ON AN UNDERLYING STRUCTURE FOR THE CONSISTENCY OF VISCOSITY SOLUTIONS

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ON AN UNDERLYING STRUCTURE FOR THE CONSISTENCY OF VISCOSITY SOLUTIONS

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ABSTRACT: To consider phenomena of physics and economics, we need solutions in weak sense, one of which is the viscosity solution. Its definition has many variations to indicate the uniqueness and the existence of solutions. This paper attempts to give a framework which unifies viscosity solutions synthetically.
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1.INTRODUCTION

The theory of physics and economics derives various models from a lot of phenomena, some of which base on the theory of partial differential equations (PDEs).

Classical solutions to elliptic type PDEs have the ordered structure which is known as the comparison principle. (There are many important results for this structure, some of them are studied by [8] and [9].) This structure makes the key of weak solutions which is called viscosity solutions. The reader is referred to [1], [2], [5], [6] and [7] for the theory of viscosity solution. This weak solution gives us effective methods for example the geometric analysis([3],[4]).

Many researchers of mathematics, even those who puzzle over the fragility of the differential structure, have recognized that the viscosity solution is a solution to PDEs. Although there is no assurance for 'solution' to have derivatives in any sense, it is dealt as a solution to equations which are made of its derivatives. However is such a 'solution' appropriate in analysis? Does the similarity among models indicate the one among phenomena? These problems are unescapable when we apply theory to models which may not be expected to have the structure in the classical sense. In this paper the author attempts to confirm the structure which builds the skeleton of the viscosity method and to supply an alternative viewpoint which helps the viscosity solution to be identified.

In order to mull over this theme we recall the definition of the viscosity solution. There are a number of ways to define it, which are equivalent with each other in usual. In this paper we adopt the one that uses local maxima and minima, for it is easy to observe the logic which is typically used in discussions on the viscosity solution.

For a little while, to explain the point of this paper, we let $F[x;u] : \Omega \times C^2(\Omega) \to \mathbb{R}$ be elliptic in the following sense; $F[x;\varphi] \leq F[x;\psi]$ if $\varphi - \psi$ has a local maximum at x. (Some one may worry if this property might denote the elliptic type PDEs. §5 concerns probing this item.) Thereunder, we call $u \in C(\Omega)$ a viscosity supersolution (resp. subsolution) of F if $F[x;\varphi] \geq 0$(resp. ≤ 0) for every $(x,\varphi) \in \Omega \times C^2(\Omega)$ which makes $u - \varphi$ maximal (resp. minimal) locally at x. We then call u a viscosity solution when u is both a viscosity supersolution and a viscosity subsolution.

This notion of viscosity solutions is thought to be generalization of classical solutions just because of the following two reasons:

- A classical solution of an elliptic PDE is a viscosity solution of the same PDE.
- A viscosity solution in the C^2 class is also a solution in classical sense.

This reason seems to be acceptable, however if the concept does not uniquely extend, it turns out that many weak solutions flood over. The point of this note is to construct the notion of the solution by binary relations. Especially we use quasiorders, which satisfy the reflexive law and the transitive law but may not generally satisfy the antisymmetric law, *i.e.*

$$\forall u, v \in \mathfrak{X} \quad u \leq v \text{ and } v \leq u \quad \Rightarrow \quad u = v.$$

This construction is helpful to observe the structure of solutions and to see why they are ‘solution’ to PDEs.

We prepare a few properties of binary relations in §2. In §3 we formalize the equation in the viscosity sense by quasiorders. In §4 we study the structure which weakens the notion of solutions. In §5 we consider the elliptic type PDEs by binary relations. A typical result is stated in Theorem 5

2. PRELIMINARY

Definition 1 *For a family of binary relations $\mathcal{F}$, $\bigcap \mathcal{F}$ denotes the binary relation that satisfies*

$$u \left(\bigcap \mathcal{F}\right) v \overset{\text{def}}{\Leftrightarrow} \forall R \in \mathcal{F} \quad uRv.$$

Definition 2 *For a family of binary relations $\mathcal{F}$, $\bigcup\mathcal{F}$ denotes the binary relation that satisfies*

$$u\left(\bigcup\mathcal{F}\right)v \overset{\text{def}}{\Leftrightarrow} \exists R \in \mathcal{F} \text{ such that } \quad uRv.$$

Definition 3 *For a binary relation R, $R^{\mathbf{c}}$ denotes the binary relation that satisfies*

$$uR^{\mathbf{c}}v \overset{\text{def}}{\Leftrightarrow} uRv \text{ does not hold.}$$

We also use $\not\lesssim$, etc. despite of this notation.

