On blow up rate for sign-changing solutions in a convex domain

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Full title: On blow up rate for sign-changing solutions in a convex domain

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SUMMARY

This paper studies a growth rate of a solution blowing up at time T of the semilinear heat equation $u_t - \Delta u - |u|^{p-1}u = 0$ in a convex domain D in $\mathbf{R}^n$ with zero-boundary condition. For a subcritical $p \in (1, (n+2)/(n-2))$ a growth rate estimate $|u(x,t)| \leq C(T-t)^{-1/(p-1)}, x \in D, t \in (0,T)$ is established with C independent of t provided that D is uniformly C^2. The estimate applies to sign-changing solutions. The same estimate has been recently established when $D = \mathbf{R}^n$ by authors. The proof is similar but we need to establish $L^h - L^k$ estimate for a time-dependent domain because of the presence of the boundary.

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On blow up rate for sign-changing solutions in a convex domain

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Dedicated to Professor Howard A. Levine on the occasion of his sixtieth birthday

SUMMARY

This paper studies a growth rate of a solution blowing up at time T of the semilinear heat equation $u_t - \Delta u - |u|^{p-1}u = 0$ in a convex domain D in $\mathbf{R}^n$ with zero-boundary condition. For a subcritical $p \in (1, (n+2)/(n-2))$ a growth rate estimate $|u(x,t)| \leq C(T-t)^{-1/(p-1)}, x \in D, t \in (0,T)$ is established with C independent of t provided that D is uniformly C^2. The estimate applies to sign-changing solutions. The same estimate has been recently established when $D = \mathbf{R}^n$ by authors. The proof is similar but we need to establish $L^h - L^k$ estimate for a time-dependent domain because of the presence of the boundary.

KEY WORDS: blowup solution, rescaled equation, semilinear heat equation, blowup rate

1. INTRODUCTION AND MAIN RESULTS

This is a continuation of our work [12] on a growth rate estimate of a blowup solution of a semilinear heat equation. We consider the initial-boundary value problem for the semilinear heat equation of the form

$$u_t - \Delta u - |u|^{p-1}\, u = 0 \quad \text{in} \quad D \times (0,T), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial D \times (0,T), \tag{1.2}$$

$$u|_{t=0} = u_0 \quad \text{at} \quad t = 0, \tag{1.3}$$

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where D is a domain in $\mathbf{R}^n$ and $p > 1$. As well-known this problem admits a local-in-time solution u for a bounded initial data u_0, *i.e.*, $u_0 \in L^\infty(D)$ under suitable assumptions on D; however, the solution u may blow up at some T in the sense that $\limsup_{t\uparrow T} ||u||_\infty(t) = \infty$, where $||u||_\infty(t) = \sup_{x\in D} |u(x,t)|$. Our main concern is a growth rate estimate of a blowup solution of (1.1)-(1.3) when p is subcritical in the sense that $1 < p < p_S(n)$ with the Sobolev critical number

$$p_S(n) = \begin{cases} (n+2)/(n-2) & n \geq 3, \\ \infty & n = 1, 2. \end{cases}$$

By a solution u of (1.1)-(1.3) we mean that $u \in C^{2,1}(D \times (0,T)) \cap C(\bar{D} \times [0,T))$ solves (1.1)-(1.3) and that u is bounded in $D \times (0, T_1)$ for any $T_1 \in (0,T)$ (if D is unbounded). Of course all blowup solution enjoys this property when $u_0 \in L^\infty(D)$. Our main result in this paper is

Theorem 1.1. Assume that $1 < p < p_S(n)$ and that D is a bounded, C^2 convex domain. Then there exists a constant C depending only on n, p, D and a bound for $T^{1/(p-1)}||u_0||_\infty$ such that any solution u of (1.1)-(1.3) fulfills

$$||u||_\infty(t) \leq C(T-t)^{-1/(p-1)} \quad \text{for} \quad t \in (0,T). \tag{1.4}$$

