On analyticity rate estimates of the solutions to the Navier –Stokes equations in Bessel–potential spaces

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On analyticity rate estimates of the solutions to the Navier–Stokes equations in Bessel–potential spaces

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Abstract

The locally–in–time solutions to the Navier–Stokes equations in $\dot{H}^{\frac{n}{2}-1}(\mathbf{R}^n)$ are regular for $t > 0$. The spatial analyticity is established by deriving rate estimates of higher order derivatives of the solutions. The solutions or the initial velocities need not be small. The estimates also provide the decay estimate on derivatives for large time. Although the basic strategy to derive estimates is similar to our previous results with Y. Giga for L^n space, we are forced to apply several tools from harmonic analysis since our space $\dot{H}^{\frac{n}{2}-1}$ is more complicated.

AMS Subject Classification. 35Q30, 35Q58, 76D03

Key words. Navier–Stokes equations, spatial analyticity, regularizing–decay rate, Gronwall inequality.

1. Introduction.

The initial value problem of Navier–Stokes equations in $\mathbf{R}^n$ $(n \geq 2)$ is written as

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } \mathbf{R}^n \times (0, T), \\ u|_{t=0} = u_0 \quad (\text{with } \nabla \cdot u_0 = 0) & \text{in } \mathbf{R}^n. \end{cases}$$

Here, $u = u(x,t) = (u^1(x,t), \dots, u^n(x,t))$ and $p = p(x,t)$ represent the unknown velocity and pressure, respectively, at $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $t > 0$; u_0 is a given initial velocity and f is a given vector–valued function which represents the external force. We have used a standard notation about derivatives; $u_t = \partial u / \partial t$, $\partial_i = \partial / \partial x_i$, $(u, \nabla) = \sum_{i=1}^n u^i \partial_i$, $\Delta = \sum_{i=1}^n \partial_i^2$, $\nabla \cdot u = \sum_{i=1}^n \partial_i u^i$ and $\nabla p = (\partial_1 p, \dots, \partial_n p)$. Hereafter, we use the standard notation of $\partial_x^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, and $|\beta| = \sum_{i=1}^n \beta_i$ for multi–index $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}_0^n$, where $\mathbf{N}_0$ is the set of all non–negative integers.

Before stating our results, we introduce the some function spaces. Let $n \geq 1$, and let $L^p = L^p(\mathbf{R}^n)$ be the usual Lebesgue space for $p \in [1, \infty]$ with norm $\| \cdot \|_p$. We sometimes suppress the domain dependence of function spaces, and do not distinguish spaces of vector–valued from scalar. For $s \in \mathbf{R}$ and $p \in [1,\infty]$, the (homogeneous) Bessel–potential space $\dot{H}^s_p$ is defined as the space $(-\Delta)^{-s/2}L^p$, equipped with the norm $\|f\|_{\dot{H}^s_p} = \|(-\Delta)^{s/2}f\|_p$. Here, $(-\Delta)^{s/2} = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is its inverse. Similarly, we define its inhomogeneous version H^s_p as $(I-\Delta)^{-s/2}L^p$. We use a standard notation $\dot{H}^s = \dot{H}^s_2$ for the sake of simplicity. The properties of these function spaces (for example, they are Banach spaces) are found in [1], [30] and [21].

In two–dimensional problem J. Leray [22] obtained the globally–in–time smooth solution to (NS), when the initial data belongs to $L^2(\mathbf{R}^2)$. In general dimensions, the locally–in–time solvability with the initial data in $H^{n/2-1}(\mathbf{R}^n)$ is already obtained by H. Fujita and T. Kato [17] and [6]. In fact, although they proved that in $H^{1/2}(\Omega)$ for $\Omega \subset \mathbf{R}^3$ bounded domain, their results can be extended in the whole space $\mathbf{R}^n$ for general dimension $n \geq 3$. Its homogeneous version is also established by [18] and [3].

Our goal is to derive several regularizing rate estimates for the higher order derivatives of the solution u of (NS), when the initial velocity belongs to $\dot{H}^{n/2-1}(\mathbf{R}^n)$ for $n \geq 2$. In this paper we rather discuss the solution of the integral equation:

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbf{P}(u\otimes u)(s)ds + \int_0^t e^{(t-s)\Delta}\mathbf{P}f(s)ds,$$

which is formally equivalent to (NS). Here, $e^{t\Delta} = G_t *$ denotes a solution–operator of the heat equation, where $G_t(x) = (4\pi t)^{-\frac{n}{2}}\exp(-\frac{|x|^2}{4t})$ is a Gauss kernel, and $*$ means the convolution with respect to x; $\mathbf{P}$ denotes the orthogonal projection onto solenoidal subspace of L^2, and it is written as an $n\times n$ matrix operator $\mathbf{P} = (\delta_{ij} + R_iR_j)_{1\leq i,j\leq n}$, where δ_{ij} denotes the Kronecker's delta, and R_i is the Riesz transform formally defined by $R_i = \partial_i(-\Delta)^{-1/2}$; $u\otimes u$ is a tensor, whose ij–component is u^iu^j. In the formal level once one finds the solution of u of (INT), (u,p) with $p = \sum_{i,j=1}^n R_iR_ju^iu^j$ solves (NS). A solution of (INT) is often said to be *mild solution*, we use this terminology.

