m_k}, z)) \frac{1}{m_k^N} \\
&\leq \liminf_{k \rightarrow \infty} \int_0^{\tau_{m_k}} \omega_1(Y_{m_k}(s, \cdot), D_{m_k}) ds = \liminf_{k \rightarrow \infty} \tau_{m_k}
\end{aligned} \tag{3.10}$$

P_1 - almost surely.

Q. E. D.

The following lemma can be proved in the same way as in [4, section 5.2] and the proof is omitted.

Lemma 4 Suppose that (A.1) holds. Let F and F_m ($m \geq 1$) be closed convex sets in $\mathbf{R}^N$ such that ∂F and ∂F_m ($m \geq 1$) are closed hypersurfaces and such that $d_H(F_m, F) \rightarrow 0$ as $m \rightarrow \infty$. Then $\omega_1(I_{F_m}(\cdot), dx)$ weakly converges to $\omega_1(F, dx)$ as $m \rightarrow \infty$, that is, for any $\varphi \in C_o(\mathbf{R}^N)$,

$$\lim_{m \rightarrow \infty} \int_{\mathbf{R}^N} \varphi(x) \omega_1(I_{F_m}(\cdot), dx) = \int_{\mathbf{R}^N} \varphi(x) \omega_1(I_F(\cdot), dx). \tag{3.11}$$

We denote by $X(\cdot, \cdot) \in C([0, \infty) : \mathcal{S})$ the P_1 -a.s. limit of $\bar{Y}_{1, m_k}(\cdot, \cdot)$ as $k \rightarrow \infty$. Then we have

Lemma 5 Suppose that (A.0)-(A.1) hold. Then there exists a solution $\{D(t)\}_{t \in [0, V_0)}$ to (1.13)-(1.14) on $[0, V_0)$ such that the following holds P_1 -almost surely:

$$X(t, x) = I_{D(t)}(x), \quad dx - a.e. \quad \text{for all } t \in [0, V_0). \tag{3.12}$$

(Proof). For $p \in \mathbf{S}^{N-1}$, let $C(x_0, r_0; p)$ denote a semi-infinite cylinder

$$\{x_0 + rp + x : r \geq 0, |x| \leq r_0, \langle x, p \rangle = 0, x \in \mathbf{R}^N\}$$

which can be obtained by moving a $(N-1)$ -dimensional ball

$$\{x_0 + x : |x| \leq r_0, \langle x, p \rangle = 0, x \in \mathbf{R}^N\}$$

in the positive direction of p .

Take $p_1, \dots, p_{k_0} \in \mathbf{S}^{N-1}$ for some $k_0 \in \mathbf{N}$ so that

$$co D \subset \cup_{i=1}^{k_0} C(x_0, r_0; p_i).$$

For $i = 1, \dots, k_0$, take $\{q_{i1}, \dots, q_{i(N-1)}\}$ so that $\{q_{i1}, \dots, q_{i(N-1)}, p_i\}$ is an orthonormal basis in $\mathbf{R}^N$, and put

$$C_{m_k}(t) := co Y_{m_k}(t, \cdot)^{-1}(1). \quad (3.13)$$

For $x = (x_k)_{k=1}^{N-1} \in \mathbf{R}^{N-1}$ for which $|x| \leq 2r_0$, put also

$$\tilde{X}_{m_k, i}(t, x) := -\sup\{r > 0 | x_0 + rp_i + \sum_{j=1}^{N-1} q_{ij}x_j \in C_{m_k}(t)\}. \quad (3.14)$$

Then $\tilde{X}_{m_k, i}(t, \cdot)$ is a bounded convex function on $\{x \in \mathbf{R}^{N-1} : |x| \leq 7r_0/4\}$ for $t \in [0, \tau_{m_k})$ if $m_k \geq 8N^{1/2}/r_0$.

It is known that the set of bounded convex functions with the same domain is compact as the set of continuous functions defined on K for every compact subset K of the interior of their domain (see [4, section 3.3]).

Therefore, by Lemma 3 and the diagonal method, there exists a subsequence $\{\tilde{X}_{\tilde{m}_k, i}(t, \cdot)\}_{k \geq 1}$ of $\{\tilde{X}_{m_k, i}(t, \cdot)\}_{k \geq 1}$ and a convex function $\tilde{X}_i(t, \cdot)$ such that for any $t \in \mathbf{Q} \cap [0, V_0)$ and $i = 1, \dots, k_0$,

$$\lim_{k \rightarrow \infty} \sup_{x \in \mathbf{R}^{N-1}, |x| \leq 3r_0/2} |\tilde{X}_{\tilde{m}_k, i}(t, x) - \tilde{X}_i(t, x)| = 0 \quad (3.15)$$

(Notice that $\{\tilde{m}_k\}_{k \geq 1}$ can be random.).

It is clear that there exists a nonincreasing family of compact convex sets $\{\tilde{C}(t)\}_{t \in \mathbf{Q} \cap [0, V_0)}$ such that for any $t \in \mathbf{Q} \cap [0, V_0)$,

$$\lim_{k \rightarrow \infty} d_H(C_{\tilde{m}_k}(t), \tilde{C}(t)) = 0, \quad (3.16)$$

$$\tilde{X}_i(t, x) = -\sup\{r > 0 | x_0 + rp_i + \sum_{j=1}^{N-1} q_{ij}x_j \in \tilde{C}(t)\}$$

for all $i = 1, \dots, k_0$, and $x = (x_k)_{k=1}^{N-1} \in \mathbf{R}^{N-1}$ for which $|x| \leq 3r_0/2$.

In particular,

$$D \subset \tilde{C}(0) \quad (\text{by (2.2)}), \quad (3.17)$$

$$\lim_{k \rightarrow \infty} \|X_{1, \tilde{m}_k}(t, \cdot) - I_{\tilde{C}(t) \circ \cap D}(\cdot)\|_{L^2([-K, K]^N)} = 0$$

for all $t \in \mathbf{Q} \cap [0, V_0)$, where $X_{1, \tilde{m}_k}$ is defined by $Y_{1, \tilde{m}_k}$ in the same way as in (2.4). When it is not confusing, we write $X_{1, \tilde{m}_k} = X_{\tilde{m}_k}$ on $(\Omega_1, \mathbf{B}_1, P_1)$ for the sake of simplicity.

The following also holds: for all $t \in [0, V_0) \cap \mathbf{Q}$,

$$\lim_{k \rightarrow \infty} \|\bar{Y}_{\tilde{m}_k}(t, \cdot) - X_{\tilde{m}_k}(t, \cdot)\|_{L^2([-K, K]^N)} = 0. \quad (3.18)$$

Indeed, if $X_m(t, x) \neq \bar{Y}_m(t, x)$, then

$$\text{dist}(x, \partial(C_m(t)^o \cap D)) \leq \frac{N^{1/2}}{m}$$

; and by (3.16), the volume of the $N^{1/2}/\tilde{m}_k$ -neighborhood of the set $\partial D \cup \partial C_{\tilde{m}_k}(t)$ converges to zero as $k \rightarrow \infty$ for $t \in [0, V_0) \cap \mathbf{Q}$.

