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Generic singularities of implicit systems of first order differential equations on the plane

A.A. Davydov*, G. Ishikawa†, S. Izumiya‡, W.-Z. Sun§

Abstract

For the implicit systems of first order ordinary differential equations on the plane there is presented the complete local classification of generic singularities of family of its phase curves up to smooth orbital equivalence. Besides the well known singularities of generic vector fields on the plane and the singularities described by a generic first order implicit differential equations, there exists only one generic singularity described by the implicit first order equation supplied by Whitney umbrella surface generically embedded to the space of directions on the plane.

1 Introduction.

An *implicit system* of first order ordinary differential equations on a smooth n -dimensional manifold is defined by a zero level, which is called *system surface*, of a smooth map F from the tangent bundle of this manifold to the n -dimensional Cartesian space. In local coordinates $x = (x_1, \dots, x_n)$ near a point of the manifold a system can be written in the standard form $F(x, \dot{x}) = 0$.

Key words: system folding, phase curve, Clairaut system

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An implicit system is with *locally bounded derivatives* if the restriction of the bundle projection to the system surface is a proper map. This restriction is called the *system folding*. Only systems with locally bounded derivatives are considered below. We identify the space of systems with locally bounded derivatives with the space of respective maps F . Then a *generic* system is a system from some open everywhere dense subset in this space, for the fine topology of Whitney.

A *solution curve* of an implicit system is defined as a differentiable map $x : t \mapsto x(t)$ from an interval of the real line to the base manifold such that the image of its natural lifting $(x(t), \dot{x}(t))$ to the tangent bundle belongs to the system surface. A *phase curve* is the image of such a map $x(t)$ and a *trajectory* is the image of its lifting.

In this paper we study the *point singularities* of the family of phase curves provided by a germ of a system surface and present the complete list of generic singularities on the plane up to smooth orbital equivalence.

For a generic system its surface is a closed smooth n -dimensional submanifold in the tangent bundle space, owing to Thom transversality theorem. Then the system folding is a smooth map between n -dimensional manifolds. Therefore the folding of a generic system can have all generic singularities like a map between n -dimensional manifolds. In fact the kernel of the bundle projection is also n -dimensional and, due to Goryunov's theorem[14], such dimension of the kernel permits all generic singularities for maps between n -dimensional manifolds.

A generic system near a regular point of its folding can be resolved with respect to derivatives. In this case, near such a point, the singularity theory of family of solutions of an implicit system is reduced to the well known theory of family of the phase curves for the generic smooth vector fields on n -dimensional manifolds [2].

For a generic system the velocity does not vanish at any singular point of the system folding. Consequently, near such a point, there is well-defined the *system 1-folding* which is the restriction of the projectivization of the tangent bundle to the system surface. Again Goryunov's theorem implies that the 1-folding of a generic system can have all singularities like a generic map from an n -dimensional manifold to an $(2n - 1)$ -dimensional one.

In particular, for the 2-dimensional case the 1-folding of a generic system can have regular points and singular points providing Whitney umbrella singularities. That implies, for a generic implicit system, the classification of point singularities of families of phase curves (see Theorem 2.1, 2.4). Be-

sides well known singularities of a generic vector fields on the plane and ones described by generic first order implicit differential equations there is only one singularity provided by the implicit differential equation on the Whitney umbrella generically embedded to the space of directions on the plane (Figures 1, 2, and 3). Up to smooth orbital equivalence the respective family of phase curve is the family of solutions of the implicit system

$$\dot{x} = \pm 1, \quad (\dot{y})^2 = x(x - y)^2$$

near the origin.

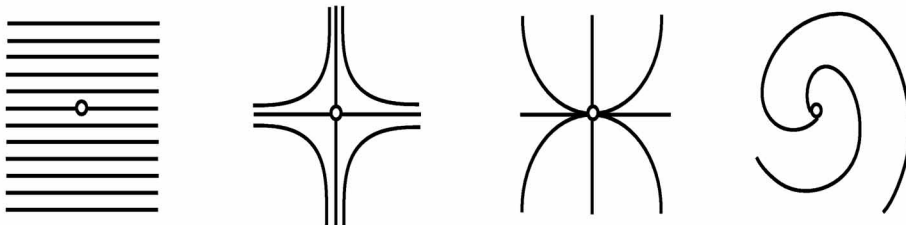


Figure 1: nonsingular point, saddle, node, and focus

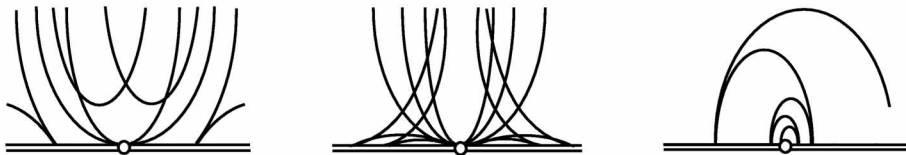


Figure 2: folded saddle, folded node, and folded focus

The family of solutions of the implicit equation $(dy/dx)^2 = x(x - y)^2$ has been well-known in another situation: V.I.Arnol'd found it by the investigation of slow motion of generic relaxation type equations with one fast and two slow variables [3][9]. However the last case and the case studied in this paper are different. At the first place, the equation surface in the space of directions on the plane is smooth in the theory of the relaxation

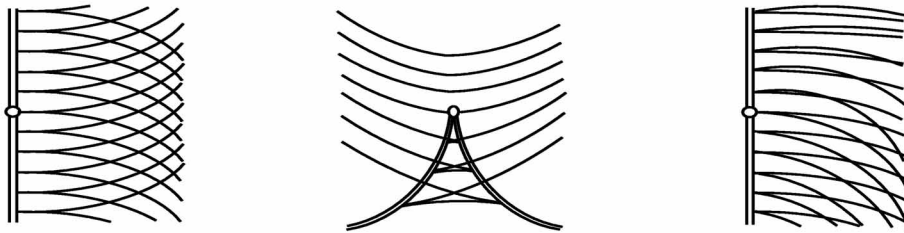


Figure 3: folded regular, pleated singular, and Whitney umbrella point

type equation, while it has the Whitney umbrella singularity for the implicit system case. At the second place, the plane distribution on the space of directions on the plane has singularities and is not the contact structure in the theory of the relaxation type equation, while it is the contact structure in the implicit system case. However if we put into correspondence to a relaxation type equation $\dot{x} = \varepsilon f(x, y, z), \dot{y} = \varepsilon g(x, y, z), \dot{z} = h(x, y, z) + \varepsilon r(x, y, z)$ (where f, g, h, r are smooth functions and ε is a small parameter) the surface $\dot{x} - f(x, y, z) = 0, \dot{y} - g(x, y, z) = 0, h(x, y, z) = 0$, then the restriction of the projection $(x, y, z, \dot{x}, \dot{y}) \mapsto (x, y, \dot{x} : \dot{y})$ to this surface is an analogue of 1-folding. In a generic case this restriction is well-defined near any critical point of the folding being here the restriction of the projection $(x, y, z) \mapsto (x, y)$ to the surface $h = 0$. That reduces Arnold's case to the considered one.

We also present the complete classification of generic singularities for Clairaut implicit systems on the plane (Figure 4). An implicit system is called *of Clairaut type* if the system surface is smooth and, for any critical point of its folding, the velocity corresponding to this point is non-zero and lies in the image of the tangent plane to the system surface under the derivative of the folding, like the classical Clairaut equations. Under a mild condition, any implicit system of Clairaut type can be approximated by a system of Clairaut type which is foliated by smooth trajectories projecting to non-singular solution curves. Actually we give the generic classification of such systems. See section 2.2 for details.