We are now in position to state our main result.

Theorem 1.1. *Assume that $n \geq 2$. Assume that f ($\in L^r$ for some $1 < r < \infty$) is conservative, i.e., $\mathbf{P}f = 0$. Assume that $u_0 \in \dot{H}^{n/2-1}$ satisfying $\nabla\cdot u_0 = 0$. Let u be a mild solution on $[0,T]$ such that*

$$u \in C([0,T);\dot{H}^{n/2-1}) \cap C((0,T);\dot{H}^{n/2-1}_p)$$

for some $T > 0$ and some $p \in (2,\infty]$. Let M_1 and M_2 be positive constants satisfying

$$\text{(1.1)} \qquad M_1 \geq \sup_{0\leq t<T}\|u(t)\|_{\dot{H}^{n/2-1}},$$

(1.2)
$$M_2 \geq \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|u(t)\|_{\dot{H}_p^{n/2-1}}.$$

Then there exist constants K_1 and K_2 depending only on n, p, M_1 and M_2 such that

(1.3)
$$\|u(t)\|_{\dot{H}_q^{n/2-1+\alpha}} \leq K_1 (K_2 \tilde{\alpha})^{\tilde{\alpha}} t^{-\tilde{\alpha}/2}$$

for all $q \in [2, \infty]$, $t \in (0, T)$ and $\alpha > 0$, where we denote by $\tilde{\alpha} = \alpha + n(\frac{1}{2} - \frac{1}{q})$.

Remark 1.1. (i) The constants M_1 and M_2 are finite as far as the solution exists. If the case $T = \infty$, then the estimates (1.3) yield the decay rates of the solution including its higher order derivatives as $t \to \infty$ since K_1 and K_2 do not depend on T explicitly, once we have bounds for M_1 and M_2 independent of T. This situation actually occurs if the initial data were enough small; see e.g. [16]. Hence, we are able to obtain (1.3) globally–in–time in this situation.

(ii) The estimates (1.3) also imply that the solution u is *uniformly analytic* in space variables independently of the location of x, i.e., we obtain its analyticity rates, where the analyticity rate means the size of the radius of convergence of the Taylor's expansion. We also note that even if f is not conservative, but f satisfies

$$\|\mathbf{P} f(t)\|_{\dot{H}^{n/2-1+\alpha}} \leq M_3 \alpha^\alpha t^{-\alpha/2}$$

for all $t \in (0, T)$ and $\alpha > 0$, then we still get the estimates (1.3). Of course, the constants K_1 and K_2 in (1.3) are depending also on M_3. Such a condition implies that f is uniformly analytic in x, and satisfies suitable (decay) rates in t.

We shall refer to several known results. In 1960's H. Fujita and T. Kato [17] and [6] introduced the notion of mild solution, and obtained the smooth solution to (NS) with regular initial velocity. In 80's T. Kato [16] (in the whole space) or Y. Giga and T. Miyakawa [11] (in a bounded domain) successfully eliminated the regularity condition, that is, they obtained the smooth solution to (NS) with $u_0 \in L^n$. T. Kato and G. Ponce [18] also constructed the mild solution in $\dot{H}_p^s(\mathbf{R}^n)$ with $s = n/p - 1$ for $1 < p < \infty$. By the way, T. Kato [16] showed that the locally–in–time solution can be extended globally, if the L^n–norm of the initial velocity is sufficiently small. M. Cannone [3] improved these results, that is, they constructed the mild solution in $\dot{B}^0_{n,\infty}(\mathbf{R}^n)$ (which is the Besov space; see e.g. [30]), and proved that the locally–in–time solution can be extended globally, if $\dot{B}^0_{n,\infty}$–norm of the initial velocity is enough small; see also [4] and [27]. For other related articles the reader is referred to [21].