For $t \in [0, V_0) \setminus \mathbf{Q}$, put

$$\tilde{C}(t) := \cap_{s \in \mathbf{Q} \cap [0, t)} \tilde{C}(s). \quad (3.19)$$

Then, by (3.17)-(3.19), the following holds P_1 -a.s.:

$$X(t, x) = I_{\tilde{C}(t) \circ \cap D}(x), \quad dx - a.e., \quad \text{for all } t \in [0, V_0), \quad (3.20)$$

since $\{\bar{Y}_{\tilde{m}_k}\}_{k \geq 1}$ is a subsequence of a convergent sequence $\{\bar{Y}_{m_k}\}_{k \geq 1}$ and since $X \in C([0, \infty) : \mathcal{S})$ is the P_1 -a.s. limit, in $D([0, \infty) : \mathcal{S})$, of $\bar{Y}_{m_k}$ as $k \rightarrow \infty$, and since $\{\tilde{C}(t)\}_{t \in [0, V_0) \cap \mathbf{Q}}$ is nonincreasing in t .

Put

$$D(t) := \tilde{C}(t)^o \cap D. \quad (3.21)$$

Then (1.13) holds for all $t \in [0, V_0)$, since $D = D(0)$ by (3.17) and since

$$D(t) \supset \{\text{co } (\tilde{C}(t)^o \cap D)\} \cap D = (\text{co } D(t)) \cap D \supset D(t) \cap D = D(t).$$

On $[0, V_0)$,

$$\omega_1(I_{\tilde{C}(t)}(\cdot), dx) = \omega_1(I_{D(t)}(\cdot), dx) \quad dt - a.e., \quad (3.22)$$

since

$$\tilde{C}(t) \setminus (\text{co } D(t))^- \subset \tilde{C}(t) \setminus D(t)^- \subset D^c$$

by (3.21), where D^c denotes a complement of D , and since

$$\int_0^{V_0} ds \omega_1(I_{\tilde{C}(s)}(\cdot), D^c) = \int_{D^c} (I_{D(0)}(x) - I_{D(V_0)}(x)) dx = 0$$

by (3.4), (3.20) and Lemma 4. Here we used the fact that (3.16) holds except for at most countably many $t \in [0, V_0)$.

Indeed, $t \mapsto C_{\tilde{m}_k}(t)$ is nonincreasing and (3.16) holds for all $t \in \mathbf{Q} \cap [0, V_0)$. Therefore, if $C_{\tilde{m}_k}(t)$ does not converge to $\tilde{C}(t)$ as $k \rightarrow \infty$, then $(\tilde{C}(t) \setminus \tilde{C}(t+))^o$ is not empty and has a positive Lebesgue measure by (3.19), where $\tilde{C}(t+) := \bigcup_{s>t} \tilde{C}(s)$. Besides, $(\tilde{C}(t) \setminus \tilde{C}(t+))^o$ are disjoint for different t .

By (3.4), Lemma 4, (3.20)-(3.22), (1.14) holds for all $t \in [0, V_0)$ since (3.16) holds except for at most countably many $t \in [0, V_0)$ as we mentioned above.

Q. E. D.

The following lemma implies the uniqueness of the solution to (1.13)-(1.14).

Lemma 6 Suppose that (A.1) hold. For $T > 0$, if $\{D_i(t)\}_{0 \leq t < T}$ ($i = 1, 2$) are solutions to (1.13)-(1.14) on $[0, T)$ for which $D_1(0) \subset D_2(0)$, then $D_1(t) \subset D_2(t)$ for all $t \in [0, T)$. In particular, for all $t \in [0, \min(\text{Vol}(D_1(0)), T))$,

$$d_H(D_1(t), D_2(t)^c) \geq d_H(D_1(0), D_2(0)^c). \quad (3.23)$$

(Proof). For each $t \geq 0$, put

$$\tilde{D}(t) := D_1(t)^- \cap D_2(t)^c, u_i(t, \cdot) := I_{D_i(t)}(\cdot) \quad u_i^-(t, \cdot) := I_{D_i(t)^-}(\cdot),$$

$$N_i(t) := \cup_{x \in \partial \tilde{D}(t) \cap \partial D_i(t)} \{p \in S^{N-1} | \sigma^+(u_i, -p, t, x) = 1\}$$

($i = 1, 2$). Then $N_2(t) \subset N_1(t)$.

Take a nondecreasing sequence $\{\eta_n\}_{n \geq 1}$ of nondecreasing C^1 -functions such that

$$\eta_n(r) = 0 \quad \text{for all } r \leq 0, \quad \eta_n(r) = 1 \quad \text{for all } r \geq \frac{1}{n}, \quad (3.24)$$

and for $r \in \mathbf{R}$, put

$$\zeta_n(r) = \int_0^r \eta_n(s) ds. \quad (3.25)$$

Then since $t \mapsto u_i(t, x)$ and $t \mapsto u_i^-(t, x)$ are respectively right and left continuous for any $x \in \mathbf{R}^N$, for $t < \min(\text{Vol}(D_1(0)), T)$ and $x \in \mathbf{R}^N$,

$$\begin{aligned} & \zeta_n(u_1^-(t, x) - u_2(t, x) - 1) - \zeta_n(u_1^-(0, x) - u_2(0, x)) \\ = & \int_0^t \zeta_n(u_1^-(s, x) - u_2(s, x) - s/t)(u_1^-(ds, x) - u_2(ds, x)) \\ & - \frac{1}{t} \int_0^t \eta_n(u_1^-(s, x) - u_2(s, x) - s/t) ds. \end{aligned} \quad (3.26)$$

Since $\zeta_n \geq 0$ and $D_1(0) \subset D_2(0)$, we have

$$\begin{aligned} 0 & \leq \int_0^t ds \int_{\mathbf{R}^N} \zeta_n(u_1^-(s, x) - u_2(s, x) - s/t) \\ & \quad \times (\omega_1(u_2(s, \cdot), dx) - \omega_1(u_1(s, \cdot), dx)) \\ & \quad - \frac{1}{t} \int_0^t ds \int_{\mathbf{R}^N} \eta_n(u_1^-(s, x) - u_2(s, x) - s/t) dx \\ & \rightarrow \int_0^t (1 - s/t) (\omega_1(u_2(s, \cdot), \tilde{D}(s)) - \omega_1(u_1(s, \cdot), \tilde{D}(s))) ds \\ & \quad - \frac{1}{t} \int_0^t ds \int_{\tilde{D}(s)} dx \quad (\text{as } n \rightarrow \infty) \\ & \leq -\frac{1}{t} \int_0^t ds \int_{\tilde{D}(s)} dx, \end{aligned} \quad (3.27)$$

which implies the first assertion of this lemma.