Remark that several singularities of the systems are studied in the *para-*

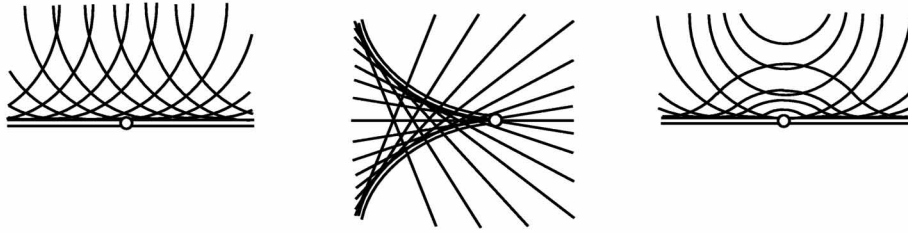


Figure 4: Clairaut fold, Clairaut cusp, and Clairaut cross cap

metric forms, not in the implicit forms. The parametric method provides, as well as the simple process of getting normal forms, also the clear understanding of moduli spaces of the singularities and the easy drawing of pictures of the phase portraits.

In section 2, we present the normal forms both in generic case and generic Clairaut case. The proofs are given in section 3 for the generic systems. We give the proof in section 4 for the generic Clairaut systems, using the theory of integral diagrams (cf. [16]).

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2 Classification of singularities.

Here we present the complete list of generic point singularities of first order implicit systems on two-dimensional manifolds up to smooth orbital equivalence for both general case and Clairaut one. Since we concern just local classification, we may treat systems on the plane $\mathbf{R}^2$.

2.1 Classification in general case.

Theorem 2.1 ([2]) *For a generic implicit system with locally bounded derivatives on the plane and for any regular point of its folding, the respective point singularity takes one of the forms listed in the second column of Table 1 near the origin up to smooth orbital equivalence. Besides the parameters of the*

Table 1

Type of singularities	Normal forms	Restrictions
Nonsingular point	$\dot{x} = 1$ $\dot{y} = 0$	
Nonresonance saddle with exponent λ	$\dot{x} = x,$ $\dot{y} = \lambda y$	$\lambda \in \mathbf{R}_- \setminus \mathbf{Q}$
Resonance saddle with exponent $-p/q$	$\dot{x} = x[1 \pm x^p y^q + ax^{2p} y^{2q}]$ $\dot{y} = -py/q$	$a \in \mathbf{R}; p, q \in \mathbf{N}, p/q$ is an irreducible fraction.
Nonresonance node with exponent λ	$\dot{x} = \varepsilon x,$ $\dot{y} = \varepsilon \lambda y$	$1 < \lambda \in \mathbf{R}_+ \setminus \mathbf{N}; \varepsilon = \pm 1$
Focus with exponent λ	$\dot{x} = \varepsilon x + \lambda y$ $\dot{y} = -\lambda x + \varepsilon y$	$\lambda \in \mathbf{R}_+, \varepsilon = \pm 1$

normal form from the second column have to satisfy the respective conditions from the third one.

Remark 2.2 In the first column of Table 1 there is pointed the standard name for the respective singularities of generic vector fields on the plane. The *smooth orbital equivalence* permits smooth change of coordinates and multiplication of vector fields by smooth positive function.

Remark 2.3 In [2] the list of the respective normal forms includes also ones for resonance node with exponent $\lambda = n \in \mathbf{N}$ ($\dot{x} = x, \dot{y} = ny + \varepsilon x^n, \varepsilon \in \{-1, 0, 1\}$), for resonance saddle with zero coefficients by few first resonance monomials ($\dot{x} = x[1 \pm u^k + au^{2k}], \dot{y} = \lambda y, 1 < k \in \mathbf{N}$) and for degenerate focus without formal first intergral starting from the positive definite quadratic form ($\dot{x} = y \pm x(r^{2k} + ar^{4k}), \dot{y} = -x \pm y(r^{2k} + ar^{4k})$). All these subcases are not generic. Ones can be removed by small perturbations of the implicit system.

Theorem 2.4 For a generic implicit system on the plane with locally bounded derivatives and any singular point of its folding the respective point singularity takes one of the forms listed in the second column of Table 2 near the origin up to smooth orbital equivalence. Besides the parameters of the normal form from the second column have to satisfy the conditions from the third one.

Table 2

Type of singularities	Normal forms	Restrictions
Folded regular point	$\dot{x} = \pm 1, (\dot{y})^2 = x$	
Folded nonresonance saddle with exponent λ	$\dot{x} = 1, (\dot{y})^2 = y - kx^2$	$\lambda \in \mathbf{R}_- \setminus \mathbf{Q};$ $k = \lambda/(2\lambda + 2)^2$
Folded resonance saddle with exponent $-p/q$	$\dot{x} = 1, (\dot{y})^2 = y - kx^2 + \varepsilon x(x^{p+q} + ax^{2p+2q})$	$a \in \mathbf{R}; p, q \in \mathbf{N}, 1 \neq p/q$ is an irreducible fraction; $k = -pq/(2p - 2q)^2$
Folded (nonresonance) node with exponent λ	$\dot{x} = 1, (\dot{y})^2 = y - kx^2$	$\lambda \in \mathbf{R}_+ \setminus \mathbf{N};$ $k = \lambda/(2\lambda + 2)^2$
Folded focus with exponent λ	$\dot{x} = 1, (\dot{y})^2 = y - kx^2$	$\lambda \in \mathbf{R}_+;$ $k = (1 + \lambda^{-2})/16$
Whitney umbrella point	$\dot{x} = \pm 1, (\dot{y})^2 = x(x - y)^2$	
Pleated singular point	$\dot{x} = 1, x = \dot{y}\varphi(y, \dot{y})$	φ is a smooth function; $\varphi(0, 0) = \varphi_{\dot{y}}(0, 0) = 0$ $\varphi_y(0, 0)\varphi_{\dot{y}\dot{y}}(0, 0) \neq 0$

Remark 2.5 By the topological orbital equivalence normal forms of folded saddle, node and focus are $\dot{x} = 1, (\dot{y})^2 = y - kx^2$ near the origin with k equals $-1, 1/20$ and $k = 1$, respectively [8][20].

2.2 Classification in the Clairaut case.

Now we denote by $T\mathbf{R}^2$ the tangent bundle of the plane $\mathbf{R}^2$, and by $\{0\}$ its zero section. We have the canonical projection from the tangent bundle, outside of the zero section, to the manifold of contact elements on the plane. We denote it by $\Pi : T\mathbf{R}^2 \setminus \{0\} \rightarrow PT^*\mathbf{R}^2$. Here $PT^*\mathbf{R}^2$ means the fiber-wise projectivization of the cotangent bundle $T^*\mathbf{R}^2$ on the plane $\mathbf{R}^2$. This projection Π actually induces the 1-folding mappings from surfaces in $T\mathbf{R}^2 \setminus \{0\}$. Remark that there does not exist a canonical isomorphism between the tangent bundle $T\mathbf{R}^2$ and the cotangent bundle $T^*\mathbf{R}^2$, but there exists the canonical isomorphism between $P\mathbf{R}^2$ and $PT^*\mathbf{R}^2$, as the manifold of contact elements on the plane, defined by mapping a tangential direction on $\mathbf{R}^2$ to the co-direction on $\mathbf{R}^2$ having the direction as the kernel. Then the classification under the smooth orbital equivalence of surfaces in $T\mathbf{R}^2 \setminus \{0\}$ is reduced, via its 1-folding and up to orientation of orbits, to the classifica-

tion under the contact diffeomorphisms on $PT^*\mathbf{R}^2$ preserving the canonical fibration $\pi : PT^*\mathbf{R}^2 \rightarrow \mathbf{R}^2$.

Let us consider a system of Clairaut type in $T\mathbf{R}^2 \setminus \{0\}$ and its 1-unfolding in $PT^*\mathbf{R}^2$. Denote by Σ_c , the locus of contact singular points, namely, the locus on the system surface consisting of points where the contact form vanishes on the corresponding point in $PT^*\mathbf{R}^2$. Also denote by Σ_π the locus of singular points on the system surface for the projection $\pi : T\mathbf{R}^2 \rightarrow \mathbf{R}^2$. Then, by definition, the system is of Clairaut type if and only if the folding mapping to $\mathbf{R}^2$ is of corank at most one and $\Sigma_c = \Sigma_\pi$.