We mention several articles related Theorem 1.1 and Remark 1.1-(ii). There are many researchers who study the regularizing or decay rate estimates of the solution to (NS), see [28, 29, 31, 32, 33, 14]. The analyticity is also studied to (NS) in several domain, see [24, 25, 15, 19, 8, 5, 13, 26]. For the details of those articles the reader is referred to [12] and papers cited there. However, it seems to be that Theorem 1.1 is the first result to obtain the regularizing, decay and analyticity rates simultaneously in $\dot{H}$ spaces. Furthermore, in Remark 1.1-(ii) we give a sufficiently condition for the external force f, so that the solution to (NS) in the whole space is uniformly analytic in x.

We now briefly explain the idea of the proof of Theorem 1.1. We first show (1.3) for $q = 2$, and then prove that for general q. We use the representation formula (INT), and take $\dot{H}^{n/2-1}$–norm of $(-\Delta)^{1/2}\partial_x^\beta u(t)$ for multi–index β. We divide the integral $\int_0^t$ into two parts as $\int_0^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t$ for $\varepsilon \in (0,1)$. Let $\delta \in (1/2, 1]$, we use an induction with respect to $|\beta|$ to get the following equivalent inequalities as (1.3):

$$\|(-\Delta)^{1/2}\partial_x^\beta u(t)\|_{\dot{H}^{n/2-1}} \le K_1'(K_2'|\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|+1}{2}}. \tag{1.4}$$

for all $t \in (0,T)$ and all $\beta \in \mathbf{N}_0^n$, where K_1' and K_2' are independent of β and t.

The estimates of linear term or $\int_0^{t(1-\varepsilon)}$ are easy, because those integrands are integrable by shifting the differential to Gauss kernel, since $e^{t\Delta}$ is a convolution type operator. In the term of $\int_{t(1-\varepsilon)}^t$ we use the assumption of induction and the Hölder type inequality in Bessel–potential spaces (Lemma 2.3). Finally, we apply the Gronwall type inequality (Lemma 2.6) and take $\varepsilon \sim 1/|\beta|$ to get (1.4) with suitable constants K_1' and K_2'. For general $q \in (2, \infty]$ we can appeal to the interpolation inequality (Lemma 2.4) to get easily, once we have the estimates (1.3) for all $\alpha > 0$ and $q = 2$. Although basic strategy is similar to that of [12], we here need to use several estimates for nonlinear terms as well as interpolation inequalities involving $\dot{H}$ spaces. This is a new feature of this paper.

With Y. Giga the author [12] obtained the (L^n version) estimates similar as (1.3) we must use the Gronwall type inequality (Lemma 2.6) three times, however, in this paper we only use once. Thus the present proof is shorter than that of [12]; however, we use several tools in harmonic analysis.

This paper is organized as follows. In Section 2 we shall introduce the some lemmas. In Section 3 we shall give two propositions for the proof of Theorem 1.1. In Proposition 3.1 we obtain the estimates (1.3) with assumptions of Theorem 1.1, in addition, $u \in C((0,T); \dot{H}_q^{n/2-1+\alpha})$ for all $\alpha > 0$ and $q \in [2, \infty]$. In Proposition 3.2 we shall show that the locally–in–time mild solution u still has this additional property.

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Before closing this Introduction, we would refer to the notations of constants in this paper. Throughout this paper we denote positive constants by C the value of which may differ from one occasion to another. The variables of constant $C(\cdot,\dots,\cdot)$ indicate the dependence of the parameters.

2. Preliminary.

In this section we prepare some lemmas to give the proof of Theorem 1.1.

We first recall the $L^p - L^q$ estimate for derivatives of the heat semigroup.

Lemma 2.1. *Assume that $n \geq 1$ and $1 \leq p \leq q \leq \infty$. Then*

$$\|(-\Delta)^{\alpha/2} e^{t\Delta} f\|_q \leq C\alpha^{\alpha/2} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}} \|f\|_p$$

for all $\alpha > 0$, $t > 0$ and $f \in L^p(\mathbf{R}^n)$. The constant $C = C(n)$ depends only on n.

The proof is standard, we only need to estimate L^r–norm of $(-\Delta)^{\alpha/2} G_t$ with for some r. It has already been shown in [12, Lemma 2.1], so we skip it. We also note that $\|\nabla f\|_2 \sim \|(-\Delta)^{1/2} f\|_2$ for all $f \in \dot{H}^1(\mathbf{R}^n)$; see e.g. [30]. Here, the meaning of $\sim$ is for equivalent norms in both hands.

There are various kinds of estimates for quadratic terms. A typical well–known inequality in the homogeneous Sobolev spaces $\dot{H}^s$ is:

Lemma 2.2 (F. Planchon [27]). *There is a positive constant $C = C(n)$ such that*

$$\|(-\Delta)^{s-n/4}(fg)\|_2 \leq C\|(-\Delta)^{s/2} f\|_2 \|(-\Delta)^{s/2} g\|_2$$

for $s < n/2$ and $f, g \in \dot{H}^s$.