Suppose that (3.23) does not hold. Then there exists $a \in (0, d_H(D_1(0), D_2(0)^c))$ such that

$$\inf\{d_H(D_1(t), D_2(t)^c) | t \in [0, \min(\text{Vol}(D_1(0)), T))\} < a.$$

Take $p_a \in \mathbf{S}^{N-1}$ and $t_a \in [0, \min(\text{Vol}(D_1(0)), T))$ so that

$$ap_a + D_1(t_a) \not\subset D_2(t_a).$$

Since $ap_a + D_1(0) \subset D_2(0)$ and $\{ap_a + D_1(t)\}_{0 \leq t < T}$ is a solution to (1.13)-(1.14) on $[0, T)$, this contradicts the first assertion of this lemma.

Q. E. D.

Take $\varphi \in C^2(\mathbf{R}^N)$ for which $D\varphi(x_o) \neq 0$ for some $x_o \in \mathbf{R}^N$.

Let I_N denote a $N \times N$ -identity matrix and put

$$f_N := \frac{D\varphi(x_o)}{|D\varphi(x_o)|}, \quad (g_1 \cdots g_N) := I_N - f_N \otimes f_N.$$

Take $\{f_1, \dots, f_{N-1}\}$ so that $\{f_1, \dots, f_N\}$ is an orthonormal basis of $\mathbf{R}^N$. Then the following holds.

Lemma 7 (i) $\langle g_i, f_N \rangle = 0$ ($1 \leq i \leq N$).

(ii) For i for which $\partial_i \varphi(x_o) := \partial \varphi(x_o) / \partial x_i \neq 0$,

$$g_i = - \sum_{k \neq i} \frac{\partial_k \varphi(x_o)}{\partial_i \varphi(x_o)} g_k.$$

(iii) $\text{span}(g_1, \dots, g_N) = \text{span}(f_1, \dots, f_{N-1})$.

(iv) $D(D\varphi(x_o)/|D\varphi(x_o)|)(\mathbf{R}^N) \subset \text{span}(g_1, \dots, g_N)$. As a mapping on $\text{span}(g_1, \dots, g_N)$, eigenvalues and eigenvectors of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are the same as those of $(g_1 \cdots g_N)(D^2\varphi(x_o)/|D\varphi(x_o)|)(g_1 \cdots g_N)$. In particular, all eigenvalues of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are real.

(v) If eigenvalues $\lambda_1 \leq \dots \leq \lambda_{N-1}$ of $-D(D\varphi(x_o)/|D\varphi(x_o)|)$ as a mapping on $\text{span}(g_1, \dots, g_N)$ are nonnegative, then

$$\prod_{i=1}^{N-1} \lambda_i = \frac{G(D\varphi(x_o), D^2\varphi(x_o))}{|D\varphi(x_o)|}. \quad (3.28)$$

(Proof). It is easy to see that (i) and (ii) hold. Take i for which $\partial_i \varphi(x_o) \neq 0$. Then, by (i) and (ii), we only have to show, to prove (iii), that $\{g_j\}_{j \neq i}$ is independent. Suppose that for $j = 1, \dots, N$,

$$\sum_{k \neq i} \lambda_k \left(\delta_{kj} - \frac{\partial_k \varphi(x_o) \partial_j \varphi(x_o)}{|D\varphi(x_o)|^2} \right) = 0. \quad (3.29)$$

Putting $j = i$ in (3.29), we obtain

$$\sum_{k \neq i} \lambda_k \frac{\partial_k \varphi(x_o) \partial_i \varphi(x_o)}{|D\varphi(x_o)|^2} = 0,$$

from which

$$\sum_{k \neq i} \lambda_k \partial_k \varphi(x_o) = 0. \quad (3.30)$$

Putting $j \neq i$ in (3.29), we obtain

$$\lambda_j - \partial_j \varphi(x_o) \sum_{k \neq i} \lambda_k \frac{\partial_k \varphi(x_o)}{|D\varphi(x_o)|^2} = 0,$$

from which $\lambda_j = 0$ for $j \neq i$, by (3.30).

We prove (iv). It is easy to see that

$$D \left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|} \right) = (g_1 \cdots g_N) \frac{D^2 \varphi(x_o)}{|D\varphi(x_o)|}. \quad (3.31)$$

Hence

$$D \left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|} \right) \left(\sum_{i=1}^N x_i g_i \right) = \lambda \sum_{i=1}^N x_i g_i$$

if and only if

$$(g_1 \cdots g_N) \frac{D^2 \varphi(x_o)}{|D\varphi(x_o)|} (g_1 \cdots g_N) \left(\sum_{i=1}^N x_i g_i \right) = \lambda \sum_{i=1}^N x_i g_i,$$

since

$$(g_1 \cdots g_N)^2 = (g_1 \cdots g_N). \quad (3.32)$$

Put $P := (f_1 \cdots f_N)$ and $Q := (f_1 \cdots f_{N-1})$. The proof of (v) is divided into the following.

(STEP I) The eigenvalues of

$$-(I_N - f_N \otimes f_N) \frac{D^2\varphi(x_o)}{|D\varphi(x_o)|} (I_N - f_N \otimes f_N) + f_N \otimes f_N$$

are those of

$$\begin{pmatrix} -Q^* D\left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right) Q & o \\ o^* & 1 \end{pmatrix}.$$

(STEP II) The eigenvalues of $Q^* D(D\varphi(x_o)/|D\varphi(x_o)|) Q$ are those of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ on $\text{span}(g_1, \dots, g_N)$.

(Proof of Step I). For $\lambda \in \mathbf{R}$, denoting by P^* the transposed matrix of P ,

$$\begin{aligned} & \det\left(- (I_N - f_N \otimes f_N) \frac{D^2\varphi(x_o)}{|D\varphi(x_o)|} (I_N - f_N \otimes f_N) + f_N \otimes f_N - \lambda I_N\right) \\ = & \det\left(- \begin{pmatrix} I_{N-1} & o \\ o^* & o \end{pmatrix} P^* \frac{D^2\varphi(x_o)}{|D\varphi(x_o)|} P \begin{pmatrix} I_{N-1} & o \\ o^* & o \end{pmatrix} + \begin{pmatrix} O & o \\ o^* & 1 \end{pmatrix} - \lambda I_N\right) \\ = & \det\left(\begin{pmatrix} -Q^* \frac{D^2\varphi(x_o)}{|D\varphi(x_o)|} Q & o \\ o^* & 1 \end{pmatrix} - \lambda I_N\right) \end{aligned}$$

since

$$P^* P = I_N, \quad P \begin{pmatrix} O & o \\ o^* & 1 \end{pmatrix} P^* = f_N \otimes f_N.$$

(3.31) completes the proof since $\langle f_i, f_N \rangle = 0$ if $i \neq N$.

(Proof of Step II). Let $x = (x_i)_{i=1}^{N-1} \in \mathbf{R}^{N-1}$ and $\lambda \in \mathbf{R}$. Suppose that

$$Q^* D\left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right) Q x = \lambda x. \quad (3.33)$$

Then

$$Q Q^* D\left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right) \left(\sum_{1 \leq i \leq N-1} x_i f_i\right) = \lambda \sum_{1 \leq i \leq N-1} x_i f_i$$

and henceforth by (3.31),

$$D\left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right) \sum_{1 \leq i \leq N-1} x_i f_i = \lambda \sum_{1 \leq i \leq N-1} x_i f_i \quad (3.34)$$

since, by (iii),

$$QQ^*(I_N - f_N \otimes f_N) = I_N - f_N \otimes f_N.$$

It is easy to see that (3.34) implies (3.33).

