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# Backward Shift Invariant Subspaces in the Bidisc

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# Backward Shift Invariant Subspaces in the Bidisc

by

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and

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**Abstract**. Suppose that $T_\phi$ is a Toeplitz operator with a symbol $\phi$ on the Hardy space $H^2$ on the bidisc. Let $N$ be a backward shift invariant subspace of $H^2$, that is, $N$ is an invariant subspace under $T_z^*$ and $T_w^*$. Let $P$ be the orthogonal projection from $H^2$ onto $N$. For $\phi$ in $H^\infty$, put $S_\phi = PT_\phi|N$. In this paper, we give a characterization of a backward shift invariant subspace which satisfies $S_z S_w^* = S_w^* S_z$.

## §1. Introduction

Let $T^2$ be the torus that is the Cartesian product of two unit circles $T$ in $\mathbf{C}$. Let $p = 1$ or $p = \infty$. The usual Lebesgue spaces, with respect to the Haar measure $m$ on $T^2$, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all $f$ in $L^p$ whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{T^2} f(z,w)\bar{z}^j \bar{w}^\ell dm(z,w)$$

are 0 as soon as at least one component of $(j,\ell)$ is negative. Then $H^p$ is called the Hardy space. As $T^2 = (z,T) \times (w,T), H^p(z,T)$ and $H^p(w,T)$ denote the one variable Hardy spaces.

Let $P_{H^2}$ be the orthogonal projection from $L^2$ onto $H^2$. For $\phi$ in $L^\infty$, the Toeplitz operator $T_\phi$ is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad (f \in H^2).$$

A closed subspace $N$ of $H^2$ is said to be backward shift invariant if

$$T_z^* N \subset N \quad \text{and} \quad T_w^* N \subset N.$$

A closed subspace $M$ of $H^2$ is said to be shift invariant if $T_z M \subset M$ and $T_w M \subset M$. The orthogonal complement of $N$ is shift invariant. Let $P_N$ and $P_M$ be the orthogonal projections from $H^2$ onto $N$ and $M$, respectively. For $\phi$ in $H^\infty$, put

$$S_\phi = P_N T_\phi P_N|N \quad \text{and} \quad V_\phi = P_M T_\phi P_M|M.$$

It is known in [2] that $V_z V_w^* = V_w^* V_z$ if and only if $M = qH^2$ for some inner function $q$ in $H^\infty$. In this paper, we are interested in backward shift invariant subspaces $N$ which satisfy $S_z S_w^* = S_w^* S_z$. Let $M = H^2 \ominus N$. We will write $P = P_N$ and $Q = I - P_N$, where $I$ is the identity operator on $H^2$. In this paper, we also study two operators

$$A = QT_z P \quad \text{and} \quad B = PT_w^* Q.$$

In §2, we show that $AB|M = V_w^* V_z - V_z V_w^*$ and $BA|N = S_z S_w^* - S_w^* S_z$. Then $AB = 0$ is equivalent to $V_z V_w^* = V_w^* V_z$, and $BA = 0$ is equivalent to $S_z S_w^* = S_w^* S_z$. Moreover we determine backward shift invariant subspaces satisfying $A = 0$ or $B = 0$. In §3, we give a characterization of backward shift invariant subspaces satisfying $BA = 0$, equivalently

$S_z S_w^* = S_w^* S_z$. And we give simple sufficient conditions to be $S_z S_w^* = S_w^* S_z$. In §4, we give a conjecture, that is, the sufficient condition is also necessary one.

Throughout this paper, for a subset $H$ of $H^2$, $[H]_2$ denotes the closed linear span of $H$ and $[H]$ the linear span of $H$.

## §2. Invariant subspace with $A = 0$ or $B = 0$

Let $N$ be a backward shift invariant subspace and $M$ be the orthogonal complement of $N$ in $H^2$. Put $P = P_N$ and $Q = I - P_N$, then $Q$ is the orthogonal projection from $H^2$ onto $M$.

**Lemma 2.1.**
*(1)* $AB = QT_w^* QT_z Q - QT_z QT_w^* Q$ *and so* $AB|M = V_w^* V_z - V_z V_w^*$.
*(2)* $BA = PT_z PT_w^* P - PT_w^* PT_z P$ *and so* $BA|N = S_z S_w^* - S_w^* S_z$
*(3)* $\ker A = \{f \in N : T_z f \in N\} \oplus M$.
*(4)* $\ker B = \{f \in M : T_w^* f \in M\} \oplus N$.