In these almost two decades an asymptotic behavior of a blowup solution near blowup point has been well studied. Here we only list some crucial earlier works by Giga and Kohn [9],[10],[11], Filippas and Kohn [5], Herrero and Velázquez [14],[15],[16] Merle and Zaag [21],[22]. As already pointed out by Giga and Kohn [9] the growth estimate like (1.4) is crucial in studying the asymptotic behavior. The bound (1.4) was first proved by Weissler [24] for a radial solution on a ball with rather restrictive initial data. The result was extended by Friedman and McLeod [6] when D is a smoothly bounded, convex domain with initial data satisfying $u_0 \geq 0$, $\Delta u_0 + u_0^p \geq 0$ but for all p including $p \geq p_S(n)$. Their assumption on initial data guarantees that $u \geq 0$ and $u_t \geq 0$ for all $t < T$ so that it excludes all possible oscillation of u in time. For a general initial data including sign-changing data the bound (1.4) was proved by Giga and Kohn [10] under more restrictive assumption of the form $p < (3n+8)/(3n-4)$ for $n \geq 2$ than $p < p_S(n)$. (They also proved (1.4) for all $p \in (1, p_S(n))$ under the assumption $u_0 \geq 0$ so that the solution is positive.) Theorem 1.1 is interpreted as an improvement of their result since we impose no assumption on sign of u_0 and no further restriction on p other than $p < p_S(n)$. Moreover, dependence of C on T and $||u_0||_\infty$ is explicit compared with earlier results. Similar result has been proved by the authors [12] for $D = \mathbf{R}^n$. The restriction $p < p_S(n)$ is optimal by the results of Filippas, Herrero and Velázquez [4] for $p = p_S(n)$ and Herrero and Velázquez [17] for a very large p. For a general domain not necessarily convex, (1.4) is proved by Fila and Souplet [3] only for a positive solution and for $p \in (1, 1 + 2/(n+1))$. For a

Neumann problem the bound (1.4) has been recently proved by Ishige and Mizoguchi [18] for a positive solution and for $p \in (1, 1+2/n)$. In these two results dependence of C on T and $||u_0||_\infty$ is also implicit. By the way the power $-1/(p-1)$ in (1.4) is optimal since we always has the converse inequality with different C; see Giga [8].

Remark 1.2. The boundedness of D in Theorem 1.1 is unnecessary if D is uniformly C^2 in the sense that ∂D has positive reach (Krantz and Parks [19]) and that all principal curvatures are bounded on ∂D. In particular (1.4) is still valid when D is a half space. (If D is a bounded C^2 domain, then ∂D always has positive reach (Krantz and Parks [19, Theorem 4.4.10]). By the way ∂D has always positive reach if D is a C^2 convex domain and all principal curvatures are bounded on ∂D. See Appendix for the proof.

The basic strategy of the proof of Theorem 1.1 is similar to that in [12] where $D = \mathbf{R}^n$ is studied. We convert the problem to establish a uniform bound for a global solution of the rescaled equation (renormalized equation)

$$\rho w_s - \nabla \cdot (\rho \nabla w) + \beta \rho w - \rho |w|^{p-1} w = 0 \quad \text{in} \quad W_a \tag{1.5}$$

with

$$\begin{aligned} W_a &= \{(y,s); s > s_0 := -\log T, e^{-s/2}y + a \in D\} = \underset{s>s_0}{\cup} \Omega_a(s) \times \{s\}, \\ \rho(y) &= \exp(-|y|^2/4),\ \beta = 1/(p-1),\ a \in D,\ \Omega_a(s) = \{y; e^{-s/2}y + a \in D\}. \end{aligned}$$

This equation is obtained from (1.1) by rescaling the variables x, t, u by y, s, w and

$$w = w^a(y,s) = (T-t)^\beta u(a + y\sqrt{T-t}, t).$$

As in [12], Remark 2.1 we may assume $T = 1$ in Theorem 1.1 by scaling so we may assume $s_0 = 0$. By a regularizing effect (cf. [12, §3.3]) we may assume that $u, \nabla u, \nabla^2 u$ and u_t are bounded and continuous on $\bar{D} \times [0, T_1]$ for each $T_1 < T$. The corresponding w now satisfies

$$\left\{ \begin{array}{lll} w, \nabla w, \nabla^2 w & \text{and} \quad (1+|y|)^{-1} w_s & \text{are bounded and} \\ \text{continuous} & \text{on} \quad \cup_{S \ge s \ge 0} \bar{\Omega}_a(s) \times \{s\} & \text{for each } S < \infty. \end{array} \right. \tag{1.6}$$

As in [12, §4] Theorem 1.1 follows from the following uniform bound for w (independent of a).