Dara [7] gives the definition of Clairaut type equations for smooth surfaces in $\mathbf{R}^3(\subset PT^*\mathbf{R}^2)$ with coordinates x, y, p , $p = \dot{y}/\dot{x}$, as follows: An implicit system $G(x, y, p) = 0$ is called *of Clairaut type in the sense of Dara* if $G_x + pG_y = AG + BG_p$ holds for some function-germs $A(x, y, p)$ and $B(x, y, p)$.

By the definition, if a system is of Clairaut type in the sense of Dara, then we have $\Sigma_c = \Sigma_\pi$. In fact, in [17], it is proved that a non-singular system $G(x, y, p) = 0$ is of Clairaut type in the sense of Dara if and only if the system possesses the system of complete solutions consisting of classical (smooth) solutions. In particular, each trajectory projects to a non-singular curve via the folding.

The converse is not true in general: For example, the system $G(x, y, p) = y - 2p^3 = 0$ is not of Clairaut type in the sense of Dara, but it satisfies the condition $\Sigma_c = \Sigma_\pi$. Moreover it has the system of complete solutions $\Gamma(t, c) = (x, y, p) = (3t^2 + c, 2t^3, t)$ and each cuspidal solution curve is tangent to the discriminant $\pi(\Sigma_\pi) = \{y = 0\}$ on the (x, y) -plane. Thus it is in fact a Clairaut type in the sense of this paper. Note that in this example the singular locus of the folding mapping is defined by $p^2 = 0$ on the system surface $\{y = 2p^3\} = \{(x, 2p^3, p) \mid (x, p) \in (\mathbf{R}^2, 0)\}$, and the singular locus has multiple components.

A system of Clairaut type is called *reduced* if the Jacobian of the folding mapping has no multiple components, or more exactly, if any differentiable function vanishing on the singular locus Σ_π is divided by the Jacobian of π restricted to the system surface. Then we see that any reduced system of Clairaut type can be approximated by a system of Clairaut type with the property that each trajectory projects to a non-singular curve via the folding. Then we have

Theorem 2.6 *A generic reduced system of Clairaut type on the plane with locally bounded derivatives takes locally one of the forms in the second column*

Table 3

Type of singularities	Normal forms	Restrictions
Nonsingular point	$\dot{x} = 1, \dot{y} = 0$	
Clairaut fold	$\dot{x} = 1, (\dot{y})^2 = y$	
Clairaut cusp	$\dot{x} = 1, y = \dot{y}\varphi(x, \dot{y})$	$\varphi(0, 0) = \varphi_{\dot{y}}(0, 0) = 0$ $\varphi_{\dot{y}\dot{y}}(0, 0)\varphi_x(0, 0) \neq 0$
Clairaut Whitney umbrella	$\dot{x} = 1, (\dot{y})^2 = x^2y$	

of Table 3 near the origin up to smooth orbital equivalence.

The Clairaut type systems can be classified by considering parametric surfaces in $PT^*\mathbf{R}^2$.

A *first order differential equation germ* (or, briefly, *equation*) is defined to be a map germ $f : (\mathbf{R}^2, 0) \rightarrow J^1(\mathbf{R}, \mathbf{R}) \subset PT^*\mathbf{R}^2$. We also say that f is *completely integrable* if there exists a submersion germ $\mu : (\mathbf{R}^2, 0) \rightarrow \mathbf{R}$ such that $d\mu \wedge f^*\theta = 0$. Here $\theta = dy - pdx$ denotes the canonical contact 1-form on $J^1(\mathbf{R}, \mathbf{R})$. We call μ an *independent first (or, complete) integral of f* and the pair $(\mu, f) : (\mathbf{R}^2, 0) \rightarrow \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}) \subset \mathbf{R} \times PT^*\mathbf{R}^2$ is called a *holonomic system with independent first integral*. We observe that $f|_{\mu^{-1}(t)}$ is a Legendrian immersion whose image is contained in $\text{Image } f$. If $\pi \circ f|_{\mu^{-1}(t)}$ are non-singular map for each $t \in (\mathbf{R}, \mu(0))$, then $\{f|_{\mu^{-1}(t)}\}_{t \in \mathbf{R}}$ is the family of graphs of non-singular solutions of $\text{Image } f$ by the previous arguments. We call such a system a *Clairaut type equation*. These situation lead us to the following definition. Let (μ, g) be a pair of a map germ $g : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ and a submersion germ $\mu : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}, 0)$. Then the diagram

$$(\mathbf{R}, 0) \xleftarrow{\mu} (\mathbf{R}^2, 0) \xrightarrow{g} (\mathbf{R}^2, 0)$$

or briefly (μ, g) , is called an *integral diagram* if there exists an equation $f : (\mathbf{R}^2, 0) \rightarrow PT^*\mathbf{R}^2$ such that (μ, f) is an equation germ with independent first integral and $\pi \circ f = g$, and we say that the integral diagram (μ, g) is *induced by f* . If f is a Clairaut type equation, then $(\mu, \pi \circ f)$ is called *of Clairaut type*. Furthermore we introduce an equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g)

and (μ', g') are *equivalent* (respectively, *strictly equivalent*) if the diagram

$$\begin{array}{ccccc} (\mathbf{R}, 0) & \xleftarrow{\mu} & (\mathbf{R}^2, 0) & \xrightarrow{g} & (\mathbf{R}^2, 0) \\ \kappa \downarrow & & \psi \downarrow & & \downarrow \phi \\ (\mathbf{R}, 0) & \xleftarrow{\mu'} & (\mathbf{R}^2, 0) & \xrightarrow{g'} & (\mathbf{R}^2, 0) \end{array}$$

commutes for some diffeomorphism germs κ, ψ and ϕ (respectively, $\kappa = id_{\mathbf{R}}$).

We give a generic classification of Clairaut type equations in terms of the notion of integral diagrams which implies Theorem 2.6:

Theorem 2.7 *For a generic Clairaut type equation*

$$(\mu, f) : (\mathbf{R}^2, 0) \rightarrow \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}),$$

the integral diagram $(\mu, \pi \circ f)$ is strictly equivalent to one of germs in the following list :

- (1) $\mu = v, g = (u, v)$; Nonsingular point.
- (2) $\mu = v - \frac{1}{2}u, g = (u, v^2)$; Clairaut fold.
- (3) $\mu_\alpha = v + \alpha \circ g$ for $\alpha \in \mathfrak{M}_{(x,y)}$, $g = (u, v^3 + uv)$; Clairaut cusp.
- (4) $\mu = v - \frac{1}{2}u^2, g = (u, \frac{1}{4}v^2)$; Clairaut Whitney umbrella.

Note that the forms (1) (2) and (4) of Theorem 2.6 are obtained from those in Theorem 2.7 as follows:

(1) : The equation is given by $f = (u, v, 0)$, namely, by $p = 0$. Setting $\dot{x} = 1$, we have $\dot{y} = 0$.

(2) : The equation is given by $f = (u, v^2, v)$, namely, by $p^2 = y$. Setting $\dot{x} = 1$, we have $(\dot{y})^2 = y$.

(4) : The equation is given by $f = (u, \frac{1}{4}v^2, \frac{1}{2}uv)$, namely by $p^2 = x^2y$. Setting $\dot{x} = 1$, we have $(\dot{y})^2 = x^2y$.