Q. E. D.

For $i = 1, \dots, N$, put

$$y_i(x) := \left((\delta_{ij} - 1) \frac{\partial_j \varphi(x)}{|D\varphi(x)|} + \delta_{ij} \varphi(x) \right)_{j=1}^N.$$

Then

Lemma 8 Suppose that all eigenvalues of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are nonpositive. Then, for $i = 1, \dots, N$,

$$\frac{\partial_i \varphi(x_o)}{|D\varphi(x_o)|} G(D\varphi(x_o), D^2\varphi(x_o)) = \det(Dy_i(x_o)). \quad (3.35)$$

(Proof). For the sake of simplicity, we assume that $i = N$.

We first consider the case when $\partial_N \varphi(x_o) \neq 0$. By (ii) in Lemma 7, it is easy to see that the following holds:

$$\left(\begin{array}{cc} I_{N-1} & o \\ -\frac{D\varphi(x_o)^*}{\partial_N \varphi(x_o)} & \end{array} \right) Dy_N(x_o) = D\left(-\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right) + \left(\begin{array}{c} O \\ -D\varphi(x_o)^* \end{array} \right). \quad (3.36)$$

By (i) and (iv) in Lemma 7, the eigenvalues and eigenvectors of $D(-D\varphi(x_o)/|D\varphi(x_o)|)$ on $\text{span}(g_1, \dots, g_N)$ are real and are also those of the left hand side (l.h.s. for short) of (3.36).

We show that all eigenvalues of the l.h.s. of (3.36) are those of $D(-D\varphi(x_o)/|D\varphi(x_o)|)$ on $\text{span}(g_1, \dots, g_N)$ and $-\partial_N \varphi(x_o)$.

By (i) and (iv) in Lemma 7, there exists an invariant subspace, which contains f_N , with an eigenvalue λ of the l.h.s. of (3.36).

Take $\ell \geq 1$ such that

$$\left(\left(\begin{array}{cc} I_{N-1} & o \\ -\frac{D\varphi(x_o)^*}{\partial_N \varphi(x_o)} & \end{array} \right) Dy_N(x_o) - \lambda \right)^\ell f_N = o. \quad (3.37)$$

Then $(-\partial_N \varphi(x_o) - \lambda)^\ell f_N \in \text{span}(g_1, \dots, g_N)$ since

$$f_N = \frac{|D\varphi(x_o)|}{\partial_N \varphi(x_o)} ((\delta_{j_N})_{j=1}^N - g_N).$$

Hence $\lambda = -\partial_N \varphi(x_o)$ by (i) in Lemma 7.

Suppose that $\partial_N \varphi(x_o) = 0$. Then, by (3.31) and (i) in Lemma 7, for $x \in \mathbf{R}^N$,

$$\langle f_N, Dy_N(x_o)x \rangle = \left\langle f_N, D\left(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\right)x \right\rangle = 0, \quad (3.38)$$

which implies that $Dy_N(x_o)(\mathbf{R}^N)$ is at most $(N-1)$ -dimensional and henceforth (3.35) holds.

Q. E. D.

4 Proof

In this section we prove the results in section 2.

(Proof of Theorem 1). By Lemmas 1-6, there exists a unique (nonrandom) solution $\{D(t)\}_{0 \leq t < V_0}$ (see (3.6) for notation) of (1.13)-(1.14) on $[0, V_0)$ such that $I_{D(\cdot)} \in C([0, V_0) : \mathcal{S})$ and that the following holds: for any $T \in [0, V_0)$ and $\gamma > 0$,

$$\lim_{m \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} d_{\mathcal{S}}(\bar{Y}_m(t, \cdot), I_{D(t)}(\cdot)) \geq \gamma\right) = 0. \quad (4.1)$$

Therefore

$$\lim_{m \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} \|\bar{Y}_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma\right) = 0, \quad (4.2)$$

since, for $m \geq 1$ and $t \in [0, T]$,

$$\begin{aligned} & \|\bar{Y}_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)}^2 \\ &= \int_{[-K, K]^N} (\bar{Y}_m(t, x) - I_{D(t)}(x))^2 dx. \end{aligned}$$

We prove that the following holds:

$$\lim_{m \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma\right) = 0. \quad (4.3)$$

For any s and t for which $0 \leq s < t \leq T$,

$$\begin{aligned} & \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \\ \leq & \|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^2([-K, K]^N)} + \|X_m(s, \cdot) - \bar{Y}_m(s, \cdot)\|_{L^2([-K, K]^N)} \\ & + \|\bar{Y}_m(s, \cdot) - I_{D(s)}(\cdot)\|_{L^2([-K, K]^N)} + \|I_{D(s)}(\cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)}. \end{aligned} \quad (4.4)$$

Let $U_{-N^{1/2}/m}(D) := \{x \in D \mid \text{dist}(x, D^c) > N^{1/2}/m\}$. Then

$$\begin{aligned} & \|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^2([-K, K]^N)}^2 = \|X_m(t, \cdot) - X_m(s, \cdot)\|_{L^1([-K, K]^N)} \quad (4.5) \\ \leq & 2^N \sum_{z \in D_m} (Y_m(s, z) - Y_m(t, z)) \frac{1}{m^N} + \text{Vol}(D \setminus U_{-N^{1/2}/m}(D)) \end{aligned}$$

(see (2.2) for notation). Indeed, if $x = (x_i)_{i=1}^N \in U_{-N^{1/2}/m}(D) \setminus (\text{co } Y_m(t, \cdot)^{-1}(1))$, then $Y_m(t, z) = 0$ for some $z = (z_i)_{i=1}^N \in \mathbf{Z}^N/m$ for which $|x_i - z_i| \leq 1/m$ for all $i = 1, \dots, N$.

In the same way as in (3.5), by (4.5), for any $\gamma > 0$, there exists $\delta > 0$ such that the following holds: for any $s \in [0, T - \delta]$,

$$\lim_{m \rightarrow \infty} P\left(\sup_{s \leq s_1 \leq s + \delta} \|X_m(s_1, \cdot) - X_m(s, \cdot)\|_{L^2([-K, K]^N)} \geq \gamma\right) = 0. \quad (4.6)$$

Since, for any $t \in [0, V_0)$, any subsequence of $\{C_m(t)\}_{m \geq 1}$ has a convergent subsequence (see the proof of Lemma 5),

$$\lim_{m \rightarrow \infty} \|\bar{Y}_m(t, \cdot) - X_m(t, \cdot)\|_{L^2([-K, K]^N)} = 0 \quad (4.7)$$

for all $t \in [0, V_0)$, P_1 -almost surely (see the discussion after (3.18)). Hence, for any $\gamma > 0$,

$$\lim_{m \rightarrow \infty} P(\|\bar{Y}_m(s, \cdot) - X_m(s, \cdot)\|_{L^2([-K, K]^N)} \geq \gamma) = 0. \quad (4.8)$$

$I_{D(\cdot)} \in C([0, V_0) : L^2([-K, K]^N))$, since

$$\|I_{D(s)}(\cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)}^2 = \int_{[-K, K]^N} I_{D(s)}(x) dx - \int_{[-K, K]^N} I_{D(t)}(x) dx,$$

and since $t \mapsto \int_{[-K, K]^N} I_{D(t)}(x) dx$ is continuous on $[0, V_0)$.