Theorem 1.3. (Uniform bound) Assume that $1 < p < p_S(n)$ and that D is uniformly C^2 convex domain. Then there exists a constant $R_0 = R_0(n, p, D) > 0$ such that the estimate

$$||w||_{L^\infty(B_{R_0} \cap \Omega_a(s))} \le C \quad \text{for} \quad s \ge 0 \tag{1.7}$$

holds for all w satisfying (1.5) (with $s_0 = 0$), (1.6) and the boundary condition

$$w = 0 \quad \text{on} \quad \underset{s>0}{\cup} \partial \Omega_a(s) \times \{s\} \tag{1.8}$$

with some constant C depending only on n, p, D and a bound for $||w_0||_\infty$, where w_0 is the initial value of w. The constant C is independent of $a \in D$. (Here B_R denotes the open ball of radius R centered at the origin of $\mathbf{R}^n$.)

Indeed, (1.8) implies that $|u(a,t)| \le C_R(1-t)^{-1/(p-1)}$ by fixing $R < R_0$ (independent of a). Since a is arbitrary, this implies (1.4) with $T = 1$. For a whole space problem we are able to choose R_0 arbitrary [12].

This uniform bound follows from the following integral bound.

Theorem 1.4. (Key integral estimate) Assume that $1 < p < p_S(n)$ and that D is uniformly C^2 convex domain. For each $q \ge 2$ there exists a constant $R_1 = R_1(n,p,q,D) > 0$ such that the estimate

$$(A_{q,R_1}) \quad \sup_{s\ge 0} \int_s^{s+1} ||w(\cdot,\tau); L^{p+1}(\Omega_a^{R_1}(\tau))||^{(p+1)q} d\tau \le \hat{C}_q, \quad a \in D$$

holds for all w satisfying (1.5),(1.6) and (1.8) with some constant $\hat{C}_q$ depending only on n, p, D and a bound for $E[w](0)$ and for $||w_0||_{BC^2(D)}$, where $\Omega_a^{R_1}(\tau) = \Omega_a(\tau) \cap B_{R_1}$.

Here $BC^m(D)$ denotes the Banach space of all C^m functions on D such that all derivatives up to m-th order is bounded in D. The quantity $E[w](\tau)$ denotes the energy of the equation (1.5) which is defined by

$$E[w](\tau) = \frac{1}{2}\int_{\Omega(\tau)} (|\nabla w|^2 + \beta |w|^2)\rho dy - \frac{1}{p+1}\int_{\Omega(\tau)} |w|^{p+1}\rho dy, \tag{1.9}$$

where $w = w(\cdot,\tau)$.

The way to derive uniform bound from integral estimate is essentially known by Giga and Kohn [10]. Actually, they proved only $(A_{2,R})$ which yields a uniform bound (1.7) but they are forced to assume that $1 < p < (3n+8)/(3n-4)$. To derive (1.7) we apply an interpolation theorem due to Cazenave and Lions [2] and an interior and boundary regularity theorem for linear parabolic equations found in Ladyzhenskaya, Solonnikov and Ural'ceva [20] as discussed by Giga and Kohn [10]. We also need some regularizing effect to complete the proof of Theorem 1.3 as discussed by the authors [12, §4]. Estimate $(A_{q,R})$ for arbitrary $q \ge 2$ enables us to prove (1.7) for all $p \in (1, p_S(n))$.

The integral estimate $(A_{2,R})$ (for all $R > 0$) is obtained by integral identities involving the energy (1.9) as proved by Giga and Kohn [10, Propositions 2.1, 2.2]. For example we have

$$\int_{\Omega_a(s)} |w_s|^2 \rho dy = -\frac{d}{ds}E[w](s) - \frac{1}{4}\int_{\partial\Omega_a(s)} (y\cdot\nu)\left|\frac{\partial w}{\partial \nu}\right|^2 \rho d\sigma,$$

where ν is the exterior unit normal vector to $\partial\Omega(s)$ and $d\sigma$ is surface area element. If D is convex, then evidently the last term is negative, so we have

$$\int_{\Omega_a(s)} |w_s|^2 \rho dy \le -\frac{d}{ds}E[w](s).$$

This in particular implies that $E[w](s)$ is decreasing in s. Using another identity obtained by multiplying w with (1.5), we also observe that $E[w](s) \geq 0$ if w is a global solution of (1.5) and (1.8). Thus under the assumption of convexity of D we have