Definition 4 *Let R_1 and R_2 be two binary relations on $\mathfrak{X}$. We say that R_2 includes R_1 (or R_1 is stronger than R_2, etc.) when they satisfy that*

$$\forall u,\, v \in \mathfrak{X} \quad uR_1v \Rightarrow \; uR_2v.$$

This shall be denoted with $R_1 \hookrightarrow R_2$.

Remark 1 Suppose that $R_1 \hookrightarrow R_2$. If R_1 is total, then R_2 is also total.

Definition 5 (quasiorder) *Let $\mathfrak{X}$ be a nonempty set. A binary relation $\precsim$ is a quasiorder on $\mathfrak{X}$ if it satisfies the following properties;*

1. *(reflexive law)* $\forall u \in \mathfrak{X} \quad u \precsim u$
2. *(transitive law)* $\forall u, v, w \in \mathfrak{X}$

$$\text{if } u \precsim v \text{ and } v \precsim w, \text{ then } u \precsim w.$$

Example 1 Every equivalent relation is a quasiorder.

Definition 6 *For a quasiorder $\precsim$, the equivalence $\sim$ defined by*

$$u \sim v \overset{\text{def}}{\Leftrightarrow} \quad u \precsim v \text{ and } v \precsim u$$

shall be called the equivalence associated with $\precsim$. $u \prec v$ denotes that $u \precsim v$ and $u \nsim v$.

Example 2 Let S be a nonempty set and let $\mathfrak{F}(S)$ be a set of functions on S. For $x \in S$ and $u,\, v \in \mathfrak{F}(S)$, let $u \precsim_x v$ denote the fulfillment of inequality $u(x) \leq v(x)$. Then $\precsim_x$ is a quasiorder on $\mathfrak{F}(S)$.

The following example is the one that appears in the definition of the viscosity solution.

Example 3 For $x \in \mathbb{R}^n$, let $\mathfrak{C}_x^r$ be the stalk of C^r functions at x. If $u \precsim v$ means that $v - u$ has a local maximum at x, then $\precsim$ is a quasiorder on $\mathfrak{C}_x^r$.

Example 4 Let $\mathfrak{C}_x^r$ be the same as in the example 3. Suppose that $u \precsim_x v$ means the following;

$$\exists f(\cdot) = o\left(|\cdot - x|^2\right) \quad \text{such that} \quad v - u - f \text{ has a local maximum at } x.$$

Therewith, $\precsim_x$ becomes a quasiorder on $\mathfrak{C}_x^r$.

Remark 2 The quasiorder in the example 3 includes the one in the example 4.

The following is related to the analysis on the geometry such as [3] and [4].

Example 5 Let X be a topological space and let $\mathcal{N}(x)$ be the neighborhood system of $x \in X$. Suppose that

$$\mathfrak{S}_x = \{S \subset X \; ; \; x \in \partial S\}\,.$$

Then, a binary relation $\precsim_x$ on $\mathfrak{S}_x$ defined as

$$S \precsim_x S' \overset{\text{def}}{\Leftrightarrow} \exists N \in \mathcal{N}(x) \text{ s.t. } S \cap N \subseteq S' \cap N \qquad \text{for } S, S' \in \mathfrak{S}_x$$

is a quasiorder.