To get above lemma we should use the equivalent $\dot{H}^s = \dot{B}^s_{2,2}$, that is, $f \in \dot{H}^s$ if and only if $\|\phi_j * f\|_2 \leq 2^{-js}\varepsilon_j \|f\|_{\dot{H}^s}$ for all j, where $\sum \varepsilon_j^2 \leq 1$; furthermore, we use the Bony's paraproduct decomposition (see [2]).

In order to obtain our theorem we must establish the estimates for bilinear terms.

Unfortunately, it seems to be difficult to get suitable estimates if we only use Lemma 2.2. We thus prepare another Hölder type inequality in Bessel–potential spaces.

Lemma 2.3 (H. Kozono–Y. Shimada [20]). *Let $\alpha, \beta > 0$, $1 < p < \infty$, and $1 \le r, s \le \infty$ satisfying $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Then there exists a positive constant $C = C(\alpha, \beta, p, r, s)$ such that*

$$\|fg\|_{\dot{H}^{\alpha}_{p}} \le C(\|f\|_{\dot{H}^{-\beta}_{r}}\|g\|_{\dot{H}^{\alpha+\beta}_{s}} + \|f\|_{\dot{H}^{\alpha+\beta}_{s}}\|g\|_{\dot{H}^{-\beta}_{r}})$$

for all $f, g \in \dot{H}^{-\beta}_{r} \cap \dot{H}^{\alpha+\beta}_{s}$.

In fact, in [20] they gave the proof of above inequality in the homogeneous Triebel–Lizorkin spaces (see e.g. [30]). Although the present version has a slight loss in sum–exponent (q of $\dot{F}^{s}_{p,q}$), this suffices to apply to our problem. In fact, they proved it of the case $r = \infty, s = p$ only, but we can modify its present version easily.

In our results (Theorem 1.1), the case $q = \infty$ is allowed. Unfortunately, there are not any useful embedding to $\dot{H}^{s}_{\infty}$ for keeping the scalings. To overcome this difficulty, we recall the following interpolation inequality.

Lemma 2.4 (S. Machihara–T. Ozawa [23]). *Let $\lambda, \mu \in \mathbf{R}$, $1 \le p, q \le r \le \infty$, $0 < \theta < 1$ satisfying $\lambda > \frac{n}{p} - \frac{n}{r}$, $\mu < \frac{n}{q} - \frac{n}{r}$, $\theta(\lambda - \frac{n}{p} + \frac{n}{r}) + (1-\theta)(\mu - \frac{n}{q} + \frac{n}{r}) = 0$. Then there exists a positive constant $C = C(\lambda, \mu, \theta, p, q, r)$ such that*

$$\|f\|_{r} \le C\|f\|^{\theta}_{\dot{H}^{\lambda}_{p}}\|f\|^{1-\theta}_{\dot{H}^{\mu}_{q}}$$

for all $f \in \dot{H}^{\lambda}_{p} \cap \dot{H}^{\mu}_{q}$.

In fact, they prove above inequality by using standard technique from the Littlewood–Paley theory. Similarly to Lemma 2.3, although the present version has a slight loss in sum–exponent, this suffices to apply to our problem.

We shall recall the estimate for multiplication of multi–sequences, which has been proved by C. Kahane [15, Lemma 2.1].

Lemma 2.5 (C. Kahane [15]). *Let $\delta > \frac{1}{2}$. Then there exists a positive constant Λ depending only on δ such that*

$$\sum_{\gamma \le \beta} \binom{\beta}{\gamma} |\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta} \le \Lambda|\beta|^{|\beta|-\delta} \qquad \text{for all } \beta \in \mathbf{N}^{n}_{0}.$$

Here, for multi–indices β and γ, $\gamma \le \beta$ means $\gamma_i \le \beta_i$ for all i, and $\binom{\beta}{\gamma} = \prod_{i=1}^{n} \frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}$. The dependence of the δ is merely similar to $\sum_{j=1}^{\infty} j^{-\delta-1/2}$ in Λ. The proof of this lemma is based on Stirling's formula; we do not present the proof here.

We recall a variant of the Gronwall type inequality, which has been proved in [7].