The form (3) of Theorem 2.6 is obtained from the parametric form (3) of Theorem 2.7 as follows: The the 1-folding map is given by $(u, v) \mapsto (x, y, p) = (u, v^3 + uv, h(u, v))$, where $p = \frac{\dot{y}}{\dot{x}}$ and

$$h(u, v) = \frac{v - (3v^2 + u) \frac{\partial \alpha}{\partial x}(u, v^3 + uv)}{1 + (3v^2 + u) \frac{\partial \alpha}{\partial y}(u, v^3 + uv)}.$$

Then u and h form another parametrization, and we have the implicit form $y = v^3 + uv = v(x, p)^3 + xv(x, p) =: \psi(x, p)$. By using a cusp-preserving diffeomorphism which maps the phase curve passing through the cusp vertex to the x -axis, we can suppose that $\psi(x, 0) = 0$. So we have $y = \psi(x, p) = p\varphi(x, p)$. By setting $\dot{x} = 1$, we have $y = \dot{y}\varphi(x, \dot{y})$.

Theorem 2.7 gives a generic classification of integral diagrams of Clairaut type under the strict equivalence. We remark that each germs from (1) to (4) are not equivalent as integral diagrams. Thus the problem is reduced to classify germs which are contained in the family (3) under the equivalence. This family is parametrized by function germs α which are called *functional moduli*. In [18], it is given a characterization of functional moduli relative to the equivalence. For a map $g : x = u, y = v^3 + uv$, define in the x, y -plane the set Δ of points where this map has three different preimages, and the set $\mathcal{D}$ like the boundary of the set Δ . In [12] Dufour has shown the following characterization theorem.

Theorem 2.8 *Let (μ_α, g) be an integral diagram of the type (3) in Theorem 2.7. Then for any α , there exists a function germ $\alpha' : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}, 0)$ such that*

- (1) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$.
- (2) $\alpha'|_{\mathcal{D}} = 0$.

Dufour has also shown that the uniqueness of functional moduli relative to the equivalence. We say that α and α' are *equivalent as moduli* if there exists $a \in \mathbf{R} \setminus \{0\}$ such that $a\alpha(x, y) = \alpha'(a^2x, a^3y)$ for any $(x, y) \in \Delta$. We remark that the above definition of the equivalence among functional moduli is slightly different from Dufour's original definition in [12]. If we adopt his definition, we cannot assert the necessity of the condition that functional moduli are equivalent. Actually, in [18], we have introduced the above definition and shown the following theorem:

Theorem 2.9 *Let (μ_α, g) and $(\mu_{\alpha'}, g)$ be integral diagrams of (3) such that $\alpha|_{\mathcal{D}} = \alpha'|_{\mathcal{D}} = 0$. Then (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent as integral diagram if and only if α and α' are equivalent as moduli.*

This theorem asserts that the equivalence classes of functional moduli α with $\alpha|_{\mathcal{D}} = 0$ are the complete invariant for generic classifications of Clairaut type equations under the equivalence relation given by the group of point transformations.

We define $\mathfrak{M}(\mathcal{D}) = \{\alpha \in \mathfrak{M}_{(x,y)} \mid \alpha|_{\mathcal{D}} = 0\}$ and $\mathcal{M}_{\text{cusp}} = \mathfrak{M}(\mathcal{D})/\sim$, where $\sim$ denotes the equivalence relation as moduli. The above theorem asserts that the moduli space for generic Clairaut type equation is $\mathcal{M}_{\text{cusp}}$.

Remark 2.10 In the paper [16], it is given the classification of first order implicit differential equations in $PT^*\mathbf{R}^2$ endowed with independent first integrals. In [16], the pleated singular points are called the *regular cusps* and their normal forms are give in the parametric forms: $(u, v) \mapsto (u^3 + uv, v)$, the parametrization of the folding map, with the first integral $\mu = \frac{3}{4}u^4 + \frac{1}{2}u^2v + \alpha(u^3 + uv, v)$, where α is any function with $\alpha(0, 0) = 0$, $\frac{\partial \alpha}{\partial y}(0, 0) = \pm 1$ (Theorem B (5) of [16]). Moreover, by a theorem of Kurokawa (Theorem A of [19]), we can take, up to the equivalence, the functional moduli α satisfying

$$\alpha(0, y) = \pm y + \frac{1}{2} \frac{\partial^2 \alpha}{\partial y^2}(0, 0) y^2, (y \leq 0).$$

Furthermore the sign ± 1 and $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0, 0)$ are invariants of the equation (cf. [19]).

Then the 1-folding map is given by $(u, v) \mapsto (x, y, p) = (u^3 + uv, v, k(u, v))$, where $p = \frac{\dot{y}}{\dot{x}}$ and

$$k(u, v) = \frac{u + \frac{\partial \alpha}{\partial x}(u^3 + uv, v)}{\frac{1}{2}u^2 - \frac{\partial \alpha}{\partial y}(u^3 + uv, v)}.$$

Remark that functions v and k provide another parametrization of the system surface. Then we have $x = u^3 + uv = u(v, k)^3 + u(v, k)v = u(y, p)^3 + u(y, p)y =: \psi(y, p)$. Note that the locus $p = 0$ defines a smooth curve that is tangent to the y -axis at the cusp vertex on (x, y) -plane. By using a cusp-preserving diffeomorphism which maps this curve to the y -axis, we may suppose $\psi(y, 0) = 0$. Thus we have the form $x = p\varphi(y, p)$. Then we have the normal form in table 2 by setting $\dot{x} = 1$.

The first integral $\mu_\alpha = \frac{3}{4}u^4 + \frac{1}{2}u^2v + \alpha(u^3 + uv, v)$ for the system with the folding map $(u, v) \mapsto (u^3 + uv, v)$ is transformed to the first integral $\mu_{\alpha'} = \frac{3}{4}u^4 + \frac{1}{2}u^2v + \alpha'(u^3 + uv, v)$ for the system with the same folding map $(u, v) \mapsto (u^3 + uv, v)$ if and only if $\alpha'(x, y) = \alpha(x, y)$ or $\alpha'(x, y) = \alpha(-x, y)$ on

the cuspidal open domain $\Delta := \{y^3 + \frac{27}{4}x^2 < 0\}$ near the origin (Proposition 6.1 of [16]).

The exact description of the moduli space of pleated singular points remains open. However the reason of the existence of the functional moduli clearly comes from that for the 3-webs on the plane by solution curves, and we see, as mentioned above, the moduli space is dominated by the function space

$$\left\{ \alpha|_{\overline{\Delta}} \mid \alpha : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}, 0), \alpha(0, y) = \pm y + \frac{1}{2} \frac{\partial^2 \alpha}{\partial y^2}(0, 0) y^2, (y \leq 0) \right\}.$$

3 Singularities in a general case.

Here we prove Theorem 2.1, 2.4. At the first place we study typical singularities of system (1-)folding, and then on that base the theorems are proved.

3.1 Generic singularities of 1-folding.

Proposition 3.1 *For a generic implicit system its surface is either empty or a smooth two dimensional submanifold of the tangent bundle space.*

Proposition 3.1 follows immediately from Thom transversality theorem.

Proposition 3.2 *The folding of a generic implicit system with locally bounded derivatives is LR-stable map and it can have singular points of type Whitney either fold or pleat only. In other words, near any its critical point, the folding takes respectively the form*

$$\text{either} \quad \begin{cases} x = u \\ y = v^2 \end{cases} \quad \text{or} \quad \begin{cases} x = u \\ y = v^3 + uv \end{cases}$$

in appropriate smooth coordinates near this point and its image under the folding with the origin at them.

Recall that the system folding is the restriction of the tangent bundle projection to the system surface. Thus due to Goryunov theorem such restriction in a generic case can have all generic singularities as a generic map between n -dimensional manifolds with the dimension of the kernel being no

greater than the dimension of the kernel of this projection [14]. But the last dimension is also equal to n , and that permits all generic singularities between the system surface and the phase space.

In the two dimensional case these singularities are Whitney fold and pleat [23], [1]. Besides our systems are with locally bounded derivatives. Therefore system foldings are proper maps. Consequently, for a generic system its folding is LR -stable map [15], [22].

Thus Proposition 3.2 is true.