Example 6 Let $\mathfrak{C}_x^r$ be the same as in the example 3 and $\precsim_x$ be a quasiorder on $\mathfrak{C}_x^r$. If $(x, u) \precsim (y, v)$ means that

$$\exists\, \sigma \in C^r(\mathbb{R}^n; \mathbb{R}^n) \quad \text{such that} \quad x = \sigma(y) \quad \text{and} \quad u \circ \sigma \precsim_y v$$

then $\precsim$ is a quasiorder on $\mathfrak{Y} = \bigcup_x \{x\} \times \mathfrak{C}_x^r$.

Proposition 1 *For any family of quasiorders $\mathcal{F}$, there is a unique quasiorder $\precsim_{\mathcal{F}}$ which satisfies the following;*

1. $\forall \precsim \in \mathcal{F} \quad \precsim_{\mathcal{F}} \hookrightarrow \precsim$
2. *If a binary relation R is stronger than any $\precsim \in \mathcal{F}$, then $R \hookrightarrow \precsim_{\mathcal{F}}$.*

Example 7 Let $\precsim$ be a quasiorder and let $u \precsim' v$ mean $v \precsim u$. Then, the equivalent $\sim$ associated with $\precsim$ satisfies

$$\sim = \bigcap \{\precsim, \precsim'\}\,.$$

Proof of Proposition 1
If $\mathcal{F}$ is a family of quasiorders, $\bigcap \mathcal{F}$ is a quasiorder. This is the one that we want.□

Proposition 2 *Let $\mathcal{F}$ be a non-empty family of binary relations. Suppose that for any R_1, $R_2 \in \mathcal{F}$ there is a quasiorder $\precsim \in \mathcal{F}$ which satisfies*

$$R_1 \hookrightarrow \precsim \qquad \textit{and} \qquad R_2 \hookrightarrow \precsim .$$

Then, there is a unique quasiorder $\precsim_{\mathcal{F}}$ which satisfies the following;

1. *$\forall R \in \mathcal{F} \quad R \hookrightarrow \precsim_{\mathcal{F}}$*
2. *If a binary relation R is weaker than any $\precsim \in \mathcal{F}$, then $\precsim_{\mathcal{F}} \hookrightarrow R$.*

Example 8 If $\mathcal{F} = \{\precsim_n\}_{n \in \mathbb{N}}$ and

$$\forall n \in \mathbb{N} \quad \precsim_n \hookrightarrow \precsim_{n+1},$$

then, $\mathcal{F}$ satisfies the assumption in the above proposition.

Proof of Proposition 2
Under the assumption of proposition 2, $\bigcup \mathcal{F}$ is a quasiorder. This is the one that we want. □

3. ORDER-RESERVING FUNCTION AND INEQUALITY IN THE VISCOSITY SENSE

In this section we formulate the inequality in the viscosity sense.

Definition 7 *Let $\precsim$ be a quasiorder on $\mathfrak{X}$. $F : \mathfrak{X} \to \mathbb{R}$ reserves the order with respect to $\precsim$ if and only if it satisfies that*

$$\forall u, v \in \mathfrak{X} \quad u \precsim v \Rightarrow F(u) \leq F(v).$$

The following define the inequality in the viscosity sense. LE and GE roughly mean $\leq$ and $\geq$ respectively.

Definition 8 *Let $\precsim$ be a quasiorder on $\mathfrak{X}$ and let $\mathfrak{A}$ be a subset of $\mathfrak{X}$. Suppose that $F : \mathfrak{A} \to \mathbb{R}$ reserves the order with respect to $\precsim$.*

1. *For $u \in \mathfrak{X}$ and $r \in \mathbb{R}$, the statement*

$$[F; u] \text{ LE } r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

means that

$$\forall \varphi \in \mathfrak{A} \quad \varphi \precsim u \Rightarrow F(\varphi) \leq r.$$

2. *For $u \in \mathfrak{X}$ and $r \in \mathbb{R}$, the statement*

$$[F; u] \ \text{GE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

means that

$$\forall \varphi \in \mathfrak{A} \quad u \precsim \varphi \Rightarrow F(\varphi) \geq r.$$

3. *The statement*

$$[F; u] \ \text{E}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

means that both

$$[F; u] \ \text{LE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

and

$$[F; u] \ \text{GE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

are true.