(4.2) and the discussion after (4.3) show that (4.3) is true.

Recall Lemmas 2-3 and the notations therein. For $T < V_0$, take $x_0 \in D(T)$ and r_0 so that $U_{4r_0}(x_0) \subset D(T)$. For sufficiently large $k \geq 1$,

$$U_{3r_0}(x_0) \subset (co Y_{m_k}(T, \cdot)^{-1}(1))^o \cap D, \quad P_1 - a.s.,$$

since

$$\lim_{k \rightarrow \infty} \|X_{m_k}(T, \cdot) - I_{D(T)}(\cdot)\|_{L^2([-K, K]^N)} = 0, \quad P_1 - a.s.$$

by Lemma 2 and (4.7) (see the discussion below (4.2)). Hence in the same way as in Lemma 3,

$$\begin{aligned} V_0 &\leq \liminf_{k \rightarrow \infty} \sum_{z \in D_{m_k}} (Y_{m_k}(T, z) - Y_{m_k}(\tau_{m_k}, z)) \frac{1}{m_k^N} \\ &\leq \liminf_{k \rightarrow \infty} (\tau_{m_k} - T) \quad P_1 - a.s., \end{aligned} \quad (4.9)$$

which implies that (4.3) holds for $T < 2V_0$. Repeating the same procedure as above and then letting $r_0 \downarrow 0$, (4.3) holds for all $T < T^* := \text{Vol}(D)$.

Put

$$D(t) = \emptyset \quad \text{for } t \geq T^*. \quad (4.10)$$

Then $I_{D(\cdot)} \in C([0, \infty) : L^2([-K, K]^N))$ and $\{D(t)\}_{t \geq 0}$ is a unique solution to (1.13)-(1.14) on $[0, \infty)$ by Lemma 6, since $t \mapsto I_{D(t)}$ is nonincreasing and since

$$\text{Vol}(D(t)) = \text{Vol}(D(0)) - t \downarrow 0, \quad \text{as } t \uparrow T^*, \quad (4.11)$$

by (1.14).

We prove (2.7). Take a sufficiently small positive ε so that

$$\text{Vol}(D(t)) \leq \left(\frac{\gamma}{4}\right)^2 \quad \text{for } t \geq t_\varepsilon := T^* - \varepsilon. \quad (4.12)$$

Then

$$P(\sup_{t \geq 0} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma) \quad (4.13)$$

$$\begin{aligned}
&\leq P\left(\sup_{0 \leq t \leq t_\varepsilon} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma\right) \\
&\quad + P\left(\sup_{t \geq t_\varepsilon} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \gamma\right) \\
&\leq 2P\left(\sup_{0 \leq t \leq t_\varepsilon} \|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \geq \frac{\gamma}{2}\right) \rightarrow 0 \quad (\text{as } m \rightarrow \infty)
\end{aligned}$$

since for $t \geq t_\varepsilon$,

$$\begin{aligned}
&\|X_m(t, \cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K]^N)} \\
&\leq \|X_m(t_\varepsilon, \cdot)\|_{L^2([-K, K]^N)} + \|I_{D(t_\varepsilon)}(\cdot)\|_{L^2([-K, K]^N)} \\
&\leq \|X_m(t_\varepsilon, \cdot) - I_{D(t_\varepsilon)}(\cdot)\|_{L^2([-K, K]^N)} + 2\|I_{D(t_\varepsilon)}(\cdot)\|_{L^2([-K, K]^N)}.
\end{aligned}$$

Q. E. D.

(Proof of Corollary 1). Since D is convex,

$$(co Y_m(t, \cdot)^{-1}(1))^o \cap D = (co Y_m(t, \cdot)^{-1}(1))^o =: D_m(t).$$

For $T < T^*$, take $x_0 \in D(T)$ and r_0 so that $U_{4r_0}(x_0) \subset D(T)$ (see (3.6) for notation). Then, for sufficiently large m , $U_{3r_0}(x_0) \subset D_m(0)$.

Consider cones

$$\text{cone}(x) := co(\{x\} \cup U_0^-) \quad (x \in D^-),$$

and for $r > 0$, put

$$V(r) := \inf_{x \in \partial D} \text{Vol}(\text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x))), \quad (4.14)$$

$$V_m(r) := \inf_{x \in \partial D_m(0)} \text{Vol}(\text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x))). \quad (4.15)$$

Then for $\gamma > 0$ and sufficiently large $m \geq 1$, by Theorem 1,

$$\begin{aligned}
&P\left(\sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma\right) \tag{4.16} \\
&\leq P\left(\|I_{D_m(T)}(\cdot) - I_{D(T)}(\cdot)\|_{L^2([-K, K])}^2 \geq V_0\right) \\
&\quad + P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) \\
&\rightarrow 0 \quad (\text{as } m \rightarrow \infty).
\end{aligned}$$

Indeed,

$$\begin{aligned} & P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) \\ & \leq P(\sup_{0 \leq t \leq T} \|I_{D_m(t)}(\cdot) - I_{D(t)}(\cdot)\|_{L^2([-K, K])}^2 \geq \min(V(\gamma), V_m(\gamma))), \end{aligned}$$

and $V_m(\gamma) \geq V(\gamma)$ for all $m \geq 1$.

Q. E. D.

(Proof of Corollary 2). For $r \in \mathbf{R}$, let $\{D_r(t)\}_{t \geq 0}$ denote a unique solution of (1.13)-(1.14) with $D_r(0) = h^{-1}((r, \infty))$ on $[0, \infty)$. Notice that

$$D_r(\cdot) := \begin{cases} \mathbf{R}^N & \text{if } r < \inf\{h(x) | x \in \mathbf{R}^N\}, \\ \emptyset & \text{if } r \geq \sup\{h(x) | x \in \mathbf{R}^N\}. \end{cases} \quad (4.17)$$

Put

$$u(t, x) := \sup\{r \in \mathbf{R} | x \in D_r(t)\}. \quad (4.18)$$

Then, for all $t \geq 0$ and $r \in \mathbf{R}$ for which $D_r(t) \neq \emptyset$, $\mathbf{R}^N$,

$$u(t, \cdot)^{-1}((r, \infty)) = D_r(t), \quad (4.19)$$

since $D_r(t) = D_{r+}(t) := \cup_{\tilde{r} > r} D_{\tilde{r}}(t)$ by (1.13).