$$0 \leq E[w](s) \leq E[w](0) \quad \text{for} \quad s \geq 0. \tag{1.10}$$

To obtain integral estimate $(A_{q,R})$ for $q > 2$ it is convenient to introduce a localized energy of the form

$$\mathcal{E}_\varphi[w](s) = \frac{1}{2} \int_{\Omega_a(s)} \varphi^2 (|\nabla w|^2 + \beta |w|^2) \rho dy - \frac{1}{p+1} \int_{\Omega_a(s)} \varphi^2 |w|^{p+1} \rho dy,$$

where φ is a cut-off function of a ball. Of course $\mathcal{E}_\varphi[w] = E[w]$ if $\varphi \equiv 1$. Such a localized energy was first introduced by the authors [12]. An argument similar to that in [12] yields bounds for $\mathcal{E}_\varphi[w]$, which is weaker than (1.10) for global energy $E[w]$.

Lemma 1.5. (Bounds for a localized energy) Assume that $p > 1$ and that D is a convex domain. Then there exist positive constants L_1 and L_2 depending only on $n, p, M_j (j = 0, 1, 2)$ and a bound for $E[w](0)$ and for $||w_0||_\infty$ such that

$$-L_2 \leq \mathcal{E}_\varphi[w](s) \leq L_1 \quad \text{for all} \quad s \geq 0 \tag{1.11}$$

for all w satisfying (1.5),(1.6) and (1.8) provided that $\varphi \in BC^2(\mathbf{R}^n)$ satisfies $||\varphi||_\infty \leq M_0$, $||\nabla \varphi||_\infty \leq M_1$, $||\Delta \varphi||_\infty \leq M_2$.

We note that the convexity of D is substantially invoked here to prove (1.10) and also (1.11).

For a given ball B_R we fix a cut-off function φ of B_R defined by

$$\varphi(x) = \phi(|x|/R), \quad x \in \mathbf{R}^n$$

with $\phi \in C^\infty[0, \infty)$ satisfying $\phi(\eta) = 1$ for $\eta \leq 1$, $\phi(\eta) = 0$ for $\eta \geq 2$ and $0 \leq \phi \leq 1$ on $[0, \infty)$; we shall fix such a φ depending on R. As in [12] thanks to a lower bound (1.11) for $\mathcal{E}_\varphi$ we observe that the following bound in an L^2- Sobolev space $W^{1,2}$ is sufficient to prove Theorem 1.5.

Theorem 1.6. (Key $W^{1,2}$- bound) Assume that $1 < p < p_S(n)$ and that D is uniformly C^2 convex domain. For each $q \geq 2$ there exists a constant $R_1 = R_1(n, p, q, D) > 0$ such that the estimate

$$(B_{q,R_1}) \quad \sup_{s \geq 0} \int_s^{s+1} ||w(\cdot, \tau); W^{1,2}(\Omega_a^{R_1}(\tau))||^{2q} d\tau \leq C_q, \quad a \in D$$

holds for all w satisfying (1.5), (1.6) and (1.8) with some constant C_q depending only on 8, n, p, D and a bound for $E[w](0)$ and for $||w_0||_{BC^2(D)}$.

It remains to prove $W^{1,2}$- bound. The basic strategy is a bootstrap argument which is basically similar to the case $D = \mathbf{R}^n$ discussed in [12, §6]. The estimate $(B_{2,R})$ for all $R > 0$ is easily proved by global theory developed by Giga and Kohn [10]; see [12, Proposition 6.1.]. Assume that $(B_{q,2R})$ holds. Then by an upper bound (1.11) for $\mathcal{E}_\varphi$ we have

$$||w(\cdot,\tau)\cdot, W^{1,2}(\Omega_a^R(\tau))||^2 \le L_3(1 + ||\varphi w \varphi w_s(\cdot,\tau); L^1(\Omega_a^{2R}(\tau))||)$$

with a constant $L_3 = L_3(n,p,R,M_1,M_2)$ as in [12, Proposition 6.3.]. We estimate φw-part by $(A_{q,2R})$ and the interpolation theorem due to Cazenave and Lions [2] and φw_s -part by $L^h - L^k$ estimate for a linear parabolic equation for which we should be careful because of the presence of the boundary. This $L^h - L^k$ estimate (Theorem 2.1) is a main technical contribution of this paper. Upper and lower bounds (1.11) for $\mathcal{E}_\varphi$ enable us to prove $(B_{\tilde{q},R})$ for all $\tilde{q} \in (q, q+2/(p+1))$. We repeat this procedure infinitely many times to obtain $(B_{q,R})$ for arbitrary $q \ge 2$ with some R and complete the proof of Theorem 1.6. A similar bootstrap argument has been developed by Quittner [23], where he established a global bound for a sign-changing global solution of (1.1)-(1.2) for any smoothly bounded domain. In that case the problem is global in space so there is no need to localize the energy.