Proposition 3 *Suppose that $s \leq r$.*

1. *If*

$$[F; u] \ \text{LE}\ s \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is true, then

$$[F; u] \ \text{LE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is also true.

2. *If*

$$[F; u] \ \text{GE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is true, then

$$[F; u] \ \text{GE}\ s \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is also true.

Proposition 4 *Suppose that $u \precsim v$.*

1. *If*

$$[F; v] \ \text{LE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is true, then

$$[F; u] \ \text{LE}\ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is also true.

2. *If*

$$[F;u]\ \mathrm{GE}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})$$

is true, then

$$[F;v]\ \mathrm{GE}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})$$

is also true.

Lemma 5 *For every $r \in \mathbb{R}$ and $u \in \mathfrak{X}$ the following two properties are equivalent.*

1.

$$[F;u]\ \mathrm{E}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})\,.$$

2.

$$\sup\{\,F(\varphi)\,;\,\varphi \precsim u,\ \varphi\in\mathfrak{A}\,\} \le r \le \inf\{\,F(\varphi)\,;\,u\precsim\varphi,\ \varphi\in\mathfrak{A}\,\}\,.$$

Remark 3 By lemma 5 we notice the relationship between classical solutions and weak ones. Namely, suppose that $F:\mathfrak{X}\to\mathbb{R}$ reserves the order with respect to $\precsim$ and let $\mathfrak{A}$ be a subset of $\mathfrak{X}$, then the following properties hold.

1. For any $u \in \mathfrak{X}$

$$[F|_{\mathfrak{A}};u]\ \mathrm{E}\ F(u) \qquad \text{with respect to } (\precsim,\mathfrak{A})\,.$$

2. If $r \in \mathbb{R}$ satisfies that

$$[F|_{\mathfrak{A}};u]\ \mathrm{E}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})$$

and if there is $\varphi\in\mathfrak{A}$ which satisfies $\varphi\sim u$, then $r = F(u)$

The first assertion refers that the solution in classical sense is a solution in weak sense. The second assertion means that the value of F in weak sense coincides with the one in classical sense where F is defined.

Lemma 6 *Let $\mathfrak{A}$, $\mathfrak{X}$ be the same as the above and F be an order-reserving function on $\mathfrak{A}$. Then, for $u\in\mathfrak{X}$ and $r\in\mathbb{R}$, either*

$$[F;u]\ \mathrm{LE}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})$$

or

$$[F;u]\ \mathrm{GE}\ r \qquad \text{with respect to } (\precsim,\mathfrak{A})$$

must hold.

Definition 9 *1. For $u \in \mathfrak{X}$,*

$$[F;u] \ \text{E} \ -\infty \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

means that

$$[F;u] \ \text{LE} \ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

holds for all $r \in \mathbb{R}$ and that there is no $s \in \mathbb{R}$ which satisfies

$$[F;u] \ \text{GE} \ s \qquad \text{with respect to } (\precsim, \mathfrak{A}) \, .$$

2. For $u \in \mathfrak{X}$,

$$[F;u] \ \text{E} \ +\infty \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

means that

$$[F;u] \ \text{GE} \ r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

holds for all $r \in R$ and that there is no $s \in \mathbb{R}$ which satisfies

$$[F;u] \ \text{LE} \ s \qquad \text{with respect to } (\precsim, \mathfrak{A}) \, .$$

Theorem 1 *For any $u \in \mathfrak{X}$, one of the following holds;*

1.

$$\exists r \in \mathbb{R} \quad \textit{such that} \quad [F;u] \ \text{E} \ r \quad \text{with respect to } (\precsim, \mathfrak{A}) \, .$$

2.

$$[F;u] \ \text{E} \ +\infty \qquad \text{with respect to } (\precsim, \mathfrak{A}) \, .$$

3.

$$[F;u] \ \text{E} \ -\infty \qquad \text{with respect to } (\precsim, \mathfrak{A}) \, .$$

Theorem 2 *Let $F : \mathfrak{A} \to \mathbb{R}$ be a function reserving the order with respect to $\precsim$ and let uR_Fv mean that*

$$\forall \varphi \in \mathfrak{A} \qquad \varphi \precsim u \ \Rightarrow \quad [F;v] \ \text{GE} \ F(\varphi) \ \text{ with respect to } (\precsim, \mathfrak{A}) \, .$$

Then, for the binary relation R_F, the following properties hold.