Lemma 2.6. *Let ψ_0 be a measurable and locally bounded function in $(0,T)$. Let $\{\psi_j\}_{j=1}^{\infty}$ be a sequence of measurable functions in $(0,T)$. Assume that $\alpha \in \mathbf{R}$ and that μ, $\nu > 0$ satisfying $\mu + \nu = 1$. Let $b_\varepsilon > 0$ be a number depending on $\varepsilon \in (0,1)$, and assume that b_ε is nonincreasing with respect to ε. Assume that there is a positive constant σ such that*

$$0 \le \psi_0(t) \le b_\varepsilon t^{-\alpha} + \sigma \int_{t(1-\varepsilon)}^{t} (t-s)^{-\mu} s^{-\nu} \psi_0(s)ds$$

and

$$0 \le \psi_{j+1}(t) \le b_\varepsilon t^{-\alpha} + \sigma \int_{t(1-\varepsilon)}^{t} (t-s)^{-\mu} s^{-\nu} \psi_j(s)ds$$

for all $j \ge 0$, $t \in (0,T)$ and $\varepsilon \in (0,1)$. Let ε_0 be a unique positive number such that $I(2\varepsilon_0) = \min\{\frac{1}{2\sigma}, I(1)\}$ with $I(\varepsilon) = \int_{1-\varepsilon}^{1} (1-\tau)^{-\mu} \tau^{-\alpha-\nu} d\tau$. Then

$$\psi_j(t) \le 2 b_{\varepsilon_0} t^{-\alpha}$$

for all $j \ge 0$ and $t \in (0,T)$.

A full proof is given in [12], so we skip it.

3. Proof of Theorem 1.1.

In this Section we shall give the proof of Theorem 1.1. We now prepare two propositions. We first prove that under the extra assumption.

Proposition 3.1. *Assume that same hypothesis of Theorem 1.1. Assume further more that the solution u satisfies*

$$u \in C((0,T); \dot{H}_q^{n/2-1+\alpha}(\mathbf{R}^n)) \tag{3.1}$$

for all $q \in [2,\infty]$ and $\alpha > 0$. Then, there exist positive constants K_1 and K_2 (depend only on n, p, M_1 and M_2) such that the estimates (1.3) hold for all $q \in [2,\infty]$, $t \in (0,T)$ and $\alpha > 0$.

We may assume that the solution u has the regularity property (3.1) without loss of generality, because the solution u in Theorem 1.1 always satisfies (3.1) as shown in Proposition 3.2. Thus, Theorem 1.1 evidently follows from Proposition 3.1.

Proof. To prove the Proposition 3.1 we divide the proof into two steps: we first get the estimates (1.3) for $q = 2$ in Step 1, we next obtain that for general q in Step 2.

(Step 1). We shall show (1.3) in the case $q=2$. By interpolation theory we have

$$\|u(t)\|_{\dot H_q^{n/2-1}} \le M_1^\theta M_2^{1-\theta} t^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}$$

for all $q \in (2,p)$, where $\theta = 1-\frac{q-2}{p-2}$. So we may assume that $p < \frac{2n}{n-2}$ without loss of generality. Let $\delta \in (1/2,1]$. We now prove that there exist positive constants K_1' (depends only on n and M_1) and K_2' (depends only on n, p, M_1, M_2 and δ) such that

$$\|(-\Delta)^{1/2}\partial_x^\beta u(t)\|_{\dot H^{n/2-1}} \le K_1'(K_2'|\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|+1}{2}} \tag{3.2}$$

for all $\beta \in \mathbf{N}_0^n$. We will obtain (3.2) by induction with respect to $|\beta|$. Let m be a positive integer. We assume that (3.2) holds for all $\beta \in \mathbf{N}_0^n$ with $|\beta| \le m-1$. We now proceed to prove (3.2) for $|\beta| = m$.

We take $\dot H^{n/2-1}$–norm of $(-\Delta)^{1/2}\partial_x^\beta u(t)$ by using the representation formula of (INT). We now divide the integral into two parts with $0<\varepsilon<1$:

$$\begin{aligned}\|(-\Delta)^{\frac12}\partial_x^\beta u(t)\|_{\dot H^{\frac n2-1}} &\le \|(-\Delta)^{\frac12}\partial_x^\beta e^{t\Delta}u_0\|_{\dot H^{\frac n2-1}} \\ &\quad + \left(\int_0^{t(1-\varepsilon)} + \int_{t(1-\varepsilon)}^t\right) \|(-\Delta)^{\frac12}\partial_x^\beta e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)(s)\|_{\dot H^{\frac n2-1}}ds \\ &\equiv A_1+A_2+A_3.\end{aligned}$$

We shall estimate each term.