Indeed, $D_r(0) = D_{r+}(0)$; and if $\tilde{r} - r$ is positive and is sufficiently small, then $D_{\tilde{r}}(t) \neq \emptyset$ by (b) in Theorem 2, and

$$\int_{\mathbf{R}^N} (I_{D_{\tilde{r}}(t)}(x) - I_{D_r(t)}(x)) dx = \int_{\mathbf{R}^N} (I_{D_{\tilde{r}}(0)}(x) - I_{D_r(0)}(x)) dx.$$

By Lemma 6 and (4.19), u is continuous.

For $m \geq 1$, put

$$\begin{aligned} k_{m,1} &:= [m \sup\{h(y) | y \in \mathbf{R}^N\}], \\ k_{m,0} &:= [m \inf\{h(y) | y \in \mathbf{R}^N\}] - 1. \end{aligned}$$

Then

$$\begin{aligned}
& \sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m} (I_{D_{\frac{k}{m}}}(t)(x) - I_{D_{\frac{k+1}{m}}}(t)(x)) \\
&= \sum_{k_{m,0} < k \leq k_{m,1}} \frac{1}{m} I_{D_{\frac{k}{m}}}(t)(x) - \frac{k_{m,1} + 1}{m} I_{D_{\frac{k_{m,1}+1}{m}}}(t)(x) + \frac{k_{m,0}}{m} I_{D_{\frac{k_{m,0}}{m}}}(t)(x).
\end{aligned} \tag{4.20}$$

Since $I_{D_{\frac{k_{m,1}+1}{m}}}(t)(x) \equiv 0$ and since $I_{D_{\frac{k_{m,0}}{m}}}(t)(x) \equiv 1$, the following holds:
for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \geq 0$,

$$\begin{aligned}
& \int_{\mathbf{R}^N} \varphi(x) \left[\sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m} (I_{D_{\frac{k}{m}}}(0)(x) - I_{D_{\frac{k+1}{m}}}(0)(x)) \right. \\
& \quad \left. - \sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m} (I_{D_{\frac{k}{m}}}(t)(x) - I_{D_{\frac{k+1}{m}}}(t)(x)) \right] dx \\
&= \int_0^t ds \left[\sum_{k_{m,0} < k \leq k_{m,1}} \frac{1}{m} \int_{\mathbf{R}^N} \varphi(x) \omega_1(I_{D_{\frac{k}{m}}}(s)(\cdot), dx) \right].
\end{aligned} \tag{4.21}$$

Letting $m \rightarrow \infty$ in (4.21), one can show that u is a solution to (1.15) by Lemma 4, since $\text{co } D_{\frac{[mr]+1}{m}}(s) \rightarrow \text{co } D_r(s)$ as $m \rightarrow \infty$ for $r \in [\inf\{u(s, y) | y \in \mathbf{R}^N\}, \sup\{u(s, y) | y \in \mathbf{R}^N\}]$, provided $D_r(s) \neq \emptyset$, $\mathbf{R}^N$.

Let $v \in C([0, \infty) \times \mathbf{R}^N)$ be a solution to (1.15) with $v(0, \cdot) = h(\cdot)$. Then for $n \geq 1$, $r \in [\inf\{h(y) | y \in \mathbf{R}^N\}, \sup\{h(y) | y \in \mathbf{R}^N\}]$, and $\varphi \in C_o(\mathbf{R}^N)$ and $t \geq 0$,

$$\begin{aligned}
& \int_{\mathbf{R}^N} \varphi(x) \{ \eta_n(v(0, x) - r) - \eta_n(v(t, x) - r) \} dx \\
&= \int_0^t ds \int_{\mathbf{R}} \frac{d\eta_n(\tilde{r} - r)}{d\tilde{r}} d\tilde{r} \int_{\mathbf{R}^N} \varphi(x) \omega_{\tilde{r}}(v(s, \cdot), dx)
\end{aligned} \tag{4.22}$$

(see (3.24) for notation). Let $n \rightarrow \infty$ in (4.22). Then we see that $\tilde{D}_r(t) := v(t, \cdot)^{-1}((r, \infty))$ is a solution to (1.14) on $[0, \infty)$ by Lemma 4 and the continuity of v .

We prove that $v(t, \cdot)^{-1}((r, \infty))$ satisfies (1.13). For $x \in (\text{co } \tilde{D}_r(t)) \cap \tilde{D}_r(0)$, take $\delta > 0$ so that $U_\delta(x) \subset (\text{co } \tilde{D}_r(t)) \cap \tilde{D}_r(0)$. Then $U_\delta(x) \subset \text{co } \tilde{D}_r(s)$ for all $s \leq t$. Hence, by (1.14), for any $\varphi \in C_o(\mathbf{R}^N)$ such that $\varphi \equiv 0$ in $U_\delta(x)^c$.

$$\int_{\mathbf{R}^N} \varphi(x) \{I_{\tilde{D}_r(0)} - I_{\tilde{D}_r(t)}\} dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x) \omega_1(I_{\tilde{D}_r(s)}(\cdot), dx) = 0, \quad (4.23)$$

which implies that $x \in (U_\delta(x) \subset) \tilde{D}_r(t)$. Hence (1.13) holds.

The uniqueness of u follows from that of $D_r(\cdot)$ for all r .

Q. E. D.

Theorem 2 is an easy consequence of Theorem 1 and Lemma 6 and we omit the proof.

(Proof of Theorem 3).

(Step I). We first show that $u(t, x) := I_{D(t)}(x)$ is a viscosity supersolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$.

Let $\psi \in \mathcal{A}((0, \infty) \times \mathbf{R}^N)$ and assume that $u - \psi$ attains a local minimum at $(t_0, x_0) \in (0, \infty) \times \mathbf{R}^N$. Without loss of generality, we may assume that $u(t_0, x_0) = \psi(t_0, x_0)$ and that $u(t, x) > \psi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbf{R}^N \setminus \{(t_0, x_0)\}$ (see [8]).

If $x_0 \notin \partial(\text{co } D(t_0)) \cap \partial D(t_0)$, then $\partial_t \psi(t_0, x_0) \geq 0$.

Indeed, $t \mapsto u(t, x_0)$ is constant if $t_0 - t$ is a sufficiently small positive number, from which $\psi(t_0, x_0) > \psi(t, x_0)$ for such t .

Suppose that $x_0 \in \partial(\text{co } D(t_0)) \cap \partial D(t_0)$. Then $u(t_0, x_0) = 0$, and $D\psi(t_0, x_0) = o$ or $\sigma^+(u, D\psi(t_0, x_0), t_0, x_0) = 1$.

Indeed, if $D\psi(t_0, x_0) \neq o$, then for y for which $y + x_0 \notin H(D\psi(t_0, x_0), x_0)$ and for $r > 0$, by the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$u(t_0, x_0 + ry) > \psi(t_0, x_0 + ry) = \psi(t_0, x_0) + r < D\psi(t_0, x_0 + \theta ry), y >> 0,$$

provided r is sufficiently small, by the continuity of $D\psi$.