1. $\precsim \hookrightarrow R_F$.
2. $\forall u, v \in \mathfrak{X} \quad uR_Fv$ *or* vR_Fu.
3. $\forall u \in \mathfrak{X} \quad uR_Fu$.
4. $\forall u, \, w \in \mathfrak{X} \; \forall v \in \mathfrak{A} \quad$ *If* uR_Fv *and* vR_Fw, *then* uR_Fw.

5. *Suppose that*

$$\forall u, v \in \mathfrak{A} \quad u \prec v \Rightarrow F(u) < F(v).$$

Then,

$$\forall u, v \in \mathfrak{A} \quad u \prec v \Rightarrow v R_F^{\mathbf{c}} u.$$

4. WEAKENING THE NOTION OF SOLUTIONS

Definition 10 *Let* $(\precsim, \mathfrak{A})$ *and* $(\precsim', \mathfrak{B})$ *be pairs composed by a quasiorder and a subset of* $\mathfrak{X}$. *Then,* $(\precsim', \mathfrak{B}) \leq (\precsim, \mathfrak{A})$ *denotes that* $\precsim' \hookrightarrow \precsim$ *and* $\mathfrak{B} \subset \mathfrak{A}$.

If $F : \mathfrak{A} \to R$ reserves the order with respect to $\precsim$, it also reserves the order with respect to any stronger quasiorder than $\precsim$. Concerning this property the following holds. This asserts that the solution constructed by $(\precsim', \mathfrak{B})$ is weaker than the one constructed by $(\precsim, \mathfrak{A})$ when $(\precsim', \mathfrak{B}) \leq (\precsim, \mathfrak{A})$.

Proposition 7 *Suppose that* $(\precsim', \mathfrak{B}) \leq (\precsim, \mathfrak{A})$

1. *If*

$$[F; u] \text{ LE } r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is true, then

$$[F|_{\mathfrak{B}}; u] \text{ LE } r \qquad \text{with respect to } (\precsim', \mathfrak{B})$$

is also true.

2. *If*

$$[F; u] \text{ GE } r \qquad \text{with respect to } (\precsim, \mathfrak{A})$$

is true, then

$$[F|_{\mathfrak{B}}; u] \text{ GE } r \qquad \text{with respect to } (\precsim', \mathfrak{B})$$

is also true.

Theorem 3 *Let* $\mathcal{F} = \{(\precsim_\lambda, \mathfrak{A}_\lambda)\}_\lambda$ *be a family of pairs composed by a quasiorder and a subset of* X. *Suppose that for* $(\precsim_1, \mathfrak{A}_1), (\precsim_2, \mathfrak{A}_2) \in \mathcal{F}$ *there is* $(\precsim, \mathfrak{A}) \in \mathcal{F}$ *which satisfies*

$$(\precsim_1, \mathfrak{A}_1) \leq (\precsim, \mathfrak{A}) \quad \textit{and} \quad (\precsim_2, \mathfrak{A}_2) \leq (\precsim, \mathfrak{A}).$$

Let $F : \mathfrak{A}_\lambda \to \mathbb{R}$ *be an order-reserve function with respect to* $\precsim_\lambda$ *and suppose that*

$$F_\lambda|_{\mathfrak{A}_\lambda \cap \mathfrak{A}_\mu} = F_\mu|_{\mathfrak{A}_\lambda \cap \mathfrak{A}_\mu}.$$

Then, there is an order-reserving function $G : \bigcup_\lambda \mathfrak{A}_\lambda \to \mathbb{R}$ *with respect to* $\bigcup \{\precsim_\lambda\}_\lambda$ *which satisfies* $G|_{\mathfrak{A}_\lambda} = F_\lambda$. *Moreover, for* $r \in \mathbb{R}$ *and* $u \in \mathfrak{X}$ *the following two are equivalent.*

1. *For any* λ
$$[F_\lambda; u] \ \text{LE}\ r \qquad \text{with respect to}\ (\precsim_\lambda, \mathfrak{A}_\lambda)\,.$$

2.
$$[G; u] \ \text{LE}\ r \qquad \text{with respect to}\ \left(\bigcup \{\precsim_\lambda\}_\lambda, \bigcup_\lambda \mathfrak{A}_\lambda\right).$$

This property also holds if LE *is replaced by* GE.