We first estimate A_1. By Lemma 2.1 we have

$$A_1 \le \|\!|(-\Delta)^{1/2}e^{\frac t2\Delta}\|\!|\ \|\!|\partial_x^\beta e^{\frac t2\Delta}\|\!|\ \|u_0\|_{\dot H^{n/2-1}} \le C_1(2|\beta|)^{|\beta|/2}t^{-\frac{|\beta|+1}{2}}$$

with some C_1 which depends only on n and M_1. Here, $\|\!|\cdot\|\!|$ stands for an operator norm in L^2. We next estimate A_2. Note that $\|f\|_{\dot H^s} = \|(-\Delta)^{s/2}f\|_2$ for $s\in\mathbf{R}$. By Lemma 2.1 and Lemma 2.2, and boundedness of $\mathbf{P}$ in L^q for $q\in(1,\infty)$ we observe

$$\begin{aligned}A_2 &\le \int_0^{t(1-\varepsilon)} \|\!|\partial_x^\beta e^{\frac{t-s}{2}\Delta}\|\!|\ \|\!|\mathbf{P}\|\!|\ \|\!|(-\Delta)e^{\frac{t-s}{2}\Delta}\nabla\|\!|\ \|(-\Delta)^{\frac n2-1+\frac n4}(u\otimes u)(s)\|_2ds. \\ &\le C|\beta|^{|\beta|/2}\int_0^{t(1-\varepsilon)}\left(\frac{t-s}{2}\right)^{-\frac{|\beta|}{2}-\frac32}\|(-\Delta)^{\frac n4-\frac12}u(s)\|_2^2ds \\ &\le CM_1^2(2|\beta|)^{|\beta|/2}\int_0^{t(1-\varepsilon)}(t-s)^{-\frac{|\beta|}{2}-\frac32}ds \\ &\le C_2(2|\beta|)^{|\beta|/2}\varepsilon^{-\frac{|\beta|}{2}-\frac32}t^{-\frac{|\beta|+1}{2}}\end{aligned}$$

with some $C_2 = C_2(n,M_1)$. Similarly, A_3 is estimated as

$$A_3 \le \int_{t(1-\varepsilon)}^t \|\!|\mathbf{P}\|\!|\ \|\!|(-\Delta)^{\frac12-\frac n2(\frac12-\frac1p)}e^{(t-s)\Delta}\nabla\|\!|\ \|(-\Delta)^{\frac n2-\frac12-\frac{n}{2p}}\partial_x^\beta(u\otimes u)(s)\|_2ds.$$

We should note that $-\frac{1}{2}+\frac{n}{2}(\frac{1}{2}-\frac{1}{p}) \in (-\frac{1}{2},0)$ since $2<p<\frac{2n}{n-2}$, and that $\frac{n}{2}-\frac{1}{2}-\frac{n}{2p}>0$ since $n\geq 2$ and $p>2$. We calculate $\partial_x^\beta(u\otimes u)$ by Leibniz rule, by Lemma 2.1 to get

$$A_3 \leq C\int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\|(-\Delta)^{\frac{n}{2}-\frac{1}{2}-\frac{n}{2p}}(\partial_x^\gamma u\otimes\partial_x^{\beta-\gamma}u)(s)\|_2 ds.$$

We now apply Lemma 2.3 to get

$$\leq C\int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s)\|_{\dot{H}^{-\delta}_{r_1}}\|\partial_x^{\beta-\gamma}u(s)\|_{\dot{H}^{n-1-\frac{n}{p}+\delta}_{r_2}} ds$$

for some $\delta>0$ and $1<r_1,r_2\leq\infty$ satisfying $\frac{1}{2}=\frac{1}{r_1}+\frac{1}{r_2}$. Here, we take these exponents as

$$\delta=\frac{n}{3p}+\frac{1}{3}-\frac{n}{6},\quad r_1=\frac{3pn}{2n+2p-pn},\quad r_2=\frac{5}{6}-\frac{2}{3n}-\frac{2}{3p}.$$

We should note that in these settings we have $\delta>0$, $2<p<r_1<\infty$ and $2<r_2<\infty$ since $p<\frac{2n}{n-2}$. Thus, we also note the following continuous embeddings (see e.g. [30]):

$$\dot{H}^{\frac{n}{2}-1+n(\frac{1}{2}-\frac{1}{p})}\subset\dot{H}_p^{\frac{n}{2}-1}\subset\dot{H}_{r_1}^{-\delta}\quad\text{and}\quad\dot{H}^{\frac{n}{2}}\subset\dot{H}_{r_2}^{n-1-\frac{n}{p}+\delta}.$$

We now divide the sum into two parts, and apply these embeddings to get

$$\begin{aligned}A_3 &\leq C\int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s)\|_{\dot{H}^{\frac{n}{2}-1+n(\frac{1}{2}-\frac{1}{p})}}\|\partial_x^{\beta-\gamma}u(s)\|_{\dot{H}^{\frac{n}{2}}}ds\\ &\quad+C\int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\|u(s)\|_{\dot{H}_p^{\frac{n}{2}-1}}\|\partial_x^\beta u(s)\|_{\dot{H}^{\frac{n}{2}}}ds\\ &\equiv A_4+A_5.\end{aligned}$$

Here, $\gamma<\beta$ means $\gamma\leq\beta$ and $|\gamma|<|\beta|$. We next estimate each term.