Proof. (1) Since $T_z Q = QT_z Q$ and $T_z T_w^* = T_w^* T_z$,

$$\begin{aligned} AB &= QT_z PT_w^* Q \\ &= QT_z T_w^* Q - QT_z QT_w^* Q \\ &= QT_w^* QT_z Q - QT_z QT_w^* Q. \end{aligned}$$

(2) Since $T_w^* P = PT_w^* P$ and $T_w^* T_z = T_z T_w^*$,

$$\begin{aligned} BA &= PT_w^* QT_z P \\ &= PT_w^* T_z P - PT_w^* PT_z P \\ &= PT_z PT_w^* P - PT_w^* PT_z P. \end{aligned}$$

The properties (3) and (4) are clear.

**Theorem 2.2.**
*(1)* $A = 0$ *if and only if* $N = H^2$ *or* $N = H^2 \ominus qH^2$ *where* $q$ *is a one variable inner function with* $q = q(w)$.
*(2)* $B = 0$ *if and only if* $M = [0]$ *or* $M = qH^2$ *where* $q$ *is a one variable inner function with* $q = q(z)$.
*(3)* $A = B = 0$ *if and only if* $N = [0]$ *or* $N = H^2$.

Proof. (2) follows from (1). We will show (1). We have $H^2 = N \oplus M$ and $T_z M \subset M$. Suppose $A = 0$. By Lemma 2.1 (3), $T_z N \subset N$. Put $N_0 = N \ominus T_z N$ and

$M_0 = M \ominus T_z M$. Then

$$H^2 = \sum_{n=0}^{\infty} \oplus (N_0 \oplus M_0) z^n = \sum_{n=0}^{\infty} \oplus H^2(w,T) z^n$$

because $zH^2 = zN \oplus zM$ and so $N_0 \oplus M_0 = H^2(w,T)$. By Lemma 2.1 (1), $V_w^* V_z = V_z V_w^*$ and so $V_z^* V_w = V_w V_z^*$ because $AB = 0$. Hence $V_w(\ker V_z^*) \subseteq \ker V_z^*$ and $\ker V_z^* = M_0$. Therefore by a theorem of Beurling [1], if $M_0 \neq [0]$, $M_0 = qH^2(w,T)$ and $q$ is a one variable inner function with $q = q(w)$. Hence $M = qH^2$ and so $N = H^2 \ominus qH^2$. If $M_0 = [0]$, then $M = [0]$, and so $N = H^2$.

## §3. Invariant subspace with $AB = 0$ or $BA = 0$

Suppose that $N$ is a backward shift invariant subspace and $M = H^2 \ominus N$. By Lemma 2.1, $AB = 0$ if and only if $V_w^* V_z = V_z V_w^*$, and $BA = 0$ if and only if $S_z S_w^* = S_w^* S_z$. Hence we know (see [2],[3],[4]) that $AB = 0$ if and only if $M = qH^2$ for some inner function $q$. In this section, we study $N$ when $BA = 0$, that is, $S_z S_w^* = S_w^* S_z$.

**Lemma 3.1.**

$$[\text{ran } A]_2 = \{M \ominus zM\} \ominus \{H^2(w,T) \cap M\}$$

*and*

$$\ker B = \{H^2(z,T) \cap M\} \oplus wM \oplus N.$$

Proof. Since $(T_w^* f, g) = (f, wg)$ if $f, g \in H^2$,

$$\{f \in M \ ; \ T_w^* f \in M\} = M \cap \{H^2 \ominus wN\} = \{H^2(z,T) \cap M\} \oplus wM,$$

because $H^2 \ominus wN = (H^2 \ominus wH^2) \oplus w(H^2 \ominus N)$ and $N = H^2 \ominus M$. Hence by Lemma 2.1 (4), $\ker B = \{f \in M \ ; \ T_w^* f \in M\} \oplus N = \{H^2(z,T) \cap M\} \oplus wM \oplus N$. By the same argument, $\ker A^* = \{H^2(w,T) \cap M\} \oplus zM \oplus N$ and so

$$\begin{aligned} [\text{ran } A]_2 &= H^2 \ominus \ker A^* \\ &= \{M \ominus zM\} \ominus \{H^2(w,T) \cap M\}. \end{aligned}$$

**Lemma 3.2.**
*(1) $A = 0$ if and only if $M = \{H^2(w,T) \cap M\} \oplus zM$.*
*(2) $B = 0$ if and only if $M = \{H^2(z,T) \cap M\} \oplus wM$.*
*(3) $BA = 0$ if and only if $\{H^2(z,T) \cap M\} \oplus wM \supseteq \{M \ominus zM\} \ominus \{H^2(w,T) \cap M\}$.*

Proof. These follow from Lemma 3.1.