Remark 3.3 A map is called *LR-stable* (= *left right stable*) if for any map being sufficiently close to it these two maps can be carried one to another by diffeomorphisms of the image space and the preimage space which are close to the identities. For example the map $(x, y) \mapsto x$ from the circle $x^2 + y^2 = 1$ to the x -axis is LR -stable (under small perturbations in C^k -topology with $k \geq 2$).

Proposition 3.4 *For a generic implicit system with locally bounded derivatives any critical point of its folding does not belong to the zero section of the tangent bundle to the phase space. In particular, near such a point the system 1-folding is well defined.*

This proposition follows immediately from Thom transversality theorem because the conditions

$$F = 0, \quad \text{rank } F_{\dot{x}} < n, \quad \dot{x} = 0$$

defines in the jet space of systems the closed Whitney stratified manifold of the codimension $2n + 1$ which is greater than the dimension $2n$ of the tangent bundle. Therefore for a generic system these conditions can not be satisfied simultaneously.

Theorem 3.5 *For a generic implicit system with locally bounded derivatives any critical point of its folding is either regular point or critical point of type Whitney umbrella for the 1-folding of this system. Besides the image of the set of critical points of the system folding under the 1-folding is generically replaced with respect to the standard contact structure in the space of directions on the phase space. In particular, it can have tangency with this structure only of the first order and only at the points being regular for the system 1-folding and critical one of type Whitney fold for the system folding.*

Remark 3.6 A generic replacement of the image with respect to the direction axis in the projectivization of the tangent bundle follows immediately from Proposition 3.2 and the following corollary of the first statement of Theorem 3.5.

Corollary 3.7 *For a generic implicit system with locally bounded derivatives any critical point of type Whitney umbrella this system 1-folding is the critical point of type Whitney fold for this system folding.*

Theorem 3.5 also implies immediately

Corollary 3.8 *For a generic implicit system with locally bounded derivatives its point singularities provided by the critical point of this system folding are described either by generic singularities of first order implicit differential equation or by such an equation provided by a generic replacement in the space of directions on the plane of germ of the Whitney umbrella at its vertex.*

Let us prove Theorem 3.5. Let the surface of a generic implicit system with locally bounded derivatives be not empty.

For such a system its folding is LR -stable due to Proposition 3.2 and in the strength of Proposition 3.4 the set of critical points of this folding does not intersect the zero section of the tangent bundle to the phase space. Therefore for any system being sufficiently close to the given one the 1-folding is well defined near this set.

Note that any regular point of this system folding is also regular point of its 1-folding because the folding provides two components of the 1-folding.

Now we again can apply Goryunov theorem [14]. Due to this theorem the 1-folding of a generic system can have all generic singularities like a generic map from two-dimensional manifold to 3-dimensional one. But any critical point of such a map is of Whitney umbrella type. Taking into account Corollary 3.7 we find that near such a point of a generic system this system 1-folding takes form

$$x = u, \quad y = v^2, \quad z = uv$$

in appropriate smooth coordinates near this critical point and local smooth coordinates in the image space fibered over the phase space $(x, y$ -space) with the origins at this point and its image, respectively.

A typical replacement (of the image of the set of critical points of the folding under the 1-folding with respect to the standard contact structure in

the space of directions on the phase space) can be obtained by small rotations of the tangent planes. Really such a rotations provides small perturbation of the system but they do not change the set of critical value of the system folding and can supply any small rotation of the field of direction defined by our system on this set. Finally if for a generic system the "replacement" is typical then it is also typical for any system being sufficiently close to the chosen one due to LR -stability of the folding of a generic system in the strength of Proposition 3.2.

Theorem 3.5 is proved.

3.2 Proofs of Theorems 2.1, 2.4.

At the first place we prove Theorem 2.1 and then Theorem 2.4.

Let P be a regular point of the folding of a generic system. Near such a point this system surface is smooth section of the tangent bundle. This section provides smooth vector field v near the image $\tilde{P}$ of this point under the system folding.

If the point P does not belong to the zero section of the tangent bundle then this field does not vanish at the point $\tilde{P}$. In that case the germ of the field v at this point is C^∞ -diffeomorphic to the germ of the constant vector field $(1, 0)$ at the origin [2]. That gives the first singularity from Table 1.

If the point P belongs to zero section of the tangent bundle then field v vanishes at the point $\tilde{P}$. Small perturbations of the studied system implies smooth small changing of the field near the point P due to LR -stability of this system folding in accordance with Proposition 3.4. But for a generic system this map ("small perturbation" $\mapsto$ "small perturbations v near $\tilde{P}$ ") is continuous and small perturbations of the system provides all small perturbations of the system surface near the point P and, hence, all small perturbations of the field v near the point $\tilde{P}$. Therefore for a generic system vector field v has at a point $\tilde{P}$ a generic singular point. Now the rest part of Theorem 2.1 follows from the classical results about normal forms of generic vector fields near singular points up to smooth orbital equivalence [2]. Theorem 2.1 is proved.

Let us prove Theorem 2.4. Due to Corollary 3.8 local singularities of a generic implicit system at singular points of its folding are described by generic singularities of first order implicit ODE except the singular points of this system 1-folding of type Whitney umbrella. But here we need to take into account that a critical point of a generic implicit system never belongs

the zero section of the tangent bundle due to Proposition 3.4. Consequently, generic singularities of implicit equations of type folded regular point, folded singular point and pleated point gives here first five singularities and the last one from Table 2.

Now due to Corollary 3.8 to finish the proof one need to get the normal form of an implicit first order ODE provided by a generic replacement of Whitney umbrella to the space of directions on the plane. For a generic system such a replacement has to have the following properties. At the Whitney umbrella vertex both the the contact plane and the vertical direction do not tangent to the image under a generic system 1-folding of set of critical points of this system folding and the "handle" of this umbrella.

Consequently, smooth local coordinate systems u, v on the system surface and x, y on the phase space with the origins at the studied point and its image under the system folding can be choosen such that this system 1-folding takes the form

$$x = v^2, \quad y = u, \quad \frac{dy}{dx} = h(u, v) \quad (1)$$

where h is a smooth function, $h(0, 0) = 0$, and dy/dx is local coordinate along the direction axis and also in these coordinates the "handle" of the Whitney umbrella is over the line $x - y = 0$.

Near the origin on the y -axis and the line $\{x - y = 0\} \cup \{x > 0\}$, the studied implicit first order equation gives two smooth direction fields $dy/dx = f_1(y)$ and $dy/dx = f_2(y)$ respectively, where f_1, f_2 are smooth functions; $f_1(0) = f_2(0) = 0$ because $h(0, 0) = 0$. Due to Hadamard lemma these functions can be presented in the form $f_1(y) = y\tilde{f}_1(y)$, $f_2(y) = y\tilde{f}_2(y)$ where $\tilde{f}_1, \tilde{f}_2$ are smooth functions. Near the origin the direction field

$$dy/dx = (y - x)\tilde{f}_1(y) + x\tilde{f}_2(y)$$

provides the semiltanious smooth extension of these two fields.

Near the origin the extended field has a first integral of the form $y + xI_1(x, y)$ where I_1 is smooth function. Taking this integral and the function $u + v^2I_1(v^2, u)$ like new coordinates y and u , respectively, one preserves the first two forms of the equation (1) but in new coordinates the function h is identically zero on the u -axis and on the set corresponding to the "handle". In new coordinates near the origin the last set can be defined by some equation $u - v^2X(v^2) = 0$ where X is a smooth function, $X(0) > 0$. Due to Hadamard lemma the function h in the last equation from (1) can be written in the form

$h(u, v) = v(u - v^2 X(v^2))H(u, v)$ where H is smooth function; $H(0, 0) \neq 0$ because at the studied point the system 1-folding has singularity of type Whitney umbrella. Consequently near the origin the rescalings $\tilde{v} = v\sqrt{X(v^2)}$ and $\tilde{x} = xX(x)$ reduce the system (1) of equations to the form ("tilde" in notations of new coordinates is omitted)

$$x = v^2, \quad y = u, \quad \frac{dy}{dx} = v(u - v^2)H(u, v) \quad (2)$$

with some new smooth function H no vanishing at the origin. It is easy to see that the direction field provided by the last equation can be lifted to the smooth direction field

$$du/dv = 2v^2(u - v^2)H(u, v). \quad (3)$$

The last direction field has first integral of the form

$$I(u, v) = u + v^3 J_1(u, v)$$

as it is easy to see.