To estimate A_4 we use the assumption by induction, then we have

$$\begin{aligned}A_4 \leq C\int_{t(1-\varepsilon)}^t &\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}(t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}K_1'\Big(K_2'\{|\gamma|-\eta\}\Big)^{|\gamma|-\eta-\delta}s^{-\frac{|\gamma|-\eta+1}{2}}\\ &\times K_1'\Big(K_2'\{|\beta-\gamma|\Big)^{|\beta-\gamma|-\delta}s^{-\frac{|\beta-\gamma|+1}{2}}ds,\end{aligned}$$

where we denote by $\eta=1-n(\frac{1}{2}-\frac{1}{p})$. Since $\eta\in(0,1)$ we obtain

$$\begin{aligned}&\leq CK_1'^2K_2'^{|\beta|-2\delta}\Big\{\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta}\Big\}\\ &\quad\times\int_{t(1-\varepsilon)}^t(t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}s^{-\frac{|\beta|}{2}-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}ds.\end{aligned}$$

We now apply Lemma 2.5, then for $\delta > 1/2$ there is a constant $\Lambda = \Lambda(\delta)$ such that

$$A_4 \le C_3 \Lambda J(\varepsilon) {K_1'}^2 {K_2'}^{|\beta|-2\delta} |\beta|^{|\beta|-\delta} t^{-\frac{|\beta|+1}{2}},$$

where $C_3 = C_3(n,p)$, and $J(\varepsilon) = \int_{1-\varepsilon}^1 (1-\tau)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \tau^{-\frac{|\beta|}{2}-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} d\tau$. On the other hand, A_5 is easily estimated from the assumption (1.2) by following:

$$A_5 \le C_4 \int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|(-\Delta)^{1/2} \partial_x^\beta u(s)\|_{\dot{H}^{n/2-1}} ds$$

with $C_4 = C_4(n,p,M_2)$. We should note that those constants C_j for $1 \le j \le 4$ are independent of β and γ. Combining these estimates and setting

$$b_\varepsilon = (C_1 + C_2)(2\beta)^{|\beta|/2}(1 + \varepsilon^{-\frac{|\beta|}{2}-\frac{3}{2}}) + C_3 \Lambda J(\varepsilon) {K_1'}^2 {K_2'}^{|\beta|-2\delta} |\beta|^{|\beta|-\delta},$$

then we obtain

$$\begin{aligned} &\|(-\Delta)^{1/2} \partial_x^\beta u(t)\|_{\dot{H}^{n/2-1}} \\ &\quad \le b_\varepsilon t^{-\frac{|\beta|+1}{2}} + C_4 \int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|(-\Delta)^{1/2} \partial_x^\beta u(s)\|_{\dot{H}^{n/2-1}} ds. \end{aligned}$$

We now apply the Gronwall type inequality (Lemma 2.6) to get

$$\|(-\Delta)^{1/2} \partial_x^\beta u(t)\|_{\dot{H}^{n/2-1}} \le 2 b_{1/|\beta|} t^{-\frac{|\beta|+1}{2}}$$

for sufficiently large β. Note that $J(1/|\beta|) \sim \sqrt{e/|\beta|} \le 2$, and $|\beta|^{3/2+\delta} \le 8 \cdot 2^{|\beta|-\delta}$. To get (3.2) we thus take K_1' and K_2' satisfying $2b_{\varepsilon_0} \le K_1'(K_2'|\beta|)^{|\beta|-\delta}$ such as $K_1' = 32(C_1 + C_2)$ and $K_2' \ge 4$ large so that $4 C_3 \Lambda K_1' {K_2'}^{-\delta} < 1/2$. Since K_1' and K_2' are independent of β (3.2) hold for all β.

Since $\|(-\Delta)^{1/2} f\|_2 \le C(n) \|\nabla f\|_2$, we may observe that (3.2) yields (1.3) for $q = 2$, indeed it suffices to take $K_1 = K_1'$ and $K_2 = C(n) K_2'$.