5. DEGENERATE ELLIPTIC TYPE EQUATION

To conclude our paper we shall review facts on elliptic PDEs with quasiorder.

Definition 11 *Let* $\mathfrak{C}^0_x$, $\mathfrak{C}^1_x$ *defined as in example 3. Let* $u \precsim^1_x v$ *mean* $u(x) \le v(x)$ *and let* $\precsim^2_x$ *be a quasiorder as example 4 and suppose*

$$\precsim^{1,2}_x = \bigcap \left\{\precsim^1_x, \precsim^2_x\right\}.$$

For (x, u), $(y, v) \in \mathfrak{C}^0 = \bigcup_x \{x\} \times \mathfrak{C}^0_x$, *suppose that* $(x, u) \precsim^i (y, v)$ *means that there is a parallel transformation* σ *such that*

$$x = \sigma(y), \quad u \circ \sigma \precsim^i_y v.$$

Then, an order-reserving function $F : \mathfrak{C}^2 = \bigcup_x \{x\} \times \mathfrak{C}^2_x \to \mathbb{R}$ *with respect to* $\precsim^{1,2}$ *is called simple and elliptic.*

Lemma 8 *Let* F *be a simple and elliptic function. Suppose that* $u(x)$, $Du(x)$, $D^2u(x)$ *equal* $v(y)$, $Dv(y), D^2v(y)$ *respectively. Then,* $F(x, u) = F(y, v)$.

Theorem 4 *Let* $\mathbb{S}^n$ *be the set of symmetric matrices of degree* n. *For* $X, Y \in \mathbb{S}^n$, *let* $X \le Y$ *mean that*

$$\forall \xi \in \mathbb{R}^n \qquad \langle X\xi, \xi\rangle \le \langle Y\xi, \xi\rangle\,.$$

Suppose that $\pi : \mathfrak{C}^2 \to \mathfrak{J} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ *is*

$$\pi(x, u) = \Big(u(x), Du(x), D^2u(x)\Big)\,.$$

Then, for $F : \mathfrak{J} \to \mathbb{R}$, *the following two conditions are equivalent;*

1. *For* $p \le q$ *and* $X \le Y$, $F(p, \xi, X) \ge F(q, \xi, Y)$.

2. $F \circ \pi$ *is simple and elliptic.*

In §2 [5] discusses the relationship between two conditions in theorem 4. If F satisfies the first condition of theorem 4 one is able to define the solution of the elliptic PDE

$$F(u(x), Du(x), D^2u(x)) = 0 \ , \ x \in \Omega$$

without the differentiability of unknown functions u. Namely, the following statement holds:

Theorem 5 *Suppose that $F : \mathfrak{J} \to \mathbb{R}$ satisfies the condition 1 in theorem 4. Then, For $u \in C^0(\mathbb{R}^n)$, the following properties are equivalent.*

1. *u is a viscosity solution of $F(u(x), Du(x), D^2u(x)) = 0$.*

2. *For every $x \in \mathbb{R}^n$, the germ of u at x, which is denoted by u_x, is satisfies*

$$[F \circ \pi; (x, u_x)] \ \mathrm{E} \ 0 \qquad \text{with respect to } \left(\precsim^{1,2}, \mathfrak{C}^2\right).$$

Here π is the same as in theorem 4.

We should remark that the second formula in theorem 5 consists not of F but of $F \circ \pi$. This is because the viscosity solution is only based on the quasiorder structure of $\mathfrak{C}^0$ not based on derivatives of unknown functions. This is the conclusion of this paper.

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