Now it is sufficient to get normal form of this integral by the changing of coordinates commuting with the involution $(u, v) \mapsto (u, -v)$ defined by the system folding. Taking new coordinate u in the form $(I(u, v) + I(u, -v))/2$ (being even with respect to v and so it is permitted by our involution) we reduce the integral to the form $I(u, v) = u + v^3 J_1(u, v^2)$ or

$$I(u, v) = u + v^3 a(u) + v^5 b(u) + v^7 c(u, v^2),$$

where a, b, c are some smooth functions; $a(0) = 0 \neq a'(0)b(0)$ because $v^2 u$ is the term of the lower degree in the right hand side of the equation (3), $b(0) \neq 0$ due to $H(0, 0) \neq 0$.

The following lemma completes the proof

Lemma 3.9 ([3], [9]) *Near the origin the (u, v) -plane a function $u + v^3 a(u) + v^5 b(u) + v^7 c(u, v^2)$ with smooth functions a, b and c , $a(0) = 0 \neq a'(0)b(0)$, is reduced to the form $u + v^3 u + v^5$ by a smooth diffeomorphism preserving the origin and commuting with the involution $(u, v) \mapsto (u, -v)$.*

Hence the function H in the equation (3) can be reduced to 1. After that this equation takes the form $du/dv = 2v^2(u - v^2)$. Near the points under consideration that implies the equation $dy/dx = v(u - v^2)$ on the system surface or the equation $(dy/dx)^2 = x(y - x)^2$ on the (x, y) -plane. That gives the rest (fifth) singularity from the list of Table 2. Theorem 2.4 is proved.

4 Clairaut type equations and Legendre singularity theory.

4.1 Legendrian unfoldings.

We briefly review the theory of one-parameter Legendrian unfoldings. We now consider the 1-jet bundle $J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ and the canonical 1-form Θ on the space. Let (t, x) be the canonical coordinate on $\mathbf{R} \times \mathbf{R}$ and (t, x, y, q, p) be the corresponding coordinate on $J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$. Then the canonical 1-form is given by $\Theta = dy - p dx - q dt = \theta - q dt$. We also have the natural projection

$$\Pi : J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}) \rightarrow \mathbf{R} \times \mathbf{R} \times \mathbf{R}$$

defined by $\Pi(t, x, y, q, p) = (t, x, y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle*. Let (μ, f) be an equation with complete integral. Then there exists a unique function germ $h : (\mathbf{R}^2, 0) \rightarrow \mathbf{R}$ such that $f^*\theta = h \cdot d\mu$. Define a map germ

$$\ell_{(\mu, f)} : (\mathbf{R}^2, 0) \rightarrow J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$$

by

$$\ell_{(\mu, f)}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u)).$$

Then we can easily show that if (μ, f) is a Clairaut type equation, then $\ell_{(\mu, f)}$ is a Legendrian immersion germ. We call $\ell_{(\mu, f)}$ *a complete Legendrian unfolding associated with (μ, f)* . By the aid of the notion of Legendrian unfoldings, Clairaut type equations are characterized as follows :

Proposition 4.1 *Let $(\mu, f) : (\mathbf{R}^2, 0) \rightarrow \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}) \subset PT^*\mathbf{R}^2$ be an equation with complete integral. Then (μ, f) is a Clairaut type equation if and only if $\ell_{(\mu, f)}$ is a Legendrian non-singular Legendrian immersion germ.*

A complete Legendrian unfolding $\ell_{(\mu, f)}$ associated to (μ, f) is called *a Legendrian unfolding of Clairaut type* if $\ell_{(\mu, f)}$ is a Clairaut type equation.

4.2 Genericity.

Returning to the study of equations with complete integral, we now establish the notion of the genericity.

Let $U \subset \mathbf{R}^2$ be an open set. We denote by $\text{Int}(U, \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}))$ the set of systems with complete integral $(\mu, f) : U \rightarrow \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R})$. We also define $L(U, J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}))$ to be the set of complete Legendrian unfoldings $\ell_{(\mu, f)} : U \rightarrow J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$.

These sets are topological spaces equipped with the Whitney C^∞ -topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows. Let P be a property of equation germs with complete integral $(\mu, f) : (\mathbf{R}^2, 0) \rightarrow \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R})$ (respectively, Legendrian unfoldings $\ell_{(\mu, f)} : (\mathbf{R}^2, 0) \rightarrow J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$). For an open set $U \subset \mathbf{R}^2$, we define $\mathcal{P}(U)$ to be the set of $(\mu, f) \in \text{Int}(U, \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}))$ (respectively, $\ell_{(\mu, f)} \in L(U, J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}))$) such that the germ at x whose representative is given by (μ, f) (respectively, $\ell_{(\mu, f)}$) has property P for any $x \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbf{R}^2$, the set $\mathcal{P}(U)$ is a generic subset in $\text{Int}(U, \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}))$ (respectively, $L(U, J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U, J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})) \rightarrow \text{Int}(U, \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}))$$

defined by $(\Pi_1)_*(\ell_{(\mu, f)}) = \Pi_1 \circ \ell_{(\mu, f)} = (\mu, f)$, where $\Pi_1 : J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}) \rightarrow J^1(\mathbf{R}, \mathbf{R})$ is the canonical projection. Then it has been shown the following fundamental theorem:

Theorem 4.2 *The continuous map*

$$(\Pi_1)_* : L(U, J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})) \rightarrow \text{Int}(U, \mathbf{R} \times J^1(\mathbf{R}, \mathbf{R}))$$

is a homeomorphism.

As in [18], we define the equivalence relation among parametric systems under the group of point transformations: Two equations $f, f' : (\mathbf{R}^2, 0) \rightarrow PT^*\mathbf{R}^2$ are *equivalent under the group of point transformations* if there exists a diffeomorphism $\phi : (\mathbf{R}^2, \pi(f(0))) \rightarrow (\mathbf{R}^2, \pi(f'(0)))$ such that the canonical lifting $\hat{\phi} : (PT^*\mathbf{R}^2, f(0)) \rightarrow (PT^*\mathbf{R}^2, f'(0))$ transforms f to f' , namely, $\hat{\phi} \circ f = f' \circ \psi$ for some diffeomorphism $\psi : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$.

We need the following basic result:

Proposition 4.3 *Let $f, f' : (\mathbf{R}^2, 0) \rightarrow PT^*\mathbf{R}^2$ be completely integrable equations with independent first integrals $\mu, \mu' : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}, 0)$ respectively. Assume $\pi \circ f$ and $\pi \circ f' : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ have nowhere dense singular sets. Then f and f' are equivalent under the group of point transformations if and only if the induced integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent.*

Proof. Assume $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent by diffeomorphisms (κ, ψ, ϕ) . Then ϕ maps integral curves $\pi(f(\mu^{-1}(\mu(u, v))))$ through $\pi(f(u, v))$ to $\pi(f'(\mu'^{-1}(\kappa(\mu(u, v)))) = \pi(f'(\mu'^{-1}(\mu'(\psi(u, v)))))$ through $\pi(f'(\psi(u, v)))$, so the tangent lines to them. Since the set of contact singular points is contained in the set of critical points of the projection π , we see $\hat{\phi} \circ f = f' \circ \psi$. This implies f and f' are equivalent under the group of point transformations. The converse implication is clear. $\square$

5 Proofs for Theorem 2.7 and Theorem 2.6.

In the case when $\ell_{(\mu, f)}$ is a Legendrian immersion germ, there exists a generating family of $\ell_{(\mu, f)}$ by the Arnol'd-Zakalyukin's theory ([1]). In this case the generating family is naturally constructed by a one-parameter family of generating families associated with (μ, ℓ) . Let $F : ((\mathbf{R} \times \mathbf{R}) \times \mathbf{R}^k, 0) \rightarrow (\mathbf{R}, 0)$ be a function germ such that $d_2 F|_{0 \times \mathbf{R} \times \mathbf{R}^k}$ is non-singular, where $d_2 F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \dots, \frac{\partial F}{\partial q_k}(t, x, q))$. We call F a *Morse family*. Then $C(F) = d_2 F^{-1}(0)$ is a smooth surface germ and $\pi_F : (C(F), 0) \rightarrow \mathbf{R}$ is a submersion germ, where $\pi_F(t, x, q) = t$. We call the submanifold $C(F)$ a *catastrophe set of F* . Define

$$\tilde{\Phi}_F : (C(F), 0) \rightarrow J^1(\mathbf{R}, \mathbf{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \rightarrow J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$$

by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that $(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F) = 0$. By definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([1]), we can show the following proposition.