(Case 1). We first consider the case when $D\psi(t_0, x_0) = o$. We may assume that there exist $f \in \mathcal{F}$ and $\varphi_1 \in C^2((0, \infty))$ such that

$$\psi(t, x) = -f(|x - x_0|) - \varphi_1(t) \quad (4.24)$$

(see [21]).

For $A > 0$ and $m \geq 2$, put

$$\psi_{m,A}(t, x) = \psi(t, x) - A\{|t - t_0|^2 + |x - x_0|^m\}. \quad (4.25)$$

Then

$$\partial_t \psi_{m,A}(t_0, x_0) = \partial_t \psi(t_0, x_0), \quad D\psi_{m,A}(t_0, x_0) = D\psi(t_0, x_0), \quad (4.26)$$

and

$$\begin{aligned} U_{m,A,\varepsilon}^+ &:= \{(t, x) \in (0, \infty) \times \mathbf{R}^N \mid \psi_{m,A}(t, x) + \varepsilon > u(t, x)\} \\ &\subset U_{(2\varepsilon/A)^{1/m}}((t_0, x_0)) \end{aligned} \quad (4.27)$$

($\varepsilon \in (0, A)$), and the following holds: for $t \geq 0$,

$$\lim_{x \rightarrow x_0} G(D\psi_{N,A}(t, x), D^2\psi_{N,A}(t, x)) = NA. \quad (4.28)$$

We argue by contradiction. We consider $\psi_{N,A}$ instead of ψ . When it is not confusing, we omit $_{N,A}$ for the sake of simplicity.

Assume that the following holds:

$$\partial_t \psi(t_0, x_0) < 0. \quad (4.29)$$

By reselecting $A > 0$ sufficiently small and $\varepsilon > 0$ sufficiently small compared to A if necessary, we may assume that

$$\partial_t \psi(t, x) + R\left(\frac{D\psi(t, x)}{|D\psi(t, x)|}\right)G(D\psi(t, x), D^2\psi(t, x)) + \varepsilon < 0 \quad \text{on } U_\varepsilon^+, \quad (4.30)$$

and that

$$U_\varepsilon^+ = \cup_{t \geq 0} \{t\} \times (\psi(t, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(t)^c). \quad (4.31)$$

We may also assume that $x \mapsto \psi(t, x)$ is strictly concave on U_ε^+ and henceforth $x \mapsto (\psi(s, x), D\psi(s, x)/|D\psi(s, x)|)$ is one-to-one on some neighborhood of $\partial\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c$, provided $\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \neq \emptyset$.

Indeed, if $\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c \neq \emptyset$, then $-\varepsilon$ is not the maximum of $\psi(s, \cdot)$ on $\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c$ and hence $D\psi(s, \cdot) \neq 0$ on some neighborhood of $\partial\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c$.

For $t \geq 0$,

$$\begin{aligned}
& \int_{\mathbf{R}^N} (\zeta_k(\eta_m(\psi(t, x) + \varepsilon) - u(t, x)) \\
& \quad - \zeta_k(\eta_m(\psi(0, x) + \varepsilon) - u(0, x))) dx \\
&= \int_{\mathbf{R}^N} dx \int_0^t (-\zeta_k(\eta_m(\psi(s, x) + \varepsilon) - u(s, x)) u(ds, x) \\
& \quad + \eta_k(\eta_m(\psi(s, x) + \varepsilon) - u(s, x)) \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s \psi(s, x) ds)
\end{aligned} \tag{4.32}$$

(see (3.24)-(3.25) for notation).

Letting $k \rightarrow \infty$ in (4.32), by (4.31),

$$\begin{aligned}
0 \leq & \int_0^t ds \left\{ \int_{\psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c} \eta_m(\psi(s, x) + \varepsilon) \omega_1(u(s, \cdot), dx) \right. \\
& \left. + \int_{\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon + 1/m)) \cap D(s)^c} \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s \psi(s, x) dx \right\}
\end{aligned} \tag{4.33}$$

For s for which $\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon + 1/m)) \cap D(s)^c \neq \emptyset$ and sufficiently large $m \geq 1$, by Lemma 8,

$$\begin{aligned}
& \int_{\psi(s, \cdot)^{-1}((-\varepsilon, -\varepsilon + 1/m)) \cap D(s)^c} \frac{d\eta_m(\psi(s, x) + \varepsilon)}{dr} \partial_s \psi(s, x) dx \\
< & - \int_{-\varepsilon}^{-\varepsilon + 1/m} \frac{d\eta_m(r + \varepsilon)}{dr} dr \int_{\left\{ \frac{-D\psi(s, x)}{|D\psi(s, x)|} : x \in \partial\psi(s, \cdot)^{-1}((r, \infty)) \cap D(s)^c \right\}} (R(p) \\
& \quad + \varepsilon \sup\{G(D\psi(s, x), D^2\psi(s, x)) : (s, x) \in U_\varepsilon^+\}^{-1}) d\mathcal{H}^{N-1}(p) \\
\rightarrow & - \int_{\cup_{r > -\varepsilon} \left\{ \frac{-D\psi(s, x)}{|D\psi(s, x)|} : x \in \partial\psi(s, \cdot)^{-1}((r, \infty)) \cap D(s)^c \right\}} (R(p) + \varepsilon \sup\{G(D\psi(s, x) \\
& \quad , D^2\psi(s, x)) : (s, x) \in U_\varepsilon^+\}^{-1}) d\mathcal{H}^{N-1}(p) \quad (\text{as } m \rightarrow \infty).
\end{aligned} \tag{4.34}$$

(4.33)-(4.34) contradicts to

$$\begin{aligned}
& \{p \in \mathbf{S}^{N-1} : \sigma^+(u, -p, s, x) = 1 \text{ for some } x \in \psi(s, \cdot)^{-1}((-\varepsilon, \infty)) \cap D(s)^c\} \\
\subset & \cup_{r > -\varepsilon} \left\{ -\frac{D\psi(s, x)}{|D\psi(s, x)|} : x \in \partial\psi(s, \cdot)^{-1}((r, \infty)) \cap D(s)^c \right\}
\end{aligned}$$

since

$$\eta_m(\psi(s, x) + \varepsilon) \rightarrow 1 \quad \text{if } x \in \psi(s, \cdot)^{-1}((-\varepsilon, \infty)), \text{ as } m \rightarrow \infty.$$

(Case 2). Next we consider the case when $\sigma^+(u, D\psi(t_0, x_0), t_0, x_0) = 1$. By (ii)-(iv) in Lemma 7, all eigenvalues of $-D(D\psi(t_0, x_0)/|D\psi(t_0, x_0)|)$ are non-negative since the function $x \mapsto \psi(t_0, x)$ takes a maximum $\psi(t_0, x_0)$ on the set $\{x_0 + y \in \mathbf{R}^N \mid |y| < y, D\psi(t_0, x_0) \cdot y \geq 0\}$.