In the next section we shall discuss $L^h - L^k$ estimate (Theorem 2.1) for a heat equation in a time-dependent domain which is essential to estimate φw_s- part.

2. HEAT EQUATION IN A TIME DEPENDENT DOMAIN

Our goal in this section is to prove $L^h - L^k$ estimate for the heat equation in W_a. The convexity of the domain is unnecessary. To simplify the notation we set

$$Q_{as_1s_2} = W_a \cap \{s_1 \le s < s_2\} = \{(y,s) \in W_a; s_1 \le s < s_2\}$$

and

$$S_{as_1s_2} = \bigcup_{s_1 \le s < s_2} \partial\Omega_a(s) \times \{s\}.$$

For f defined in $Q_{as_1s_2}$ we define for $\tau \in [s_1, s_2)$

$$||f||_k(\tau) = (\int_{\Omega_a(\tau)} |f|^k dy)^{1/k}, \ ||f||_{BC^2}(\tau) = \max_{0 \le k \le 2} \sup_{y \in \Omega_a(\tau)} |\nabla^k f(y,\tau)|.$$

Theorem 2.1. Assume that $1 < h,\ k < \infty$ and $\Lambda > 0$ and that D is a uniformly C^2 domain in $\mathbf{R}^n$. Then there exists constants $R_*(h,k,n,D) > 0$ and $C_* = C_*(h,k,n,D,\Lambda)$ such that the estimate

$$\int_{s_1}^{s_2} \{||w_s||_k^h(s) + ||\nabla^2 w||_k^h(s)\} ds \le C_*(\int_{s_1}^{s_2} ||f||_k^h(s) ds + ||w||_{BC^2}^h(s_1)) \tag{2.1}$$

holds for all $w \in C^{2,1}(Q_{as_1s_2}) \cap C(\bar{Q}_{as_1s_2})$ satisfying

$$w_s - \Delta w = f \quad \text{in} \quad Q_{as_1s_2}, \tag{2.2}$$

$$w = 0 \quad \text{on} \quad S_{as_1s_2}, \tag{2.3}$$

$$\text{supp } w \subset B_{R_*} \times [s_1, s_2) \cap Q_{as_1s_2} \tag{2.4}$$

with $s_2 > s_1 \geq 0,\ s_2 - s_1 \leq \Lambda$. and for all $a \in D$. (In particuler constants R_* and C_* are independent of $a \in D$.)

To prove Theorem 2.1 one way would be to use $L^h - L^k$ estimate of $u_t + A(t)u = f$, where $A(t)$ is a time-dependent elliptic operator by changing $\Omega_a(s)$ to D. Although such an estimate is established in an abstract level by Giga, Giga and Sohr [7] and Yamamoto [25], our main purpose is to estimate w_s not u_t for a fixed domain which is not comparable. So we shall prove (2.1) be localizing the problem and by reducing it to a half space $L^h - L^k$ estimate.

For this purpose we recall an $L^h - L^k$ estimate for the heat equation in the whole space $\mathbf{R}^n$ and the half space $\mathbf{R}^n_+ = \{(x', x_n); x_n > 0\}$; see e.g. book of Amann [1] or a paper by Giga and Sohr [13].

Lemma 2.2. Assume that $\Omega = \mathbf{R}^n$ or $\mathbf{R}^n_+$ and that $1 < h,\ k < \infty$ and $R > 0$. Then there exist constants $C_1 = C_1(h, k, n)$ and $C_2 = C_2(h, k, n, R)$ such that

$$\int_{t_1}^{t_2} \{||v_t||_k^h(t) + ||\nabla^2 v||_k^h(t)\} dt \leq C_1 \int_{t_1}^{t_2} ||v_t - \Delta v||_k^h(t) dt + C_2 ||v||_{BC^2}^h(t_1) \tag{2.5}$$

for all $v \in C^{2,1}(\Omega \times [t_1, t_2)) \cap C(\bar{\Omega} \times [t_1, t_2))$ satisfying $v = 0$ on $\partial\Omega \times [t_1, t_2)$(if $\partial\Omega \neq \emptyset$) with supp $v(t_1) \subset B_R$.