Proposition 5.1 *All Legendrian unfolding germs are constructed by the above method.*

Let (μ, f) be a Clairaut type equation. By Proposition 4.1, $\ell_{(\mu, f)}$ is a Legendrian immersion. Then we can choose a family of function germs

$$F : (\mathbf{R} \times \mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$$

such that $\text{Image } j_1^1 F_t = f(\mu^{-1}(t))$ for any $t \in \mathbf{R}$, where $F_t(x) = F(t, x)$. We remark that the map germ

$$j_1^1 F : (\mathbf{R} \times \mathbf{R}, 0) \rightarrow J^1(\mathbf{R}, \mathbf{R})$$

defined by $j_1^1 F(t, x) = j_1^1 F_t(x)$ is not necessary an immersion germ. In this case we have $(C(F), 0) = (\mathbf{R} \times \mathbf{R}, 0)$ and

$$\Phi_F = j_1^1 F : (\mathbf{R} \times \mathbf{R}, 0) \rightarrow J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}),$$

so that it is a complete Legendrian unfolding associated with $(\pi_1, j_1^1 F)$. Thus the generating family of a Legendrian unfolding of Clairaut type is given by the above germ.

In order to prove Theorem 2.7, we now introduce equivalence relations among Legendrian unfoldings. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are $\mathcal{R}^+$ -equivalent if there exist a diffeomorphism germ $\Psi : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0) \rightarrow (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0)$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ and a diffeomorphism germ $\Phi : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. We remark that if (μ, g) and (μ', g') are $\mathcal{R}^+$ -equivalent by the above diffeomorphisms, then we have $\mu(u) + \alpha \circ g(u) = \mu' \circ \Phi(u)$ and $\psi \circ g(u) = g' \circ \Phi(u)$ for any $u \in (\mathbf{R}^2, 0)$. Thus the diagram $(\mu + \alpha \circ g, g)$ is strictly equivalent to (μ', g') .

We now define the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu, f)}, \ell_{(\mu', f')} : (\mathbf{R}^2, 0) \rightarrow (J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}), z_0)$ be Legendrian unfoldings. We say that $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are $S.P^+$ -Legendrian equivalent (respectively, $S.P$ -Legendrian equivalent) if there exist a contact diffeomorphism

germ $K : (J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}), z_0) \rightarrow (J^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}), z'_0)$, a diffeomorphism germ $\Phi : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ and a diffeomorphism germ $\Psi : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), \Pi(z_0)) \rightarrow (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), \Pi(z'_0))$ of the form $\Psi(t, x, y) = (t + \alpha(x, y), \psi(x, y))$ (respectively, $\Psi(t, x, y) = (t, \psi(x, y))$) such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \mathcal{L} = \mathcal{L}' \circ \Phi$. It is clear that if $\ell_{(\mu, f)}$ and $\ell_{(\mu', f')}$ are $S.P^+$ -Legendrian equivalent (respectively, $S.P$ -Legendrian equivalent), then $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are $\mathcal{R}^+$ -equivalent (respectively, strictly equivalent). By Theorem 1.1 in [18], the converse is also true for generic (μ, f) and (μ', f') . The notion of the stability of Legendrian unfoldings with respect to $S.P^+$ -Legendrian equivalence (respectively, $S.P$ -Legendrian equivalence) is analogous to the usual notion of the stability of Legendrian immersion germs with respect to Legendrian equivalence (cf. Part III in [1]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. For the purpose, we use some notations and results in [1]. Let $\tilde{F}, \tilde{G} : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0) \rightarrow (\mathbf{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type. We say that $\tilde{F}$ and $\tilde{G}$ are $P\text{-}\mathcal{C}^+$ -equivalent (respectively, $P\text{-}\mathcal{C}$ -equivalent) if there exists a diffeomorphism germ $\Phi : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0) \rightarrow (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0)$ of the form $\Phi(t, x, y) = (t + \alpha(x, y), \phi_1(x, y), \phi_2(x, y))$ (respectively, $\Phi(t, x, y) = (t, \phi_1(x, y), \phi_2(x, y))$) such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(t, x, y)}} = \langle G \rangle_{\mathcal{E}_{(t, x, y)}}$ where $\langle G \rangle_{\mathcal{E}_{(t, x, y)}}$ is the ideal generated by G in the local ring of function germs $\mathcal{E}_{(t, x, y)}$ of (t, x, y) -variables. We also say that $\tilde{F}(t, x, y)$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|_{\mathbf{R} \times 0}$ if

$$\mathcal{E}_t = \langle \frac{df}{dt} \rangle_{\mathbf{R}} + \langle f \rangle_{\mathcal{E}_t} + \langle \frac{\partial F}{\partial x} |_{\mathbf{R} \times \{(0, 0)\}}, \frac{\partial F}{\partial y} |_{\mathbf{R} \times \{(0, 0)\}}, 1 \rangle_{\mathbf{R}}$$

(respectively,

$$\mathcal{E}_t = \langle f \rangle_{\mathcal{E}_t} + \langle \frac{\partial F}{\partial x_1} |_{\mathbf{R} \times \{0\}}, \dots, \frac{\partial F}{\partial x_n} |_{\mathbf{R} \times \{0\}}, 1 \rangle_{\mathbf{R}}).$$

By the similar arguments like as those of Theorems 20.8 and 21.4 in [1], we can show the following :

Theorem 5.2 *Let $\tilde{F}, \tilde{G} : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0) \rightarrow (\mathbf{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type Φ_F, Φ_G respectively. Then*

- (1) Φ_F and Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian equivalent if and only if $\tilde{F}$ and $\tilde{G}$ are $P\text{-}\mathcal{C}^+$ (respectively, $\mathcal{C}$)-equivalent.
- (2) Φ_F is $S.P^+$ (respectively, $S.P$)-Legendrian stable if and only if $\tilde{F}$ is a $P\text{-}\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|_{\mathbf{R} \times \{0\}}$.

The following theorem is a corollary of Damon's general versality theorem in [6].

Theorem 5.3 *Let $\tilde{F}, \tilde{G} : (\mathbf{R} \times (\mathbf{R} \times \mathbf{R}), 0) \rightarrow (\mathbf{R}, 0)$ be generating families of Legendrian unfoldings of Clairaut type such that Φ_F, Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian stable. Then Φ_F, Φ_G are $S.P^+$ (respectively, $S.P$)-Legendrian equivalent if and only if $f = F|_{\mathbf{R} \times \{0\}}$, $g = G|_{\mathbf{R} \times \{0\}}$ are $\mathcal{C}$ -equivalent (i.e. $\langle f \rangle_{\mathcal{E}_t} = \langle g \rangle_{\mathcal{E}_t}$).*

Then the classification theory of function germs by the $\mathcal{C}$ -equivalence is quite useful for our purpose. For each function germ $f : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$, we set

$$\begin{aligned}\mathcal{C}\text{-cod}(f) &= \dim_{\mathbf{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t}, \\ \mathcal{C}^+\text{-cod}(f) &= \dim_{\mathbf{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{df}{dt} \right\rangle_{\mathbf{R}}, \\ \mathcal{K}\text{-cod}(f) &= \dim_{\mathbf{R}} \mathcal{E}_t / \langle f \rangle_{\mathcal{E}_t} + \left\langle \frac{df}{dt} \right\rangle_{\mathcal{E}_t}.\end{aligned}$$

Then we have the following well-known classification (cf. [21]).