For $A > 0$, all eigenvalues of $-D(D\psi_{2,A}(t_0, x_0)/|D\psi_{2,A}(t_0, x_0)|)$ as a mapping on the set $\{y \in \mathbf{R}^N \mid |y| < y, D\psi_{2,A}(t_0, x_0) \cdot y \geq 0\}$ are greater than or equal to $2A/|D\psi(t_0, x_0)|$ (see (3.31)-(3.32)) since, in Lemma 7, 1 and $f_1, \dots, f_{N-1}$ are a eigenvalue and eigenvectors of $(g_1 \cdots g_N)$, respectively.

We argue by contradiction. Assume that the following holds:

$$\partial_t \psi(t_0, x_0) + R\left(\frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|}\right)G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) < 0. \quad (4.35)$$

We consider $\psi_{2,A}$ instead of ψ . When it is not confusing, we omit $_{2,A}$ for the sake of simplicity. By reselecting $A, \varepsilon > 0$ if necessary, we may assume that (4.30)-(4.31) hold.

One can also assume, in $U_{(2\varepsilon/A)^{1/2}}((t_0, x_0))$, that $\partial_i \psi(s, x) \neq 0$ and all eigenvalues of $-D(D\psi(s, x)/|D\psi(s, x)|)$ as a mapping on the set $\{y \in \mathbf{R}^N \mid |y| < y, D\psi(s, x) \cdot y \geq 0\}$ are greater than or equal to $A/|D\psi(t_0, x_0)|$, and $x \mapsto y_i(s, x)$ is one-to-one for some $i \in \{1, \dots, N\}$ by the inverse function theorem and (v) in Lemma 7, and Lemma 8.

In the same way as in (4.32)-(4.34), we obtain a contradiction.

(Step II). We show that $u^-(t, x) = I_{D(t)^-}(x)$ is a viscosity subsolution of (1.20).

Let $\psi \in \mathcal{A}((0, \infty) \times \mathbf{R}^d : \mathbf{R}^d)$ and assume that $u^- - \psi$ attains a maximum at $(t_0, x_0) \in (0, \infty) \times \mathbf{R}^d$. We may assume as well that $u^-(t_0, x_0) = \psi(t_0, x_0)$, so that $u^-(t, x) < \psi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbf{R}^d \setminus \{(t_0, x_0)\}$ (see [8]).

Since $t \mapsto u^-(t, x)$ is nonincreasing, $\partial_t \psi(t_0, x_0) \leq 0$.

Hence we only have to consider the case when the following holds: $D\psi(t_0, x_0) \neq 0$, and

$$\sigma^-(u^-, D\psi(t_0, x_0), t_0, x_0) = 1, \quad R\left(\frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|}\right)G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) > 0.$$

In particular, $u^-(t_0, x_0) = 1$. By adding to ψ the function $(t, x) \mapsto A\{|t - s|^2 + |x - y|^2\}$, with a sufficiently small $A > 0$, if necessary, we may assume that

$$U_\varepsilon^- := \{(t, x) \in (0, \infty) \times \mathbf{R}^d \mid \psi(t, x) - \varepsilon < u^-(t, x)\} \quad (\varepsilon > 0) \quad (4.36)$$

is contained in the set $U_{(\varepsilon/A)^{1/2}}((t_0, x_0))$.

We argue by contradiction. Assume that the following holds:

$$\partial_t \psi(t_0, x_0) + R \left(\frac{D\psi(t_0, x_0)}{|D\psi(t_0, x_0)|} \right) G(D\psi(t_0, x_0), D^2\psi(t_0, x_0)) > 0. \quad (4.37)$$

By reselecting $\varepsilon > 0$ if necessary, we may assume that

$$\partial_t \psi(t, x) + R \left(\frac{D\psi(t, x)}{|D\psi(t, x)|} \right) G(D\psi(t, x), D^2\psi(t, x)) - \varepsilon > 0, \quad (4.38)$$

and $u^-(t, x) = 1$ on U_ε^- by the continuity of ψ .

Put $\tilde{\eta}_m(r) = \eta_m(r + 1/m)$ for $r \in \mathbf{R}$ and $m \geq 1$. In the same way as in (Step I), considering $u^-(t, x) - \tilde{\eta}_m(\psi(t, x) - 1 - \varepsilon)$ instead of $\eta_m(\psi(t, x) + \varepsilon) - u(t, x)$, we obtain a contradiction.

Q. E. D.

(Proof of Corollary 3).

We first show that u is a viscosity supersolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$. Let $\varphi \in \mathcal{A}((0, \infty) \times \mathbf{R}^N)$ and assume that $u - \varphi$ attains a minimum at $(t_0, x_0) \in (0, \infty) \times \mathbf{R}^N$. We may assume that $u(t_0, x_0) = \varphi(t_0, x_0)$, so that $u(t, x) > \varphi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbf{R}^N \setminus \{(t_0, x_0)\}$ (see [8]). By subtracting a constant, we may assume that $\varphi \leq u < 0$.

Put $r_0 := \varphi(t_0, x_0)$ and

$$u_r(t, x) := I_{u^{-1}(t, \cdot)((r, 0))}(x) \quad (r < 0). \quad (4.39)$$

Then

$$u_{r_0}(t, x) \geq \frac{\varphi(t, x)}{|r_0|} + 1 \quad \text{for all } (t, x) \in (0, \infty) \times \mathbf{R}^N, \quad (4.40)$$

where the equality holds if and only if $(t, x) = (t_0, x_0)$.

Since u_r is a viscosity supersolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$ by Corollary 2 and Theorem 3, and since

$$\sigma^+(u_{r_0}, D(\varphi(t_0, x_0)/|r_0| + 1), t_0, x_0) = \sigma^+(u, D\varphi(t_0, x_0), t_0, x_0),$$

(1.25) holds.

Next we show that u is a viscosity subsolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$. Let $\varphi \in \mathcal{A}((0, \infty) \times \mathbf{R}^d)$ and assume that $u - \varphi$ attains a maximum at $(t_1, x_1) \in (0, \infty) \times \mathbf{R}^d$. We may assume as well that $u(t_1, x_1) = \varphi(t_1, x_1)$, so that $u(t, x) < \varphi(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbf{R}^d \setminus \{(t_1, x_1)\}$ (see [8]).

By adding a constant, we may assume that $\varphi \geq u > 0$.

Put $r_1 := \varphi(t_1, x_1)$ and

$$u_r^-(t, x) := I_{u^{-1}(t, \cdot)([r, \infty))}(x).$$

Then

$$u_{r_1}^-(t, x) \leq \frac{\varphi(t, x)}{r_1} \quad \text{for all } (t, x) \in (0, \infty) \times \mathbf{R}^N,$$

where the equality holds if and only if $(t, x) = (t_1, x_1)$.

Since u_r^- is a viscosity subsolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$ by Corollary 2 and Theorem 3, and since

$$\sigma^-(u_{r_1}^-, D(\varphi(t_1, x_1)/r_1), t_1, x_1) = \sigma^-(u, D\varphi(t_1, x_1), t_1, x_1),$$

(1.27) holds.

Q. E. D.

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