For a subset $H$ of $H^2$, let $H_k = \sum_{i+j=k} z^i w^j H$ for $k \geq 0$.

**Theorem 3.3**. *Let $N$ be a backward shift invariant subspace of $H^2$ and $M$ its orthogonal complement. Suppose $N \neq H^2$.*

*(1) $S_z S_w^* = S_w^* S_z$ if and only if $M = H + M_1$ and if and only if $M = \sum_{j=0}^{k-1} H_j + M_k$ for any $k \geq 1$, where $H = H_0 = H^2(z,T) \cap M + H^2(w,T) \cap M$. If $S_z S_w^* = S_w^* S_z$, then $H \neq [0]$.*

*(2) When $M \cap H^2(z,T) = [0]$ or $M \cap H^2(w,T) = [0]$, $S_z S_w^* = S_w^* S_z$ if and only if $M = qH^2 + M_k$ for any $k \geq 1$ where $q$ is a one variable inner function such that $M \cap H^2(z,T) = qH^2(z,T)$ or $M \cap H^2(w,T) = qH^2(w,T)$.*

*(3) When $M \cap H^2(z,T) \neq [0]$ and $M \cap H^2(w,T) \neq [0]$, $S_z S_w^* = S_w^* S_z$ if and only if $M = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions such that $M \cap H^2(z,T) = q_1 H^2(z,T)$ and $M \cap H^2(w,T) = q_2 H^2(w,T)$.*

Proof. (1) Since $S_z S_w^* = S_w^* S_z$ is equivalent to $BA = 0$, $S_z S_w^* = S_w^* S_z$ if and only if $M = H + M_1$ by Lemma 3.2 (3). It is easy to see that $M = H + M_1 = \sum_{j=0}^{k-1} H_j + M_k$ for any $k \geq 1$. If $H = [0]$, then $M = M_k$ and hence $M = [0]$. This contradicts $N \neq H^2$.

(2) We may assume that $M \cap H^2(z,T) = [0]$ and $M \cap H^2(w,T) \neq [0]$. By a theorem of Beurling [1], $M \cap H^2(w,T) = qH^2(w,T)$ for some one variable inner function $q = q(w)$. By (1), $S_z S_w^* = S_w^* S_z$ if and only if $M = qH^2(w,T) + M_1$ if and only if $M = q \sum_{j=0}^{k-1} \oplus H^2(w,T) z^j + M_k$ for any $k \geq 1$. This is equivalent to $M = qH^2 + M_k$ for any $k \geq 1$. For, $M_k \supset qz^k H^2$.

(3) By a theorem of Beurling, $M \cap H^2(z,T) = q_1 H^2(z,T)$ and $M \cap H^2(w,T) = q_2 H^2(w,T)$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions. By (1), $S_z S_w^* = S_w^* S_z$ if and only if $M = q_1 H^2(z,T) + q_2 H^2(w,T) + M_1$ if and only if

$$M = q_1 \sum_{j=0}^{k-1} \oplus H^2(z,T) w^j + q_2 \sum_{j=0}^{k-1} \oplus H^2(w,T) z^j + M_k$$

for any $k \geq 1$. This is equivalent to $M = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$. For, $M_k \supseteq q_1 w^k H^2 + q_2 z^k H^2$.

**Corollary 3.4**.

*(1) $AB = BA = 0$ if and only if $A = 0$ or $B = 0$.*

*(2) If $N = H^2 \ominus qH^2$ and $q$ is an inner function and $S_z S_w^* = S_w^* S_z$, then $q$ is a one variable.*

Proof. (1) If $AB = BA = 0$, then by Lemma 2.1 (1) $V_w^* V_z = V_z V_w^*$ and so $M = qH^2$ for some inner function $q$ (see [2], [4]). On the other hand, by Theorem 3.3 (1), $M \cap H^2(z,T) \neq [0]$ or $M \cap H^2(w,T) \neq [0]$ because $S_z S_w^* = S_w^* S_z$. Hence $q$ is one variable. By Theorem 2.2, $A = 0$ or $B = 0$.

(2) is clear by (1).