In the literature the last term is often replaced by some Besov norm of $v(t_1)$ which is dominated by $BC^2(\Omega)$ norm by restricting support of $v(t_1)$. The second term $||\nabla^2 v||_h^k$ is often replaced by a weaker norm $||\Delta v||_k^h$ but it is equivalent for $\Omega = \mathbf{R}^n$ or $\mathbf{R}^n_+$ with the Dirichlet condition by the Calderón - Zygmund inequality.

We shall first establish an $L^h - L^k$ estimate for the heat equation when D is close to a half space. For $\xi \in BC^2(\mathbf{R}^{n-1})$ with $\xi(0) < 0$ let D be the domain of the form

$$D = \{(y', y_n) \in \mathbf{R}^n; y_n > \xi(y')\}.$$

Evidently D contains the origin and $\Omega_0(s) \uparrow \mathbf{R}^n$ as $s \to \infty$, where

$$\Omega_0(s) = e^{s/2} D = \{e^{s/2} y; y \in D\}.$$

We simply write $Q_{s_1s_2}$ instead of $Q_{0s_1s_2}$.

By the definition we observe that

$$Q_{s_1s_2} = \{(y', y_n, s) \in \mathbf{R}^n \times \mathbf{R};\ y_n > \xi^s(y'),\ s_1 \leq s < s_2\}$$

with $\xi^s(y') := e^{s/2}\xi(e^{-s/2}y')$. Since $\xi(0) < 0$, the minimum time for $B_R \subset \Omega_0(s)$ defined by

$$s_*(R,\xi) = \inf\{s \geq 0; B_R \subset e^{s/2}D\} \tag{2.6}$$

is finite. If

$$\sup_{\mathbf{R}^{n-1}} |\nabla\xi| \leq 1, \quad \text{so that} \quad \sup_{\mathbf{R}^{n-1}} |\nabla\xi^s| \leq 1,$$

we observe, by definition of s_* and geometry, that

$$\sup_{|y'|<R} |\xi^s(y')| \leq (\sqrt{2}+1)R \quad \text{for} \quad s \in [0, s_*]. \tag{2.7}$$

Thus

$$\sup_{|y'|<R} |\xi^s(y')| \leq e^{(s-s_*)/2}(\sqrt{2}+1)R \quad \text{for all} \quad s \geq s_*. \tag{2.8}$$

Lemma 2.3. Assume that $1 < h,\ k < \infty,\ 0 < \Lambda,\ N,\ R < \infty$. Then there exist constants $\delta = \delta(h,k,n) \in (0,1),\ C_3 = C_3(h,k,n),\ C_4 = C_4(N,\Lambda,R,h,k,n)$ and $C_5 = C_5(h,k,n,R)$ such that

$$\int_{s_1}^{s_2} \{||w_s||_k^h(s) + ||\nabla^2 w||_k^h(s)\}ds \leq C_3 \int_{s_1}^{s_2} ||f||_h^k(s)ds + C_4 \int_{s_1}^{s_2} ||w||_k^h(s)ds + C_5||w||_{BC^2}^h(s_1) \tag{2.9}$$

for all $w \in C^{2,1}(Q_{s_1 s_2}) \cap C(\bar{Q}_{s_1 s_2})$ satisfying (2.2)-(2.4) (with $R_* = R$) provided that $s_2 - s_1 \leq \Lambda,\ 0 \leq s_1 \leq s_2,\ \sup_{\mathbf{R}^{n-1}} |\nabla\xi| \leq \delta,\ \sup_{\mathbf{R}^{n-1}} |\nabla^2\xi| \leq N$.

Proof. If $s_1 \geq s_*$ where s_* is defind in (2.6), we apply $L^h - L^k$ estimate (2.5) for $\mathbf{R}^n$ (Lemma 2.2) to get (2.9) with $C_4 = 0$ without any restriction on δ. So we may assume that $s_1 < s_*$.