(Step 2). We shall prove (3.2) for $2 < q \le \infty$. By Lemma 2.4 we have

$$\|u(t)\|_{\dot{H}_q^{n/2-1+\alpha}} \le C_5 \|u(t)\|_{\dot{H}^{n/2-1+\alpha+\lambda}}^\theta \|u(t)\|_{\dot{H}^{n/2-1+\alpha+\mu}}^{1-\theta}$$

for $0 < \theta < 1$, $\lambda > \frac{n}{2} - \frac{n}{q}$ and $\mu < \frac{n}{2} - \frac{n}{q}$ satisfying $\theta(\lambda - \frac{n}{2} + \frac{n}{q}) + (1-\theta)(\mu - \frac{n}{2} + \frac{n}{q}) = 0$. Here, $C_5 = C_5(n, \theta, \lambda, \mu, q)$ is a positive constant independent of α. We now take $\theta = \frac{1}{2}$, $\lambda = n - \frac{2n}{q}$ and $\mu = 0$, then we have

$$\begin{aligned} \|u(t)\|_{\dot{H}_q^{n/2-1+\alpha}} &\le C_5 \|u(t)\|_{\dot{H}^{n/2-1+\alpha+n-\frac{2n}{q}}}^{\frac{1}{2}} \|u(t)\|_{\dot{H}^{n/2-1+\alpha}}^{\frac{1}{2}} \\ &\le C_5 \Big[K_1 \Big\{ K_2 \Big(\alpha + n - \frac{2n}{q} \Big) \Big\}^{\alpha+n-\frac{2n}{q}} t^{-\frac{\alpha}{2}-\frac{n}{2}+\frac{n}{q}} \Big]^{\frac{1}{2}} \Big[K_1 (K_2 \alpha)^\alpha t^{-\frac{\alpha}{2}} \Big]^{\frac{1}{2}} \\ &\le C_5 K_1 (2K_2 \tilde{\alpha})^{\tilde{\alpha}} t^{-\tilde{\alpha}/2}. \end{aligned}$$

To get (3.1) for all q we now retake K_1 for $(C_5+1)K_1$, and K_2 for $2K_2$. Therefore, this is now complete the proof of Proposition 3.1. $\square$

Proposition 3.2. *Assume that f is conservative. Let $u_0 \in \dot{H}^{n/2-1}(\mathbf{R}^n)$ with $\nabla \cdot u_0 = 0$. Then there exist $T>0$ and a unique mild solution u satisfying*

$$\|u(t)\|_{\dot{H}_q^{n/2-1+\alpha}} \leq \tilde{K}_1(\tilde{K}_2\tilde{\alpha})^{\tilde{\alpha}/2}t^{-\tilde{\alpha}/2} \tag{3.3}$$

for all $t \in (0,T)$, $\alpha > 0$ and $q \in [2,\infty]$. Here, we denote $\tilde{\alpha} = \alpha + n(1/2-1/q)$, and note that $\tilde{K}_1$ and $\tilde{K}_2$ are positive constants depending only on n, T and $\|u_0\|_{\dot{H}^{n/2-1}}$.

Proof. We recall that the mild solution $u(t)$ is the limit function for successive approximation $\{u_j(t)\}_{j\geq 1}$ by iteration as follows: let $u_1(t) = e^{t\Delta}u_0$, and for all $j \geq 1$

$$u_{j+1}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u_j \otimes u_j)(s)ds.$$

Actually, we could prove that there exists $T_0 > 0$ such that

$$\sup_j \|u_j(t)\|_{\dot{H}_q^{n/2-1}} \leq \tilde{C}_1\|u_0\|_{\dot{H}^{n/2-1}}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}$$

for $q \in [2,\infty]$ and $t \in [0,T_0]$, see [6]. Here, $\tilde{C}_1$ is a positive constant depending only on n and T_0. Note that u_j converges to u in $(0,T)$. Let $\alpha > 0$, and we show (3.3) for the case $q=2$. We define $\psi_j(t) = \|u_j(t)\|_{\dot{H}^{n/2-1+\alpha}}$, and apply the estimates similar to prove Proposition 3.1: then, there are $\tilde{b}_\varepsilon$ (similar to b_ε in Step 1) and $\tilde{C}_2$ (depends only on n, T_0 and $\|u_0\|_{\dot{H}^{n/2-1}}$) such that

$$\psi_{j+1}(t) \leq \tilde{b}_\varepsilon t^{-\alpha/2} + \tilde{C}_2\int_{t(1-\varepsilon)}^t (t-s)^{-1+\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}\psi_j(s)ds$$

for all $j \geq 1$ and $t \in (0,T)$. We now apply Lemma 2.6 to obtain that $\psi_j(t) \leq \tilde{K}_1(\tilde{K}_2\alpha)^\alpha t^{-\alpha/2}$ for all $j \geq 1$ with $\tilde{K}_1$ and $\tilde{K}_2$ suitably. For the case $q \in (2,\infty]$ we appeal to Lemma 2.4 (same as Step 2 in the proof of Proposition 3.1), therefore, the proof is now complete. $\square$

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