**Corollary 3.5**. *Let $N$ be a backward shift invariant subspace and $N \neq H^2$.*
*(1) If $S_z S_w^* = S_w^* S_z$, then $N \subseteq H^2 \ominus qH^2$ for some one variable inner function $q$.*
*(2) If $N = H^2 \ominus qH^2$ for some one variable inner function $q$, then $S_z S_w^* = S_w^* S_z$.*

Proof. (1) By Theorem 3.3, if $S_z S_w^* = S_w^* S_z$ then $M \supseteq qH^2$ for some one variable inner function $q$. Hence $N \subseteq H^2 \ominus qH^2$. (2) is clear by Theorem 3.3 (3).

**Corollary 3.6**. *Suppose that $A \neq 0$ and $B \neq 0$.*
*(1) If $S_z S_w^* = S_w^* S_z$, then $N \subseteq (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions.*
*(2) If $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions, then $S_z S_w^* = S_w^* S_z$.*

Proof. By Theorem 2.2, we can prove (1) as in the proof of Corollary 3.5 (1). (2) Since $q_1 H^2 + q_2 H^2 = [q_1, q_2] + (q_1 H_1^2 + q_2 H_1^2)$, $M = [q_1, q_2] + (zM + wM) = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$. It is easy to see that $M \cap H^2(z,T) = q_1 H^2(z,T)$ and $M \cap H^2(w,T) = q_2 H^2(w,T)$. Hence by Theorem 3.3 (3) $S_z S_w^* = S_w^* S_z$.

## §4. Conjecture

By Corollary 3.5 (2) and Corollary 3.6, if $N = H^2$, $N = H^2 \ominus qH^2$ for some one variable inner function $q$ or $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ for some one variable inner functions $q_1 = q_1(z)$ and $q_2 = q_2(w)$, then $S_z S_w^* = S_w^* S_z$. Because of Theorem 3.3, we have the following conjecture. In this section, we study this conjecture.

**Conjecture**. If $S_z S_w^* = S_w^* S_z$, then $N = H^2$, $N = H^2 \ominus qH^2$ for some one variable inner function $q$ or $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$, where $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable inner functions.

**Proposition 4.1**. *If $M = q_1 H^2 + q_2 H^2 + M_k$ for any $k \geq 1$, where $M \cap H^2(z,T) = q_1 H^2(z,T)$ and $M \cap H^2(w,T) = q_2 H^2(w,T)$, then $M = q_1(H^2 \ominus w^k H^2) + q_2(H^2 \ominus w^k H^2) + w^k M$ for any $k \geq 1$. The converse is also true.*

Proof. Since $wM \supseteq q_2 w H^2(w,T)$, by Lemma 3.2 (3) and Theorem 3.3 (3),

$$q_1 H^2(z,T) \oplus wM \supseteq K_2 \oplus q_2 w H^2(w,T),$$

where $M \ominus zM = K_2 \oplus q_2 H^2(w,T)$. Thus

$$q_1 H^2(z,T) \oplus wM \supseteq \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2 w H^2(w,T)\} z^j.$$

Since

$$M = \sum_{j=0}^{\infty} \oplus (M \ominus zM) z^j = \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2 H^2(w,T)\} z^j,$$

we have

$$q_1 H^2(z,T) + q_2 H^2(z,T) + wM \supseteq M.$$

Hence $M = q_1 H^2(z,T) + q_2 H^2(z,T) + wM$. This leads our assertion.

**Corollary 4.2.** *If $M = qH^2 + M_k$ for any $k \geq 1$ where $M \cap H^2(z,T) = qH^2(z,T)$ and $M \cap H^2(w,T) = [0]$, then $M = qH^2$.*

Proof. By Proposition 4.1 and its proof, $M = q(H^2 \ominus w^k H^2) + w^k M$ for any $k \geq 1$. This implies that $M = qH^2$, because $q(H^2 \ominus w^k H^2)$ is orthogonal to $w^k M$.

It is not difficult to prove that $q_1 H^2 + q_2 H^2$ is closed when $q_1 = q_1(z)$ and $q_2 = q_2(w)$ are one variable. Hence our conjecture is equivalent to the following one. If $S_z S_w^* = S_w^* S_z$, then $M = [0]$, $M = qH^2$ or $M = q_1 H^2 + q_2 H^2$. Even if $N$ is of finite dimension, $S_z S_w^* \neq S_w^* S_z$ may happen. In fact, $N = \{1, z, w\}$ is such an example.

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