We introduce a time-dependent change of coordinates to make $S_{s_1 s_2}$ flat and time-independent: let $v(x,t) = w(y,s)$ with

$$x_n = y_n - \xi^s(y'), \quad x' = y', \quad t = s.$$

Since $\nabla_{y'}\xi^s(y') = (\nabla\xi)(e^{-s/2}y')$, the Jacobian J of the mapping $(y', y_n) \mapsto (x', x_n)$ fulfills $|J-1| \leq 1/2$ if $\kappa = \sup_{\mathbf{R}^{n-1}} |\nabla\xi|$ is small, say $\kappa \leq \delta_0 < 1$, where $\delta_0 = \delta_0(n)$. We shall assume that $\kappa \leq \delta_0$. Then the norms $||v||_k(t)$ and $||w||_k(s)$ is comparable in the sense that there is a constant $\lambda = \lambda(k,n)$

$$\lambda^{-1}||w||_k(s) \leq ||v||_k(t) \leq \lambda||w||_k(s). \tag{2.10}$$

Since w solves (2.2) and (2.3), v solves a linear parabolic equation in $\mathbf{R}_+^n \times (s_1, s_2)$ with $v = 0$ on the boundary. The parabolic equation v solves is easily calculated. We note that

$$\begin{aligned} \frac{\partial}{\partial s} &= \frac{\partial}{\partial t} + \frac{\partial x_n}{\partial s}\frac{\partial}{\partial x_n}, \\ \nabla_{y'} &= \frac{\partial}{\partial x'} + \nabla_{y'}x_n\frac{\partial}{\partial x_n}, \quad \frac{\partial}{\partial y_n} = \frac{\partial}{\partial x_n} \end{aligned}$$

with
$$\frac{\partial x_n}{\partial s} = -\frac{1}{2}\xi^s(y') + \frac{1}{2}y' \cdot \nabla\xi(e^{-s/2}y'), \quad \nabla_{y'} x_n = -\nabla\xi(e^{-s/2}y').$$
By (2.7) and (2.8) we see that
$$\left|\frac{\partial x_n}{\partial s}\right| \leq C' = C'(\Lambda, R) \quad \text{on} \quad B_R \times [s_1, s_2] \cap Q_{s_1,s_2} \tag{2.11}$$
of $s_2 - s_1 \leq \Lambda$. Thus (2.2) is transformed into
$$v_t = \Delta v + p(x,t)\frac{\partial^2}{\partial x_n^2}v + \sum_{j=1}^{n-1} 2p_j(x,t)\frac{\partial^2}{\partial x_j \partial x_n}v + g(x,t)\frac{\partial}{\partial x_n}v + \tilde{f} \quad \text{in} \quad \mathbf{R}^n_+ \times (s_1, s_2)$$
with $|p(x,t)| \leq \kappa^2$, $|p_j(x,t)| \leq \kappa$ $(j = 1, \ldots, n-1)$, $|g(x,t)| \leq C' + N$, where N is a bound for $|\nabla^2\xi|$ in $\mathbf{R}^{n-1}$ and $\tilde{f}(x,t) = f(y,s)$. We now apply (2.5) with $\Omega = \mathbf{R}^n_+$ and obtain that
$$\int_{t_1}^{t_2} \{||v_t||_k^h + ||\nabla^2 v||_k^h\}dt \leq C_1 \left(\int_{t_1}^{t_2} (\kappa^h ||\nabla^2 v||_k^h + C''||\nabla v||_k^h + ||\tilde{f}||_k^h)dt\right) + C_2 ||v||_{BC^2}^h(t_1)$$
with $C'' = (C' + N)^h$. We take $\delta \in (0, \delta_0)$ small so that $C_1\delta^h < 1/2$ to absorb the first term in RHS to LHS if $\kappa < \delta$. By (2.10) and (2.11) we see that
$$||w_s||_k(s) \leq \lambda(||v_t||_k(t) + C'||\nabla v||_k(t)), \ ||\nabla^2 w||_k \leq \lambda(1+\kappa)||\nabla^2 v||_k + \lambda N||\nabla v||_k.$$
These observations yields (2.1) if we use an interpolation
$$||\nabla v||_k \leq \varepsilon||\nabla^2 v||_k + C_\varepsilon||v||_k$$
for small $\varepsilon > 0$. □

Proof of Theorem 2.1. Since the principal curvature of ∂D is bounded on ∂D , for a given $\delta > 0$ there exists a small number $r_0 > 0$ such that for any $x_0 \in \partial D$ the hypersurface ∂D in $B_{4r_0}(x_0)$ is represented as the graph $x_n = \xi(x')$ up to rotation with some $\xi \in C^2(\overline{B_{4r_0}(x_0')})$ satisfying
$$\nabla\xi(x_0') = 0 \quad \text{and} \quad \sup_{|x'-x_0'| \leq 4r_0} |\nabla\xi(x')| \leq \delta/2.$$
Here $B_R(x_0)$ is an open ball centered at $x_0 \in \mathbf{R}^n$ and $x_0 = (x_0', x_{0n})$. Note that $r_0 > 0$ can be taken independent of x_0. We may assume that D is located in $x_n > \xi(x')$.