Lemma 5.4 *Let $f : (\mathbf{R}, 0) \rightarrow (\mathbf{R}, 0)$ be a function germ with $\mathcal{K}\text{-cod}(f) < \infty$. Then f is $\mathcal{C}$ -equivalent to the map germ $t^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By the direct calculation, we have

$$\begin{aligned}\mathcal{C}\text{-cod}(t^{\ell+1}) &= \ell + 1, \\ \mathcal{C}^+\text{-cod}(t^{\ell+1}) &= \ell.\end{aligned}$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^+$)-versal deformations of the above germs by the usual method as follows :

The $\mathcal{C}$ -versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1} t^i.$$

The $\mathcal{C}^+$ -versal deformation :

$$t^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1} t^i.$$

We now ready to give a proof of Theorem 2.7.

Proof of Theorem 2.7 Let (μ, f) be a Clairaut type equation and the corresponding Legendrian unfolding $\ell_{(\mu, f)}$. By the previous arguments, we have a function germ $F : (\mathbf{R} \times \mathbf{R}, 0) \longrightarrow (\mathbf{R}, 0)$ such that $\text{Image } j_1^1 F = \text{Image } \ell_{(\mu, f)}$. Therefore we consider generic property of $F(t, x)$. By definition $j_1^1 F$ is an immersion germ if and only if

$$\left(\frac{\partial F}{\partial t}(0), \frac{\partial^2 F}{\partial t \partial x}(0) \right) \neq 0.$$

Under this condition, we have the characterization of the fold point and the cusp point of $\pi \circ j_1^1 F$ as follows (cf., [13][15]):

- (A) $\pi \circ j_1^1 F$ is the fold germ if and only if $\frac{\partial F}{\partial t}(0) = 0$ and $\frac{\partial^2 F}{\partial t^2}(0) \neq 0$.
- (B) $\pi \circ j_1^1 F$ is the cusp germ if and only if

$$\frac{\partial F}{\partial t}(0) = \frac{\partial^2 F}{\partial t^2}(0) = 0 \text{ and } \frac{\partial^2 F}{\partial t \partial x}(0) \frac{\partial^3 F}{\partial t^3}(0) \neq 0.$$

When $j_1^1 F$ is not a immersion germ, we have the following characterization of the cross cap:

- (C) $j_1^1 F$ is a cross cap germ if and only if

$$\frac{\partial F}{\partial t}(0) = \frac{\partial^2 F}{\partial t \partial x}(0) = 0 \text{ and } \frac{\partial^3 F}{\partial t \partial x^2}(0) \frac{\partial^2 F}{\partial t^2}(0) \neq 0.$$

In the first place, we give normal forms under the assumption that the conditions (a),(b),(c). Let assume that the condition (C) holds. In this case the function germ has the following form:

$$F(t, x) = at^2 + bx^2 + ctx^2 + h(t, x),$$

where $a \neq 0, c \neq 0$ and $h(0, 0) = 0$. Since $F(t, 0) = at^2 + h(t, 0)$ is $\mathcal{C}$ -equivalent to t^2 , $F(t, x)$ is $P\text{-}\mathcal{C}$ -equivalent to a deformation of t^2 . By the previous arguments, the $\mathcal{C}$ -versal deformation of t^2 is $t^2 + v_1 t + v_2$. Therefore, $F(t, x)$ is $P\text{-}\mathcal{C}$ -equivalent to the function germ of the form:

$$G(t, x) = t^2 + t\phi_1(x) + \phi_2(x).$$

Since $j_1^1 G$ is also a cross cap germ, we have

$$\phi_1(x) = \alpha x^2 + \text{higher order term},$$

with $\alpha \neq 0$. By a local diffeomorphism of the variable x , we have $\phi_1(x) = x^2$. This means that $F(t, x)$ is P - $\mathcal{C}$ -equivalent to the germ of the form $t^2 + tx^2 + \phi(x)$. Hence, we might put that $F(t, x) = t^2 + tx^2 + \phi(x)$. In this case.

$$j^1 F(t, x) = (t, x, t^2 + tx^2 + \phi(x), 2t + x^2, 2tx + \phi'(x)).$$

The corresponding integral diagram is

$$\mu(u_1, u_2) = u_1, \quad g(u_1, u_2) = (u_2, u_1^2 + u_2^2 + \phi(u_2)).$$

On the (x, y) -plane, we have a diffeomorphism germ $\Psi : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$ defined by $\Psi(x, y) = (x, \frac{1}{4}(y + \frac{1}{4}x^4 - \phi(x)))$. Then we have $\Psi \circ g(u_1, u_2) = (u_2, \frac{1}{4}(u_1 + \frac{1}{2}u_2^2)^2)$. This is the normal form (4) in Theorem 2.7, after setting $(u, v) = (u_2, u_1 + \frac{1}{2}u_2^2)$.

For the case (A), we can apply almost the same arguments as the above and get the normal form of (2) in Theorem 2.7.

For the case (B), the situation is a rather different. In this case the function $F(t, 0)$ is $\mathcal{C}$ -equivalent to t^3 . The $\mathcal{C}$ -versal deformation of t^3 is $t^3 + v_1 t^2 + v_2 t + v_3$, then the above arguments cannot work in this case. However, the $\mathcal{C}^+$ -versal deformation of t^3 is $t^3 + v_1 t + v_2$. Thus we can apply almost the same arguments as the above and the corresponding integral diagram is $\mathcal{R}^+$ -equivalent to

$$\mu(u_1, u_2) = u_2, \quad g(u_1, u_2) = (u_1, u_2^3 + u_1 u_2).$$

This means that the diagram is strictly equivalent to the normal form (3) in Theorem 2.7.

We now show that the set of function $F(t, x)$ satisfying the conditions (A), (B), (C) or (R) at any point are generic in the space of all functions (equipped with the Whitney C^∞ -topology). Here the condition (R) is that $\frac{\partial F}{\partial t}(0) \neq 0$. Let $J^3(2, 1)$ be the set of 3-jets of function germs $h : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$. We consider the following two algebraic subset of $J^3(2, 1)$:

$$\begin{aligned} \Sigma_1 &= \left\{ j^3 h(0) \mid \frac{\partial h}{\partial t}(0) = \frac{\partial^2 h}{\partial t \partial x}(0) = \frac{\partial^2 h}{\partial t^2}(0) \frac{\partial^3 h}{\partial t \partial x^2}(0) = 0 \right\}, \\ \Sigma_2 &= \left\{ j^3 h(0) \mid \frac{\partial h}{\partial t}(0) = \frac{\partial^2 h}{\partial t^2}(0) = \frac{\partial^2 h}{\partial t \partial x}(0) \frac{\partial^3 h}{\partial t^3}(0) = 0 \right\}. \end{aligned}$$

We consider the union $W = \Sigma_1 \cup \Sigma_2$, then it is also an algebraic subset of $J^3(2, 1)$. We can stratify the algebraic set W by submanifolds whose codimensions are at least 3. By Thom's jet transversality theorem, $j^3 F(\mathbf{R}^2) \cap (\mathbf{R}^2 \times \mathbf{R} \times W) = \emptyset$ for a generic function $F(t, x)$. We can easily show that the conditions (A),(B),(C) or (R) are satisfied for such a function $F(t, x)$. This completes the proof of Theorem 2.7 and in particular we have Theorem 2.6. $\square$

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