Let r_1 be the reach of ∂D. We may assume that $4r_0 \leq r_1$ by taking $r_0 = r_0(\delta)$ smaller so that for any $x \in B_{4r_0}(x_0)$ there is a unique $z \in \partial D$ such that $\mathrm{dist}(x,z) = \mathrm{dist}(x, \partial D)$.

We now take $\delta = \delta(h,k,n)$ as in Lemma 2.3 and fix $r_0 = r_0(\delta, D)$. We divide D into two parts
$$D_1 = \{a \in D; \mathrm{dist}(a, \partial D) \geq 2r_0\}, \quad D_2 = \{a \in D; \mathrm{dist}(a, \partial D) < 2r_0\}$$

and take $R_* = r_0$. If a is not close to ∂D in the sense that $a \in D_1$, then $\overline{B_{R_*}(a)} \times [s_1, s_2] \subset Q_{as_1s_2}$ so one is able to apply $L^h - L^k$ estimate (2.5) for $\mathbf{R}^n$ (Lemma 2.2) to get (2.1). For $a \in D_2$ there is a unique $x_0 \in \partial D$ such that $\eta = \mathrm{dist}(a, x_0) = \mathrm{dist}(a, \partial D)$. By rotation and translation we may assume that $a = 0$ and $x_0 = (0, -\eta)$. We take ξ as above and represent $D \cap B_{4r_0}(x_0)$ by $x_n > \xi(x')$, $x' \in B_{4r_0}(0)$. We extend ξ outside $\bar{B}_{4r_0} \subset \mathbf{R}^{n-1}$ such that

$$\sup_{x' \in \mathbf{R}^{n-1}} |\nabla \xi(x')| \leq \delta.$$

Since $B_{R_*} \times [s_1, s_2] \cap S_{as_1s_1} \subset B_{4r_0}(x_0) \times [s_1, s_2] \cap S_{as_1s_2}$, we are able to apply $L^h - L^k$ estimate in Lemma 2.3. Applying a Poincaré type inequality

$$\int_{s_1}^{s_2} ||w||_k^h \mathrm{ds} \leq C_0 \left(\int_{s_1}^{s_2} ||w_s||_k^h \mathrm{ds} + ||w||_k^h(s_1) \right)$$

to (2.9) yields (2.1). □

Using Theorem 2.1 we are able to estimate $w_s\varphi$ term as in [12, Lemma 6.4.]. The only difference is that R should be small so that $2R \leq R_* = R_*(p_1, \theta\tilde{q}\alpha', n, D)$. We also needs a boundary version of a regularing effect [12, Lemma 6.6.] for the proof; see also Giga and Kohn [10, Proposition 3.4.].

APPENDIX

Lemma A. Let D be a C^2 convex domain in $\mathbf{R}^n$ with $n \geq 2$. Assume that all principal curvatures of ∂D is bounded. Then ∂D has positive reach.

Proof. Let $\kappa > 0$ be a bound of all principle curvatures of ∂D. We shall assert that for each point $x_0 \in \partial D$ there is a ball B of radius $1/\kappa$ contained in D such that $\partial B \cap \partial D = \{x_0\}$. This is an interior ball condition which guarantees the positive reach of ∂D. For $n = 2$ such a property is easily proved since ∂D is a convex curve. For $n \geq 3$ it can be proved by induction on n. If there is $x_0 \in \partial D$, $y_0 \in \partial D$ such that B is tangent to ∂D at x_0 and that $y_0 \in B$ although B is contained in D near x_0 by the rotation of the curvature. Let H be a hyperplane containing x_0 and y_0. Then $D \cap H$ does not satisfy the interior ball condition (of radius $1/\kappa$) which contradicts the induction assumption for $n - 1$. □

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